O N E

FUNCTIONS, LIMITS, AND CONTINUITY

1.1 FUNCTIONS AND THEIR GRAPHS

Real Number can be expressed as a decimal.
A *rational number* is a repeating or terminating decimal.

Inequalities $a < b$ (a is less than b) if $b - a$ is positive;
$a > b$ (a is greater than b) if $a - b$ is positive.
$a \le b$ (a is less than or equal to b) if $a < b$ or $a = b$; $a \ge b$ if $a > b$ or $a = b$.

Intervals

$$\begin{array}{lll} (a,b) = \{x \mid a < x < b\} & [a,b] = \{x \mid a \le x \le b\} & (a,b] = \{x \mid a < x \le b\} \\ [a,b) = \{x \mid a \le x < b\} & (a,+\infty) = \{x \mid x > a\} & (-\infty,b) = \{x \mid x < b\} \\ [a,+\infty) = \{x \mid x \ge a\} & (-\infty,b] = \{x \mid x \le b\} & (-\infty,+\infty) = \mathrm{R} \end{array}$$

Absolute Value $|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}, \ |a-b| = \begin{cases} a-b & \text{if } a \ge b \\ b-a & \text{if } a < b \end{cases}.$

Sign, Step Function $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}. \ \mathrm{U}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

Triangle Inequality $|a+b| \le |a| + |b|$

Graphing *Sketch* mean by hand; *plot* mean using a graphics calculator. Our plots label the axes.

1.1.1 Definition A *function* is a set of ordered pairs of numbers (x, y) in which no two distinct ordered pairs have the same first number. The set of all admissible values of x is called the *domain* of the function and the set of resulting values of y is called the *range* of the function. The numbers x and y are *variables.* A function like $|x|$ is said to be defined *in pieces.*

1.1.2 Definition If f is a function, then the *graph* of f is the set of all points (x, y) in R^2 for which (x, y) is an ordered pair in f, that is, the graph of the equation $y = f(x)$.

A function determines a correspondence between the domain and the range. For each number x in the domain there corresponds one and only one number y in the range. When addition, subtraction, and multiplication are the only operations required to calculate y, as in a polynomial function, the domain is the set of all real numbers. If division is required to calculate y, then the domain does not contain any replacement for x that results in division by zero. Thus, for a *rational function,* which is the quotient of two polynomials, the domain is the set of all real numbers, except for those replacements of x that result in a value of zero for the denominator of the fraction. Because the range of a function is a set of *real* numbers, and the square root of a negative number is not real, if the formula that defines a function contains the square root sign, then x must satisfy the inequality obtained by making the expression under the radical sign greater than or equal to zero.

Greatest Integer $\llbracket x \rrbracket = n$ if $n \le x < n+1$ where n is an integer. Note: $-\llbracket -n \rrbracket = n + 1$ if $n < x \le n + 1$.

3. (a) $\{(x,y) \mid y = x^2\}$ is a function; the domain is $(-\infty,+\infty)$
(b) $\{(x,y) \mid x = y^2\}$ is not a function; it contains $(4,2)$ and $(4,-2)$
(c) $\{(x,y) \mid y = x^3\}$ is a function; the domain is $(-\infty,+\infty)$
(d) $\{(x,y) \mid x = y^3\}$ is a function; the domain is $(-\infty,+\infty)$

4. (a) $\{(x,y) \mid y = (x-1)^2+2\}$ is a function; the domain is $(-\infty,+\infty)$
(b) $\{(x,y) \mid x = (y+1)^2-2\}$ is not a function; it contains $(2,1)$ and $(2,-3)$
(c) $\{(x,y) \mid y = (x+2)^3-1\}$ is a function; the domain is $(-\infty,+\infty)$
(d) $\{(x,y) \mid x = (y+1)^3-2\}$ is a function; the domain is $(-\infty,+\infty)$

5. $f(x) = 2x-1$
(a) $f(3) = 2(3)-1 = 5$
(b) $f(-2) = 2(-2)-1 = -5$
(c) $f(0) = 2(0)-1 = -1$
(d) $f(a+1) = 2(a+1)-1 = 2a+1$
(e) $f(x+1) = 2(x+1)-1 = 2x+1$
(f) $f(2x) = 2(2x)-1 = 4x-1$
(g) $2f(x) = 2(2x-1) = 4x-2$
(h) $f(x+h) = 2(x+h)-1 = 2x+2h-1$
(i) $f(x)+f(h) = 2x-1+2h-1 = 2x+2h-2$
(j) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(2x+2h-1)-(2x-1)}{h} = \dfrac{2h}{h} = 2$

6. $f(x) = \frac{3}{x}$
(a) $f(1) = \frac{3}{1} = 3$
(b) $f(-3) = \dfrac{3}{-3} = -1$
(c) $f(6) = \frac{3}{6} = \frac{1}{2}$
(d) $f(\frac{1}{3}) = \dfrac{3}{\frac{1}{3}} = 9$
(e) $f(3/a) = \dfrac{3}{3/a} = a$ if $a \neq 0$
(f) $f(3/x) = \dfrac{3}{3/x} = x$ if $x \neq 0$
(g) $\dfrac{f(3)}{f(x)} = \dfrac{3/3}{3/x} = \frac{x}{3}$ if $x \neq 0$
(h) $f(x-3) = \dfrac{3}{x-3}$
(i) $f(x)-f(3) = \frac{3}{x} - \frac{3}{3} = \frac{3}{x} - 1$
(j) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3/(x+h)-3/x}{h} = \dfrac{3[x-(x+h)]}{hx(x+h)} = \dfrac{-3}{x(x+h)}$

7. $f(x) = 2x^2+5x-3$
(a) $f(-2) = 2(-2)^2+5(-2)-3 = -5$
(b) $f(-1) = 2(-1)^2+5(-1)-3 = -6$
(c) $f(0) = -3$
(d) $f(3) = 2(3)^2+5(3)-3 = 30$
(e) $f(h+1) = 2(h+1)^2+5(h+1)-3 = 2h^2+9h+4$
(f) $f(2x^2) = 2(2x^2)^2+5(2x^2)-3 = 8x^4+10x^2-3$
(g) $f(x^2-3) = 2(x^2-3)^2+5(x^2-3)-3 = 2x^4-7x^2$
(h) $f(x+h) = 2(x+h)^2+5(x+h)-3 = 2x^2+4xh+2h^2+5x+5h-3$
(i) $f(x)+f(h) = 2x^2+5x-3+2h^2+5h-3 = 2x^2+2h^2+5x+5h-6$
(j) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(2x^2+4xh+2h^2+5x+5h-3)-(2x^2+5x-3)}{h} = \dfrac{4xh+2h^2+5h}{h} = 4x+2h+5$

8. $g(x) = 3x^2-4$
(a) $g(-4) = 3(-4)^2-4 = 44$
(b) $g(\frac{1}{2}) = 3(\frac{1}{2})^2-4 = -\frac{13}{4}$
(c) $g(x^2) = 3(x^2)^2-4 = 3x^4-4$
(d) $g(3x^2-4) = 3(3x^2-4)^2-4 = 3(9x^4-24x^2+16)-4 = 27x^4-72x^2+44$
(e) $g(x-h) = 3(x-h)^2-4 = 3(x^2-2xh+h^2)-4 = 3x^2-6xh+3h^2-4$
(f) $g(x)-g(h) = (3x^2-4)-(3h^2-4) = 3x^2-3h^2$
(g) $\dfrac{g(x+h)-g(x)}{h} = \dfrac{[3(x+h)^2-4]-(3x^2-4)}{h} = \dfrac{3(x^2+2xh+h^2)-3x^2}{h} = \dfrac{6xh+3h^2}{h} = 6x+3h$

9. $\mathrm{F}(x) = \sqrt{x+9}$
(a) $\mathrm{F}(x+9) = \sqrt{(x+9)+9} = \sqrt{x+18}$
(b) $\mathrm{F}(x^2-9) = \sqrt{(x^2-9)+9} = \sqrt{x^2} = |x|$
(c) $\mathrm{F}(x^4-9) = \sqrt{(x^4-9)+9} = \sqrt{x^4} = x^2$
(d) $\mathrm{F}(x^2+6x) = \sqrt{x^2+6x+9} = |x+3|$
(e) $\mathrm{F}(x^4-6x^2) = \sqrt{x^4-6x^2+9} = |x^2-3|$
(f) $\dfrac{\mathrm{F}(x+h)-\mathrm{F}(x)}{h} = \dfrac{\sqrt{x+h+9}-\sqrt{x+9}}{h} \cdot \dfrac{\sqrt{x+h+9}+\sqrt{x+9}}{\sqrt{x+h+9}+\sqrt{x+9}} = \dfrac{(x+h+9)-(x+9)}{h(\sqrt{x+h+9}+\sqrt{x+9})} = \dfrac{1}{\sqrt{x+h+9}+\sqrt{x+9}}$

10. $G(x) = \sqrt{4-x}$

(a) $G(4-x) = \sqrt{4-(4-x)} = \sqrt{x}$

(b) $G(4-x^2) = \sqrt{4-(4-x^2)} = \sqrt{x^2} = |x|$

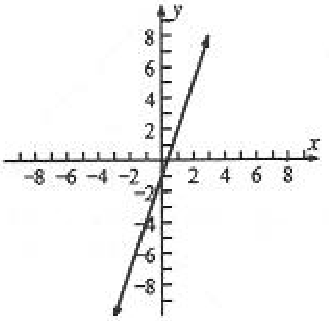
(c) $G(4-x^4) = \sqrt{4-(4-x^4)} = \sqrt{x^4} = x^2$

(d) $G(4x-x^2) = \sqrt{4-(4x-x^2)} = \sqrt{x^2-4x+4} = |x-2|$

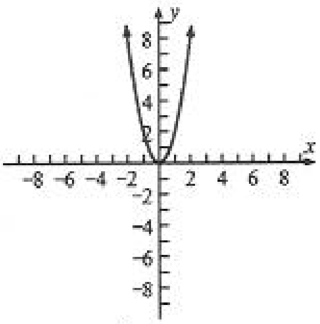
(e) $G(-x^4-4x^2) = \sqrt{4-(-x^4-4x^2)} = \sqrt{x^4+4x^2+4} = x^2+2$

(f) $\dfrac{G(x+h)-G(h)}{h} = \dfrac{\sqrt{4-x-h}-\sqrt{4-x}}{h} \cdot \dfrac{\sqrt{4-x-h}+\sqrt{4-x}}{\sqrt{4-x-h}+\sqrt{4-x}} = \dfrac{(4-x-h)-(4-x)}{h(\sqrt{4-x-h}+\sqrt{4-x})} = \dfrac{-1}{\sqrt{4-x-h}+\sqrt{4-x}}$

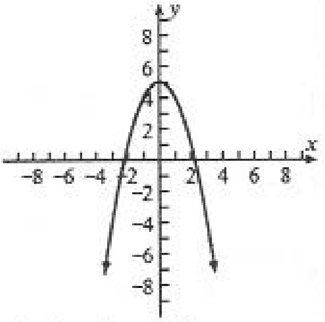
11. $f(x) = 3x-1$
▷ D: $(-\infty, +\infty)$
R: $(-\infty, +\infty)$



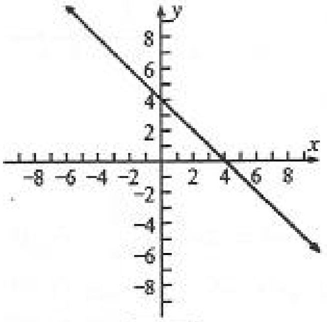
13. $2x^2$
▷ D: $(-\infty, +\infty)$
R: $[0, +\infty)$



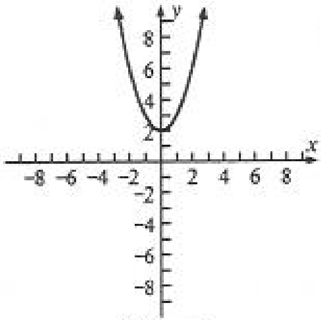
15. $g(x) = 5-x^2$
▷ D: $(-\infty, +\infty)$
R: $(-\infty, 5]$



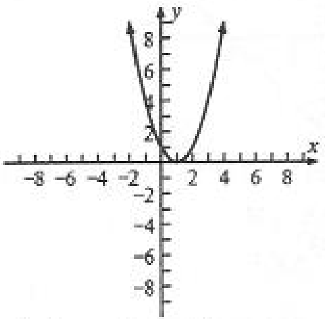
12. $g(x) = 4-x$
▷ D: $(-\infty, +\infty)$
R: $(-\infty, +\infty)$



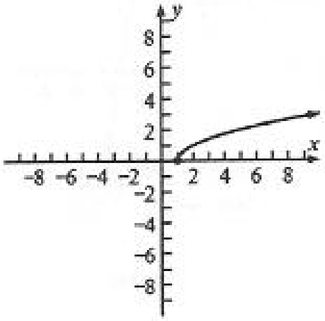
14. $G(x) = x^2+2$
▷ D: $(-\infty, +\infty)$
R: $[2, +\infty)$



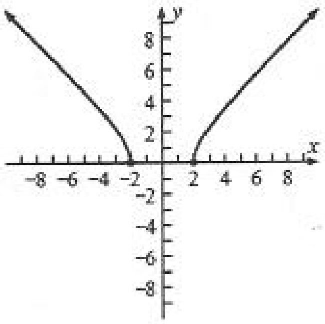
16. $f(x) = (x-1)^2$
▷ D: $(-\infty, +\infty)$
R: $[0, \infty)$



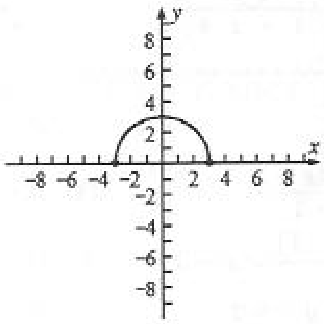
17. $G(x) = \sqrt{x-1}$
▷ D: $x-1 \geq 0$,
$[1, +\infty)$
R: $[0, +\infty)$



19. $f(x) = \sqrt{x^2-4}$
▷ D: $x^2-4 \geq 0$,
$(-\infty, -2] \cup [2, +\infty)$
R: $[0, +\infty)$

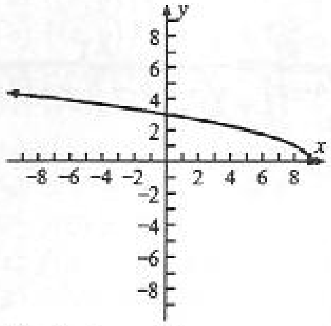


21. $g(x) = \sqrt{9-x^2}$
▷ D: $9-x^2 \geq 0$
$[-3, 3]$
R: $[0, 3]$



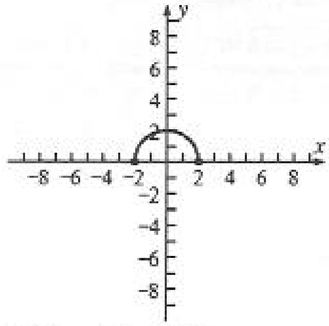
18. $F(x) = \sqrt{9-x}$

▷ D: $9 - x \geq 0$
$(-\infty, 9]$
R: $[0, +\infty)$



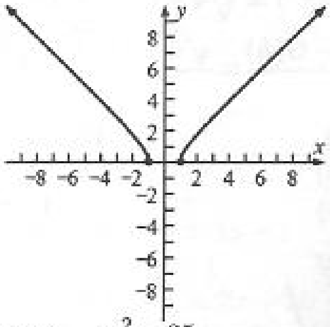
20. $g(x) = \sqrt{4-x^2}$

▷ D: $4 - x^2 \geq 0$
$[-2, 2]$
R: $[0, 2]$



22. $f(x) = \sqrt{x^2-1}$

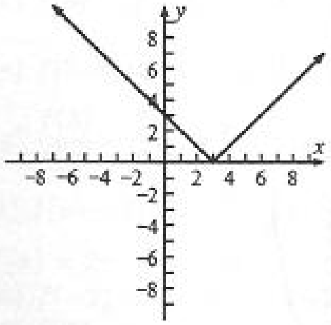
▷ D: $x^2 - 1 \geq 0$
$(-\infty, -1] \cup [1, +\infty)$
R: $[0, +\infty)$



23. $h(x) = |x - 3|$

▷ D: $(-\infty, +\infty)$

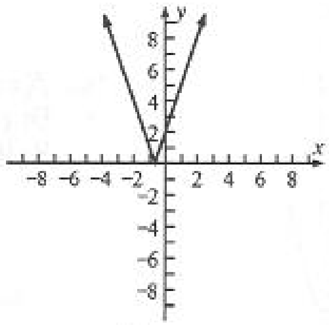
R: $[0, +\infty)$



25. $F(x) = |3x + 2|$

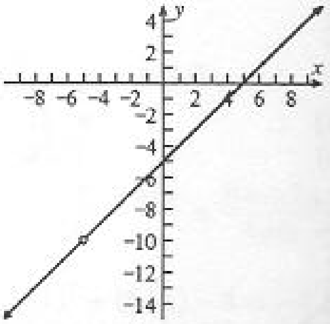
▷ D: $(-\infty, +\infty)$

R: $[0, +\infty)$



27. $H(x) = \dfrac{x^2 - 25}{x + 5}$

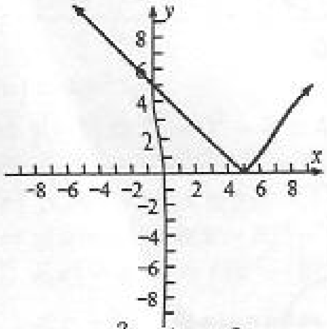
▷ $= \dfrac{(x-5)(x+5)}{x+5} = x - 5$
D: $x \neq -5$, R: $y \neq -10$



24. $H(x) = |5 - x|$

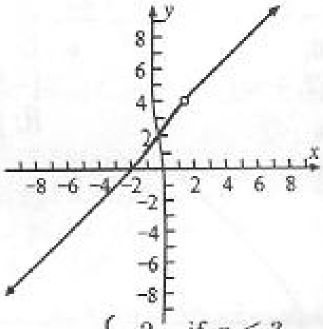
▷ D: $(-\infty, +\infty)$

R: $[0, +\infty)$



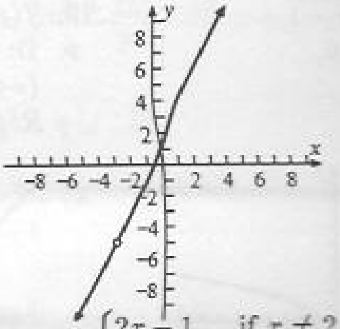
26. $G(x) = \dfrac{x^2 - 4}{x - 2}$

▷ $= \dfrac{(x-2)(x+2)}{x-2} = x + 2$
D: $x \neq 2$, R: $y \neq 4$



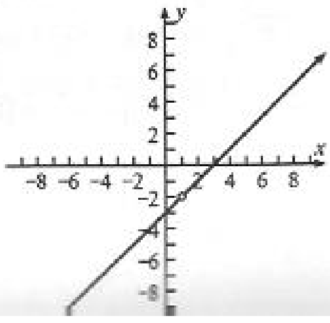
28. $f(x) = \dfrac{2x^2 + 7x + 3}{x + 3}$

▷ $= \dfrac{(2x+1)(x+3)}{x+3} = 2x + 1$
D: $x \neq -3$, R: $y \neq -5$



29. $f(x) = \dfrac{x^2 - 4x + 3}{x - 1}$

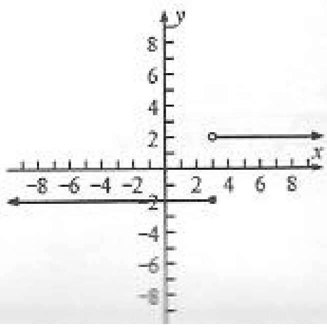
▷ $= \dfrac{(x-3)(x-1)}{x-1} = x - 3$
D: $x \neq 1$, R: $y \neq -2$



31. $f(x) = \begin{cases} -2 & \text{if } x \leq 3 \\ 2 & \text{if } 3 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

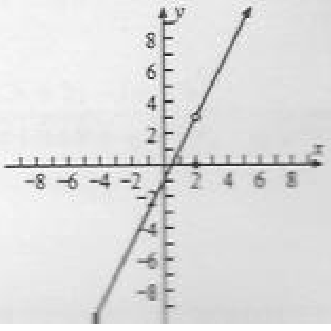
R: $\{-2, 2\}$



33. $g(x) = \begin{cases} 2x - 1 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $y \neq 3$



30. $g(x)=\dfrac{(x^2-4)(x-3)}{x^2-x-6}$

▹ $=\dfrac{(x-2)(x+2)(x-3)}{(x+2)(x-3)}$

$= x-2$. D: $x \neq -2,\ 3$

R: $y \neq -4,\ 1$

32. $g(x)=\begin{cases}-4 & \text{if } x<-2\\ -1 & \text{if } -2\le x\le 2\\ 3 & \text{if } 2<x\end{cases}$

▹ D: $(-\infty,+\infty)$

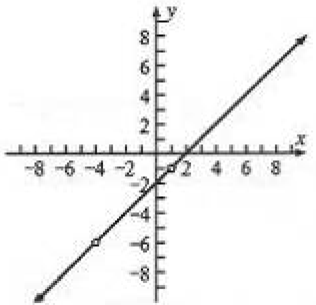
R: $\{-4,-1,3\}$

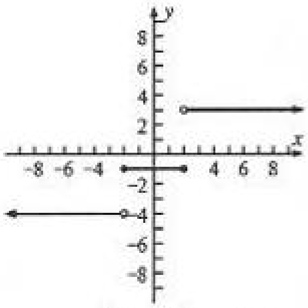
34. $f(x)=\begin{cases}3x+2 & \text{if } x\neq 1\\ 8 & \text{if } x=1\end{cases}$

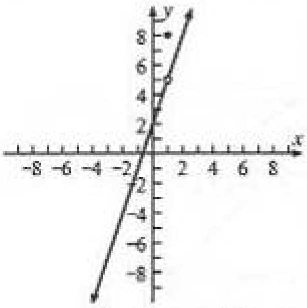
▹ $f(x)=3x+2$

D: $(-\infty,+\infty)$

R: $y\neq 5$







35. $F(x)=\begin{cases}x^2-4 & \text{if } x\neq 3\\ -2 & \text{if } x=3\end{cases}$

▹ D: $(-\infty,+\infty)$, R: $[-4,+\infty)$

Note: $F(-3)=5$

37. $G(x)=\begin{cases}1-x^2 & \text{if } x<0\\ 3x+1 & \text{if } 0\le x\end{cases}$

▹ D: $(-\infty,+\infty)$

R: $(-\infty,1)\cup[1,+\infty)$

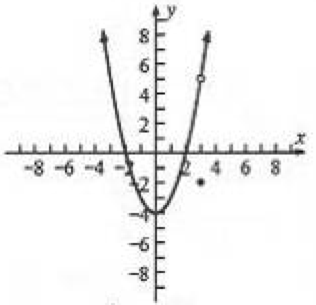
$=(-\infty,+\infty)$

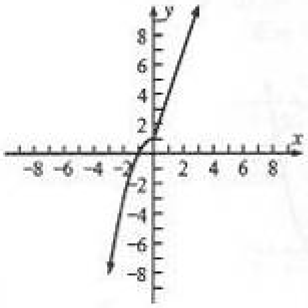
39. $g(x)=\begin{cases}6x+7 & \text{if } x\le -2\\ 4-x & \text{if } -2<x\end{cases}$

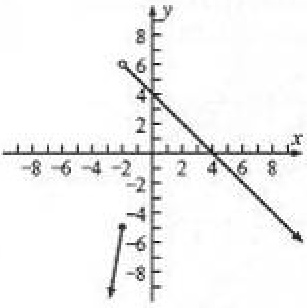
▹ D: $(-\infty,+\infty)$

R: $(-\infty,-5]\cup(-\infty,6)$

$=(-\infty,6)$







36. $G(x)=\begin{cases}9-x^2 & \text{if } x\neq -3\\ 4 & \text{if } x=-3\end{cases}$

▹ D: $(-\infty,+\infty)$, R: $(-\infty,9]$

Note: $G(3)=0$

38. $F(x)=\begin{cases}x^2-4 & \text{if } x<3\\ 2x-1 & \text{if } 3\le x\end{cases}$

▹ D: $(-\infty,+\infty)$

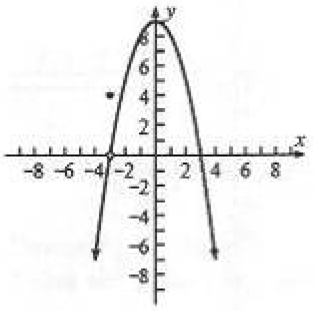
R: $[-4,+\infty)\cup[5,+\infty)$

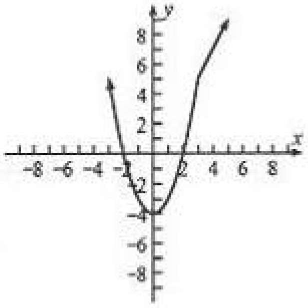
$=[-4,+\infty)$

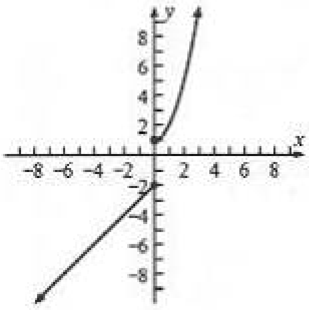
40. $f(x)=\begin{cases}x-2 & \text{if } x\le 0\\ x^2+1 & \text{if } 0<x\end{cases}$

▹ D: $(-\infty,+\infty)$

R: $(-\infty,-2]\cup(1,+\infty)$



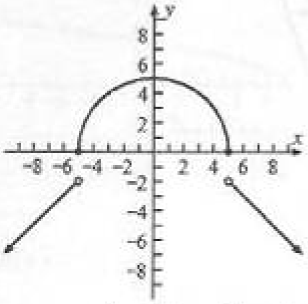




41. $h(x) = \begin{cases} x+3 & \text{if } x < -5 \\ \sqrt{25-x^2} & \text{if } -5 \le x \le 5 \\ 3-x & \text{if } 5 < x \end{cases}$

▹ D: $(-\infty, +\infty)$

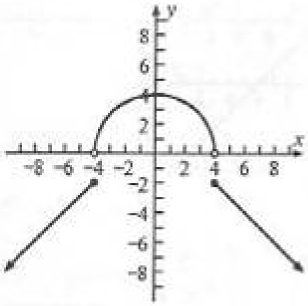
R: $(-\infty,-2) \cup [0,5] \cup (-\infty,-2)$
$= (-\infty,-2) \cup [0,5]$



42. $\mathrm{H}(x) = \begin{cases} x+2 & \text{if } x \le -4 \\ \sqrt{16-x^2} & \text{if } -4 < x < 4 \\ 2-x & \text{if } 4 \le x \end{cases}$

▹ D: $(-\infty, +\infty)$

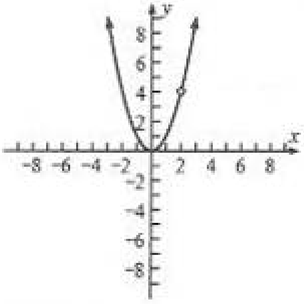
R: $(-\infty,-2] \cup (0,4) \cup (-\infty,-2]$
$= (-\infty,-2] \cup (0,4)$



43. $\mathrm{F}(x) = \dfrac{x^3-2x^2}{x-2}$

▹ $= \dfrac{x^2(x-2)}{x-2} = x^2$

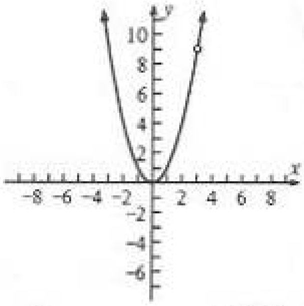
D: $x \ne 2$, R: $[0,+\infty)$
Note: $\mathrm{F}(-2) = 4$



44. $\mathrm{G}(x) = \dfrac{x^3+3x^2}{x+3}$

▹ $= \dfrac{x^2(x+3)}{x+3} = x^2$

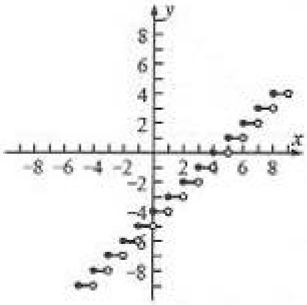
D: $x \ne -3$, R: $[0,+\infty)$
Note: $\mathrm{G}(3) = 9$



45. $f(x) = [\![x-4]\!]$

▹ D: $(-\infty, +\infty)$

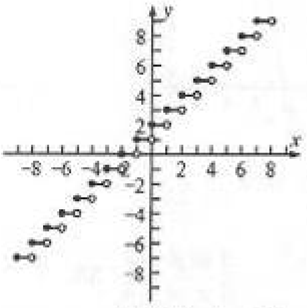
R: {integers}



46. $\mathrm{F}(x) = [\![x+2]\!]$

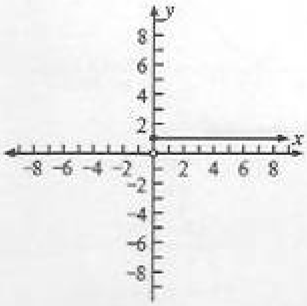
▹ D: $(-\infty, +\infty)$

R: {integers}



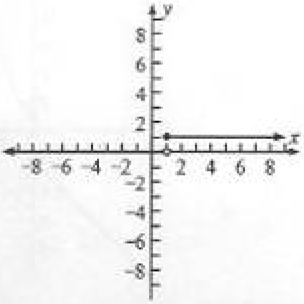
47. (a) $\mathrm{U}(x)$

▹ $= \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$



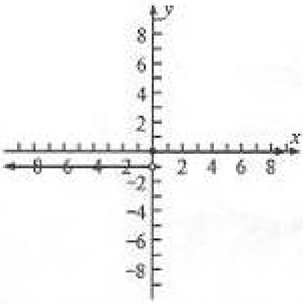
(b) $\mathrm{U}(x-1)$

$= \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$



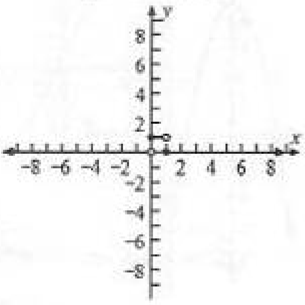
(c) $\mathrm{U}(x) - 1$

$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$



(d) $\mathrm{U}(x) - \mathrm{U}(x-1)$

$= \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$



48. (a) $x\,\mathrm{U}(x)$

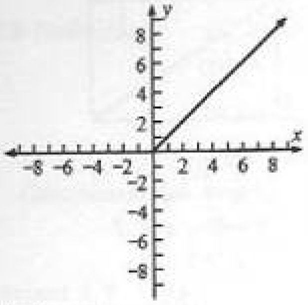
▹ $= \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$

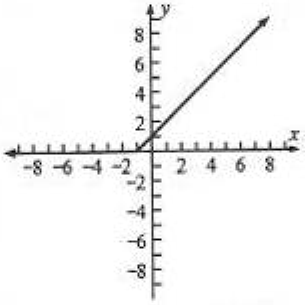
(b) $(x+1)\mathrm{U}(x+1)$

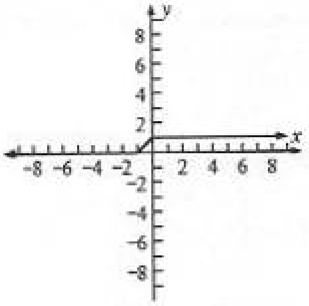
$= \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } x \ge -1 \end{cases}$

(c) $(x+1)\mathrm{U}(x+1) - x\mathrm{U}(x)$

$= \begin{cases} 0 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$







49. (a) $\operatorname{sgn} x$

▹ $= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

(b) $x \operatorname{sgn} x$

$= \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$

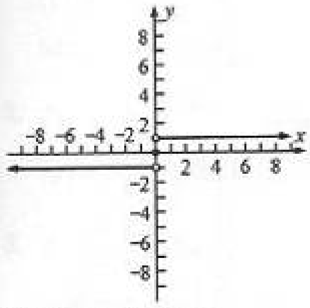
$= |x|$

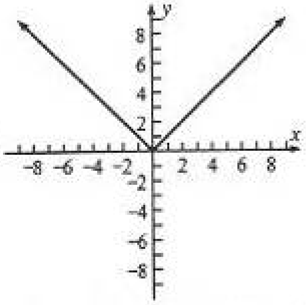
(c) $2 - x \operatorname{sgn} x$

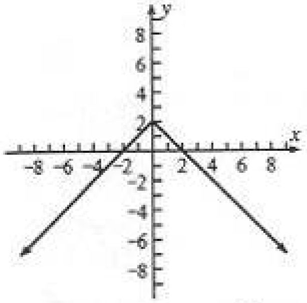
$= \begin{cases} 2+x & \text{if } x < 0 \\ 2-x & \text{if } x \ge 0 \end{cases}$

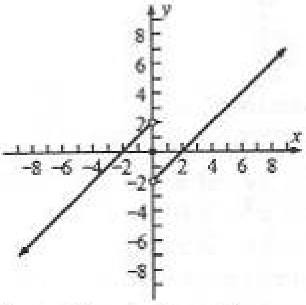
(d) $x - 2 \operatorname{sgn} x$

$= \begin{cases} x+2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x-2 & \text{if } x > 0 \end{cases}$









50. (a) $\operatorname{sgn}(x+1)$

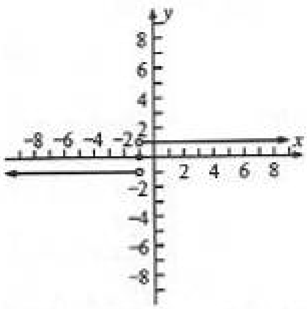
▹ $= \begin{cases} -1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 1 & \text{if } x > -1 \end{cases}$

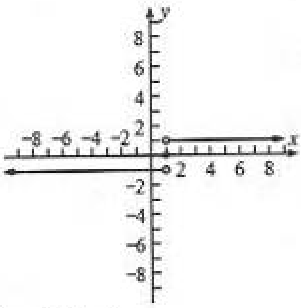
(b) $\operatorname{sgn}(x-1)$

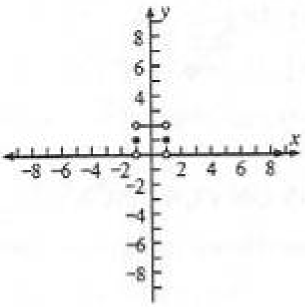
$= \begin{cases} -1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$

(c) $\operatorname{sgn}(x+1) - \operatorname{sgn}(x-1)$

$= \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 1 \\ 1 & \text{if } x = -1 \text{ or } x = 1 \\ 2 & \text{if } -1 < x < 1 \end{cases}$



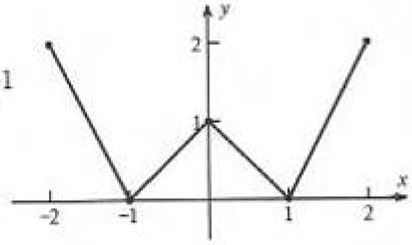




51. Define $f(x)$ piecewise for the graph of the figure.

▹ Using the point-slope formula for each segment give

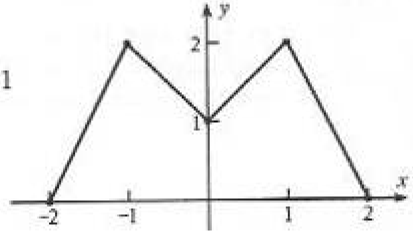
$$f(x) = \begin{cases} 2 - 2(x+2) & \text{if } -2 \le x \le -1 \\ 0 + 1(x+1) & \text{if } -1 < x \le 0 \\ 1 - 1x & \text{if } 0 < x \le 1 \\ 0 + 2(x-1) & \text{if } 1 < x \le 2 \end{cases} = \begin{cases} -2x - 2 & \text{if } -2 \le x \le -1 \\ x+1 & \text{if } -1 < x \le 0 \\ 1 - x & \text{if } 0 < x \le 1 \\ 2x - 2 & \text{if } 1 < x \le 2 \end{cases}$$



52. Define $f(x)$ piecewise for the graph of the figure.

▹ Using the point-slope formula for each segment give

$$f(x) = \begin{cases} 0 + 2(x+2) & \text{if } -2 \le x \le -1 \\ 2 - 1(x+1) & \text{if } -1 < x \le 0 \\ 1 + 1x & \text{if } 0 < x \le 1 \\ 2 - 2(x-1) & \text{if } 1 < x \le 2 \end{cases} = \begin{cases} 2x + 4 & \text{if } -2 \le x \le -1 \\ 1 - x & \text{if } -1 < x \le 0 \\ 1 + x & \text{if } 0 < x \le 1 \\ 4 - 2x & \text{if } 1 < x \le 2 \end{cases}$$



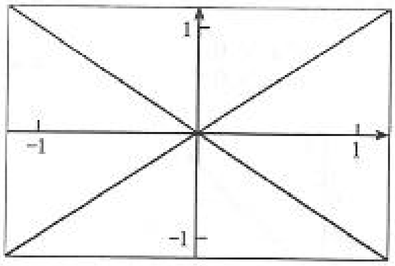
53. Define the graph of the figure as the union of the graphs of two functions f_1 and f_2.

▷ $f_1 = x$, $f_2 = -x$ or $f_1 = |x|$, $f_2 = -|x|$

54. Define the graph of the letter Z as the union of f_1, f_2, f_3.

▷ Use relational operators to get segments, rather than lines.

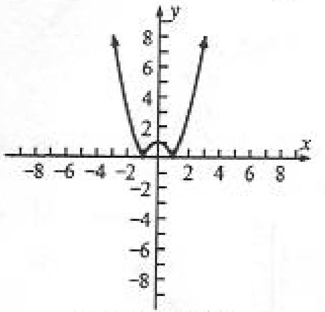
$f_1 = .8(x > -.8)(x < .8)$
$f_2 = x(x > -.8)(x < .8)$
$f_3 = -.8(x > -.8)(x < .8)$



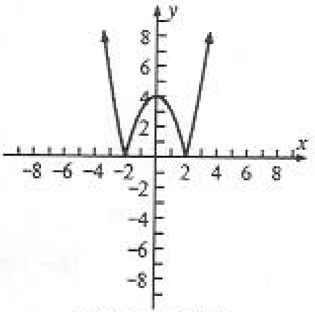
In Exercises 55–58, define the function piecewise and sketch the graph.

55. $f(x) = |x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x \le -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \\ x^2 - 1 & \text{if } x \ge 1 \end{cases}$

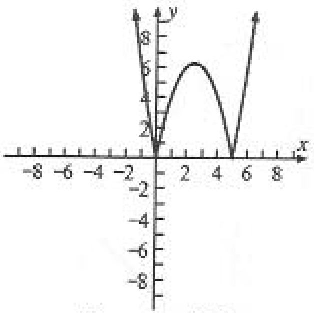
56. $g(x) = |4 - x^2| = \begin{cases} x^2 - 4 & \text{if } x \le -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \\ x^2 - 4 & \text{if } x \ge 2 \end{cases}$



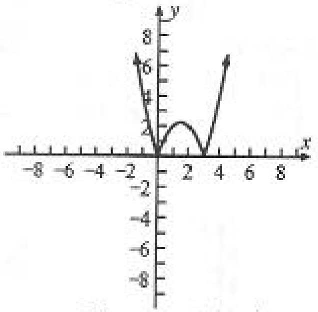
Exercise 55



Exercise 56



Exercise 57



Exercise 58

57. $g(x) = |x| \cdot |5 - x| = |x(5 - x)|$

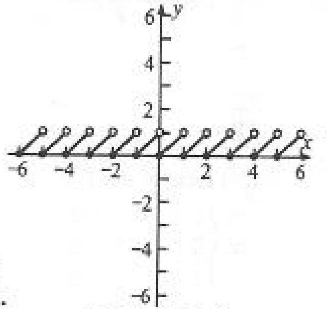
$= \begin{cases} x^2 - 5x & \text{if } x < 0 \\ 5x - x^2 & \text{if } 0 \le x \le 5 \\ x^2 - 5x & \text{if } x > 5 \end{cases}$

58. $g(x) = |x| \cdot |x - 3| = |x(x - 3)|$

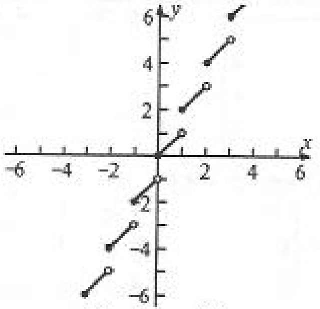
$= \begin{cases} x^2 - 3x & \text{if } x < 0 \\ 3x - x^2 & \text{if } 0 \le x \le 3 \\ x^2 - 3x & \text{if } x > 3 \end{cases}$

In Exercises 59 and 60, sketch the graph of the function and determine its domain and range. Check by plotting.

59. $h(x) = x - [\![x]\!]$. D: $(-\infty, +\infty)$; R: $[0, 1)$



Exercise 59



Exercise 60

60. $F(x) = x + [\![x]\!]$. D: $(-\infty, +\infty)$; R: $[2k, 2k + 1)$, where k is any integer.

61. Define two other functions whose graphs resemble two different letters of the alphabet.

▷ U: $f(x) = 1 - \sqrt{1 - x^2}$, V: $f(x) = |x|$

1.2 OPERATIONS ON FUNCTIONS AND TYPES OF FUNCTIONS

1.2.1 Definition Given the two functions f and g:

(i) their *sum*, denoted by $f + g$, is the function defined by
$$(f + g)(x) = f(x) + g(x)$$

(ii) their *difference*, denoted by $f - g$, is the function defined by
$$(f - g)(x) = f(x) - g(x)$$

(iii) their *product*, denoted by $f \cdot g$, is the function defined by
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

(iv) their *quotient*, denoted by f/g, is the function defined by
$$(f/g)(x) = f(x)/g(x)$$

In each case the *domain* of the resulting function consists of those values of x common to the domains of f and g, with the additional requirement in case (iv) that the values of x for which $g(x) = 0$ are excluded.

1.2.2 Definition Given two functions f and g, the *composite function*, denoted by $f \circ g$ and read f of g, is defined by $(f \circ g)(x) = f(g(x))$ and the *domain* of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

The composition of functions is associative, that is $(f \circ g) \circ h = f \circ (g \circ h)$.

1.2.3 Definition (i) A function is said to be an *even* function if for every x in the domain of f, $f(-x) = f(x)$.
(ii) A function is said to be an *odd* function if for every x in the domain of f, $f(-x) = -f(x)$.

In both parts (i) and (ii) it is understood that $-x$ is in the domain of f whenever x is.

Even powers of x are even functions; odd powers and odd roots of x are odd functions.

Combinations even ± even = even, odd ± odd = odd, even × even = even, odd × odd = even, even × odd = odd even/even = even, odd/odd = even, even/odd = odd, odd/even = odd. See Exercise 52.

Exercises 1.2

In *Exercises 1–10*, define the following functions and determine their domain D:
(a) $f + g$ (b) $f - g$ (c) $f \cdot g$ (d) f/g (e) g/f

▸ We adopt the common practice of omitting "(x)". Values not in the implicit domain are bolded.

1. $f = x - 5$, $g = x^2 - 1$ (a) $f + g = x^2 + x - 6$, D: $(-\infty,\infty)$ (b) $f - g = -x^2 + x - 4$, D: $(-\infty,\infty)$
(c) $f \cdot g = (x-5)(x^2-1) = x^3 - 5x^2 - x + 5$, D: $(-\infty,\infty)$
(d) $f/g = \frac{x-5}{x^2-1}$, D: $x \neq \pm 1$ (e) $g/f = \frac{x^2-1}{x-5}$, D: $x \neq 5$

2. $f = \sqrt{x}$, $g = x^2 + 1$ (a) $f + g = \sqrt{x} + x^2 + 1$, D: $[0,+\infty)$ (b) $f - g = \sqrt{x} - x^2 - 1$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(x^2+1)$, D: $[0,+\infty)$ (d) $f/g = \frac{\sqrt{x}}{x^2+1}$, D: $[0,+\infty)$ (e) $g/f = \frac{x^2+1}{\sqrt{x}}$, D: $(0,+\infty)$

3. $f = \frac{x+1}{x-1}$, $g = \frac{1}{x}$ (a) $f + g = \frac{x+1}{x-1} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{x-1}{x-1} = \frac{x^2+2x-1}{x(x-1)}$, D: $x \neq 0, 1$
(b) $f - g = \frac{x+1}{x-1} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x-1}{x-1} = \frac{x^2+1}{x(x-1)}$, D: $x \neq 0, 1$ (c) $f \cdot g = \frac{x+1}{x-1} \cdot \frac{1}{x} = \frac{x+1}{x(x-1)}$, D: $x \neq 0, 1$
(d) $\frac{f}{g} = \frac{x+1}{x-1} \div \frac{1}{x} = \frac{x+1}{x-1} \cdot x = \frac{x^2+x}{x-1}$, $x \neq 0, 1$ (e) $\frac{g}{f} = \frac{1}{x} \div \frac{x+1}{x-1} = \frac{1}{x} \cdot \frac{x-1}{x+1} = \frac{x-1}{x(x+1)}$, D: $x \neq -1, 0, \mathbf{1}$

4. $f = \sqrt{x}$, $g = 4 - x^2$ (a) $f + g = \sqrt{x} + 4 - x^2$, D: $[0,+\infty)$ (b) $f - g = \sqrt{x} - 4 + x^2$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(4 - x^2)$, D: $[0,+\infty)$ (d) $f/g = \frac{\sqrt{x}}{4-x^2}$, D: $[0,2) \cup (2,+\infty)$ (e) $g/f = \frac{4-x^2}{\sqrt{x}}$, D: $(0,+\infty)$

5. $f = \sqrt{x}$, $g = x^2 - 1$ (a) $f + g = \sqrt{x} + x^2 - 1$, D: $[0,+\infty)$ (b) $f - g = \sqrt{x} - x^2 + 1$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(x^2-1)$, D: $[0,+\infty)$ (d) $f/g = \frac{\sqrt{x}}{x^2-1}$, D: $[0,1) \cup (1,+\infty)$ (e) $g/f = \frac{x^2-1}{\sqrt{x}}$, D: $(0,+\infty)$

6. $f = |x|$, $g = |x-3|$ (a) $f + g = |x| + |x-3|$, D: $(-\infty,+\infty)$ (b) $f - g = |x| - |x-3|$, D: $(-\infty,+\infty)$
(c) $f \cdot g = |x||x-3|$, D: $(-\infty,+\infty)$ (d) $f/g = \frac{|x|}{|x-3|}$, D: $x \neq 3$ (e) $g/f = \frac{|x-3|}{|x|}$, D: $x \neq 0$

7. $f = x^2 + 1$, $g = 3x - 2$ (a) $f + g = x^2 + 3x - 1$, D: $(-\infty,+\infty)$ (b) $f - g = x^2 - 3x + 3$, D: $(-\infty,+\infty)$
(c) $f \cdot g = (x^2+1)(3x-2) = 3x^3 - 2x^2 + 3x - 2$, D: $(-\infty,+\infty)$ (d) $f/g = \frac{x^2+1}{3x-2}$, D: $x \neq \frac{2}{3}$
(e) $g/f = \frac{3x-2}{x^2+1}$, D: $(-\infty,+\infty)$

8. $f = \sqrt{x+4}$, $g = x^2 - 4$ (a) $f + g = \sqrt{x+4} + x^2 - 4$, D: $[-4,+\infty)$ (b) $f - g = \sqrt{x+4} - x^2 + 4$,
D: $[-4,+\infty)$ (c) $f \cdot g = \sqrt{x+4}(x^2-4)$, D: $[-4,+\infty)$ (d) $f/g = \frac{\sqrt{x+4}}{x^2-4}$, D: $[-4,-2) \cup (-2,2) \cup (2,+\infty)$
(e) $g/f = \frac{x^2-4}{\sqrt{x+4}}$, D: $(-4,+\infty)$

9. $f=\frac{1}{x+1}$, $g=\frac{x}{x-2}$ (a) $f+g=\frac{1}{x+1}\cdot\frac{x-2}{x-2}+\frac{x}{x-2}\cdot\frac{x+1}{x+1}=\frac{x^2+2x-2}{(x-2)(x+1)}$, d: $x\neq-1,2$

(b) $f-g=\frac{1}{x+1}\cdot\frac{x-2}{x-2}-\frac{x}{x-2}\cdot\frac{x+1}{x+1}=\frac{-x^2-2}{(x+1)(x-2)}$, D: $x\neq-1,2$

(c) $f\cdot g=\frac{x}{(x+1)(x-2)}$, D: $x\neq-1,2$ (d) $\frac{f}{g}=\frac{1}{x+1}\div\frac{x}{x-2}=\frac{1}{x+1}\cdot\frac{x-2}{x}=\frac{x-2}{x(x+1)}$, D: $x\neq-1,0,2$

(e) $g/f=\frac{x}{x-2}\div\frac{1}{x+1}=\frac{x}{x-2}\cdot\frac{x+1}{1}=\frac{x^2+x}{x-2}$, D: $x\neq-1,2$

10. $f=x^2$, $g=\frac{1}{\sqrt{x}}$ (a) $f+g=x^2+\frac{1}{\sqrt{x}}$, D: $(0,+\infty)$ (b) $f-g=x^2-\frac{1}{\sqrt{x}}$, D: $(0,+\infty)$

(c) $f\cdot g=\frac{x^2}{\sqrt{x}}=x^{3/2}$, $(0,+\infty)$ (d) $\frac{f}{g}=x^2\div\frac{1}{\sqrt{x}}=x^{5/2}$, D: $(0,+\infty)$ (e) $\frac{g}{f}=\frac{1}{\sqrt{x}}\div x^2=x^{-5/2}$, D: $(0,+\infty)$

In *Exercises 11–14*, compute $(f\circ g)(c)$ by two methods: (a) Find $g(c)$ first; (b) Find $(f\circ g)(x)$ first.

11. $f(x)=3x^2-4x$, $g(x)=2x-5$, $c=4$ ▹ (a) $g(4)=2\cdot4-5=3$, $f(3)=3\cdot3^2-4\cdot3=15$
(b) $f(g(x))=3(2x-5)^2-4(2x-5)=3(4x^2-20x+25)-8x+20=12x^2-68x+95$
$f(g(4))=12\cdot4^2-68\cdot4+95=15$

12. $f(x)=\sqrt{x^2-36}$, $g(x)=x^2-3x$, $c=5$ ▹ (a) $g(5)=5^2-3\cdot5=10$, $f(10)=\sqrt{10^2-36}=8$
(b) $f(g(x))=\sqrt{(x^2-3x)^2-36}=\sqrt{x^4-6x^3+9x^2-36}$, $f(g(5))=\sqrt{5^4-6\cdot5^3+9\cdot5^2-36}=8$

13. $f(x)=\frac{1}{x-1}$, $g(x)=\frac{2}{x^2+1}$, $c=\frac{1}{2}$ ▹ (a) $g(\frac{1}{2})=\frac{2}{(\frac{1}{2})^2+1}=\frac{8}{5}$, $f(\frac{8}{5})=\frac{1}{\frac{8}{5}-1}=\frac{5}{3}$

(b) $f(g(x))=\frac{1}{\frac{2}{x^2+1}-1}=\frac{x^2+1}{2-(x^2+1)}=\frac{x^2+1}{1-x^2}$, $f(g(\frac{1}{2}))=\frac{(\frac{1}{2})^2+1}{1-(\frac{1}{2})^2}=\frac{\frac{5}{4}}{\frac{3}{4}}=\frac{5}{3}$

14. $f(x)=\frac{2\sqrt{x+3}}{x}$, $g(x)=\frac{2x+5}{x^4}$, $c=-2$ ▹ (a) $g(-2)=\frac{2(-2)+5}{(-2)^4}=\frac{1}{16}$,

$f(\frac{1}{16})=\frac{2\sqrt{\frac{1}{16}+3}}{\frac{1}{16}}=32\sqrt{\frac{49}{16}}=32\cdot\frac{7}{4}=56$ (b) $f(g(x))=\frac{2\sqrt{\frac{2x+5}{x^4}+3}}{\frac{2x+5}{x^4}}=\frac{x^4}{2x+5}\cdot2\sqrt{\frac{3x^4+2x+5}{x^4}}$

$=\frac{2x^2}{2x+5}\sqrt{3x^4+2x+5}$, $f(g(-2))=\frac{2(-2)^2}{2(-2)+5}\sqrt{3(-2)^4+2(-2)+5}=8\sqrt{49}=56$

In *Exercises 15–24*, find (a) $f\circ g$ (b) $g\circ f$ (c) $f\circ f$ (d) $g\circ g$ and give their domains D.

15. $f(x)=x-2$, $g(x)=x+7$
▹ (a) $f(g(x))=(x+7)-2=x+5$, D: $(-\infty,+\infty)$ (b) $g(f(x))=(x-2)+7=x+5$, D: $(-\infty,+\infty)$
(c) $f(f(x))=(x-2)-2=x-4$ D: $(-\infty,+\infty)$ (d) $g(g(x))=(x+7)+7=x+14$, D: $(-\infty,+\infty)$

16. $f(x)=3-2x$, $g(x)=6-3x$
▹ (a) $f(g(x))=3-2(6-3x)=6x-9$, D: $(-\infty,+\infty)$ (b) $g(f(x))=6-3(3-2x)=6x-3$, D: $(-\infty,+\infty)$
(c) $f(f(x))=3-2(3-2x)=4x-3$, D: $(-\infty,+\infty)$ (d) $g(g(x))=6-3(6-3x)=9x-12$, D: $(-\infty,+\infty)$

17. $f(x)=x-5$, $g(x)=x^2-1$ ▹ (a) $f(g(x))=(x^2-1)-5=x^2-6$, D: $(-\infty,+\infty)$
(b) $g(f(x))=(x-5)^2-1=x^2-10x+24$, D: $(-\infty,+\infty)$
(c) $f(f(x))=(x-5)-5=x-10$, D: $(-\infty,+\infty)$ (d) $g(g(x))=(x^2-1)^2-1=x^4-2x^2$, D: $(-\infty,+\infty)$

18. $f(x)=\sqrt{x}$, $g(x)=x^2+1$
▹ (a) $f(g(x))=\sqrt{x^2+1}$, D: $(-\infty,+\infty)$ (b) $g((x))=(\sqrt{x})^2+1=x+1$, D: $[0,+\infty)$
(c) $f(f(x))=\sqrt{\sqrt{x}}=\sqrt[4]{x}$, D: $[0,+\infty)$ (d) $g(g(x))=(x^2+1)^2+1=x^4+2x^2+2$, D: $(-\infty,+\infty)$

19. $f(x)=\sqrt{x-2}$, $g(x)=x^2-2$ ▹ (a) $f(g(x))=\sqrt{(x^2-2)-2}=\sqrt{x^2-4}$, D: $(-\infty,-2]\cup[2,+\infty)$
(b) $g(f(x))=(\sqrt{x-2})^2-2=x-2-2=x-4$, D: $[2,+\infty)$
(c) $f(f(x))=\sqrt{\sqrt{x-2}-2}$, D: $x-2\geq4$, $[6,+\infty)$ (d) $g(g(x))=(x^2-2)^2-2=x^4-4x^2+2$, D: $(-\infty,+\infty)$

20. $f(x) = x^2 - 1$, $g(x) = \frac{1}{x}$

▹ (a) $f(g(x)) = \left(\frac{1}{x}\right)^2 - 1 = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}$, D: $x \neq 0$ (b) $g(f(x)) = \frac{1}{x^2-1}$, D: $x \neq \pm 1$

(c) $f(f(x)) = (x^2-1)^2 - 1 = x^4 - 2x^2$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = \frac{1}{1/x} = x$, D: $x \neq 0$

21. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$

▹ (a) $f(g(x)) = \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$ (b) $g(f(x)) = \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$

(c) $f(f(x)) = \frac{1}{1/x} = x$, D: $x \neq 0$ (d) $g(g(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$

22. $f(x) = \sqrt{x}$, $g(x) = -\frac{1}{x}$ ▹ (a) $f(g(x)) = \sqrt{-\frac{1}{x}}$, D: $(-\infty, 0)$ (b) $g(f(x)) = -\frac{1}{\sqrt{x}}$, D: $(0, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$ (d) $g(g(x)) = -\frac{1}{-1/x} = x$, D: $x \neq 0$

23. $f(x) = |x|$, $g(x) = |x+2|$

▹ (a) $f(g(x)) = \big||x+2|\big| = |x+2|$, D: $(-\infty, +\infty)$ (b) $g(f(x)) = \big||x|+2\big| = |x|+2$, D: $(-\infty, +\infty)$

(c) $f(f(x)) = ||x|| = |x|$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = \big||x+2|+2\big| = |x+2|+2$, D: $(-\infty, +\infty)$

24. $f(x) = \sqrt{x^2-1}$, $g(x) = \sqrt{x-1}$

▹ (a) $f(g(x)) = \sqrt{(\sqrt{x-1})^2 - 1} = \sqrt{x-1-1} = \sqrt{x-2}$, D: $[2, +\infty)$

(b) $g(f(x)) = \sqrt{\sqrt{x^2-1} - 1}$, D: $\sqrt{x^2-1} \geq 1$, $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$

(c) $f(f(x)) = \sqrt{(\sqrt{x^2-1})^2 - 1} = \sqrt{x^2-1-1} = \sqrt{x^2-2}$, D: $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$

(d) $g(g(x)) = \sqrt{\sqrt{x-1} - 1}$, D: $x - 1 > 1$, $[2, +\infty)$

25. $f(x) = \sqrt{x}$ (a) $f(x^2) = \sqrt{x^2} = |x|$, D: $(-\infty, +\infty)$ (b) $[f(x)]^2 = (\sqrt{x})^2 = x$, D: $[0, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$ (d) $f(f(-x)) = \sqrt[4]{-x}$, D: $(-\infty, 0]$

26. $f(x) = \frac{1}{1-x}$ (a) $f(x^2) = \frac{1}{1-x^2}$, D: $x \neq \pm 1$ (b) $[f(x)]^2 = \frac{1}{(1-x)^2}$, D: $x \neq 1$

(c) $f(f(x)) = \frac{1}{1 - 1/(1-x)} = \frac{1-x}{-x} = \frac{x-1}{x}$, D: $x \neq 0, 1$ (d) $f(f(-x)) = \frac{x+1}{x}$, D: $x \neq 0, -1$

In *Exercises 27–32*, express the function as $f(g(x))$ in two ways.

▹ Express the function using *of* twice; break at the first or the second.

27. $\sqrt{x^2-4}$ ▹ square root of the difference of ⋯, $f = \sqrt{x}$, $g = x^2 - 4$ or $f = \sqrt{x-4}$, $g = x^2$

28. $(9+x^2)^{-2}$ ▹ -2 power of the sum of ⋯. $f = x^{-2}$, $g = 9 + x^2$ or $f = (9+x)^{-2}$, $g = x^2$

29. $\left(\frac{1}{x-2}\right)^3$ ▹ 3 power of the reciprocal of ⋯. $f = x^3$, $g = \frac{1}{x-2}$ or $f = \left(\frac{1}{x}\right)^3$, $g = x - 2$

30. $\frac{4}{\sqrt[3]{x^3+3}}$ ▹ quotient of 4 and cube root of ⋯. $f = \frac{4}{x}$, $g = \sqrt[3]{x^3+3}$ or $f = \frac{1}{\sqrt[3]{x}}$, $g = x^3 + 3$

31. $(x^2+4x-5)^4$ ▹ 4 power of difference of ⋯. $f = x^4$, $g = x^2 + 4x - 5$ or $f = (x-5)^4$, $g = x^2 + 4x$

32. $\sqrt{|x|+4}$ ▹ square root of sum of ⋯. $f = \sqrt{x}$, $g = |x| + 4$ or $f = \sqrt{x+4}$, $g = |x|$

In *Exercises 33–38*, plot the function and determine if it is even, odd, or neither.

33. (a) $2x^4 - 3x^2 + 1$ is even (sum of even powers) (b) $5x^5 + 1$ is neither (odd + even)

34. (a) $x^2 + 2x + 2$ is neither (even + odd) (b) $x^6 - 1$ is even (even − even)

35. (a) $5x^3 - 7x$ is odd (odd + odd) (b) $g(x) = |x|$, $g(-x) = |-x| = |x| = g(x)$ even

36. (a) $4x^5 + 3x^3$ is odd (odd + odd) (b) $x^3 + 1$ is neither (odd + even)

37. (a) $\sqrt[3]{x}$ is odd (odd root) (b) $5x^4 - 4$ is even (even − even)

38. (a) $|x|/x$ is odd (even ÷ odd) (b) $2|x| + 3$ is even (even + even)

In *Exercises 39 and 40*, determine if the function is even, odd, or neither.

39. (a) $\frac{y^3-y}{y^2+1}$ is odd (odd ÷ even) (b) $\frac{x^2-1}{x^2+1}$ is even (even ÷ even) (c) $\frac{|x|}{x^2+1}$ is even (even ÷ even)

40. (a) $\frac{x^2-5}{2x^3+x}$ is odd (even ÷ odd) (b) $g(z)=\frac{z-1}{z+1}$, $g(-z)=\frac{-z-1}{-z+1}=\frac{z+1}{z-1}$ neither

(c) $f(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases}$. $f(-x)=\begin{cases}-1 & \text{if } x>0\\ 1 & \text{if } x\leq 0\end{cases}$ and $-f(x)=\begin{cases}1 & \text{if } x<0\\ -1 & \text{if } x\geq 0\end{cases}$ neither

In *Exercises 41–44*, define the function in intervals, plot it, and determine if it is even, odd, or neither.

41. $\frac{|x|}{x}=\begin{cases}x/x=1 & \text{if } x>0\\ -x/x=-1 & \text{if } x<0\end{cases}$ is odd (even ÷ odd) **42.** $x|x|=\begin{cases}x\cdot x=x^2 & \text{if } x\geq 0\\ x(-x)=-x^2 & \text{if } x<0\end{cases}$ is even (odd · even)

43. $|x-2|-|x+2|=\begin{cases}-(x-2)+(x+2)=4 & \text{if } x<-2\\ -(x-2)-(x+2)=-2x & \text{if } -2\leq x<2\\ (x-2)-(x+2)=-4 & \text{if } x\geq 2\end{cases}$ is odd

44. $\frac{|x+1|-|x-1|}{x}=\begin{cases}[-(x+1)+(x-1)]/x=-2/x & \text{if } x<-1\\ [(x+1)-(x-1)]/x=2x/x=2 & \text{if } x\in[-1,0)\cup(0,1]\\ [(x+1)-(x-1)]/x=2/x & \text{if } x>1\end{cases}$ is even

45. $f\circ g\neq g\circ f$. See Exercises 16–24, (a) versus (b).

In *Exercises 46–50*, show that f and g are inverse functions by showing that $f(g(x))=x$ and $g(f(x))=x$.

46. $f=2x-3$, $g=\frac{1}{2}(x+3)$. $f\circ g=2[\frac{1}{2}(x+3)]-3=(x+3)-3=x$, $g\circ f=\frac{1}{2}[(2x-3)+2]=\frac{1}{2}(2x)=x$

47. $f=\frac{1}{x+1}$, $g=\frac{1-x}{x}$. $f\circ g=\frac{1}{(1-x)/x+1}\cdot\frac{x}{x}=\frac{x}{1-x+x}=x$, $g\circ f=\frac{1-1/(x+1)}{1/(x+1)}\cdot\frac{x+1}{x+1}=\frac{x+1-1}{1}=x$

48. $f=x^2$, $x\geq 0$; $g=\sqrt{x}$. $f\circ g=(\sqrt{x})^2=x$, $g\circ f=\sqrt{x^2}=x$

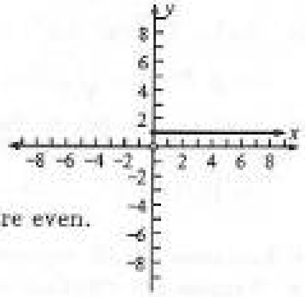
49. $f=x^2$, $x\leq 0$, $g=-\sqrt{x}$. $f\circ g=(-\sqrt{x})^2=x$, $g\circ f=-\sqrt{x^2}=-(-x)=x$

50. $f=(x-1)^3$, $g=1+\sqrt[3]{x}$. $f\circ g=[(1+\sqrt[3]{x})-1]^3=(\sqrt[3]{x})^3=x$, $g\circ f=1+\sqrt[3]{(x-1)^3}=1+(x-1)=x$

51. Find formulas for sgn(U(x)) and U(sgn(x)) and sketch their graph.

▹ $\text{sgn}(\text{U}(x))=\begin{cases}-1 & \text{if U}(x)<0\\ 0 & \text{if U}(x)=0\\ 1 & \text{if U}(x)=1\end{cases}=\left.\begin{cases}-1 & \text{never}\\ 0 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases}\right\}=\text{U}(x).$

$\text{U}(\text{sgn}(x))=\begin{cases}1 & \text{if sgn } x\geq 0\\ 0 & \text{if sgn } x<0\end{cases}=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}=\text{U}(x)$



52. If f and g are odd functions, prove that $f+g$ and $f-g$ are odd and $f\cdot g$ and f/g are even.

▹ $f(-x)=-f(x)$, $g(-x)=g(x)$.

$(f+g)(-x)=f(-x)+g(-x)=-f(x)-g(x)=-[f(x)+g(x)]=-(f+g)(x)$

$(f-g)(-x)=f(-x)-g(-x)=-f(x)+g(x)=-[f(x)-g(x)]=-(f-g)(x)$

$(f\cdot g)(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=(f\cdot g)(x)$

$(f/g)(-x)=f(-x)/g(-x)=-f(x)/-g(x)=f(x)/g(x)=(f/g)(x)$

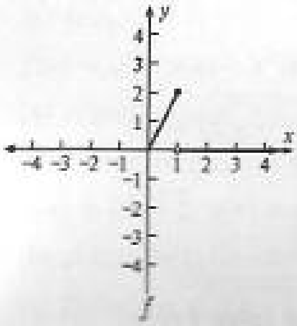
53. If f is any function and g is even, $f(g(-x))=f(g(x))$ so $f\circ g$ is even.

If f and g are both odd, $f(g(-x))=f(-(g(x))=-f(g(x))$ so $f\circ g$ is odd.

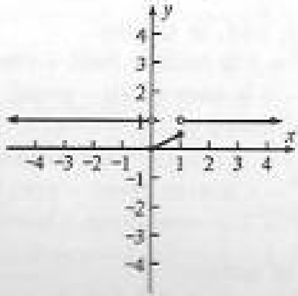
If f is even and g is odd, $f(g(-x))=f(-g(x))=f(g(x))$ so $f\circ g$ is even.

54. Find formulas for $(f\circ g)(x)$. Sketch the graphs of f, g, and $f\circ g$.

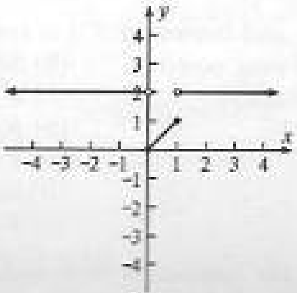
▹ $f(g(x))=\begin{cases}2g(x) & \text{if } 0\leq g(x)\leq 1\\ 0 & \text{otherwise}\end{cases}=\begin{cases}x & \text{if } 0\leq x\leq 1\\ 2 & \text{otherwise}\end{cases}$



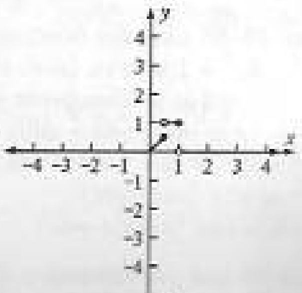
f



g



$f\circ g$



$g\circ f$

55. Find formulas for $(g \circ f)(x)$ and sketch its graph.

▷ $g(f(x)) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \frac{1}{2}f(x) & \text{if } 0 < f(x) \le 1 \\ 1 & \text{if otherwise} \end{cases} = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x > 1 \\ x & \text{if } 0 < x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

56. If $f(x) = x^2 + 2x + 2$, find two functions g for which $f(g(x)) = x^2 - 4x + 5$.

▷ Assume g has the form $ax + b$. Then $f(g) = (ax+b)^2 + 2(ax+b) + 2 = a^2x^2 + (2ab + 2a)x + (b^2 + 2b + 2)$. $a^2 = 1$, $a = \pm 1$. If $a = 1$, $2b + 2 = -4$, $b = -3$. If $a = -1$, $-2b - 2 = -4$, $b = 1$. Both satisfy $b^2 + 2b + 2 = 5$. $g(x) = x - 3$ or $g(x) = 1 - x$.

57. If $f(x) = x^2$, find two functions g for which $f(g(x)) = 4x^2 - 12x + 9 = (2x-3)^2$. $g = 2x - 3$ or $-2x + 3$.

58. Prove that if f and g are linear functions, then so is $f \circ g$.

▷ Let $f = ax + b$, $g = cx + d$. $f(g) = a(cx + d) + b = acx + (ad + b) = \mathrm{A}x + \mathrm{B}$ with $\mathrm{A} = ac$, $\mathrm{B} = ad + b$.

59. Find the function $f(x)$ which is odd and even.

▷ $f(-x) = -f(x)$ and $f(-x) = f(x)$. Then $-f(x) = f(x)$ so $f(x) \equiv 0$.

60. If $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$, $h(x) = -x$, is $(f \circ g)(x) = h(x)$?

▷ $(f \circ g)(x) = (g \circ f)(x) = -x$ if $x \ne 0$ while $h(x) = -x$ for all x.

61. Explain why $\mathrm{F}(x) = \sqrt{x+1}/\sqrt{x-4}$ and $\mathrm{G}(x) = \sqrt{(x+1)/(x-4)}$ are not the same.

▷ Dom $\mathrm{F} = \{x + 1 \ge 0\} \cap \{x - 4 > 0\} = (4, +\infty)$ but dom $\mathrm{G} = (-\infty, -1] \cup (4, +\infty)$

1.3 FUNCTIONS AS MATHEMATICAL MODELS

Variables	should be described using units: x feet is the length.
y is Proportional to x	Directly: $y = kx$. Inversely: $y = k/x$. Jointly proportional to x and z: $y = kxz$.
Extremum of	$ax^2 + bx + c$ occurs when $x = -b/2a$; this can be obtained by completing the square.
Maximum Product	of a set of positive numbers of constant sum is when factors are equal. See Ex. 28.
Minimum Sum	of a set of positive numbers of constant product is when terms are equal. See Ex. 22.

Exercises 1.3

In each exercise, obtain a function as a mathematical model of the situation. Be sure to write a conclusion.

1. A payroll of p dollars is directly proportional to the number w of workers, and a crew of 12 workers earns \$810. (a) Find $p(w)$. (b) What is the payroll for a crew of 15 workers?

▷ p = kw. 810 = k(12), $k = 67.5$ (a) $p(w) = 67.5w$ (b) $p(15) = 67.5 \cdot 15 = 1012.50$

2. A person's brain weight b lb is directly proportional to his body weight w lb, and a person weighing 150 lb has a 4 lb brain (a) $b(w)$. (b) Find the brain weight of a 176 lb person.

▷ $b = kw$. $4 = k(150)$, $k = \frac{2}{75}$. (a) $b(w) = \frac{2}{75}w$ (b) $b(176) = \frac{2}{75} \cdot 176 \approx 4.69$

3. The period p sec of a pendulum is directly proportional to the square root of the number x of feet in its length, and an 8 ft pendulum has a 2 sec period. (a) Find $p(x)$. (b) Find the period of a 2 ft pendulum.

▷ $p = k\sqrt{x}$. $2 = k\sqrt{8}$, $k = 1/\sqrt{2}$ (a) $p(x) = \sqrt{x/2}$ (b) $p(2) = \sqrt{1} = 1$. The period is 1 sec.

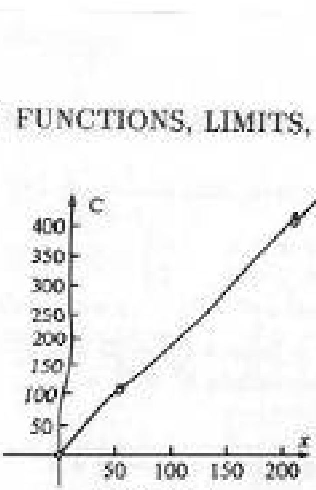
4. The frequency f per sec of a vibration is directly proportional to the square root of the tension t kg, and is 864/sec when the tension is 24 kg. (a) Find $f(t)$. (b) Find the frequency under a tension of 6 kg.

▷ $f = k\sqrt{t}$. $864 = k\sqrt{24} = 12k/\sqrt{6}$, $k = 72\sqrt{6}$ (a) $f(t) = 72\sqrt{6t}$ (b) $f(6) = 72 \cdot 6 = 432$

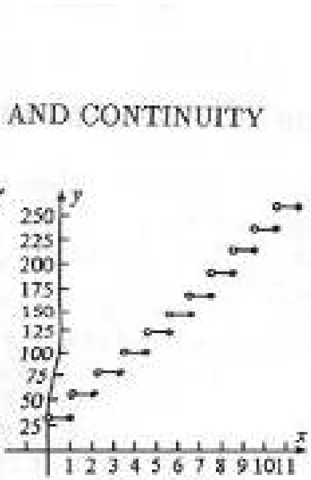
5. $\mathrm{C}(x)$ dollars is the cost of shipping x lb. $\mathrm{C}(x) = \begin{cases} 2.2x & \text{if } 0 < x \le 50 \\ 2.1x & \text{if } 50 < x \le 200 \\ 2.05x & \text{if } x > 200 \end{cases}$. $\mathrm{C}(50) = 2.2 \times 50 = 110$,

$\mathrm{C}(51) = 2.1 \times 51 = 107.10$, $\mathrm{C}(52) = 2.1 \times 52 = 109.20$, $\mathrm{C}(53) = 2.1 \times 53 = 111.30$, $\mathrm{C}(200) = 2.1 \times 200 = 420$
$\mathrm{C}(202) = 2.05 \times 202 = 414.10$, $\mathrm{C}(204) = 2.05 \times 204 = 418.20$, $\mathrm{C}(206) = 2.05 \times 206 = 422.30$

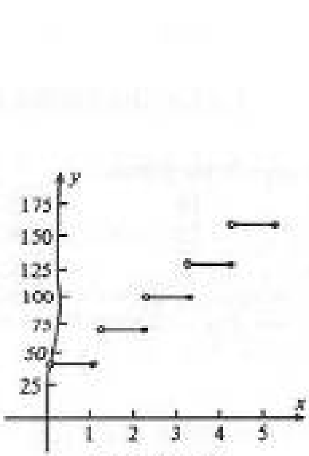
6. $y(x)$ cents is the cost of mailing x ounces. $y = 9 - 23[\![-x]\!]$. $y(1.6) = 9 - 23[\![-1.6]\!] = 9 - 23(-2) = 55$
$y(2) = 9 - 23[\![-2]\!] = 9 - 23(-2) = 55$
$y(2.1) = 9 - 23[\![-2.1]\!] = 9 - 23(-3) = 78$
$y(8.4) = 9 - 23[\![-8.4]\!] = 9 - 23(-9) = 216$
$y(11) = 9 - 23[\![-11]\!] = 9 - 23(-11) = 262$



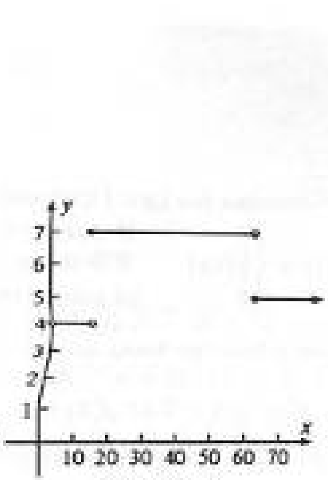
Exercise 5



Exercise 6



Exercise 7



Exercise 8

7. $y(x)$ cents is the cost of an x minute call. $y(x) = 10 - 30[\![-x]\!]$. $y(0.5) = 10 - 30[\![-0.5]\!] = 10 - 30(-1) = 40$
$y(2) = 10 - 30[\![-2]\!] = 10 - 30(-2) = 70$ $y(2.5) = 10 - 30[\![-2.5]\!] = 10 - 30(-3) = 100$
$y(3) = 10 - 30[\![-3]\!] = 10 - 30(-3) = 100$ $y(3.5) = 10 - 30[\![-3.5]\!] = 10 - 30(-4) = 130$
$y(5) = 10 - 30[\![-5]\!] = 10 - 30(-5) = 160$

8. The adult admission price is \$7, while the price for cildren under 12 years is \$4 and the price for seniors at least 60 yers is \$5. (a) Find a model of price as a function of age. (b) Sketch the graph of your function in (a).

▸ $y(x)$ dollars is the admission for age x years. $y(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \le x < 60 \\ 5 & \text{if } 60 \le x \end{cases}$.

9. The demand for a toy is $f(p) = \dfrac{5000}{p^2}$ where p dollars is the price. After t months the price is $g(t) = \frac{1}{20}t^2 + \frac{7}{20}t + 5$. (a) The demand after t months is $f(g(t)) = \dfrac{5000}{(\frac{1}{20}t^2 + \frac{7}{20}t + 5)^2} = \dfrac{2{,}000{,}000}{(t^2 + 7t + 100)^2}$.
(b) The demand after 5 months is $\dfrac{2{,}000{,}000}{(5^2 + 7\cdot 5 + 100)^2} = \dfrac{2000000}{160^2} = 78.125 \approx 78$

10. The number of large fish is $f(x) = \sqrt{20x + 150}$ where x is number of medium fish. The number if medium fish is $g(w) = \sqrt{w + 5000}$ where w is the number of small fish. (a) As a function of w the number of large fish is $f(g(w)) = \sqrt{20(\sqrt{w} + 5000) + 150}$. (b) There are 9 million small fish.
$f(g(9000000)) = \sqrt{20(3000 + 5000) + 150} = 550$. There are 550 large fish.

11. If r cm is the radius of a sphere and A cm^2 is the surface area, then $A(r) = 4\pi r^2$. The radius is increasing at 3 cm/sec, so $r = f(t) = 3t$. (a) After t sec the area is $A(f(t)) = 4\pi(3t)^2 = 36\pi t^2$ cm^2
(b) After 4 sec the area is $36\pi \cdot 4^2 = 576\pi \approx 1809.6$ cm^2

12. If r ft is the radius of a sphere and V ft^3 is the volume, then $V(r) = \frac{4}{3}\pi r^3$. The radius is 2 ft and decreases at 4.5 in/min $= \frac{3}{8}$ ft/min so $r = f(t) = 2 - \frac{3}{8}t$. (a) After t minutes the volume is
$V(f(t)) = \frac{4}{3}\pi(2 - \frac{3}{8}t)^3$. (b) After 3 minutes the volume is $\frac{4}{3}\pi(2 - \frac{3}{8}\cdot 3)^3 = 2.81$ ft^3.

13. A field of length x m is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - 2x) = 120 - x$ m and its area is $a(x) = (120 - x)x = 120x - x^2$, $0 \le x \le 120$. $a(x) = 3600 - (x - 60)^2$ is greatest when $x = 60$ and the field is 60 m × 60 m.

14. A garden of length x ft is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - 2x) = 50 - x$ ft and its area is $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$. $a(x) = 625 - (x - 25)^2$ is greatest when $x = 25$ and the garden is 25 ft × 25 ft

15. A field of length x m parallel to a river is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - x)$ m and its area is $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$. $a(x) = -\frac{1}{2}(x^2 - 240x + 120^2 + \frac{1}{2}\cdot 14400 = -\frac{1}{2}(x - 120)^2 + 7200$ is greatest when $x = 120$ and the field is 120 m × 60 m.

16. A garden of length x ft parallel to a house is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - x)$ ft and its area is $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$. $a(x) = -\frac{1}{2}(x^2 - 100x + 50^2) + 1250 = -\frac{1}{2}(x - 50)^2 + 1250$ is greatest when $x = 50$ and the garden is 50 ft × 25 ft.

17. x in. squares are cut from the corners of 8 in. by 15 in. sheet and the sides are turned up. (a) Find the volume V(x) in^3. (b) Find dom(V). (c) Maximize the volume graphically.

▸ $V = \ell wh = (8 - 2x)(15 - 2x)x$, $0 \le x \le 4$. The maximum volume is 90.74 in^3 when $x = \frac{5}{3} \approx 1.7$ (in.)

18. x cm squares are cut from the corners of 12 cm square and the sides are turned up. (a) Find the volume $V(x)$ cm^3. (b) Find dom(V). (c) Maximize the volume graphically.

▷ $V = \ell wh = (12-2x)^2 x = \frac{1}{4}\cdot 4x(12-2x)(12-2x)$ (sum = 24) $0 \le x \le 6$. The maximum volume is 128 cm^3 when $12 - 2x = 4x$, $x = 2$. Cut 2 cm squares.

19. x in. squares are cut from the corners of 12 in. by 15 in. sheet and the sides are turned up. (a) Find the volume $V(x)$ in^3. (b) Find dom(V). (c) Maximize the volume graphically.

▷ $V = \ell wh = (12-2x)(15-2x)x$, $0 \le x \le 6$. The maximum volume is 177 in^3 when $x = 2.21$ (in.)

20. x cm squares are cut from the corners of 40 cm by 50 cm sheet and the sides are turned up. (a) Find the volume $V(x)$ cm^3. (b) Find dom(V). (c) Maximize the volume graphically.

▷ $V = \ell wh = (40-2x)(50-2x)x$, $0 \le x \le 20$. The maximum volume is 6564.2 cm^3 when $x = 7.36$ (cm)

21. 60 in^3 is the volume of a cylinder, r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, $2\pi r^2$ in^2 is the area of the ends. The cost is \$$k$/in^2 for the sides and \$$2k$/in^2 for the ends. The total cost is $C(r) = k\cdot 120/r + 2k\cdot 2\pi r^2 = k(120/r + 4\pi r^2)$, $\quad r > 0$. $\quad C = k(60/r + 60/r + 4\pi r^2)$ $\quad$(product $= 14400\pi$) $\quad$ is minimum when $60/r = 4\pi r^2$, $r^3 = 15/\pi$, $r \approx 1.68$ in.

22. 60 in^3 is the volume of a cylinder, r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, πr^2 in^2 is the area of the bottom. The total surface is $S(r) = 120/r + \pi r^2$, $r > 0$. $S = 60/r + 60/r + \pi r^2$ (product $= 3600\pi$) is minimum when $60/r = \pi r^2$, $r^3 = 60/\pi$, $r \approx 2.67$ in.

23. A page with margins of 1.5 in. at the top and bottom and 1 in. at the sides is to contain 24 in^2 of print. (a) Find the total area of the page, $A(x)$ in^2, when the width of the printed region is x in. (b) What is the domain D of A? (c) Approximate to the nearest hundredth of an inch the size of the smallest page.

▷ (a) The length of the printed region is $\frac{24}{x}$ in. $A(x) = (x+2)\left(\frac{24}{x} + 3\right) = 30 + \left(3x + \frac{48}{x}\right)$ (product = 144)

(b) D: $x > 0$ (c) $A_{min} = 54$ when $3x = \frac{48}{x}$, $x = 4$. $x + 2 = 4 + 2 = 6$, $\frac{24}{x} + 3 = \frac{24}{4} + 3 = 9$. The smallest page is 6 in. wide and 9 in. long.

24. A lot with walkways 22 ft wide at the front and back and 15 ft at the sides is to contain a 13,200 ft.^{2} building. (a) Find the total area of the lot, $A(x)$ ft^2, when the width of the front is x ft. (b) What is the domain D of A? (c) Approximate to the nearest hundredth of a foot the size of the smallest lot.

▷ (a) The length of the building is $\frac{13{,}200}{x}$ ft. $A(x) = (x+30)\left(\frac{13200}{x} + 44\right) = 14520 + (44x + 396000/x)$

(b) D: $x > 0$ (c) Because the product of the variable terms is 1,742,000, the smallest lot has area 22,868.4 ft^2 when $44x = 396{,}000/x$, $x = \sqrt{9000} \approx 94.87$. The field is 124.87 ft by 183.14 ft.

25. A box of length x in. with square cross section has 100 in. as the sum of its length and girth. (a) Find the volume V in^3 as a function of x. (b) What is the domain D of V? (c) Approximate to the nearest inch the dimensions of the largest box.

▷ (a) The width of the box is $\frac{1}{4}(100-x)$ in. $V = x[\frac{1}{4}(100-x)]^2 = \frac{1}{32}\cdot 2x(100-x)(100-x)$ (sum = 200)

(b) D: $20 \le x \le 100$ (length $\ge$ width) (c) $V_{max} = \frac{1}{32}(\frac{200}{3})^2 \approx 9259$ when $100 - x = 2x$, $x = \frac{100}{3}$.
The largest box is about 33 by 17 by 17 in.

26. The growth rate f bacteria/min of a colony is jointly proportional to the number x of bacteria and the number $1{,}000{,}000 - x$ of capacity. $f(x) = kx(1{,}000{,}000 - x)$, $f(1000) = 60 = 1000k \cdot 999{,}000$, $k = \frac{60}{999{,}000{,}000}$ $= \frac{1}{16{,}650{,}000}$. $f = x(1{,}000{,}000 - x)/16{,}650{,}000$, $0 \le x \le 1{,}000{,}000$. $f(100{,}000) = 100{,}000 \cdot 900{,}000/16{,}650{,}000$ ≈ 5405 bacteria/min. $f(x) = k(-x^2 + 1{,}000{,}000x)$ is maximum when $x = -1{,}000{,}000/-2 = 500{,}000$.

27. The growth rate f infected/day of an epidemic is jointly proportional to the number x of infected and the number $5{,}000 - x$ of capacity. $f(x) = kx(5{,}000 - x)$, $f(100) = 9 = 100k \cdot 4900$, $k = 9/490{,}000$.
$f(x) = 9x(5{,}000 - x)/490{,}000$, $0 \le x \le 5{,}000$. $f(200) = 9 \cdot 200 \cdot 4800/490{,}000 \approx 17.6 \approx 18$ people/day.
$f(x) = k(-x^2 + 5{,}000x)$ is maximum when $x = -5{,}000/-2 = 2500$.

28. The base of a pyramidal tent is $2x$ m square and a triangular side has height $2.5 - x$ m. The height h of the tent satisfies $h^2 + x^2 = (2.5 - x)^2$, $h = \sqrt{6.25 - 5x}$. The volume is $V = \frac{1}{3}(2x)^2\sqrt{6.25 - 5x}$, $0 \le x \le 1.25$.
$V(0.8) = \frac{1}{3}(2\cdot 0.8)^2\sqrt{6.25 - 5\cdot 0.8} = \frac{1}{3}\cdot 1.6^2\sqrt{2.25} = 1.28$ m^3. $V^2 = \frac{25}{9}x^4(1 - \frac{4}{5}x) = \frac{1}{9}5^6(\frac{1}{5}x)(\frac{1}{5}x)(\frac{1}{5}x)(\frac{1}{5}x)(1 - \frac{4}{5}x)$
Because the sum of the variable factors is 1, V is maximum when $\frac{1}{5}x = 1 - \frac{4}{5}x$, $x = 1$

1.4 GRAPHICAL INTRODUCTION TO LIMITS OF FUNCTIONS

The formal definition of limit is given in the next section:

1.4.1 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. The *limit of $f(x)$ as x approaches a is L*, written as

$$\lim_{x \to a} f(x) = \mathrm{L}$$

if the following statement is true:
Given any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad \left|f(x) - \mathrm{L}\right| < \epsilon$$

Note that $0 < |x-a|$ implies that $x \neq a$. This hypothesis must be used whenever $f(a)$ is not defined as in Exercises 29 and 30.

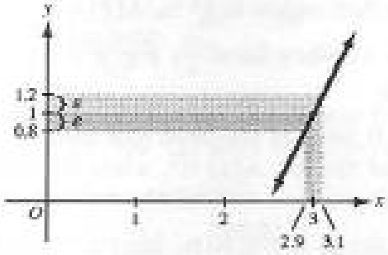
To find δ we usually factor the expression $\left|f(x)-\mathrm{L}\right|$ into the form $|x-a| \cdot \left|g(x)\right|$. If $f(x)$ is a first degree polynomial, then $\left|g(x)\right|$ is a constant, and we find δ as in Exercise 28. However, if $f(x)$ is not a first degree polynomial, then $\left|g(x)\right|$ is not a constant, and we must find an upper bound for $\left|g(x)\right|$, as in Exercises 36. We may often find this upper bound if we assume that $\delta \leq 1$.

Finally, we note that the definition does not tell us how to find the limit L. In Section 1.5 we have theorems that can be used to find L.

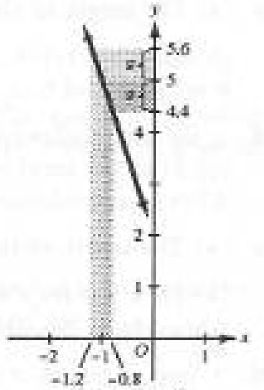
Exercises 1.4

1. $f(x) = 2x - 5$, $a = 3$, $\mathrm{L} = 1$, $\epsilon = 0.2$.
▹ From the figure, $\delta = .1$

2. $f(x) = 2 - 3x$, $a = -1$, $\mathrm{L} = 5$, $\epsilon = 0.6$.
▹ From the figure, $\delta = .4$



Exercise 1



Exercise 2

3 and 25. $f(x) = x - 1$, $a = 4$, $\mathrm{L} = 3$, $\epsilon = 0.03$.
▹ $x_1 - 1 = 3 - 0.03$, $x_1 - 4 = -0.03$. $x_2 - 1 = 3 + 0.03$, $x_2 - 4 = 0.03$. Choose $\delta = 0.03$.
$\left|f(x) - \mathrm{L}\right| = \left|(x-1) - 3\right| = |x - 4| < \epsilon$ when $|x-4| < \epsilon = 0.03 = \delta$.

4 and 26. $f(x) = x + 2$, $a = 3$, $\mathrm{L} = 5$, $\epsilon = 0.02$
▹ $x_1 + 2 = 5 - 0.02$, $x_1 - 3 = -0.02$. $x_2 + 2 = 5 + 0.02$, $x_2 - 3 = 0.02$. Choose $\delta = 0.02$.
$\left|f(x) - \mathrm{L}\right| = \left|(x+2) - 5\right| = |x-3| < \epsilon$ when $|x-3| < \epsilon = 0.02 = \delta$

5. $f(x) = 2x + 4$, $a = 3$, $\mathrm{L} = 10$, $\epsilon = 0.01$
▹ $2x_1 + 4 = 10 - 0.01, 2x_1 - 6 = -.01$, $x_1 - 3 = -.005$. $2x_2 + 4 = 10 + .01$, $2x_2 - 6 = 0.01$, $x_2 - 3 = .005$. $\delta = .005$

6. $f(x) = 3x - 1$, $a = 2$, $\mathrm{L} = 5$, $\epsilon = 0.1$
▹ $3x_1 - 1 = 5 - 0.1$, $3x_1 - 6 = -0.1$, $x_1 - 2 = -\frac{1}{30}$. $3x_2 - 1 = 5 + 0.1$, $3x_2 - 6 = 0.1$, $x_2 - 2 = \frac{1}{30}$. Choose $\delta = \frac{1}{30}$.

7 and 27. $f(x) = 5x - 3$, $a = 1$, $\mathrm{L} = 2$, $\epsilon = 0.05$
▹ $5x_1 - 3 = 2 + 0.05$, $5x_1 - 5 = 0.05$, $x_1 - 1 = 0.01$. $5x_2 - 3 = 2 + 0.05$, $5x_2 - 5 = 0.05$, $x_2 - 1 = 0.01$. $\delta = 0.01$.
$\left|(5x-3) - 2\right| = |5x - 5| = 5|x-1| < \epsilon$ when $|x - 1| = \frac{1}{5}\epsilon = .01 = \delta$

8 and 28. $f(x) = 4x - 5$, $a = 2$, $\mathrm{L} = 3$, $\epsilon = 0.001$
▹ $4x_1 - 5 = 3 - .001$, $4x_1 - 8 = -.001$, $x_1 - 2 = -.00025$. $4x_2 - 5 = 3 + .001$, $4x_2 - 8 = .001$, $x_2 - 2 = .00025 = \delta$.
$\left|(4x-5) - 3\right| = |4x - 8| = 4|x-2| < \epsilon$ when $|x-2| < \frac{1}{4}\epsilon = 0.00025 = \delta$

9. $f(x) = 3 - 4x$, $a = -1$, $\mathrm{L} = 7$, $\epsilon = 0.02$. f is decreasing.
▹ $3 - 4x_1 = 7 + .02$, $-4 - 4x_1 = .02$, $x_1 + 1 = -.005$. $3 - 4x_2 = 7 - .02$, $-4 - 4x_2 = -.02$, $x_2 + 1 = .005$. $\delta = .005$

10. $f(x) = 2 + 5x$, $a = -2$, $\mathrm{L} = -8$, $\epsilon = 0.002$
▹ $2 + 5x_1 = -8 - .002, 5x_1 + 10 = -.002, x_1 + 2 = -.0004$. $2 + 5x_2 = -8 + .002, 5x_2 + 10 = .002, x_2 + 2 = .0004 = \delta$

11. $f(x) = \frac{x^2-4}{x+2}$, $a = -2$, L $= -4$, $\epsilon = 0.01$. ▷ $f(x) = x - 2$ if $x \neq -2$

▷ $x_1 - 2 = -4 - .01$, $x_1 - 2 = -4 - .01$, $x_1 + 2 = -.01$. $x_2 - 2 = -4 + .01$, $x_2 - 2 = -4 + .01$, $x_2 + 2 = .01 = \delta$

12. $f(x) = \frac{9x^2-1}{3x-1}$, $a = \frac{1}{3}$, L $= 2$, $\epsilon = 0.01$ ▷ $f(x) = 3x + 1$ if $x \neq \frac{1}{3}$

$3x_1 + 1 = 2 - .01$, $3x_1 - 1 = -.01$, $x_1 - \frac{1}{3} = -\frac{1}{300}$. $3x_2 + 1 = 2 + .01$, $3x_2 - 1 = .01$, $x_2 - \frac{1}{3} = \frac{1}{300}$. $\delta = \frac{1}{300}$

13 and 29. $f(x) = \frac{4x^2-4x-3}{2x+1}$, $a = -\frac{1}{2}$, L $= -4$, $\epsilon = 0.03$ ▷ $f(x) = 2x - 3$ if $x \neq -\frac{1}{2}$

$2x_1 - 3 = -4 - .03$, $2x_1 + 1 = -.03$, $x_1 + \frac{1}{2} = -.015$. $2x_1 - 3 = -4 + .03$, $2x_1 + 1 = .03$, $x_1 + \frac{1}{2} = .015$. $\delta = .015$

29. $\left|(2x-3) - (-4)\right| = |2x+1| = 2\left|x + \frac{1}{2}\right| < \epsilon$ when $0 < \left|x + \frac{1}{2}\right| < \frac{1}{2}\epsilon = .015 = \delta$

14 and 30. $f(x) = \frac{3x^2-8x-3}{x-3}$, $a = 3$, L $= 10$, $\epsilon = .05$ ▷ $f(x) = 3x + 1$ if $x \neq 3$

$3x_1 + 1 = 10 - .05$, $3x_1 - 9 = -.05$, $x_1 - 3 = -\frac{1}{60}$. $3x_2 + 1 = 10 + .05$, $3x_2 - 9 = .05$, $x_2 - 3 = \frac{1}{60}$. $\delta = \frac{1}{60}$

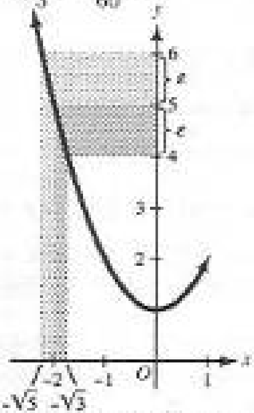
30. $\left|(3x+1) - 10\right| = |3x - 9| = 3|x - 3| < \epsilon$ when $0 < |x - 3| < \frac{1}{3}\epsilon = \frac{1}{60} = \delta$

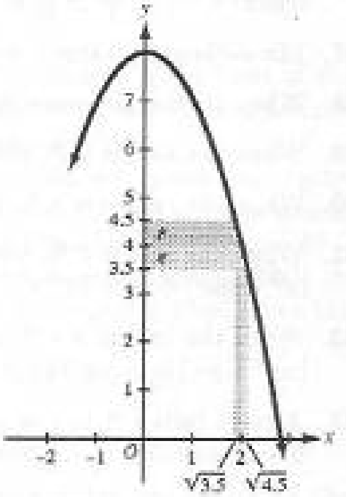
15. $f(x) = x^2 + 1$, $a = -2$, L $= 5$, $\epsilon = 1$

▷ From the figure, $\delta = \min(-2 + \sqrt{6}, -\sqrt{3} + 2)$
$= \min(0.449, 0.268) = 0.268$

16. $f(x) = 9 - x^2$, $a = 2$, L $= 5$, $\epsilon = 0.5$

▷ From the figure, $\delta = \min(2 - \sqrt{7/2}, \sqrt{9/2} - 2)$
$= \min(0.129, 0.121) = 0.121$





Note. For nonlinear f, the values of δ obtained from inequalities are usually smaller than those obtained directly. The factoring is really done first to determine what bounds we need.

17 and 31. $f(x) = x^2$, $a = 3$, L $= 9$, $\epsilon = 0.5$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 = 9 - 0.5 = 8.5$,
$x_1 = \sqrt{8.5} = 2.915$, $3 - x_1 = .085$. ${x_2}^2 = 9 + 0.5 = 9.5$, $x_2 = \sqrt{9.5} = 3.082$. $x_2 - 3 = .082$. $\delta = .082$

31. Choose $\delta < 1$ so $-1 < x - 3 < 1$, $-5 < x + 3 < 7$. Then $\left|x^2 - 9\right| = |x + 3||x - 3| < 7|x - 3| < \epsilon$ when $|x - 3| < \frac{1}{7}\epsilon = \frac{1}{14} = \delta$

18 and 32. $f(x) = x^2$, $a = 0.5$, L $= 0.25$, $\epsilon = 0.1$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 = .25 - .1 = .15$,
$x_1 = \sqrt{.15} = 0.387$, $.5 - x_1 = .113$. ${x_2}^2 = .25 + .1 = .35$, $x_2 = \sqrt{.35} = 0.592$, $x_2 - .5 = .092$. $\delta = .092$

32. Choose $\delta < 1$ so $-1 < x - .5 < 1$, $0 < x + .5 < 2$. Then $\left|x^2 - .25\right| = |x + .25||x - .25| < 2|x - .5| < \epsilon$ if $|x - .5| < \frac{1}{2}\epsilon = .05 = \delta$

19. $f(x) = x^2$, $a = -1$, L $= 1$, $\epsilon = .2$ ▷ $a < 0 \Rightarrow x_1, x_2 < 0$. f is decreasing. ${x_1}^2 = 1 + .2 = 1.2$
$x_1 = -\sqrt{1.2} = -1.095$, $-1 - x_1 = .095$. ${x_2}^2 = 1 - .2 = -.8$, $x_2 = -\sqrt{.8} = -.894$, $x_1 - (-1) = .106$. $\delta = .095$

20. $f(x) = x^2 - 5$, $a = 1$, L $= -4$, $\epsilon = 0.15$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 - 5 = -4 - .15$, ${x_1}^2 = .85$
$x_1 = \sqrt{.85} = .922$, $1 - x_1 = .078$. ${x_2}^2 - 5 = -4 + .15$, ${x_2}^2 = 1.15$, $x_2 = \sqrt{1.15} = 1.072$, $x_2 - 1 = .072$. $\delta = .072$

21 and 33. $f(x) = x^2 - 2x + 1$, $a = 2$, L $= 1$, $\epsilon = 0.4$ ▷ $x_1, x_2 > 1$. ${x_1}^2 - 2x_1 + 1 = 1 - .4$, $(x_1 - 1)^2 = .6$
$x_1 - 1 = \sqrt{.6}$, $x_2 - 2 = \sqrt{.6} - 1 = -.225$. ${x_2}^2 - 2x_2 + 1 = 1 + .4$, $(x_2 - 1)^2 = 1.4$, $x_1 - 1 = \sqrt{1.4}$,
$x_2 - 2 = \sqrt{1.4} - 1 = .183$. $\delta = .183$

33. Choose $\delta < 1$ so $-1 < x - 2 < 1$, $1 < x < 3$. $\left|(x^2 - 2x + 1) - 1\right| = \left|x^2 - 2x\right| = |x||x - 2| < 3|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{3}\epsilon = \frac{2}{15} = \delta$.

22 and 34. $f(x) = x^2 + 4x + 4$, $a = -1$, L $= 1$, $\epsilon = .08$ ▷ $x_1, x_2 > -2$. ${x_1}^2 + 4x_1 + 4 = 1 - .08$, $(x_1 + 2)^2 = .92$
$x_1 + 2 = \sqrt{.92}$, $x_1 + 1 = \sqrt{.92} - 1 = -.041$. ${x_2}^2 + 4x_2 + 4 = 1 + .08$, $(x_2 + 2)^2 = 1.08$, $x_2 + 2 = \sqrt{1.08}$.
$x_2 + 1 = \sqrt{1.08} - 1 = .0392$

34. Choose $\delta < 1$ so $-1 < x + 1 < 1, 1 < x + 3 < 3$. $\left|(x^2 + 4x + 4) - 1\right| = \left|x^2 + 4x + 3\right| = |x + 3||x + 1| < 3|x + 1| < \epsilon$ when $|x + 1| < \frac{1}{3}\epsilon = \frac{2}{75} = \delta$

23 and 35. $f(x) = 2x^2 + 5x + 3$, $a = -3$, $L = 6$, $\epsilon = .6$ ▹ $x_1, x_2 < -2.5$. f is decreasing. $2{x_1}^2 + 5x_1 + 3 = 6 +$
$2{x_1}^2 + 5x_1 - 3.6 = 0$, $x_1 = (-5 - \sqrt{5^2 + 4 \cdot 2 \cdot 3.6})/4 = -3.084$. $-3 - x_1 = .084$. $2x_2 + 5x_2 + 3 = 6 - .6$,
$2x_2 + 5x_2 - 2.4 = 0$, $x_2 = (-5 - \sqrt{5^2 + 4 \cdot 2 \cdot 2.4})/4 = -2.912$. $x_2 + 3 = .088$. $2x_2 + 5x_2$. $\delta = .084$

35. Choose $\delta < 1$ so $-1 < x + 3 < 1$, $-2 < 2x + 6 < 2$, $-9 < 2x - 1 < -5$. $\left|(2x^2 + 5x + 3) - 6\right| =$
$\left|2x^2 + 5x - 3\right| = |2x - 1||x + 3| < 9|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{9}\epsilon = \frac{1}{15} = \delta$

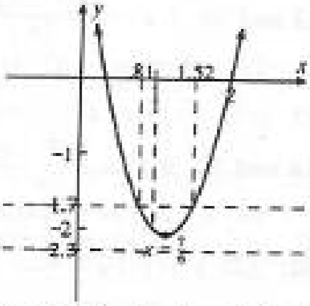
24 and 36. $f(x) = 3x^2 - 7x + 2$, $a = 1$, $L = -2$, $\epsilon = .3$

▹ $x_1, x_2 < \frac{7}{6}$. f is decreasing. $3{x_1}^2 - 7x_1 + 2 = -2 + .3$, $3{x_1}^2 - 7x_1 + 3.7 = 0$,

$x_1 = (7 - \sqrt{7^2 - 4 \cdot 3 \cdot 3.7})/6 = .809$, $1 - x_1 = .191$. Because

$f(x) \geq f(\frac{7}{6}) > -2 - .3$, we may take $x_2 = (7 + \sqrt{7^2 - 4 \cdot 3 \cdot 3.7})/6 = 1.524$,

$x_2 - 1 = .524$. $\delta = .191$.



36. Choose $\delta < 1$ so $-1 < x - 1 < 1$, $-3 < 3x - 3 < 3$, $-4 < 3x - 4 < 2$.
$\left|(3x^2 - 7x + 2) - (-2)\right| = \left|3x^2 - 7x + 4\right| = |3x - 4||x - 1| < 4|x - 1| < \epsilon$
when $|x - 1| < \frac{1}{4}\epsilon = \frac{3}{40} = \delta$.

37. $15x$ dollars are earned working x hours. $|15x - 120| = 15|x - 8| < .25$ when $|x - 8| < \frac{1}{60}$. Within 1 min of 8 hr.

38. When the temperature is x degrees the volume is $\frac{4}{7}x$ cm^3. $79.95 < \frac{4}{7}x < 80.05$ when $139.9125 < x < 140.0875$.

39. When the side is x ft, the fence is $4x$ ft. $39.96 < 4x < 40.04$ $-.04 < 4x - 40 < .04$, $|x - 10| < .01$. Within .01 ft.

40. When the radius is r ft, the circumference is $2\pi r$ ft. $|2\pi r - 6\pi| < .1$, $|r - 3| < 1/20\pi$. Within .0159 ft.

41. When the side is x ft, the area is x^2 ft^2. Assume $-1 < x - 10 < 1$, $19 < x + 10 < 21$.
$\left|x^2 - 100\right| = |x + 10||x - 10| < 21|x - 10| < .5$ when $|x - 10| < \frac{1}{42}$. Within $\frac{2}{7}$ in.

42. When the radius is r ft. the area is πr^2 ft^2. Assume $-1 < r - 3 < 1$, $5 < r + 3 < 7$.
$\left|\pi r^2 - 9\pi\right| = \pi|r + 3||r - 3| < 7\pi|r - 3| < 0.2$ when $|r - 3| < 1/35\pi \approx 0.0091$. Within 0.0091 ft.

43. A body falls s ft in t sec, $s = kt^2$. $t = 2$: $64 = k2^2$, $k = 16$. $s = 16t^2$. $398 < 16t^2 < 402$, $t_1 = \sqrt{398/16} = 4.9875$,
$5 - t_1 = .0125$. $t_2 = \sqrt{402/16} = 5.0125$, $t_2 - 5 = .0125$. Within .0125 sec.

44. A wind has pressure f psi when its velocity is v mph, $f = kv^2$. $v = 20$: $2 = k20^2$, $k = \frac{1}{200}$. $f = \frac{1}{200}v^2$.
$4.45 < \frac{1}{200}v^2 < 4.55$, $890 < v^2 < 910$, $29.833 < v < 30.166$, $-.167 < v - 30 < .166$. Within .166 mph.

1.5 DEFINITION OF THE LIMIT OF A FUNCTION AND LIMIT THEOREMS

The definition of limit was given in Section 1.4. The following theorems may often be used to find the limit of a function. "L.T." means "Limit Theorem".

L.T.1: Linear Function If m and b are any constants, $\lim_{x \to a} (mx + b) = ma + b$.

L.T.2: Constant If c is a constant, then for any number a, $\lim_{x \to a} c = c$.

L.T.3: Identity Function $\lim_{x \to a} x = a$.

L.T.4: Sum/Difference If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$

$$\Leftrightarrow \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

L.T.5: Sum/Difference If $\lim_{x \to a} f_1(x) = L_1$, $\lim_{x \to a} f_2(x) = L_2$, …, and $\lim_{x \to a} f_n(x) = L_n$, then
$\lim_{x \to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = L_1 \pm L \pm \cdots \pm L_n$

$$\Leftrightarrow \lim_{x \to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = \lim_{x \to a} f_1(x) \pm \lim_{x \to a} f_2(x) \pm \cdots \pm \lim_{x \to a} f_n(x)$$

L.T.6: Product If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$

$$\Leftrightarrow \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

It follows from L.T.2 that if c is a constant, $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$.

L.T.7: Product If $\lim_{x \to a} f_1(x) = L_1$, $\lim_{x \to a} f_2(x) = L_2$, …, and $\lim_{x \to a} f_n(x) = L_n$, then
$\lim_{x \to a} [f_1(x) f_2(x) \cdots f_n(x)] = L_1 L \cdots L_n$

$$\Leftrightarrow \lim_{x \to a} [f_1(x) f_2(x) \cdots f_n(x)] = \lim_{x \to a} f_1(x) \cdot \lim_{x \to a} f_2(x) \cdot \cdots \cdot \lim_{x \to a} f_n(x)$$

L.T.8: Power If $\lim_{x\to a} f(x) = \mathrm{L}$ and n is any positive integer, then $\lim_{x\to a} [f(x)]^n = \mathrm{L}^n$

$$\Leftrightarrow \lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$$

L.T.9: Quotient If $\lim_{x\to a} f(x) = \mathrm{L}$ and $\lim_{x\to a} g(x) = \mathrm{M}$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\mathrm{L}}{\mathrm{M}}$ if $\mathrm{M} \neq 0$

$$\Leftrightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

L.T.10: Root If n is a positive integer, and $\lim_{x\to a} f(x) = \mathrm{L}$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\mathrm{L}}$

$$\Leftrightarrow \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

with the restriction that if n is even, $\mathrm{L} > 0$.

Substitution Limit Theorems 1–10 imply that the limit as x approaches a of any formula (that is, excluding functions defined in pieces) can be found by substitution, if substitution is meaningful.

Fractions Note that we cannot use Limit Theorem 9 to find the limit of a fraction if the limit of the denominator is zero. However, when the limit of both the numerator and denominator is zero at $x = a$ we may be able to apply one of the following methods:

(i) If the numerator and denominator are both polynomials, we apply the *factor theorem* which states that if $x = a$ is a zero of a polynomial, then $x - a$ is a factor. The common factor $x - a$ may then be canceled. This is illustrated in Exercise 32.

(ii) If the denominator is a polynomial and a term of the numerator is a radical, we may multiply by its conjugate, which is not zero at a, and apply method (i) to the product. This is shown in Exercises 20 and 32.

In so doing, we use the following theorem, a consequence of Definition 1.5.1.

Theorem A If $\lim_{x\to a} g(x) = \mathrm{L}$ and $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \mathrm{L}$.

We cannot use Limit Theorems 1–10 to find the limit of a fraction for which the limit of the denominator is zero and for which the limit of the numerator is not zero. In Section 1.7 we have a theorem for finding the limit of such a fraction.

Exercises 1.5

---

1. We wish to determine a $\delta > 0$ such that

if $0 < |x - 2| < \delta$ then $|7 - 7| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 2| < \delta$ then $0 < \epsilon$

Therefore for any $\epsilon > 0$, any choice of $\delta > 0$ will suffice; in particular take $\delta = \epsilon$.

2. We wish to determine a $\delta > 0$ such that

if $0 < |x - 5| < \delta$ then $|(-4) - (-4)| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 5| < \delta$ then $0 < \epsilon$

Therefore for any $\epsilon > 0$, any choice of $\delta > 0$ will suffice; in particular take $\delta = \epsilon$.

3. We wish to determine a $\delta > 0$ such that

if $0 < |x - 4| < \delta$ then $|(2x + 1) - 9| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 4| < \delta$ then $2|x - 4| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 4| < \delta$ then $|x - 4| < \frac{1}{2}\epsilon$

Hence take $\delta = \frac{1}{2}\epsilon$; then $0 < |x - 4| < \delta \Rightarrow 2|x - 4| < 2(\frac{1}{2}\epsilon) \Rightarrow |(2x + 1) - 9| < \epsilon$.

4. We wish to determine a $\delta > 0$ such that

if $0 < |x - 1| < \delta$ then $|(4x + 3) - 7| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 1| < \delta$ then $4|x - 1| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 1| < \delta$ then $|x - 1| < \frac{1}{4}\epsilon$

Hence take $\delta = \frac{1}{4}\epsilon$; then $0 < |x - 1| < \delta \Rightarrow 4|x - 1| < 4(\frac{1}{4}\epsilon) \Rightarrow |(4x + 3) - 7| < \epsilon$.

5. We wish to determine a $\delta > 0$ such that
if $0 < |x-3| < \delta$ then $\left|(7-3x)-(-2)\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $3|x-3| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $|x-3| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x-3| < \delta \Rightarrow 3|x-3| < 3(\frac{1}{3}\epsilon) \Rightarrow \left|(7-3x)-(-2)\right| < \epsilon$.

6. We wish to determine a $\delta > 0$ such that
if $0 < |x+4| < \delta$ then $\left|(2x+7)+1\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x+4| < \delta$ then $2|x+4| < \epsilon$
$\Leftrightarrow$ if $0 < |x+4| < \delta$ then $|x+4| < \frac{1}{2}\epsilon$
Hence take $\delta = \frac{1}{2}\epsilon$; then $0 < |x+4| < \delta \Rightarrow 2|x+4| < 2(\frac{1}{2}\epsilon) \Rightarrow \left|(2x+7)-1\right| < \epsilon$.

7. We wish to determine a $\delta > 0$ such that
if $0 < |x+2| < \delta$ then $\left|(1+3x)-(-5)\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $3|x+2| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $|x+2| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x+2| < \delta \Rightarrow 3|x+2| < 3(\frac{1}{3}\epsilon) \Rightarrow \left|(1+3x)-(-5)\right| < \epsilon$.

8. We wish to determine a $\delta > 0$ such that
if $0 < |x+2| < \delta$ then $\left|(1+3x)-(-5)\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $3|x+2| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $|x+2| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x+2| < \delta \Rightarrow 3|x+2| < 3(\frac{1}{3}\epsilon) \Rightarrow \left|(1+3x)-(-5)\right| < \epsilon$.

9. We wish to determine a $\delta > 0$ such that
if $0 < |x+1| < \delta$ then $\left|\frac{x^2-1}{x+1}-(-2)\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x+1| < \delta$ then $|x+1| < \epsilon$
Hence, take $\delta = \epsilon$; then $0 < |x+1| < \delta \Rightarrow 0 < |x+1| < \epsilon \Rightarrow \left|\frac{x^2-1}{x+1}-(-2)\right| < \epsilon$.

10. We wish to determine a $\delta > 0$ such that
if $0 < |x-3| < \delta$ then $\left|\frac{x^2-9}{x-3}-6\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $|x-3| < \epsilon$
Hence, take $\delta = \epsilon$; then $0 < |x-3| < \delta \Rightarrow 0 < |x-3| < \epsilon \Rightarrow \left|\frac{x^2-9}{x-3}-6\right| < \epsilon$.

11. $\lim\limits_{x\to 5} (3x-7) \overset{\text{L.T.1}}{=} 3(5)-7 = 8$

12. $\lim\limits_{x\to -4} (5x+2) \overset{\text{L.T.1}}{=} 5(-4)+2 = -18$

13. $\lim\limits_{x\to 2} (x^2+2x-1) \overset{\text{L.T.4}}{=} \lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} (2x-5) \overset{\substack{\text{L.T.8}\\ \text{L.T.1}}}{=} (\lim\limits_{x\to 2} x)^2 + [2(2)-1] \overset{\text{L.T.3}}{=} (2)^2+3 = 7$

14. $\lim\limits_{x\to 3} (2x^2-4x+5) \overset{\text{L.T.4}}{=} \lim\limits_{x\to 3} 2x^2 + \lim\limits_{x\to 3} (-4x+5) \overset{\substack{\text{L.T.1}\\ \text{L.T.7}}}{=} \lim\limits_{x\to 3} 2 \cdot \lim\limits_{x\to 3} x \cdot \lim\limits_{x\to 3} x + (-4)3+5 \overset{\text{L.T.1}}{=} 2\cdot 3\cdot 3-7 = 11$

15. $\lim\limits_{x\to -2} (x^3+8) \overset{\text{L.T.4}}{=} \lim\limits_{x\to -2} x^3 + \lim\limits_{x\to -2} 8 \overset{\substack{\text{L.T.8}\\ \text{L.T.3}}}{=} \left(\lim\limits_{x\to -2} x\right)^3 + 8 \overset{\text{L.T.2}}{=} (-2)^3+8 = 0$

16. $\lim\limits_{y\to -1} (y^3-2y^2+3y-4) \overset{\text{L.T.1}}{=} \lim\limits_{y\to -1} y^3 - \lim\limits_{y\to -1} 2y^2 + \lim\limits_{y\to -1} (3y-4)$
$\overset{\text{L.T.7}}{=} \lim\limits_{y\to -1} y \cdot \lim\limits_{y\to -1} y \cdot \lim\limits_{y\to -1} y - \lim\limits_{y\to -1} 2 \cdot \lim\limits_{y\to -1} y \cdot \lim\limits_{y\to -1} y + \lim\limits_{y\to -1} (3y-4) \overset{\text{L.T.1}}{=} (-1)^3-2(-1)^2+3(-1)-4 = -10$

17. $\lim\limits_{x\to 3} \frac{4x-5}{5x-1} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 3} (4x-5)}{\lim\limits_{x\to 3} (5x-1)} \overset{\text{L.T.1}}{=} \frac{4(3)-5}{5(3)-1} = \frac{1}{2}$

18. $\lim\limits_{x\to 2} \frac{3x+4}{8x-1} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 2} (3x+4)}{\lim\limits_{x\to 2} (8x-1)} \overset{\text{L.T.1}}{=} \frac{3(2)+4}{8(2)-1} = \frac{10}{15} = \frac{2}{3}$

19. $\lim\limits_{t\to 2}\frac{t^2-5}{2t^3+6} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{t\to 2}(t^2-5)}{\lim\limits_{t\to 20}(2t^3+6)} \overset{\text{L.T.4}}{=} \frac{\lim\limits_{t\to 2} t^2-\lim\limits_{t\to 2} 5}{\lim\limits_{t\to 2}(2t^3)+\lim\limits_{t\to 2}(6)} \overset{\substack{\text{L.T.2}\\\text{L.T.6}}}{=} \frac{\lim\limits_{t\to 2} t^2-5}{(\lim\limits_{t\to 2} 2)(\lim\limits_{t\to 2} t^3)+6} \overset{\substack{\text{L.T.2}\\\text{L.T.8}}}{=} \frac{(\lim\limits_{t\to 2} t)^2-5}{2(\lim\limits_{t\to 2} t)^3+6} =$

$\overset{\text{L.T.3}}{=} \frac{(2)^2-5}{2(2)^3+6} = -\frac{1}{22}$

20. $\lim\limits_{x\to -1}\frac{2x+1}{x^2-3x+4} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\lim\limits_{x\to -1}(x^2-3x+4)} \overset{\text{L.T.4}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\lim\limits_{x\to -1} x^2+\lim\limits_{x\to -1}(-3x+4)} \overset{\text{L.T.8}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\left(\lim\limits_{x\to -1} x\right)^2+\lim\limits_{x\to -1}(-3x+4)}$

$\overset{\text{L.T.1}}{=} \frac{2(-1)+1}{(-1)^2+(-3)(-1)+4} = \frac{-1}{8}$

21. $\lim\limits_{r\to 1}\sqrt{\frac{8r+1}{r+3}} \overset{\text{L.T.10}}{=} \sqrt{\lim\limits_{r\to 1}\frac{8r+1}{r+3}} \overset{\text{L.T.9}}{=} \sqrt{\frac{\lim\limits_{r\to 1}(8r+1)}{\lim\limits_{r\to 2}(r+3)}} \overset{\text{L.T.1}}{=} \sqrt{\frac{8(1)+1}{1+3}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

22. $\lim\limits_{x\to 2}\sqrt{\frac{x^2+3x+4}{x^3+1}} \overset{\text{L.T.10}}{=} \sqrt{\lim\limits_{x\to 2}\frac{x^2+3x+4}{x^3+1}} \overset{\text{L.T.9}}{=} \sqrt{\frac{\lim\limits_{x\to 2}(x^2+3x+4)}{\lim\limits_{x\to 2} x^3+1}} \overset{\text{L.T.4}}{=} \sqrt{\frac{\lim\limits_{x\to 2} x^2+\lim\limits_{x\to 2}(3x+4)}{\lim\limits_{x\to 2} x^3+\lim\limits_{x\to 2} 1}}$

$\overset{\text{L.T.8}}{=} \sqrt{\frac{(\lim\limits_{x\to 2} x)^2+\lim\limits_{x\to 2}(3x+4)}{(\lim\limits_{x\to 2} x)^3+\lim\limits_{x\to 2} 1}} \overset{\text{L.T.1}}{=} \sqrt{\frac{2^2+3\cdot 2+4}{2^3+1}} = \sqrt{\frac{14}{9}} = \frac{1}{3}\sqrt{14}$

23. $\lim\limits_{x\to 4}\sqrt[3]{\frac{x^2-3x+4}{2x^2-x-1}} \overset{\text{L.T.10}}{=} \sqrt[3]{\lim\limits_{x\to 4}\frac{x^2-3x+4}{2x^2-x-1}} \overset{\text{L.T.9}}{=} \sqrt[3]{\frac{\lim\limits_{x\to 4}(x^2-3x+4)}{\lim\limits_{x\to 4}(2x^2-x-1)}} \overset{\text{L.T.4}}{=} \sqrt[3]{\frac{\lim\limits_{x\to 4} x^2+\lim\limits_{x\to 4}(-3x+4)}{\lim\limits_{x\to 4} 2x^2+\lim\limits_{x\to 4}(-x-1)}}$

$\overset{\substack{\text{L.T.7}\\\text{L.T.1}\\\text{L.T.8}}}{=} \sqrt[3]{\frac{(\lim\limits_{x\to 4} x)^2+(-8)}{\lim\limits_{x\to 4} 2(\lim\limits_{x\to 4} x)^2+(-5)}} \overset{\substack{\text{L.T.3}\\\text{L.T.2}}}{=} \sqrt[3]{\frac{(4)^2-8}{2(4)^2-5}} = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$

24. $\lim\limits_{x\to -3}\sqrt[3]{\frac{5+2x}{5-x}} \overset{\text{L.T.10}}{=} \sqrt[3]{\lim\limits_{x\to -3}\frac{5+2x}{5-x}} \overset{\text{L.T.9}}{=} \sqrt[3]{\frac{\lim\limits_{x\to -3}(5+2x)}{\lim\limits_{x\to -3}(5-x)}} \overset{\text{L.T.1}}{=} \sqrt[3]{\frac{5+2(-3)}{5-(-3)}} = \sqrt[3]{\frac{-1}{8}} = -\frac{1}{2}$

25. $f(x)=\frac{x-2}{x^2-4}$ (a)

x	1	1.5	1.9	1.99	1.999	3	2.5	2.1	2.01	2.001
f	.3333	.2857	.2564	.2506	.2501	.2000	.2222	.2439	.2494	.2499

(b) $\lim\limits_{x\to 2}\frac{x-2}{x^2-4} = \lim\limits_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2}\frac{1}{x+2} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2}(x+2)} \overset{\text{L.T.1}}{=} \frac{1}{4}$

26. $f(x)=\frac{2x^2+3x-2}{x^2-6x-16}$ (a)

x	-3	-2.5	-2.1	-2.01	-2.001	-1	-1.5	-1.9	-1.99	-1.999
f	.6364	.5714	.5149	.5015	.5002	.3333	.4211	.4848	.4985	.4998

(b) $\lim\limits_{x\to -2}\frac{2x^2+3x-2}{x^2-6x-16} = \lim\limits_{x\to -2}\frac{(x+2)(2x-1)}{(x+2)(x-8)} = \lim\limits_{x\to -2}\frac{2x-1}{x-8} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -2}(2x-1)}{\lim\limits_{x\to -2}(x-8)} \overset{\text{L.T.1}}{=} \frac{2(-2)-1}{-2-8} = \frac{-2}{-10} = \frac{1}{2}$

27. $f(x)=\frac{x^2+5x+6}{x^2-x-12}$ (a)

x	-4	-3.5	-3.1	-3.01	-3.001	-3.0001	-2	-2.5	-2.9	-2.99	-2.999	-2.9999
f	.2500	.2000	.1549	.1441	.1430	.1429	.0000	.0769	.1304	.1416	.1427	.1428

(b) $\lim\limits_{x\to -3}\frac{x^2+5x+6}{x^2-x-12} = \lim\limits_{x\to -3}\frac{(x+2)(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to -3}\frac{x+2}{x-4} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -3}(x+2)}{\lim\limits_{x\to -3}(x-4)} \overset{\text{L.T.1}}{=} \frac{-3+2}{-3-4} = \frac{-1}{-7} = \frac{1}{7}$

28. $f(x)=\frac{2x-3}{4x^2-9}$ (a)

x	1	1.4	1.49	1.499	1.4999	2	1.6	1.51	1.501	1.5001
f	.2	.1724	.1672	.1667	.1667	.1429	.1613	.1661	.1666	.1667

(b) $\lim\limits_{x\to 1.5}\frac{2x-3}{4x^2-9} = \lim\limits_{x\to 1.5}\frac{2x-3}{(2x-3)(2x+3)} = \lim\limits_{x\to 1.5}\frac{1}{2x+3} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 1.5} 1}{\lim\limits_{x\to 1.5}(2x+3)} \overset{\text{L.T.1}}{=} \frac{1}{2\cdot 1.5+3} = \frac{1}{6}$

29. $f(x)=\dfrac{3-\sqrt{x}}{9-x}$ (a)

x	8	8.5	8.9	8.99	8.999	10	9.5	9.1	9.01	9.001
f	.1716	.1690	.1671	.1667	.1667	.1623	.1644	.1662	.1666	.1667

(b) $$\lim_{x\to9}\frac{3-\sqrt{x}}{9-x}=\lim_{x\to9}\frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})}=\lim_{x\to9}\frac{9-x}{(9-x)(3+\sqrt{x})}=\lim_{x\to9}\frac{1}{3+\sqrt{x}}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to9}1}{\lim\limits_{x\to9}(3+\sqrt{x})}$$

$$\overset{\substack{\text{L.T.4}\\\text{L.T.2}}}{=}\frac{1}{\lim\limits_{x\to9}3+\lim\limits_{x\to9}\sqrt{x}}\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\frac{1}{3+\sqrt{\lim\limits_{x\to9}x}}\overset{\text{L.T.3}}{=}\frac{1}{3+\sqrt{9}}=\frac{1}{6}$$

30. $f(x)=\dfrac{2-\sqrt{4-x}}{x}$ (a)

x	-1	-0.5	-0.1	-0.01	-.001	1	0.5	0.1	0.01	0.001
f	.2361	.2426	.2485	.2498	.2500	.2679	.2583	.2516	.2502	.2500

$$\lim_{x\to0}\frac{2-\sqrt{4-x}}{x}\cdot\frac{2+\sqrt{4-x}}{2+\sqrt{4-x}}=\lim_{x\to0}\frac{4-(4-x)}{x(2+\sqrt{4-x})}=\lim_{x\to0}\frac{1}{2+\sqrt{4-x}}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to0}1}{\lim\limits_{x\to0}(2+\sqrt{4-x})}$$

$$\overset{\text{L.T.4}}{=}\frac{\lim\limits_{x\to0}1}{\lim\limits_{x\to0}2+\lim\limits_{x\to0}\sqrt{4-x}}\overset{\text{L.T.10}}{=}\frac{\lim\limits_{x\to0}1}{\lim\limits_{x\to0}2+\sqrt{\lim\limits_{x\to0}(4-x)}}\overset{\text{L.T.1}}{=}\frac{1}{2+\sqrt{4}}=\frac{1}{4}$$

31. $$\lim_{x\to7}\frac{x^2-49}{x-7}=\lim_{x\to7}\frac{(x-7)(x+7)}{x-7}=\lim_{x\to7}x+7\overset{\text{L.T.1}}{=}7+7=14$$

32. $$\lim_{z\to-5}\frac{z^2-25}{z+5}=\lim_{z\to-5}\frac{(z-5)(z+5)}{z+5}=\lim_{z\to-5}z-5\overset{\text{L.T.1}}{=}-5-5=-10$$

33. $$\lim_{x\to-3/2}\frac{4x^2-9}{2x+3}=\lim_{x\to-3/2}\frac{(2x+3)(2x-3)}{2x+3}=\lim_{x\to-3/2}2\text{x}-3\overset{\text{L.T.1}}{=}2(-\tfrac{3}{2})-3=-6$$

34. $$\lim_{x\to1/3}\frac{3x-1}{9x^2-1}=\lim_{x\to1/3}\frac{3x-1}{(3x-1)(3x+1)}=\lim_{x\to1/3}\frac{1}{3x+1}\overset{\text{L.T.1}}{=}\frac{1}{3(1/3)+1}=\frac{1}{2}$$

35. $$\lim_{s\to4}\frac{3s^2-8s-16}{2s^2-9s+4}=\lim_{s\to4}\frac{(3s+4)(s-4)}{(2s-1)(s-4)}=\lim_{s\to4}\frac{3s+4}{2s-1}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{s\to4}(3s+4)}{\lim\limits_{s\to4}(2s-1)}\overset{\text{L.T.1}}{=}\frac{3(4)+4}{2(4)-1}=\frac{16}{7}$$

36. $$\lim_{x\to4}\frac{3x^2-17x+20}{4x^2-25x+36}=\lim_{x\to4}\frac{(x-4)(3x-5)}{(x-4)(4x-9)}=\lim_{x\to4}\frac{3x-5}{4x-9}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to4}(3x-5)}{\lim\limits_{x\to4}(4x-9)}\overset{\text{L.T.1}}{=}\frac{3\cdot4-5}{4\cdot4-9}=\frac{7}{7}=1$$

37. $$\lim_{y\to-2}\frac{y^3+8}{y+2}=\lim_{y\to-2}\frac{(y+2)(y^2-2y+4)}{y+2}=\lim_{y\to-2}(y^2-2y+4)\overset{\text{L.T.4}}{=}\lim_{y\to-2}y^2+\lim_{y\to-2}(-2y+4)$$

$$\overset{\substack{\text{L.T.8}\\\text{L.T.1}}}{=}(\lim_{y\to-2}y)^2+[(-2)(-2)+4]\overset{\text{L.T.3}}{=}(-2)^2+8=12$$

38. $$\lim_{s\to1}\frac{s^3-1}{s-1}=\lim_{s\to1}\frac{(s-1)(s^2+s+1)}{s-1}=\lim_{s\to1}(s^2+s+1)\overset{\text{L.T.4}}{=}\lim_{s\to1}s^2+\lim_{s\to1}(s+1)\overset{\text{L.T.8}}{=}(\lim_{s\to1}s)^2+\lim_{s\to1}(s+1)$$

$$\overset{\text{L.T.1}}{=}1^2+1+1=3$$

39. $$\lim_{y\to-3}\sqrt{\frac{y^2-9}{2y^2+7y+3}}=\lim_{y\to-3}\sqrt{\frac{(y-3)(y+3)}{(2y+1)(y+3)2}}=\lim_{y\to-3}\sqrt{\frac{y-3}{2y+1}}\overset{\text{L.T.10}}{=}\sqrt{\lim_{y\to-3}\frac{y-3}{2y+1}}$$

$$\overset{\text{L.T.9}}{=}\sqrt{\frac{\lim\limits_{y\to-3}(y-3)}{\lim\limits_{y\to-3}(2y+1)}}\overset{\text{L.T.1}}{=}\sqrt{\frac{-6}{-5}}=\sqrt{\frac{6}{5}}=\tfrac{1}{5}\sqrt{30}$$

40. $$\lim_{t\to3/2}\sqrt{\frac{8t^3-27}{4t^2-9}}=\lim_{t\to3/2}\sqrt{\frac{(2t-3)(4t^2+6t+9)}{(2t-3)(2t+3)}}=\lim_{t\to3/2}\sqrt{\frac{4t^2+6t+9}{2t+3}}\overset{\substack{\text{L.T.10}\\\text{L.T.9}}}{=}\sqrt{\frac{\lim\limits_{t\to3/2}(4t^2+6t+9)}{\lim\limits_{t\to3/2}(2t+1)}}$$

$$\overset{\substack{\text{L.T.4}\\\text{L.T.7}}}{=}\sqrt{\frac{\lim\limits_{t\to3/2}4\cdot\lim\limits_{t\to3/2}t\cdot\lim\limits_{t\to3/2}t+\lim\limits_{t\to3/2}(6t+9)}{\lim\limits_{t\to3/2}(2t+1)}}\overset{\text{L.T.1}}{=}\sqrt{\frac{4(\frac{3}{2})^2+6\cdot\frac{3}{2}+9}{2\cdot\frac{3}{2}+1}}=\sqrt{\frac{27}{4}}=\tfrac{3}{2}\sqrt{3}$$

41. $$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to 1}1}{\lim\limits_{x\to 1}(\sqrt{x}+1)}\overset{\substack{\text{L.T.2}\\\text{L.T.4}}}{=}\frac{1}{\lim\limits_{x\to 1}\sqrt{x}+\lim\limits_{x\to 1}1}$$

$$\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\frac{1}{\sqrt{\lim\limits_{x\to 1}x}+1}\overset{\text{L.T.3}}{=}\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

42. $$\lim_{x\to -1}\frac{\sqrt{x+5}-2}{x+1}\cdot\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}=\lim_{x\to -1}\frac{(x+5)-4}{(x+1)(\sqrt{x+5}+2)}=\lim_{x\to -1}\frac{1}{\sqrt{x+5}+2}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to -1}1}{\lim\limits_{x\to -1}(\sqrt{x+5}+2)}$$

$$\overset{\text{L.T.4}}{=}\frac{\lim\limits_{x\to -1}1}{\lim\limits_{x\to -1}\sqrt{x+5}+\lim\limits_{x\to -1}2}\overset{\text{L.T.10}}{=}\frac{\lim\limits_{x\to -1}1}{\sqrt{\lim\limits_{x\to -1}(x+5)}+\lim\limits_{x\to -1}2}\overset{\text{L.T.1}}{=}\frac{1}{\sqrt{-1+5}+2}=\frac{1}{4}$$

43. $$\lim_{h\to 0}\frac{\sqrt{h+2}-\sqrt{2}}{h}=\lim_{h\to 0}\frac{(\sqrt{h+2}-\sqrt{2})(\sqrt{h+2}+\sqrt{2})}{h(\sqrt{h+2}+\sqrt{2})}=\lim_{h\to 0}\frac{(h+2)-2}{h(\sqrt{h+2}+\sqrt{2})}=\lim_{h\to 0}\frac{1}{\sqrt{h+2}+\sqrt{2}}$$

$$\overset{\text{L.T.9}}{=}\frac{\lim\limits_{h\to 0}1}{\lim\limits_{h\to 0}(\sqrt{h+2}+\sqrt{2})}\overset{\substack{\text{L.T.2}\\\text{L.T.4}}}{=}\frac{1}{\lim\limits_{h\to 0}\sqrt{h+2}+\lim\limits_{h\to 0}\sqrt{2}}\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\frac{1}{\sqrt{\lim\limits_{h\to 0}(h+2)}+\sqrt{2}}\overset{\text{L.T.1}}{=}\frac{1}{\sqrt{2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}=\tfrac{1}{4}\sqrt{2}$$

44. $$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(\sqrt[3]{x})^3-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)[(\sqrt[3]{x})^2+\sqrt[3]{x}+1]}=\lim_{x\to 1}\frac{1}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}$$

$$\overset{\substack{\text{L.T.4}\\\text{L.T.9}}}{=}\frac{\lim\limits_{x\to 1}1}{\lim\limits_{x\to 1}(\sqrt[3]{x})^2+\lim\limits_{x\to 1}\sqrt[3]{x}+\lim\limits_{x\to 1}1}\overset{\substack{\text{L.T.9}\\\text{L.T.10}}}{=}\frac{\lim\limits_{x\to 1}1}{(\sqrt[3]{\lim\limits_{x\to 1}x})^2+\sqrt[3]{\lim\limits_{x\to 1}x}+\lim\limits_{x\to 1}1}\overset{\text{L.T.1}}{=}\frac{1}{1+1+1}=\frac{1}{3}$$

45. $$\lim_{x\to -1}\frac{2x^2-x-3}{x^3+2x^2+6x+5}=\lim_{x\to -1}\frac{(2x-3)(x+1)}{(x^2+x+5)(x+1)}=\lim_{x\to -1}\frac{2x-3}{x^2+x+5}\overset{\text{L.T.9}}{=}\frac{\lim\limits_{x\to -1}(2x-3)}{\lim\limits_{x\to -1}(x^2+x+5)}$$

$$\overset{\substack{\text{L.T.4}\\\text{L.T.1}}}{=}\frac{-5}{\lim\limits_{x\to -1}x^2+\lim\limits_{x\to -1}(x+5)}\overset{\substack{\text{L.T.1}\\\text{L.T.8}}}{=}\frac{-5}{(\lim\limits_{x\to -1}x)^2+4}\overset{\text{L.T.3}}{=}\frac{-5}{(-1)^2+4}=-1$$

46. $$\lim_{x\to -2}\frac{x^3-x^2-x+10}{x^2+3x+2}=\lim_{x\to -2}\frac{(x+2)(x^2-3x+5)}{(x+2)(x+1)}=\lim_{x\to -2}\frac{x^2-3x+5}{x+1}\overset{\substack{\text{L.T.9}\\\text{L.T.4}}}{=}\frac{\lim\limits_{x\to -2}x^2+\lim\limits_{x\to -2}(-3x+5)}{\lim\limits_{x\to -2}(x+1)}$$

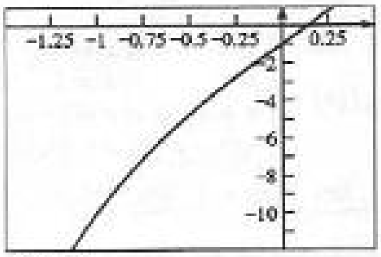
$$\overset{\text{L.T.8}}{=}\frac{(\lim\limits_{x\to -2}x)^2+\lim\limits_{x\to -2}(-3x+5)}{\lim\limits_{x\to -2}(x+1)}\overset{\text{L.T.1}}{=}\frac{(-2)^2-3(-2)+5}{-2+1}=\frac{15}{-1}=-15$$

47. $$\lim_{x\to 2}f(x)=\lim_{x\to 2}(x^2+5x-3)\overset{\text{L.T.4}}{=}\lim_{x\to 2}x^2+\lim_{x\to 2}(5x-3)$$

$$\overset{\text{L.T.8}}{=}(\lim_{x\to 2}x)^2+\lim_{x\to 2}(5x-3)\overset{\text{L.T.1}}{=}2^2+5(2)-3=f(2)$$

48. $$\lim_{x\to -1}\mathrm{F}(x)=\lim_{x\to -1}(2x^3+7x-1)\overset{\text{L.T.4}}{=}\lim_{x\to -1}2x^3+\lim_{x\to -1}(7x-1)$$

$$\overset{\text{L.T.7}}{=}\lim_{x\to -1}2\cdot\lim_{x\to -1}x\cdot\lim_{x\to -1}x+\lim_{x\to -1}(7x-1)\overset{\text{L.T.1}}{=}2(-1)^3+7(-1)-1=10=\mathrm{F}(-1)$$



49. $g(x)=\frac{x^2-1}{x-1}$. $g(1)$ does not exist because 0/0 is not defined. $\lim\limits_{x\to 1}g(x)=\lim\limits_{x\to 1}(x+1)=1+1=2$

50. $\mathrm{G}(x)=\frac{x-1}{x^2-1}$. $\mathrm{G}(1)$ does not exist because 0/0 is not defined. $\lim\limits_{x\to 1}\mathrm{G}(x)=\lim\limits_{x\to 1}\frac{1}{x+1}=\frac{1}{1+1}=\frac{1}{2}$

51. $h(x)=\dfrac{\sqrt{x+9}-3}{x}$. $h(0)$ does not exist because 0/0 is not defined.

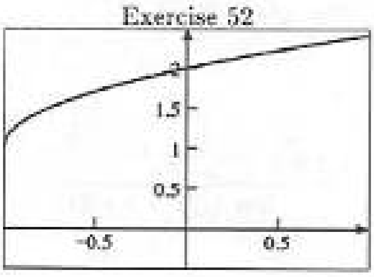
$$\lim_{x\to 0}h(x)=\frac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x(\sqrt{x+9}+3)}=\lim_{x\to 0}\frac{x+9-9}{x(\sqrt{x+9}+3)}=\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

52. $\mathrm{H}(x)=\dfrac{x}{\sqrt{x+1}-1}$. $\mathrm{H}(0)$ does not exist because 0/0 is not defined.

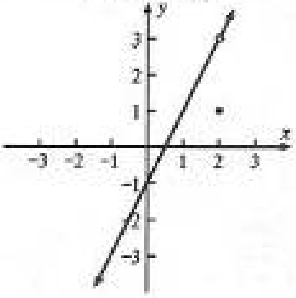
$$\lim_{x\to 0}\mathrm{H}(x)=\frac{x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{(x+1)-1}=\lim_{x\to 0}(\sqrt{x+1}+1)=1+1=2$$

53. $f(x) = \begin{cases} 2x-1 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$. $\lim_{x\to 2} f(x) = \lim_{x\to 2}(2x-1) = 2\cdot 2 - 1 = 3 \neq 1 = f(2)$

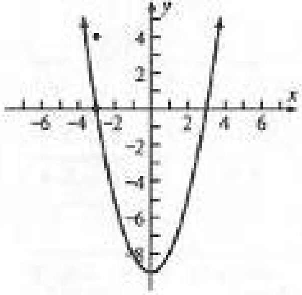
Exercise 52



Exercise 53



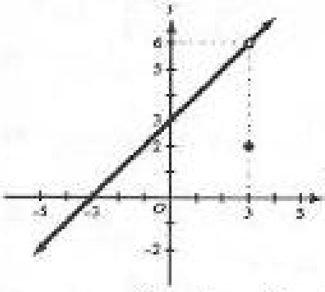
Exercise 54



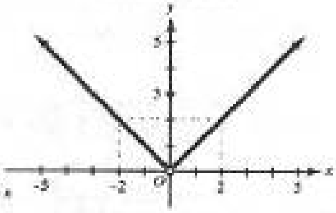
54. $f(x) = \begin{cases} x^2-9 & \text{if } x \neq -3 \\ 4 & \text{if } x = -3 \end{cases}$. $\lim_{x\to -3} f(x) = \lim_{x\to -3}(x^2-9) = (-3)^2 - 9 \neq 4 = f(-3)$

In Exercises 55–58, answer parts (a)–(c) from the graph of f sketched in the accompanying figure.

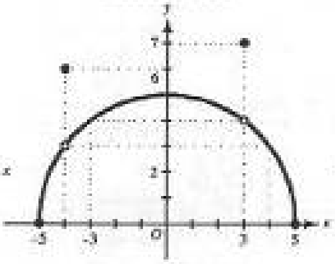
Exercise 55



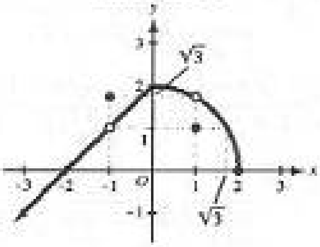
Exercise 56



Exercise 57



Exercise 58



55. $f(x) = \begin{cases} x+3 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$. $f(-3) = 0$, $f(0) = 3$, $f(3) = 2$. $\lim_{x\to -3} f(x) = 0$. $\lim_{x\to 0} f(x) = 3$. $\lim_{x\to 3} f(x) = 6$

56. $f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$. $f(-2) = 2$, $f(0) = 3$, $f(2) = 2$. $\lim_{x\to -2} f(x) = 2$, $\lim_{x\to 0} f(x) = 0$. $\lim_{x\to 2} f(x) = 2$

57. $f(x) = \begin{cases} 5 & \text{if } x = -4 \\ 6 & \text{if } x = 3 \\ \sqrt{25-x^2} & \text{if } x \in [-5,-4) \cup (-4,3) \cup (3,5] \end{cases}$. $f(-4) = 5$, $f(-3) = 4$, $f(3) = 6$, $f(4) = 3$

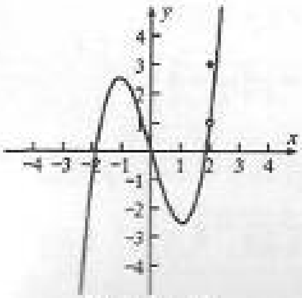
$\lim_{x\to -4} f(x) = 3$, $\lim_{x\to -3} f(x) = 4$, $\lim_{x\to 3} f(x) = 4$, $\lim_{x\to 4} f(x) = 3$

58. $f(x) = \begin{cases} \sqrt{3} & \text{if } x = -1 \\ 1 & \text{if } x = 1 \\ x+2 & \text{if } x \in (-\infty,-1) \cup (-1,0) \\ \sqrt{4-x^2} & \text{if } x \in [0,1) \cup (1,2] \end{cases}$. $f(-1) = \sqrt{3}$, $f(0) = 2$, $f(1) = 1$, $f(\sqrt{3}) = 1$.

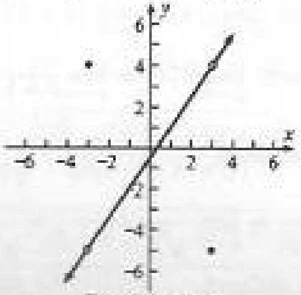
$\lim_{x\to -1} f(x) = 1$, $\lim_{x\to 0} f(x) = 2$, $\lim_{x\to 1} f(x) = \sqrt{3}$, $\lim_{x\to \sqrt{3}} f(x) = 1$

In Exercises 59–62, sketch the graph of a function f with domain $(-\infty,+\infty)$ satisfying the given properties.

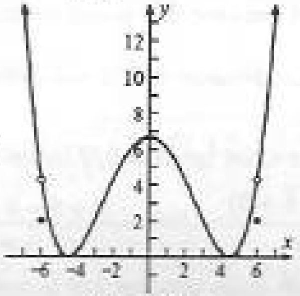
60. $f(-3) = 4$; $f(3) = -5$; $\lim_{x\to -3} f(x) = -5$; $\lim_{x\to 3} f(x) = 4$; $\lim_{x\to a} f(x) = f(a)$ if $a \neq \pm 3$; range$(f) = (-\infty,+\infty)$



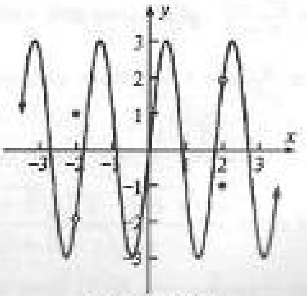
Exercise 59



Exercise 60



Exercise 61



Exercise 62

63. $\lim\limits_{x\to a} f(x) = \mathrm{L} \Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|f(x) - \mathrm{L}\right| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|[f(x) - \mathrm{L}] - 0\right| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow \lim\limits_{x\to a}[f(x) - \mathrm{L}] = 0$

64. Let $t = x - a$, $x = t + a$.
$\lim\limits_{x\to a} f(x) = \mathrm{L} \Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|f(x) - \mathrm{L}\right| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|f(t + a) - \mathrm{L}\right| < \epsilon$ when $0 < \left|t - 0\right| < \delta$

$\Leftrightarrow \lim\limits_{t\to 0} f(t + a) = \mathrm{L}$

65. $\lim\limits_{x\to a} \mathrm{P}(x) = \lim\limits_{x\to a}(c_n x^n + \cdots + c_0) \overset{\text{L.T.5}}{=} \lim\limits_{x\to a} c_n x^n + \cdots + \lim\limits_{x\to a} a_0 \overset{\text{L.T.7}}{=} \lim\limits_{x\to a} c_n \cdot \lim\limits_{x\to a} x \cdots \lim\limits_{x\to a} x + \cdots + \lim\limits_{x\to a} a_0$
$\overset{\text{L.T.1}}{=} c_n a^n + \cdots + a_0 = \mathrm{P}(a)$. Let $\mathrm{R}(x) = \mathrm{P}(x)/\mathrm{Q}(x)$. $\lim\limits_{x\to a} \mathrm{R}(x) = \mathrm{R}(a)$ provided $\lim\limits_{x\to a} \mathrm{Q}(x) \neq 0$.

66. We are given $\lim\limits_{x\to a} f(x)$ exists. If $\lim\limits_{x\to a} g(x)$ did exist, then by L.T.4, so would $\lim\limits_{x\to a}[f(x) + g(x)]$, contradicting the other hypothesis.

1.5 SUPPLEMENT

1.5.12 Theorem If a is any real number except 0, $\lim\limits_{x\to a}\frac{1}{x} = \frac{1}{a}$.

1.5.13 Theorem If $a > 0$ and n is a positive integer, or $a \leq 0$ and n is an odd positive integer, $\lim\limits_{x\to a} \sqrt[n]{x} = \lim\limits_{x\to a} \sqrt[n]{a}$.

1.5.16 Theorem If $\lim\limits_{x\to a} f(x) = \mathrm{L}_1$ and $\lim\limits_{x\to a} f(x) = \mathrm{L}_2$, then $\mathrm{L}_1 = \mathrm{L}_2$.

Supplementary Exercises 1.5

▹ The key to a short solution is to factor in the second line before completing the first line.

1. $\lim\limits_{x\to 1} x^2 = 1$ ▹ Choose $\delta \leq 1$ so $-1 < x - 1 < 1 \Rightarrow 1 < x + 1 < 3$.
Then $\left|x^2 - 1\right| = |x + 1||x - 1| < 3|x - 1| < \epsilon$ when $|x - 1| < \frac{1}{3}\epsilon$. $\delta = \min(1, \frac{1}{3}\epsilon)$.

2. $\lim\limits_{x\to -3} x^2 = 9$ ▹ Choose $\delta \leq 1$ so $-1 < x + 3 < 1 \Rightarrow -7 < x - 3 < -5$.
Then $\left|x^2 - 9\right| = |x - 3||x + 3| < 7|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{7}\epsilon$. $\delta = \min(1, \frac{1}{7}\epsilon)$.

3. $\lim\limits_{x\to 5}(x^2 - 3x) = 10$ ▹ Choose $\delta \leq 1$ so $-1 < x - 5 < 1 \Rightarrow 6 < x + 2 < 8$.
Then $\left|x^2 - 3x - 10\right| = |x + 2||x - 5| < 8|x - 5| < \epsilon$ when $|x - 5| < \frac{1}{8}\epsilon$. $\delta = \min(1, \frac{1}{8}\epsilon)$.

4. $\lim\limits_{x\to 2}(x^2 + 2x - 1) = 7$ ▹ Choose $\delta \leq 1$ so $-1 < x - 2 < 1 \Rightarrow -5 < x + 4 < 7$.
Then $\left|x^2 + 2x - 1 - 7\right| = |x + 4||x - 2| < 7|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{7}\epsilon$. $\delta = \min(1, \frac{1}{7}\epsilon)$.

5. $\lim\limits_{x\to -3}(5 - x - x^2) = -1$ ▹ Choose $\delta \leq 1$ so $-1 < x + 3 < 1 \Rightarrow -6 < x - 2 < -4$.
Then $\left|5 - x - x^2 + 1\right| = \left|x^2 + x - 6\right| = |x - 2||x + 3| < 6|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{6}\epsilon$. $\delta = \min(1, \frac{1}{6}\epsilon)$.

6. $\lim\limits_{x\to -1}(3 + 2x - x^2) = 0$ ▹ Choose $\delta \leq 1$ so $-1 < x + 1 < 1 \Rightarrow -5 < x - 3 < -3$.
Then $\left|3 + 2x - x^2 - 0\right| = \left|x^2 - 2x - 3\right| = |x - 3||x + 1| < 5|x + 1| < \epsilon$ when $|x + 1| < \frac{1}{5}\epsilon$. $\delta = \min(1, \frac{1}{5})$.

7. $\lim\limits_{x\to 2}(6x^2 - 13x + 5) = 3$ ▹ Choose $\delta \leq 1$ so $-1 < x - 2 < 1 \Rightarrow -6 < 6x - 12 < 6 \Rightarrow 5 < 6x - 1 < 17$.
Then $\left|6x^2 - 13x + 5 - 3\right| = |6x - 1||x - 2| < 17|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{17}\epsilon$. $\delta = \min(1, \frac{1}{17}\epsilon)$.

8. $\lim\limits_{x\to 1}(4x^2 - 13x + 12) = 3$ ▹ Choose $\delta \leq 1$ so $-1 < x - 1 < 1 \Rightarrow -4 < 4x - 4 < 4 \Rightarrow -9 < 4x - 9 < -1$
Then $\left|4x^2 - 13x + 12 - 3\right| = |4x - 9||x - 1| < 9|x - 1| < \epsilon$ when $|x - 1| < \frac{1}{9}\epsilon$. $\delta = \min(1, \frac{1}{9}\epsilon)$.

10. We wish to prove by mathematical induction that if $\lim\limits_{x\to a} f_1(x) = \mathrm{L}_1$, $\lim\limits_{x\to a} f_2(x) = \mathrm{L}_2, \ldots,$ and $\lim\limits_{x\to a} f_n(x) = \mathrm{L}_n$,
then $\lim\limits_{x\to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = \mathrm{L}_1 \pm \mathrm{L}_2 \pm \cdots \pm \mathrm{L}_n$ (1)

Proof: By L.T.4, $\lim\limits_{x\to a} [f_1(x) \pm f_2(x)] = \mathrm{L}_1 \pm \mathrm{L}_2$. Therefore, Equation (1) holds when n = 2.

Assume Equation (1) holds when n = k; that is
$\lim\limits_{x\to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] = \mathrm{L}_1 \pm \mathrm{L}_2 \pm \cdots \pm \mathrm{L}_k$ (2)

We wish to prove that Equation (1) holds when $n = k + 1$. From L.T.4
$\lim\limits_{x\to a} \{[f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] \pm f_{k+1}(x)\} = \lim\limits_{x\to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] \pm \lim\limits_{x\to a} [f_{k+1}(x)]$
From Equation (2) and because by hypothesis $\lim\limits_{x\to a} f_{k+1}(x) = L_{k+1}$ the right-hand side of the above equation is $(L_1 \pm L_2 \pm \cdots \pm L_k) \pm L_{k+1}$. Therefore, Equation (1) holds for $n = k + 1$, and hence for every integer n.

11. We are given $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = 0$. In order to prove that $\lim\limits_{x\to a} [f(x) \cdot g(x)] = 0$ we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) \cdot g(x)| < \epsilon$ (1)
Because $\lim\limits_{x\to a} f(x) = L$ it follows from Definition 2.1.1 that there is a $\delta_1 > 0$ such that
if $0 < |x - a| < \delta_1$ then $|f(x) - L| < 1$ (2)
Because $|a| - |b| \le |a - b|$ it follows that $|f(x)| - |L| \le |f(x) - L|$. Hence, from statement (2) there is a $\delta_1 > 0$ such that
if $0 < |x - a| < \delta_1$ then $|f(x)| < 1 + |L|$
Because $\lim\limits_{x\to a} g(x) = 0$ it follows from Definition 2.1.1 that there is a $\delta_2 > 0$ such that
if $0 < |x - a| < \delta_2$ then $|g(x)| < \dfrac{\epsilon}{1 + |L|}$
Let $\delta = \min(\delta_1, \delta_2)$; it follows that
if $0 < |x - a| < \delta$ then $|f(x) \cdot g(x)| = |f(x)||g(x)| < (1 + |L|)\dfrac{\epsilon}{1 + |L|} = \epsilon$
Therefore statement (1) is proved and so $\lim\limits_{x\to a} f(x)g(x) = 0$.

12. We are given $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M \Rightarrow \lim\limits_{x\to a} [f(x) - L] = 0$ and $\lim\limits_{x\to a} [g(x) - M] = 0$. Hence
$\lim\limits_{x\to a} f(x) \cdot g(x) = \lim\limits_{x\to a} \{[f(x) - L]g(x) + L[g(x) - M] + L \cdot M\}$
$\overset{\text{L.T.1}}{=} \lim\limits_{x\to a} \{[f(x) - L]g(x)\} + \lim\limits_{x\to a} \{L[g(x) - M]\} + \lim\limits_{x\to a} L \cdot M \overset{\text{Ex. 11}}{=} 0 + 0 + L \cdot M$

13. We wish to prove by mathematical induction that if $\lim\limits_{x\to a} f_1(x) = L_1$, $\lim\limits_{x\to a} f_2(x) = L_2, \ldots,$ and $\lim\limits_{x\to a} f_n(x) = L_n$, then $\lim\limits_{x\to a} [f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x)] = L_1 \cdot L_2 \cdot \cdots \cdot L_n$ (1)
Proof: By L.T.6, $\lim\limits_{x\to a} [f_1(x) \cdot f_2(x)] = L_1 \cdot L_2$. Therefore, Equation (1) holds when $n = 2$.
Assume Equation (1) holds when $n = k$; that is
$\lim\limits_{x\to a} [f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_k(x)] = L_1 \cdot L_2 \cdot \cdots \cdot L_k$ (2)
We wish to prove that Equation (1) holds when $n = k + 1$. From L.T.6
$\lim\limits_{x\to a} \{[f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_k(x)] \cdot f_{k+1}(x)\} = \lim\limits_{x\to a} [f_1(x) \cdot f_2(x) \cdot \cdots \pm f_k(x)] \cdot \lim\limits_{x\to a} [f_{k+1}(x)]$
From Equation (2) and because by hypothesis $\lim\limits_{x\to a} f_{k+1}(x) = L_{k+1}$ the right-hand side of the above equation is $(L_1 \cdot L_2 \cdot \cdots \cdot L_k) \cdot L_{k+1}$. Therefore, Equation (1) holds for $n = k + 1$, and hence for every integer n.

14. Suppose $a < 0$ and so $-a > 0$. By what was proved, $\lim\limits_{y\to -a} \dfrac{1}{y} = \dfrac{1}{-a} \Rightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\dfrac{1}{y} - \dfrac{1}{-a}\right| < \epsilon$ when $0 < |y - (-a)| < \delta$. Replacing y by $-x$, we get the equivalent conclusion
$\left|\dfrac{1}{-x} - \dfrac{1}{-a}\right| < \epsilon$ when $0 < |-x - (-a)| < \delta \Leftrightarrow \left|\dfrac{1}{x} - \dfrac{1}{a}\right| < \epsilon$ when $0 < |x - a| < \delta \Leftrightarrow \lim\limits_{x\to a} \dfrac{1}{x} = \dfrac{1}{a}$.

15. Suppose $a < 0$ so $-a > 0$. By what was proved, $\lim\limits_{y\to -a} \sqrt[n]{y} = \sqrt[n]{-a} \Rightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\sqrt[n]{y} - \sqrt[n]{-a}\right| < \epsilon$ when $0 < |y - (-a)| < \delta$. Replacing y by $-x$, we get the equivalent conclusion
$\left|\sqrt[n]{-x} - \sqrt[n]{-a}\right| < \epsilon$ when $0 < |-x - (-a)| < \delta \overset{n \text{ is odd}}{\Leftrightarrow} \left|\sqrt[n]{x} - \sqrt[n]{a}\right| < \epsilon$ when $0 < |x - a| < \delta \Leftrightarrow \lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
The case $a = 0$ follows from $a < 0$ and $a > 0$ and one;sided limits, discussed in the next section.

1.6 ONE-SIDED LIMITS

1.6.1 Definition Let f be a function that is defined at every number in some open interval (a, c). Then *limit of $f(x)$, as x approaches a from the right, is L,* written

$$\lim_{x\to a^+} f(x) = \mathrm{L}$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < x - a < \delta \text{ then } \left|f(x) - \mathrm{L}\right| < \epsilon$$

1.6.2 Definition Let f be a function that is defined at every number in some open interval (c, a). Then *limit of $f(x)$, as x approaches a from the left, is L,* written

$$\lim_{x\to a^-} f(x) = \mathrm{L}$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < a - x < \delta \text{ then } \left|f(x) - \mathrm{L}\right| < \epsilon$$

Limit Theorems 1–10 hold if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$".

1.6.3 Theorem $\lim_{x\to a} f(x)$ exists and equals L if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and equal L.

Exercises 1.6

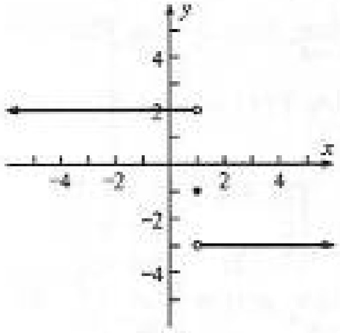
In Exercises 1–22, sketch the graph. Then find the indicated limit or state why it does not exist.

1. $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ -1 & \text{if } x = 1 \\ -3 & \text{if } 1 < x \end{cases}$

▹ (a) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-3) = -3$

(b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 = 2$

(c) $\lim_{x\to 1} f(x)$ does not exist because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

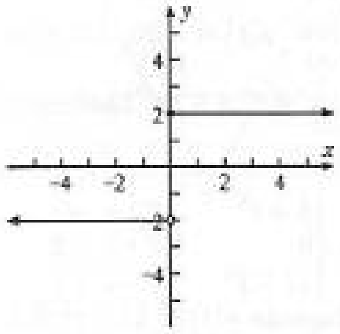


2. $f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

▹ (a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 = 2$

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -2 = -2$

(c) $\lim_{x\to 0} f(x)$ does not exist because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

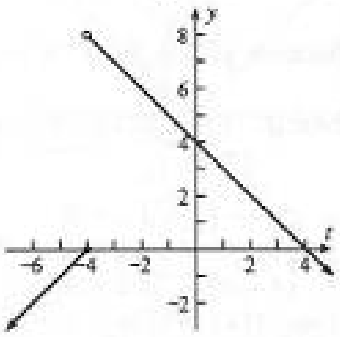


3. $f(t) = \begin{cases} t + 4 & \text{if } t \leq -4 \\ 4 - t & \text{if } -4 < t \end{cases}$

▹ (a) $\lim_{t\to -4^+} f(t) = \lim_{t\to -4^+} (4 - t) = 8$

(b) $\lim_{t\to -4^-} f(t) = \lim_{t\to -4^-} (t + 4) = 0$

(c) $\lim_{t\to -4} f(t)$ does not exist because $\lim_{t\to -4^-} f(t) \neq \lim_{t\to 4^+} f(t)$.

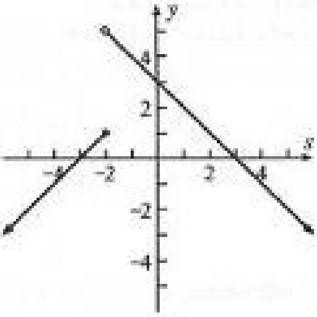


4. $g(s) = \begin{cases} s+3 & \text{if } s \le -2 \\ 3-s & \text{if } -2 < s \end{cases}$

▷ (a) Because $g(s) = 3 - s$ if $s > -2$ then $\lim\limits_{s \to -2^+} g(s) = \lim\limits_{s \to -2^+} (3-s) = 5$.

(b) Because $g(s) = s + 3$ if $s < -2$, then $\lim\limits_{s \to -2^-} g(s) = \lim\limits_{s \to -2^-} (s+3) = 1$.

(c) Because $\lim\limits_{s \to -2^+} g(s) \ne \lim\limits_{s \to -2^-} g(s)$, by Theorem 1.6.3 $\lim\limits_{s \to -2} g(s)$ does not exist.

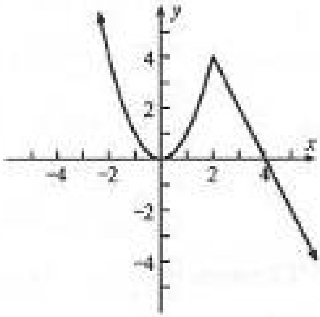


5. $F(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 8-2x & \text{if } 2 < x \end{cases}$

▷ (a) $\lim\limits_{x \to 2^+} F(x) = \lim\limits_{x \to 2^+} (8 - 2x) = 4$

(b) $\lim\limits_{x \to 2^-} F(x) = \lim\limits_{x \to 2^-} x^2 = 4$

(c) $\lim\limits_{x \to 2} F(x) = 4$ by Theorem 1.6.3.

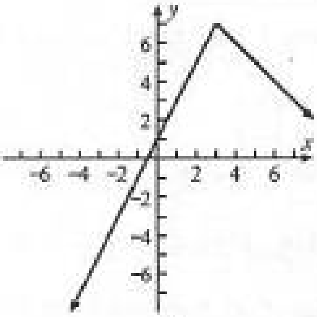


6. $h(x) = \begin{cases} 2x+1 & \text{if } x < 3 \\ 10-x & \text{if } x \ge 3 \end{cases}$

▷ (a) $\lim\limits_{x \to 3^+} h(x) = \lim\limits_{x \to 3^+} (10 - x) = 10 - 3 = 7$

(b) $\lim\limits_{x \to 3^-} h(x) = \lim\limits_{x \to 3^-} (2x + 1) = 2(3) + 1 = 7$

(c) $\lim\limits_{x \to 3} F(x) = 7$ by Theorem 1.6.3.

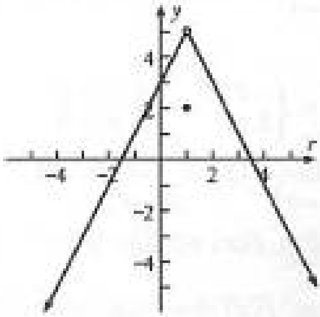


7. $g(r) = \begin{cases} 2r+3 & \text{if } r < 1 \\ 2 & \text{if } r = 1 \\ 7-2r & \text{if } 1 < r \end{cases}$

▷ (a) $\lim\limits_{r \to 1^+} g(r) = \lim\limits_{r \to 1^+} (7 - 2r) = 5$

(b) $\lim\limits_{r \to 1^-} g(r) = \lim\limits_{r \to 1^-} (2r + 3) = 5$

(c) $\lim\limits_{r \to 1} g(r) = 5$ by Theorem 1.6.3.

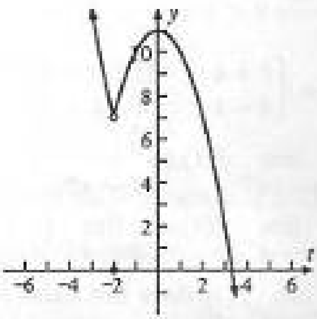


8. $g(t) = \begin{cases} 3+t^2 & \text{if } t < -2 \\ 0 & \text{if } t = -2 \\ 11-t^2 & \text{if } -2 < t \end{cases}$

▷ (a) Because $g(t) = 11 - t^2$ if $t > -2$, then $\lim\limits_{t \to -2^+} g(t) = \lim\limits_{t \to -2^+} (11 - t^2) = 7$.

(b) Because $g(t) = 3 + t^2$ if $t < -2$, then $\lim\limits_{t \to -2^-} g(t) = \lim\limits_{t \to -2^-} (3 + t^2) = 7$.

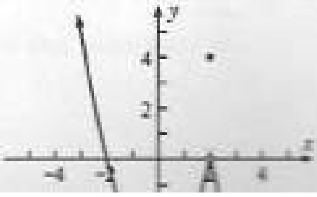
(c) Because $\lim\limits_{t \to -2^+} g(t) = \lim\limits_{t \to -2^-} g(t) = 7$, then by Theorem 1.6.3 $\lim\limits_{t \to -2} g(t) = 7$.



9. $f(x) = \begin{cases} x^2-4 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 4-x^2 & \text{if } 2 < x \end{cases}$

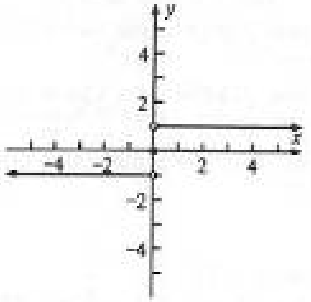
▷ (a) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4 - x^2) = 0$

(b) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 - 4) = 0$



16. $S(x) = |\operatorname{sgn} x|$, where $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

▹ $$S(x) = |\operatorname{sgn} x| = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

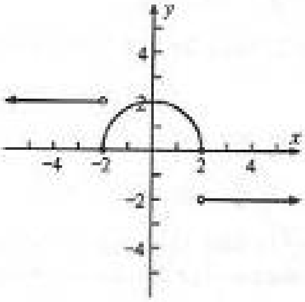


(a) Because $\operatorname{sgn} x = 1$ if $x > 0$, then $|\operatorname{sgn} x| = 1$ if $x > 0$, and
$$\lim_{x\to 0^+} S(x) = \lim_{x\to 0^+} |\operatorname{sgn} x| = \lim_{x\to 0^+} 1 = 1$$

(b) Because $\operatorname{sgn} x = -1$ if $x < 0$, then $|\operatorname{sgn} x| = 1$ if $x < 0$, and
$$\lim_{x\to 0^-} S(x) = \lim_{x\to 0^-} |\operatorname{sgn} x| = \lim_{x\to 0^-} 1 = 1$$

(c) Because $\lim_{x\to 0^+} S(x) = \lim_{x\to 0^-} S(x) = 1$, by Theorem 1.6.3 $\lim_{x\to 0} S(x) = 1$.

17. $f(x) = \begin{cases} 2 & \text{if } x < -2 \\ \sqrt{4 - x^2} & \text{if } -2 \le x \le 2 \\ -2 & \text{if } 2 < x \end{cases}$



▹ (a) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 2 = 2$

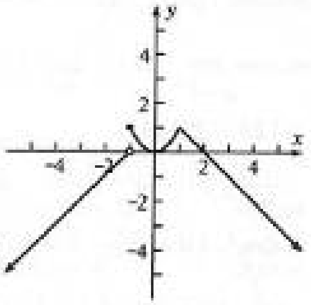
(b) $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \sqrt{4 - x^2} = 0$

(c) $\lim_{x\to -2} f(x)$ does not exist because $\lim_{x\to -2^-} f(x) \ne \lim_{x\to -2^+} f(x)$.

(d) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{4 - x^2} = 0$ (e) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-2) = -2$

(f) $\lim_{x\to 2^+} f(x)$ does not exist because $\lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} f(x)$.

18. $f(x) = \begin{cases} x + 1 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



▹ (a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x + 1) = -1 + 1 = 0$

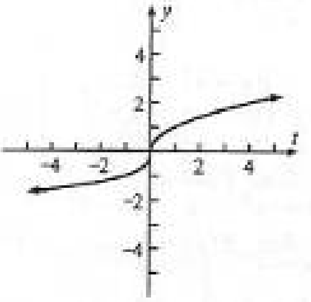
(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 = (-1)^2 = 1$

(c) $\lim_{x\to -1} f(x)$ does not exist because $\lim_{x\to -1^-} f(x) \ne \lim_{x\to -1^+} f(x)$.

(d) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$ (e) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2 - x) = 2 - 1 = 1$

(f) $\lim_{x\to 1} f(x) = 1$ by Theorem 1.6.3.

19. $f(t) = \begin{cases} \sqrt[3]{t} & \text{if } t < 0 \\ \sqrt{t} & \text{if } 0 \le t \end{cases}$

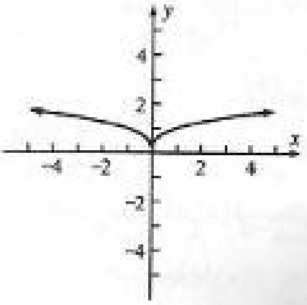


▹ (a) $\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} \sqrt{t} = 0$

(b) $\lim_{t\to 0^-} f(t) = \lim_{t\to 0^-} \sqrt[3]{t} = 0$

(c) $\lim_{t\to 0} f(t) = 0$ by Theorem 1.6.3

20. $g(x) = \begin{cases} \sqrt[3]{-x} & \text{if } x \le 0 \\ \sqrt[3]{x} & \text{if } 0 < x \end{cases}$

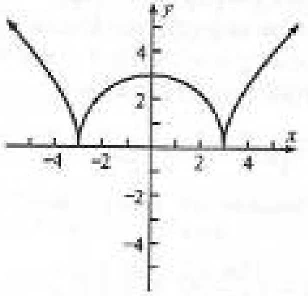


▹ (a) $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \sqrt[3]{x} = 0$

(b) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \sqrt[3]{-x} = 0$

(c) Because $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 0$, then by Theorem 1.6.3 $\lim_{x\to 0} g(x) = 0$.

21. $F(x) = \begin{cases} \sqrt{x^2 - 9} & \text{if } x \le -3 \text{ or } x \ge 3 \\ \sqrt{9 - x^2} & \text{if } -3 < x < 3 \end{cases}$



▸ (a) $\lim\limits_{x \to -3^-} F(x) = \lim\limits_{x \to -3^-} \sqrt{x^2 - 9} = 0$

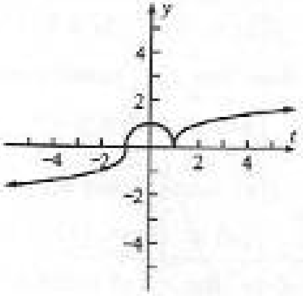
(b) $\lim\limits_{x \to -3^+} F(x) = \lim\limits_{x \to -3^+} \sqrt{9 - x^2} = 0$

(c) $\lim\limits_{x \to -3} F(x) = 0$ by Theorem 1.6.3.

(d) $\lim\limits_{x \to 3^-} F(x) = \lim\limits_{x \to 3^-} \sqrt{9 - x^2} = 0$ (e) $\lim\limits_{x \to 3^+} F(x) = \lim\limits_{x \to 3^+} \sqrt{x^2 - 9} = 0$

(f) $\lim\limits_{x \to 3} F(x) = 0$ by Theorem 1.6.3.

22. $G(t) = \begin{cases} \sqrt[3]{t + 1} & \text{if } t \le -1 \\ \sqrt{1 - t^2} & \text{if } -1 < t < 1 \\ \sqrt[3]{t - 1} & \text{if } t \ge 1 \end{cases}$



▸ (a) $\lim\limits_{t \to -1^-} G(t) = \lim\limits_{t \to -1^-} \sqrt[3]{t + 1} = \sqrt[3]{-1 + 1} = 0$

(b) $\lim\limits_{t \to -1^+} G(t) = \lim\limits_{t \to -1^+} \sqrt{1 - t^2} = \sqrt{1 - (-1)^2} = 0$

(c) $\lim\limits_{t \to -1} G(t) = 0$ by Theorem 1.6.3.

(d) $\lim\limits_{t \to 1^-} G(t) = \lim\limits_{t \to 1^-} \sqrt{1 - t^2} = \sqrt{1 - 1^2} = 0$

(e) $\lim\limits_{x \to 1^+} G(t) = \lim\limits_{t \to 1^+} \sqrt[3]{t - 1} = \sqrt[3]{1 - 1} = 0$

(f) $\lim\limits_{t \to 1} G(t) = 0$ by Theorem 1.6.3.

23. $F(x) = x - 2 \operatorname{sgn} x = \begin{cases} x + 2 & \text{if } x < 0 \\ x & \text{if } x = 0 \\ x - 2 & \text{if } 0 < x \end{cases}$

(a) $\lim\limits_{x \to 0^+} F(x) = \lim\limits_{x \to 0^+} (x - 2) = -2$ (b) $\lim\limits_{x \to 0^-} F(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$

(c) $\lim\limits_{x \to 0} F(x)$ does not exist because $\lim\limits_{x \to 0^-} F(x) \ne \lim\limits_{x \to 0^+} F(x)$.

24. Let h be defined by $h(x) = \operatorname{sgn} x - U(x)$, where $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$ and $U(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

▸ $h(x) = \begin{cases} -1 - 0 = -1 & \text{if } x < 0 \\ 0 - 1 = -1 & \text{if } x = 0 \\ 1 - 1 = 0 & \text{if } x > 0 \end{cases}$

(a) $h(x) = 0$ if $x > 0$, and $\lim\limits_{x \to 0^+} h(x) = 0$. (b) $h(x) = -1$ if $x < 0$, and $\lim\limits_{x \to 0^-} h(x) = -1$.

(c) Because $\lim\limits_{x \to 0^+} h(x) \ne \lim\limits_{x \to 0^-} h(x)$, then $\lim\limits_{x \to 0} h(x)$ does not exist.

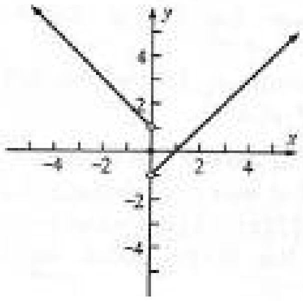
25. (a) $\lim\limits_{x \to 2^+} [\![x]\!] = 2$ (b) $\lim\limits_{x \to 2^-} [\![x]\!] = 1$

(c) $\lim\limits_{x \to 2} [\![x]\!]$ does not exist because $\lim\limits_{x \to 2^-} [\![x]\!] \ne \lim\limits_{x \to 2^+} [\![x]\!]$.

26. (a) $\lim\limits_{x \to 4^+} [\![x - 3]\!] = [\![1^+]\!] = 1$ (b) $\lim\limits_{x \to 4^-} [\![x - 3]\!] = [\![1^-]\!] = 0$

(c) $\lim\limits_{x \to 4} [\![x]\!]$ does not exist because $\lim\limits_{x \to 4^-} [\![x]\!] \ne \lim\limits_{x \to 4^+} [\![x]\!]$.

27. $h(x) = (x - 1) \operatorname{sgn} x = \begin{cases} 1 - x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x - 1 & \text{if } 0 < x \end{cases}$



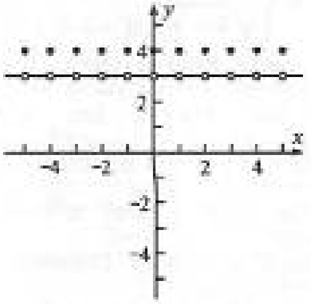
(a) $\lim\limits_{x \to 0^+} h(x) = \lim\limits_{x \to 0^+} (x - 1) = -1$

(b) $\lim\limits_{x \to 0^-} h(x) = \lim\limits_{x \to 0^-} (1 - x) = 1$

(c) $\lim\limits_{x \to 0} h(x)$ does not exist because $\lim\limits_{x \to 0^-} h(x) \ne \lim\limits_{x \to 0^+} h(x)$.

28. Let $G(x) = [\![x]\!] + [\![4 - x]\!]$.

▷ Let n be any integer. If $x = n$, then $G(x) = n + (4 - n) = 4$.
If $n < x < n + 1$ then $4 - n > 4 - x > 4 - (n + 1) = 3 - n$
so $G(x) = n + (3 - n) = 3$.



(a) $\lim\limits_{x\to 3^+} G(x) = \lim\limits_{x\to 3^+} 3 = 3$ (b) $\lim\limits_{x\to 3^-} G(x) = \lim\limits_{x\to 3^-} 3 = 3$

(c) Because $\lim\limits_{x\to 3^+} G(x) = \lim\limits_{x\to 3^-} G(x) = 3$, by Theorem 1.6.3 $\lim\limits_{x\to 3} G(x) = 3$.

29. $f(x) = \begin{cases} 3x + 2 & \text{if } x < 4 \\ 5x + k & \text{if } 4 \le x \end{cases}$

▷ $\lim\limits_{x\to 4} f(x)$ exists if and only if $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x)$.
$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} (3x + 2) = 14$ $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} (5x + \mathrm{k}) = 20 + \mathrm{k}$
Therefore $\lim\limits_{x\to 4} f(x)$ exists if and only if $20 + k = 14$; hence $k = -6$.

30. $f(x) = \begin{cases} kx - 3 & \text{if } x \le -1 \\ x^2 + k & \text{if } x > -1 \end{cases}$

▷ $\lim\limits_{x\to -1} f(x)$ exists if and only if $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x)$.
$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (kx - 3) = -k - 3$ $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (x^2 + k) = 1 + \mathrm{k}$
Therefore $\lim\limits_{x\to -1} f(x)$ exists if and only if $-k - 3 = 1 + k$; hence $k = -2$.

31. $f(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ ax + b & \text{if } -2 < x < 2 \\ 2x - 6 & \text{if } 2 \le x \end{cases}$

▷ If $\lim\limits_{x\to -2} f(x)$ exists then $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^+} f(x)$; that is, $4 = -2\mathrm{a} + \mathrm{b}$.
If $\lim\limits_{x\to 2} f(x)$ exists then $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$; that is, $2\mathrm{a} + \mathrm{b} = -2$.
Solving these equations simultaneously, we get $\mathrm{a} = -\frac{3}{2}$, $\mathrm{b} = 1$.

32. Given $f(x) = \begin{cases} 2x - a & \text{if } x < -3 \\ ax + 2b & \text{if } -3 \le x \le 3 \\ b - 5x & \text{if } 3 < x \end{cases}$

Find the values of a and b so that $\lim\limits_{x\to -3} f(x)$ and $\lim\limits_{x\to 3} f(x)$ both exist.

▷ Because of the two one-sided limits

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (2x - a) = -6 - a$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (ax + 2b) = -3a + 2b$$

if $\lim\limits_{x\to -3} f(x)$ exists, by Theorem 1.6.3 we must have

$$-6 - a = -3a + 2b$$
$$a - b = 3$$

Because of the two one-sided limits

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax + 2b) = 3a + 2b$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (b - 5x) = b - 15$$

if $\lim\limits_{x\to 3} f(x)$ exists, by Theorem 1.6.3 we must have

$$3a + 2b = b - 15$$
$$3a + b = -15$$

Solving these equations simultaneously, we obtain $a = -3$ and $b = -6$.

33. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } 0 < x \end{cases}$; $|f(x)| = 1$ if $x \ne 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (1) = 1$
Because $\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist. But $\lim\limits_{x\to 0} |f(x)| = \lim 1 = 1$.

34. If $\delta = \min(\delta_1, \delta_2)$ then the following statements are equivalent.
$\lim\limits_{x\to a} f(x) = \mathrm{L}$
⇔ For every $\epsilon \ge 0$ there is a $\delta > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < |x - a| < \delta$
⇔ For every $\epsilon \ge 0$ there is a $\delta_1 > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < x - a < \delta$ and there is a $\delta_2 > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < -(x - a) < \delta$
⇔ $\lim\limits_{x\to a^+} f(x) = \mathrm{L}$ and $\lim\limits_{x\to a^-} f(x) = \mathrm{L}$.

In Exercises 35 and 36, evaluate the limits from the graph of function f sketched in the accompanying figure.

35. (a) $\lim_{x\to -1^+} f(x) = 0$ (b) $\lim_{x\to 0^-} f(x) = 3$ (c) $\lim_{x\to 0^+} f(x) = 0$ (d) $\lim_{x\to 0} f(x)$ does not exist (e) $\lim_{x\to 2^-} f(x) = 5$

(f) $\lim_{x\to 2^+} f(x) = 5$ (g) $\lim_{x\to 2} f(x) = 5$ (h) $\lim_{x\to 3^-} f(x) = 2$ (i) $\lim_{x\to 3^+} f(x) = 2$ (j) $\lim_{x\to 3} f(x) = 2$ (k) $\lim_{x\to 5^-} f(x) = 0$

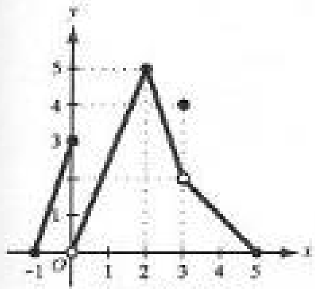
36. (a) $\lim_{x\to 0^+} f(x) = 4$ (b) $\lim_{x\to 1^-} f(x) = 2$ (c) $\lim_{x\to 1^+} f(x) = 2$

(d) Because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$, by Theorem 1.6.3, $\lim_{x\to 1} f(x) = 3$

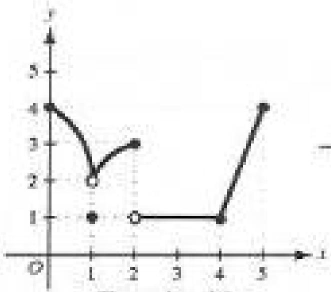
(e) $\lim_{x\to 2^-} f(x) = 3$ (f) $\lim_{x\to 2^+} f(x) = 1$. (g) Because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$ does not exist.

(h) $\lim_{x\to 4^-} f(x) = 1$ (i) $\lim_{x\to 4^+} f(x) = 1$ (j) Because $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = 1$, by Theorem 1.6.3, $\lim_{x\to 4} f(x) = 1$

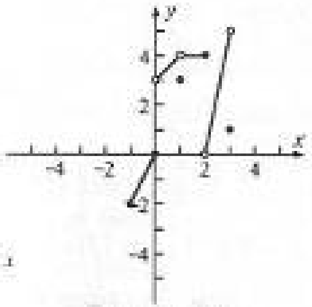
(k) $\lim_{x\to 5^-} f(x) = 4$



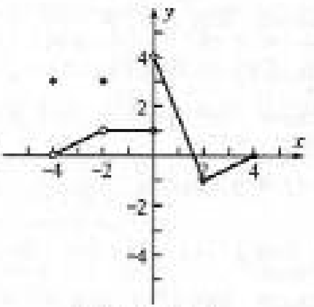
Exercise 35



Exercise 36



Exercise 37



Exercise 38

39. $f(x)$ dollars is the cost of shipping x lb. $f(x) = \begin{cases} 2.2x & \text{if } 0 < x \le 50 \\ 2.1x & \text{if } 50 < x \le 200 \\ 2.05x & \text{if } x > 200 \end{cases}$.

▹ (a) $\lim_{x\to 50^-} f(x) = \lim_{x\to 50^-} 2.2x = 2.2 \cdot 50 = 110$ (b) $\lim_{x\to 50^+} f(x) = \lim_{x\to 50^+} 2.1x = 2.1 \cdot 50 = 105$

(c) $\lim_{x\to 200^-} f(x) = \lim_{x\to 200^-} 2.1x = 2.1 \cdot 200 = 420$ (d) $\lim_{x\to 200^+} f(x) = \lim_{x\to 200^+} 2.05x = 2.05 \cdot 200 = 410$

40. $\mathrm{F}(x)$ cents is the cost of mailing x ounces. $\mathrm{F}(x) = 9 - 23\llbracket -x \rrbracket$.

▹ (a and b) If $0 < x \le 1$ then $\mathrm{F}(x) = 9 - 23(-1) = 32$ and so $\lim_{x\to 0^+} \mathrm{F}(x) = \lim_{x\to 0^+} 32 = 32$ and $\lim_{x\to 1^-} \mathrm{F}(x) = 32$

(c and d) If $1 < x \le 2$ then $\mathrm{F}(x) = 9 - 23(-2) = 55$ and so $\lim_{x\to 1^+} \mathrm{F}(x) = \lim_{x\to 1^+} 55 = 55$ and $\lim_{x\to 2^-} \mathrm{F}(x) = 55$

(e) If $9 < x \le 10$ then $\mathrm{F}(x) = 9 - 23(-10) = 239$ and so $\lim_{x\to 10^-} \mathrm{F}(x) = \lim_{x\to 10^-} 239 = 239$

(f and g) If $10 < x \le 11$ then $\mathrm{F}(x) = 9 - 23(-11) = 262$ and so $\lim_{x\to 10^+} \mathrm{F}(x) = \lim_{x\to 10^+} 262 = 262$ and $\lim_{x\to 11^-} \mathrm{F}(x) = 262$

41. $g(x)$ cents is the cost of an x minute call. $g(x) = 10 - 30\llbracket -x \rrbracket$.

▹ (a) $\lim_{x\to 1^-} g(x) = 10 - 30\llbracket -(1^-) \rrbracket = 10 - 30(-1) = 40$ (b) $\lim_{x\to 1^+} g(x) = 10 - 30\llbracket -(1^+) \rrbracket = 10 - 30(-2) = 70$

(c) $\lim_{x\to 2^-} g(x) = 10 - 30\llbracket -(2^-) \rrbracket = 10 - 30(-2) = 70$ (d) $\lim_{x\to 5^-} g(x) = 10 - 30\llbracket -(5^-) \rrbracket = 10 - 30(-5) = 160$

42. $\mathrm{G}(x)$ dollars is the admission for age x years. $\mathrm{G}(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \le x < 60 \\ 5 & \text{if } 60 \le x \end{cases}$.

▹ (a) $\lim_{x\to 12^-} \mathrm{G}(x) = \lim_{x\to 12^-} 4 = 4$ (b) $\lim_{x\to 12^+} \mathrm{G}(x) = \lim_{x\to 12^+} 7 = 7$

(c) $\lim_{x\to 60^-} \mathrm{G}(x) = \lim_{x\to 60^-} 7 = 7$ (d) $\lim_{x\to 60^+} \mathrm{G}(x) = \lim_{x\to 60^+} 5 = 5$

43. $f(x) = \begin{cases} x^2+3 & \text{if } x \le 1 \\ x+1 & \text{if } 1 < x \end{cases}$ and $g(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2 & \text{if } 1 < x \end{cases}$

▸ (a) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x^2+3) = 1^2+3 = 4$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x+1) = 1+1 = 2$

(b) $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} x^2 = 1^2 = 1$ and $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} 2 = 2$

(c) $f(x)\cdot g(x) = \begin{cases} (x^2+3)x^2 & \text{if } x \le 1 \\ (x+1)(2) & \text{if } x > 1 \end{cases}$

(d) $\lim\limits_{x\to 1^-}[f(x)\cdot g(x)] = (1^2+3)(1^2) = 4$ and $\lim\limits_{x\to 1^+}[f(x)\cdot g(x)] = (1+1)(2) = 4$

Therefore, $\lim\limits_{x\to 1}[f(x)\cdot g(x)]$ exists and is 4.

44. $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x-1 & \text{if } x \ge 1 \end{cases}$ and $g(x) = \begin{cases} 1-x & \text{if } x < 1 \\ 1+x & \text{if } x \ge 1 \end{cases}$

(a) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x+1) = 1+1 = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x-1) = 1-1 = 0$

Because $\lim\limits_{x\to 1^-} f(x) \ne \lim\limits_{x\to 1^+} f(x)$, then $\lim\limits_{x\to 1} f(x)$ does not exist.

$\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-}(1-x) = 1-1 = 0$ and $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+}(1+x) = 1+1 = 2$

Because $\lim\limits_{x\to 1^-} g(x) \ne \lim\limits_{x\to 1^+} g(x)$, then $\lim\limits_{x\to 1} g(x)$ does not exist.

(b) $(f+g)(x) = \begin{cases} (x+1)+(1-x) & \text{if } x < 1 \\ (x-1)+(1+x) & \text{if } x \ge 1 \end{cases} = \begin{cases} 2 & \text{if } x < 1 \\ 2x & \text{if } x \ge 1 \end{cases}$

(c) $\lim\limits_{x\to 1^-}[f(x)+g(x)] = \lim\limits_{x\to 1^-} 2 = 2$ and $\lim\limits_{x\to 1^+}[f(x)+g(x)] = \lim\limits_{x\to 1^+} 2x = 2\cdot 1 = 2$

Because $\lim\limits_{x\to 1^-}[f(x)+g(x)] = \lim\limits_{x\to 1^+}[f(x)+g(x)] = 2$, then $\lim\limits_{x\to 1}[f(x)+g(x)] = 2$

(d) Because $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to 1} g(x)$ do not exist, the hypotheses of Limit Theorem 4 are not satisfied.

1.7 INFINITE LIMITS

1.7.1 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. *As x approaches a, $f(x)$ increases without bound,* which is written

$$\lim_{x\to a} f(x) = +\infty$$

if for any number $N > 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } f(x) > N$$

1.7.2 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. *As x approaches a, $f(x)$ decreases without bound,* which is written

$$\lim_{x\to a} f(x) = -\infty$$

if for any number $N < 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } f(x) < N$$

We may also define one-sided limits that are infinite. The following theorems involve such definitions.

Limit Theorem 11 If r is any positive integer, then

$$\text{(i) } \lim_{x\to 0^+}\frac{1}{x^r} = +\infty$$

$$\text{(ii) } \lim_{x\to 0^-}\frac{1}{x^r} = \begin{cases} -\infty & \text{if } r \text{ is odd} \\ +\infty & \text{if } r \text{ is even} \end{cases}$$

Limit Theorem 12 If a is any real number, and if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = c$, where c is a constant not equal to 0, then

(i) if $c > 0$ and if $f(x) \to 0$ through positive values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = +\infty$

(ii) if $c > 0$ and if $f(x) \to 0$ through negative values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$

(iii) if $c < 0$ and if $f(x) \to 0$ through positive values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$

(iv) if $c < 0$ and if $f(x) \to 0$ through negative values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = +\infty$

The theorem is also valid if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$".

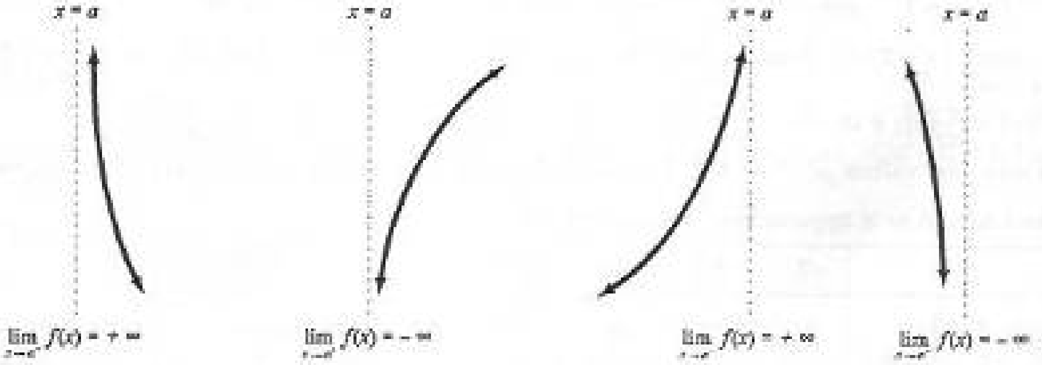
We can now usually find the limit of a fraction if either the numerator or the denominator has limit zero. If the numerator has limit zero and the denominator has a limit that is not zero, then by Limit Theorem 9 the limit of the fraction is zero. If the denominator has limit zero and the numerator has a limit that is not zero, then one of the cases of Limit Theorem 12 is usually satisfied and the limit of the fraction is $+\infty$ or $-\infty$, depending on which case. If both the numerator and denominator have limit zero, then one of the methods of Section 1.5 often works:

(i) Factor the numerator and denominator and cancel the common factors.
(ii) Rationalize either the numerator or denominator and proceed as in (i).

We will learn additional methods as we proceed through the book, but it is important to realize that some limits simply don't exist.

1.7.8 Definition The line $x = a$ is said to be a *vertical asymptote* of the graph of the function f if at least one of the following statements is true. Each statement is illustrated by the graph below it.

(i) $\lim_{x\to a^+} f(x) = +\infty$ (ii) $\lim_{x\to a^+} f(x) = -\infty$ (iii) $\lim_{x\to a^-} f(x) = +\infty$ (iv) $\lim_{x\to a^-} f(x) = -\infty$



Exercises 1.7

In Exercises 1–12, do the following: (a) Use a calculator to tabulate values of $f(x)$ for the specified values of x, and from these values make a statement regarding the apparent behavior of $f(x)$. (b) Support your answer in part (a) by plotting the graph of f. (c) Confirm your answer in part (a) analytically by computing the indicated limit.

▶ Let 0^+ and 0^- denote quantities that approach 0 through positive and negative values, respectively. If p and q are positive and negative numbers, then Limit Theorem 12 can be restated as below. See Solutions 9 and 10.
(i) $p/0^+ = +\infty$ (ii) $p/0^- = -\infty$ (iii) $q/0^+ = -\infty$ (iv) $q/0^- = +\infty$

1. (a) $f(x) = \dfrac{1}{x-5}$.

x	6	5.5	5.1	5.01	5.001	5.0001
$f(x)$	1	2	10	100	1000	10,000

(b) $\lim_{x\to 5^+} 1 = 1$ and $\lim_{x\to 5^+} (x-5) = 0$ through positive values. Hence, $\lim_{x\to 5^+} \dfrac{1}{x-5} = +\infty$.

2. (a) $f(x) = \dfrac{1}{x-5}$.

x	4	4.5	4.9	4.99	4.999	4.9999
$f(x)$	−1	−2	−10	−100	−1000	−10000

(b) $\lim_{x\to 5^-} 1 = 1$ and $\lim_{x\to 5^-} (x-5) = 0$ through negative values. Hence, $\lim_{x\to 5^-} \dfrac{1}{x-5} = -\infty$.

3. (a) $f(x) = \dfrac{1}{(x-5)^2}$.

x	6	5.5	5.1	5.01	5.001	5.0001
$f(x)$	1	4	100	10,000	1,000,000	100,000,000

x	4	4.5	4.9	4.99	4.999	4.9999
$f(x)$	1	4	100	10,000	1,000,000	100,000,000

(b) $\lim\limits_{x\to 5} 1 = 1$ and $\lim\limits_{x\to 5} (x-5)^2 = 0$ through positive values. Hence, $\lim\limits_{x\to 5} \dfrac{1}{(x-5)^2} = +\infty$.

4. (a) $f(x) = \dfrac{x+2}{1-x}$; x is 0, 0.5, 0.9, 0.99, 0.999, 0.9999; (b) $\lim\limits_{x\to 1^-} \dfrac{x+2}{1-x}$

▸ (a) From the values of $f(x)$ for the specified values of x given in the table, $f(x)$ appears to be increasing without bound as x approaches 1 from the left.

x	0	0.5	0.9	0.99	0.999	0.9999
$f(x) = \dfrac{x+2}{1-x}$	2	5	29	299	2999	29999

(b) $\lim\limits_{x\to 1^-}(x+2) = 3$ and $\lim\limits_{x\to 1^-}(1-x) = 0$. Because $x \to 1^-$, then $x < 1$ and $1 - x > 0$. Thus, $1 - x$ approaches 0 through positive values. By Limit Theorem 12(i), we have $\lim\limits_{x\to 1^-}\dfrac{x+2}{1-x} = +\infty$

5. (a) $f(x) = \dfrac{x+2}{1-x}$.

x	2	1.5	1.1	1.01	1.001	1.0001
$f(x)$	-4	-7	-31	-301	-3001	$-30,001$

(b) $\lim\limits_{x\to 1^+} (x+2) = 3$ and $\lim\limits_{x\to 1^+} (1-x) = 0$ through negative values. Hence, $\lim\limits_{x\to 1^+} \dfrac{x+2}{1-x} = -\infty$.

6. (a) $f(x) = \dfrac{x+2}{(x-1)^2}$.

x	0	0.5	0.9	0.99	0.999	0.9999
$f(x)$	2	10	290	29,900	2,999,000	299,900,000

x	2	1.5	1.1	1.01	1.001	1.0001
$f(x)$	4	14	310	30,100	3,001,000	300,010,000

(b) $\lim\limits_{x\to 5} 1 = 1$ and $\lim\limits_{x\to 5} (x-5)^2 = 0$ through positive values. Hence, $\lim\limits_{x\to 3} \dfrac{1}{(x-5)^2} = +\infty$.

7. (a) $f(x) = \dfrac{x-2}{x+1}$.

x	0	-0.5	-0.9	-0.99	-0.999	-0.9999
$f(x)$	-2	-5	-29	-299	-2999	$-29,999$

(b) $\lim\limits_{x\to -1^+} (x-2) = -3$ and $\lim\limits_{x\to -1^+} (x+1) = 0$ through positive values. Hence, $\lim\limits_{x\to -1^+} \dfrac{x-2}{x+1} = -\infty$.

8. (a) $f(x) = \dfrac{x-2}{x+1}$; x is -2, -1.5, -1.1, -1.01, -1.001, -1.0001; (b) $\lim\limits_{x\to -1^-} \dfrac{x-2}{x+1}$

▸ (a) From the values of $f(x)$ for the specified values of x given in the table, $f(x)$ appears to be increasing without bound as x approaches -1 from the left.

x	-2	-1.5	-1.1	-1.01	-1.001	-1.0001
$f(x) = \dfrac{x-2}{x+1}$	4	7	31	301	3001	30001

(b) $\lim\limits_{x\to -1^-}(x-2) = -3$ and $\lim\limits_{x\to -1^-}(x+1) = 0$. Because $x \to -1^-$, then $x < -1$ and $x + 1 < 0$. Thus, $x + 1$ approaches 0 through negative values. By Limit Theorem 12(iv), we have $\lim\limits_{x\to -1^-}\dfrac{x-2}{x+1} = +\infty$

9. (a) $f(x) = \dfrac{x}{x+4}$.

x	-5	-4.5	-4.1	-4.01	-4.001	-4.0001
$f(x)$	5	9	41	401	4001	40,001

(b) $\lim\limits_{x\to -4^-} x = -4$ and $\lim\limits_{x\to -4^-} (x+4) = 0$ through negative values. Hence, $\lim\limits_{x\to -4^-} \dfrac{x}{x+4} = +\infty$.
Alternatively, we could write the entire solution as: $\lim\limits_{x\to -4^-}\dfrac{x}{x+4} = \dfrac{-4}{0^-} = +\infty$.

10. (a) $f(x) = \dfrac{x}{x-4}$.

x	5	4.5	4.1	4.01	4.001	4.0001
$f(x)$	5	9	41	401	4001	40,001

(b) $\lim\limits_{x\to 4^+}\dfrac{x}{x-4} = \dfrac{4}{0^+} = +\infty$

11. (a) $f(x) = \dfrac{4x}{9-x^2}$.

x	-4	-3.5	-3.1	-3.01	-3.001	-3.0001
$f(x)$	2.3	4.3	20.3	200.3	2000.5	20,037

(b) $\lim\limits_{x\to -3^-} 4x = -12$ and $\lim\limits_{x\to -3^-} (9-x^2) = 0$ through negative values. Hence, $\lim\limits_{x\to -3^-} \dfrac{4x}{9-x^2} = +\infty$.

12. (a) $f(x) = \dfrac{4x^2}{9-x^2}$; x is 4, 3.5, 3.1, 3.01, 3.001, 3.0001; (b) $\lim\limits_{x\to 3^+} \dfrac{4x^2}{9-x^2}$

▸ (a) The table gives the values of $f(x)$ for the specified values of x. From the table, $f(x)$ appears to be decreasing without bound as x approaches 3 from the right.

x	4	3.5	3.1	3.01	3.001	3.0001
$f(x) = \dfrac{4x^2}{9-x^2}$	−9.14	−15.1	−63.0	−603	−6003	−60003

(b) $\lim\limits_{x\to 3^+}(4x^2) = 36$ and $\lim\limits_{x\to 3^+}(9-x^2) = 0$. Furthermore, since $x \to 3^+$, then $x > 3$ and $9 - x^2 < 0$. Thus, $9 - x^2$ approaches 0 through negative values. By Limit Theorem 12(ii), $\lim\limits_{x\to 3^+}\dfrac{4x^2}{9-x^2} = -\infty$

In Exercises 13–32, find the limit and support your answer by plotting the graph of the function.

13. $\lim\limits_{t\to 2^+}\dfrac{t+2}{t^2-4} = \lim\limits_{t\to 2^+}\dfrac{t+2}{(t+2)(t-2)} = \lim\limits_{t\to 2^+}\dfrac{1}{t-2} = \dfrac{1}{0^+} = +\infty.$

14. $\lim\limits_{t\to 2^-}\dfrac{-t+2}{(t-2)^3} = \lim\limits_{t\to 2^-}\dfrac{-(t-2)}{(t-2)^3} = \lim\limits_{t\to 2^-}\dfrac{-1}{(t-2)^2} = \dfrac{-1}{0^+} = -\infty$

15. $\lim\limits_{t\to 2^-}\dfrac{t+2}{t^2-4} = \lim\limits_{t\to 2^-}\dfrac{t+2}{(t+2)(t-2)} = \lim\limits_{t\to 2^-}\dfrac{1}{t-2} = \dfrac{1}{0^-} = -\infty.$

16. $\lim\limits_{x\to 0^+}\dfrac{\sqrt{3+x^2}}{x} = +\infty$

▸ $\lim\limits_{x\to 0^+}\sqrt{3+x^2} = \sqrt{3}$ and $\lim\limits_{x\to 0^+} x = 0$. Moreover, since $x \to 0^+$, x approaches 0 through positive values. Thus by Limit Theorem 12(i), $\lim\limits_{x\to 0^+}\dfrac{\sqrt{3+x^2}}{x} = +\infty$

17. $\lim\limits_{x\to 0^-}\sqrt{3+x^2} = \sqrt{3}$ and $\lim\limits_{x\to 0^-} x = 0$ through negative values. Therefore, $\lim\limits_{x\to 0^-}\dfrac{\sqrt{3+x^2}}{x} = -\infty.$

18. $\lim\limits_{x\to 0}\dfrac{\sqrt{3+x^2}}{x^2} = \dfrac{\sqrt{3}}{0^+} = +\infty$

19. $\lim\limits_{x\to 3^+}\dfrac{\sqrt{x^2-9}}{x-3} = \lim\limits_{x\to 3^+}\dfrac{\sqrt{x-3}\sqrt{x+3}}{\sqrt{x-3}\sqrt{x-3}} = \lim\limits_{x\to 3^+}\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$; $\lim\limits_{x\to 3^+}\sqrt{x+3} = \sqrt{6}$ and $\lim\limits_{x\to 3^+}\sqrt{x-3} = 0$ through positive values. Therefore, $\lim\limits_{x\to 3^+}\dfrac{\sqrt{x+3}}{\sqrt{x-3}} = +\infty.$

20. $\lim\limits_{x\to 4^-}\dfrac{\sqrt{16-x^2}}{x-4}$

▸ Because both the numerator and denominator have limit 0, we cannot use Limit Theorem 12. We must factor. Because $x \to 4^-$, then $x < 4$ or, equivalently, $4 - x > 0$. Thus,

$$x - 4 = -(4-x) = -\sqrt{(4-x)^2}$$

Therefore

$$\begin{aligned}\lim_{x\to 4^-}\frac{\sqrt{16-x^2}}{x-4} &= \lim_{x\to 4^-}\frac{\sqrt{(4-x)(4+x)}}{-\sqrt{(4-x)^2}}\\ &= \lim_{x\to 4^-}\frac{\sqrt{4-x}\cdot\sqrt{4+x}}{-\sqrt{4-x}\cdot\sqrt{4-x}}\\ &= \lim_{x\to 4^-}\frac{\sqrt{4+x}}{-\sqrt{4-x}} \qquad (1)\end{aligned}$$

Because $\lim\limits_{x\to 4^-}\sqrt{4+x} = \sqrt{8}$ and $\lim\limits_{x\to 4^-} -\sqrt{4-x} = 0$ and $-\sqrt{4-x}$ approaches 0 through negative values, by Limit Theorem 12(ii) and Eq. (1) we have $\lim\limits_{x\to 4^-}\dfrac{\sqrt{16-x^2}}{x-4} = -\infty$

21. $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim\limits_{x\to 0^+}\frac{x-1}{x^2}$; $\lim\limits_{x\to 0^+}(x-1)=-1$ and $\lim\limits_{x\to 0^+}x^2=0$ through positive values.

Therefore, $\lim\limits_{x\to 0^+}\frac{x-1}{x^2}=-\infty$.

22. $\lim\limits_{x\to 0^+}\frac{x^2-3}{x^3+x^2}=\lim\limits_{x\to 0^+}\frac{x^2-3}{x^2(x+1)}=\frac{-3}{0^+}=-\infty$

23. $\lim\limits_{x\to 0^-}(2-4x^3)=2$ and $\lim\limits_{x\to 0^-}(5x^2+3x^3)=0$ through positive values. Hence, $\lim\limits_{x\to 0^-}\frac{2-4x^3}{5x^2+3x^3}=+\infty$.

24. $\lim\limits_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)$

▷
$$\begin{aligned}\lim_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)&=\lim_{s\to 2^-}\left(\frac{s+2}{(s-2)(s+2)}-\frac{3}{(s-2)(s+2)}\right)\\&=\lim_{s\to 2^-}\frac{s-1}{(s-2)(s+2)}\end{aligned}$$

Moreover, $\lim\limits_{s\to 2^-}(s-1)=1$ and $\lim\limits_{s\to 2^-}(s-2)(s+2)=0$. Because $s\to 2^-$, then $s-2<0$; thus $(s-2)(s+2)$ approaches 0 through negative values. Therefore, by Limit Theorem 12(ii), $\lim\limits_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)=-\infty$

25. $\lim\limits_{t\to -4^-}\left(\frac{2}{t^2+3t-4}-\frac{3}{t+4}\right)=\lim\limits_{t\to -4^-}\left(\frac{2}{(t+4)(t-1)}-\frac{3(t-1)}{(t+4)(t-1)}\right)=\lim\limits_{t\to -4^-}\frac{5-3t}{(t+4)(t-1)}$.

$\lim\limits_{t\to -4^-}(5-3t)=17$ and $\lim\limits_{t\to -4^-}(t+4)(t-1)=0$ through positive values. Therefore, $\lim\limits_{t\to -4^-}\frac{5-3t}{(t+4)(t-1)}=+\infty$.

26. $\lim\limits_{x\to 1^-}\frac{2x^3-5x^2}{x^2-1}=\frac{-3}{0^-}=+\infty$

27. $\lim\limits_{x\to 3^-}(\llbracket x\rrbracket-x)=\lim\limits_{x\to 3^-}\llbracket x\rrbracket-3=\lim\limits_{x\to 3^-}(2)-3=-1$ and $\lim\limits_{x\to 3^-}(3-x)=0$ through positive values.

Therefore, $\lim\limits_{x\to 3^-}\frac{\llbracket x\rrbracket-x}{3-x}=-\infty$.

28. $\lim\limits_{x\to 1^-}\frac{\llbracket x^2\rrbracket-1}{x^2-1}$

▷ Since $x\to 1^-$, then $x<1$. If $0<x<1$, then $\llbracket x^2\rrbracket=0$. Thus

$$\lim_{x\to 1^-}\llbracket x^2\rrbracket-1=-1$$

Because $x<1$ and $x^2-1<0$ if $-1<x<1$, then

$$\lim_{x\to 1^-}(x^2-1)=0$$

and x^2-1 approaches 0 through negative values. Therefore, by Limit Theorem 12(iv),

$$\lim_{x\to 1^-}\frac{\llbracket x^2\rrbracket-1}{x^2-1}=+\infty$$

29. $\lim\limits_{x\to 3^-}\frac{x^3+9x^2+20x}{x^2+x-12}=\lim\limits_{x\to 3^-}\frac{x(x+5)(x+4)}{(x-3)(x+4)}=\lim\limits_{x\to 3^-}\frac{x(x+5)}{x-3}$

$\lim\limits_{x\to 3^-}x(x+5)=24$ and $\lim\limits_{x\to 3^-}(x-3)=0$ through negative values. Hence, $\lim\limits_{x\to 3^-}\frac{x(x+5)}{x-3}=-\infty$.

30. $\lim\limits_{x\to -2^+}\frac{6x^2+x-2}{2x^2+3x-2}=\lim\limits_{x\to -2^+}\frac{6x^2+x-2}{(x+2)(2x-1)}=\frac{20}{0^+(-5)}=-\infty$

31. $\lim\limits_{x\to1^+}\dfrac{x-1}{\sqrt{2x-x^2}-1}=\lim\limits_{x\to1^+}\dfrac{(x-1)(\sqrt{2x-x^2}+1)}{(\sqrt{2x-x^2}-1)(\sqrt{2x-x^2}+1)}=\lim\limits_{x\to1^+}\dfrac{(x-1)(\sqrt{2x-x^2}+1)}{2x-x^2-1}$

$$=\lim_{x\to1^+}\frac{(x-1)(\sqrt{2x-x^2}+1)}{-(x-1)^2}=\lim_{x\to1^+}\frac{\sqrt{2x-x^2}+1}{1-x}$$

$\lim\limits_{x\to1^+}(\sqrt{2x-x^2}+1)=2$ and $\lim\limits_{x\to1^+}(1-x)=0$ through negative values. Hence, $\lim\limits_{x\to1^+}\dfrac{\sqrt{2x-x^2}+1}{1-x}=-\infty$.

32. $\lim\limits_{x\to2^-}\dfrac{x-2}{2-\sqrt{4x-x^2}}$

▸ Because the numerator and denominator both have limit 0, Limit Theorem 12 does not apply. We rationalize the denominator.

$$\begin{aligned}\lim_{x\to2^-}\frac{x-2}{2-\sqrt{4x-x^2}}&=\lim_{x\to2^-}\frac{x-2}{2-\sqrt{4x-x^2}}\cdot\frac{2+\sqrt{4x-x^2}}{2+\sqrt{4x-x^2}}\\&=\lim_{x\to2^-}\frac{(x-2)(2+\sqrt{4x-x^2})}{4-(4x-x^2)}\\&=\lim_{x\to2^-}\frac{(x-2)(2+\sqrt{4x-x^2})}{(x-2)^2}\\&=\lim_{x\to2^-}\frac{2+\sqrt{4x-x^2}}{x-2}\qquad(1)\end{aligned}$$

Now $\lim\limits_{x\to2^-}(2+\sqrt{4x-x^2})=4$ and $\lim\limits_{x\to2^-}(x-2)=0$. Furthermore, $x-2$ approaches 0 through negative values. Thus, by Limit Theorem 12(ii) and Eq. (1) we conclude that $\lim\limits_{x\to2^-}\dfrac{x-2}{2-\sqrt{4x-x^2}}=-\infty$

33. $f(x)=\dfrac{x^2+x-6}{x^2-6x+8}=\dfrac{(x-2)(x+3)}{(x-2)(x-4)}=\dfrac{x+3}{x-4}$ if $x\neq2$.

$\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=-\frac{5}{2}$. $\lim\limits_{x\to4^-}f(x)=\dfrac{7}{0^-}=-\infty$. $\lim\limits_{x\to4^+}f(x)=\dfrac{7}{0^+}=+\infty$

34. $f(x)=\dfrac{x^2+2x-3}{x^2+x-6}=\dfrac{(x+3)(x-1)}{(x+3)(x-2)}=\dfrac{x-1}{x-2}$ if $x\neq-3$.

$\lim\limits_{x\to-3^-}f(x)=\lim\limits_{x\to-3^+}f(x)=\frac{2}{1}=2$. $\lim\limits_{x\to2^-}f(x)=\dfrac{2}{0^-}=-\infty$. $\lim\limits_{x\to2^+}f(x)=\dfrac{2}{0^+}=+\infty$

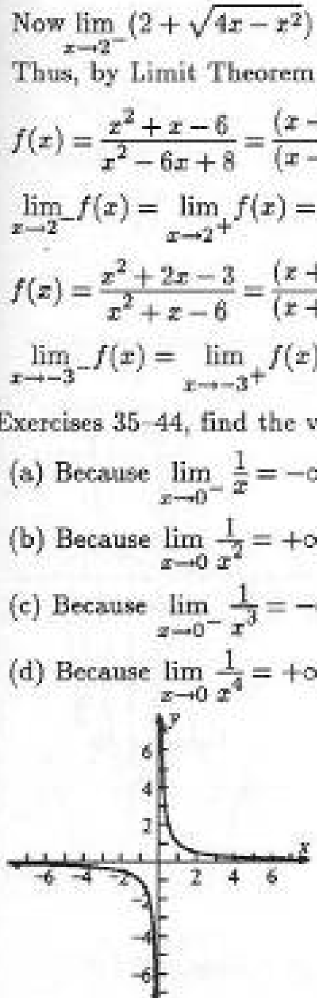
In Exercises 35–44, find the vertical asymptote(s) of the graph of the function, and sketch the graph.

35. (a) Because $\lim\limits_{x\to0^-}\frac{1}{x}=-\infty$ or because $\lim\limits_{x\to0^+}\frac{1}{x}=+\infty$, $x=0$ is a vertical asymptote.

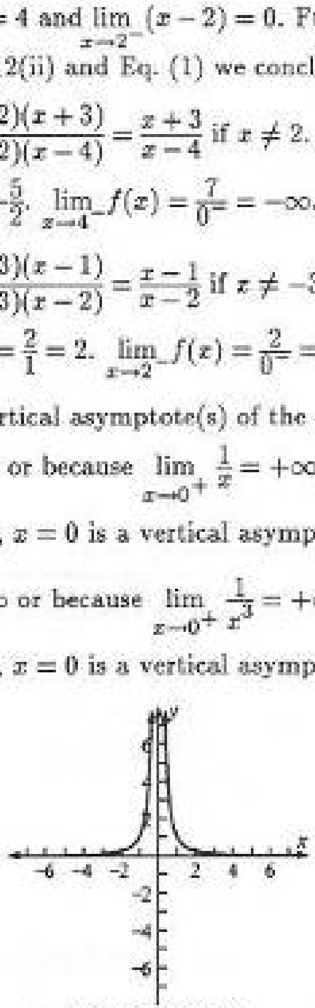
(b) Because $\lim\limits_{x\to0}\frac{1}{x^2}=+\infty$, $x=0$ is a vertical asymptote.

(c) Because $\lim\limits_{x\to0^-}\frac{1}{x^3}=-\infty$ or because $\lim\limits_{x\to0^+}\frac{1}{x^3}=+\infty$, $x=0$ is a vertical asymptote.

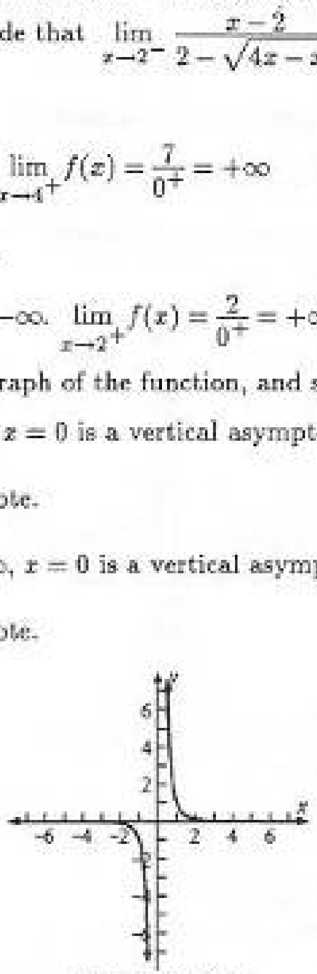
(d) Because $\lim\limits_{x\to0}\frac{1}{x^4}=+\infty$, $x=0$ is a vertical asymptote.



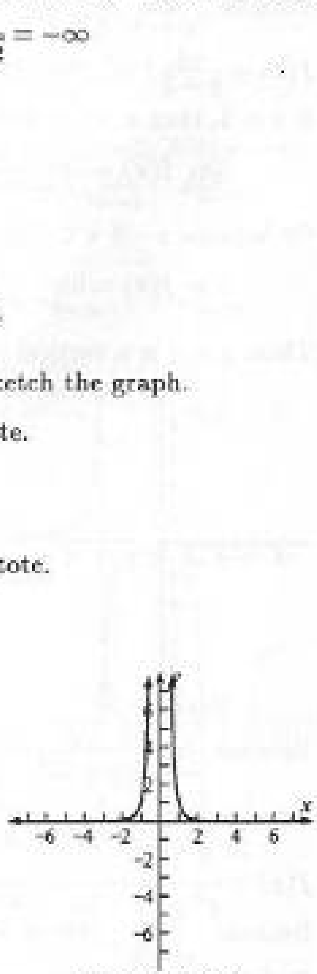
Exercise 35(a)



Exercise 35(b)



Exercise 35(c)



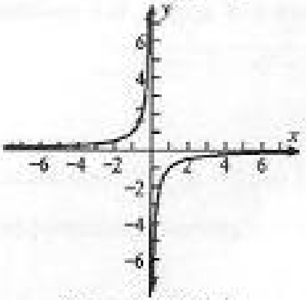
Exercise 35(d)

36. (a) Because $\lim\limits_{x\to 0^-} -\dfrac{1}{x} = +\infty$ by Limit Theorem 12(iv) or because $\lim\limits_{x\to 0^+} -\dfrac{1}{x} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

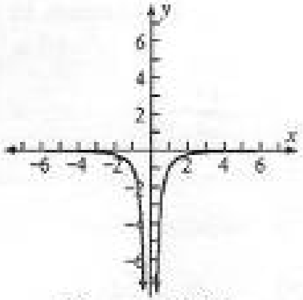
(b) Because $\lim\limits_{x\to 0} -\dfrac{1}{x^2} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

(c) Because $\lim\limits_{x\to 0^-} -\dfrac{1}{x^3} = +\infty$ by Limit Theorem 12(iv) or because $\lim\limits_{x\to 0^+} -\dfrac{1}{x^3} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

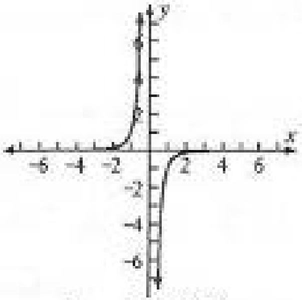
(d) Because $\lim\limits_{x\to 0} -\dfrac{1}{x^4} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.



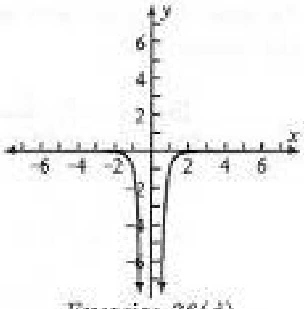
Exercise 36(a)



Exercise 36(b)



Exercise 36(c)



Exercise 36(d)

37. Because $\lim\limits_{x\to 4^-} \dfrac{2}{x-4} = -\infty$ or because $\lim\limits_{x\to 0^+} \dfrac{2}{x-4} = +\infty$, $x = 4$ is a vertical asymptote.

38. Because $\lim\limits_{x\to -1^-} \dfrac{3}{x+1} = \dfrac{3}{0^-} = -\infty$ or because $\lim\limits_{x\to -1^+} \dfrac{3}{x+1} = \dfrac{3}{0^+} = +\infty$, $x = -1$ is a vertical asymptote.

39. Because $\lim\limits_{x\to -3^-} \dfrac{-2}{x+3} = +\infty$ or because $\lim\limits_{x\to -3^+} \dfrac{-2}{x+3} = -\infty$, $x = -3$ is a vertical asymptote.

40. $f(x) = \dfrac{-4}{x-5}$

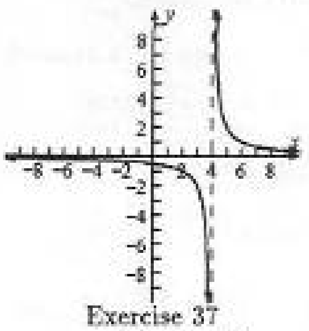
▸ If $x = 5$, then $x - 5 = 0$. Furthermore, because $x - 5 > 0$ if $x > 5$, and $-4 < 0$, by Limit Theorem 12(iii)

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{-4}{x-5} = -\infty$$

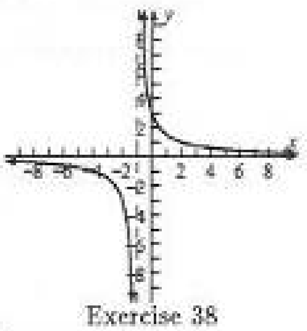
Or because $x - 5 < 0$ if $x < 5$, by Limit Theorem 12(iv)

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{-4}{x-5} = -\infty$$

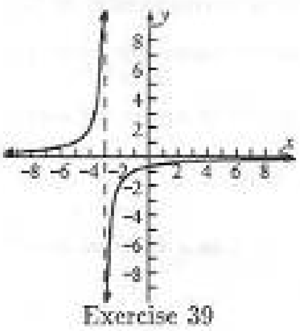
Thus, $x = 5$ is a vertical asymptote.



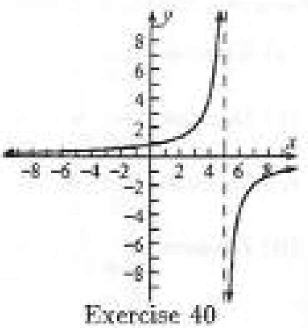
Exercise 37



Exercise 38



Exercise 39



Exercise 40

41. Because $\lim\limits_{x\to -3} \dfrac{-2}{(x+3)^2} = \dfrac{-2}{0^+} = -\infty$, $x = -3$ is a vertical asymptote.

42. Because $\lim\limits_{x\to 5} \dfrac{4}{(x-5)^2} = \dfrac{4}{0^+} = +\infty$, $x = 5$ is a vertical asymptote.

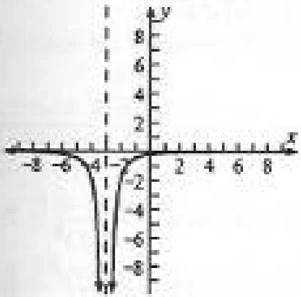
43. $f(x) = \dfrac{5}{x^2 + 8x + 15} = \dfrac{5}{(x+3)(x+5)}$

Because $\lim\limits_{x\to -5^-} f(x) = +\infty$ or because $\lim\limits_{x\to -5^+} f(x) = -\infty$, $x = -5$ is a vertical asymptote.

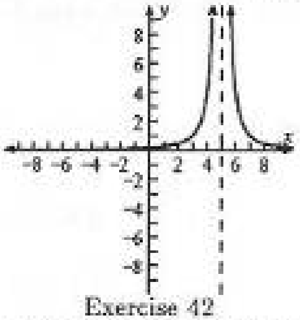
Because $\lim\limits_{x\to -3^-} f(x) = -\infty$ or because $\lim\limits_{x\to -3^+} f(x) = +\infty$, $x = -3$ is a vertical asymptote.

44. $f(x) = \dfrac{1}{x^2+5x-6}$

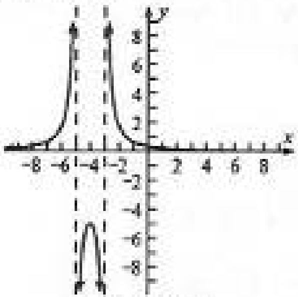
▸ $f(x) = \dfrac{1}{(x+6)(x-1)}$.
$\lim\limits_{x\to-6^-}(x+6) = 0$ and $\lim\limits_{x\to-6^-}(x-1) = -7$. Because $x \to 6^-$, then $x+6<0$; thus $(x+6)(x-1)$ approaches 0 through negative values. Therefore, by Limit Theorem 12(ii) $\lim\limits_{x\to-6^-}\dfrac{1}{x^2+5x-6} = -\infty$. Thus $x=-6$ is a vertical asymptote.
$\lim\limits_{x\to1^+}(x+6) = 7$ and $\lim\limits_{x\to1^+}(x-1) = 0$. Because $x\to1^+$, then $x-1>0$; thus $(x+6)(x-1)$ approaches 0 thru positive values. Hence, by Limit Theorem 12(i) $\lim\limits_{x\to1^+}\dfrac{1}{x^2+5x-6} = +\infty$. Thus $x=1$ is a vertical asymptote.



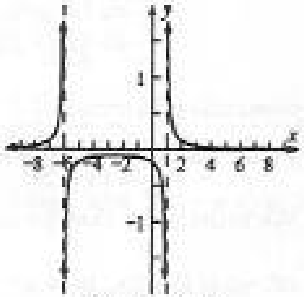
Exercise 41



Exercise 42



Exercise 43



Exercise 44

In Exercises 45 and 46, evaluate the limits from the graph of function f sketched in the accompanying figure,

45. (a) $\lim\limits_{x\to-2^+}f(x) = 0$ (b) $\lim\limits_{x\to-1^-}f(x) = -\infty$ (c) $\lim\limits_{x\to-1^+}f(x) = +\infty$ (d) $\lim\limits_{x\to0}f(x) = 0$ (e) $\lim\limits_{x\to1^-}f(x) = +\infty$
(f) $\lim\limits_{x\to1^+}f(x) = +\infty$ (g) $\lim\limits_{x\to1}f(x) = +\infty$ (h) $\lim\limits_{x\to2^-}f(x) = 1$ (i) $\lim\limits_{x\to2^+}f(x) = -\infty$ (j) $\lim\limits_{x\to3^-}f(x) = 0$

46. (a) $\lim\limits_{x\to-4^+}f(x) = +\infty$ (b) $\lim\limits_{x\to-2^-}f(x) = 2$ (c) $\lim\limits_{x\to-2^+}f(x) = +\infty$ (d) $\lim\limits_{x\to0}f(x) = 0$ (e) $\lim\limits_{x\to2^-}f(x) = -\infty$
(f) $\lim\limits_{x\to2^+}f(x) = 0$ (g) $\lim\limits_{x\to3^-}f(x) = -\infty$ (h) $\lim\limits_{x\to3^+}f(x) = -\infty$ (i) $\lim\limits_{x\to3}f(x) = -\infty$ (j) $\lim\limits_{x\to4^+}f(x) = 0$

In Exercises 47 and 48, sketch the graph of a function f satisfying the given properties.

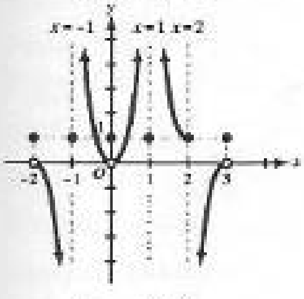
47. The domain of f is $[-5,5]$. $f(-5)=0$; $f(-3)=2$; $f(-1)=0$; $f(0)=0$; $f(1)=0$; $f(3)=-2$; $f(5)=-4$;
$\lim\limits_{x\to-5^+}f(x) = +\infty$; $\lim\limits_{x\to-3}f(x) = 0$; $\lim\limits_{x\to-1^-}f(x) = +\infty$; $\lim\limits_{x\to-1^-}f(x) = -\infty$; $\lim\limits_{x\to0}f(x) = 0$; $\lim\limits_{x\to1} = +\infty$;
$\lim\limits_{x\to3}f(x) = 0$; $\lim\limits_{x\to5^-}f(x) = -\infty$

▸ The figure shows the graph of $\begin{cases} -(x+3)^2/[(x+5)(x+1)] & \text{if } -5<x<-1 \\ x/(1-x^2) & \text{if } -1<x<1 \\ (x-3)/[(x-1)(x-5)] & \text{if } 1<x<5 \end{cases}$

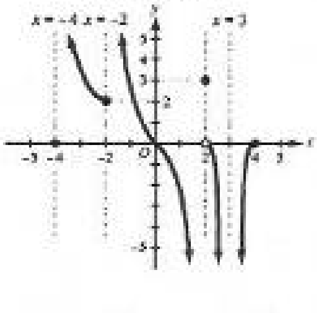
48. The domain of f is $[-2,2]$. $f(-2)=0$; $f(-1)=0$; $f(0)=5$; $f(1)=-5$; $f(2)=3$; $\lim\limits_{x\to-2^+}f(x) = -\infty$;
$\lim\limits_{x\to-1^-}f(x) = +\infty$; $\lim\limits_{x\to-1^+}f(x) = -\infty$; $\lim\limits_{x\to0^-}f(x) = 0$; $\lim\limits_{x\to0^+}f(x) = +\infty$; $\lim\limits_{x\to1}f(x) = -\infty$; $\lim\limits_{x\to2^-}f(x) = 3$

▸ The figure shows the graph of $\begin{cases} -x(x+1.5)/[(x+2)(x+1)] & \text{if } x\in(-2,-1)\cup(-1,0) \\ 8(x-.5)(x-1.5)/[x(x-1)^2] & \text{if } x\in(0,1)\cup(1,2] \end{cases}$

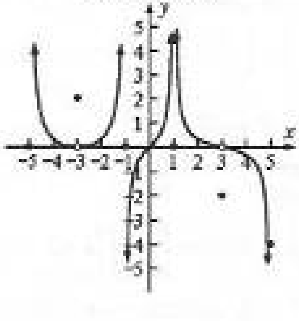
Exercise 45



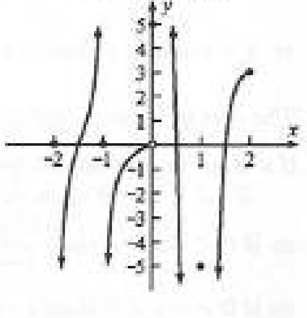
Exercise 46



Exercise 47



Exercise 48



49. $\lim\limits_{t\to0^+} n\left(\dfrac{r}{t} + \dfrac{epk}{1000}\right) = +\infty$ because r and n are positive.

50. $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{2-x}$. (a) $\lim\limits_{x\to2^-}\dfrac{1}{x-2} = \dfrac{1}{0^-} = -\infty$ so $\lim\limits_{x\to2}f(x)$ does not exist. $\lim\limits_{x\to2^-}\dfrac{1}{2-x} = \dfrac{1}{0^+} = +\infty$ so $\lim\limits_{x\to2}g(x)$ does not exist. (b) $f(x)+g(x) = 0$ if $x \neq 2$. (c) $\lim\limits_{x\to2}[f(x)+g(x)] = 0$

(d) Because $\lim_{x\to 2} f(x)$ and $\lim_{x\to 2} g(x)$ do not exist, the hypotheses of Limit Theorem 4 are not satisfied.

51. $\lim_{v\to c^-} \dfrac{m_0}{\sqrt{1-(v/c)^2}} = \dfrac{m_0}{0^+} = +\infty$. $\lim_{v\to c^+} \dfrac{m_0}{\sqrt{1-(v/c)^2}}$ does not exist because $1-(v/c)^2 < 0$ if $v > c$. The limit as $v \to c$ does not exist because neither 1-sided limit exists.

1.7 SUPPLEMENT

Theorem B $\lim_{x\to a} f(x) = +\infty$ if and only if $\lim_{x\to a} -f(x) = -\infty$.

Proof Let $\mathrm{M} = -\mathrm{N}$. $\lim_{x\to a} f(x) = +\infty$

$\Leftrightarrow$ for every $\mathrm{N} > 0$ there is a $\delta > 0$ such that $f(x) > \mathrm{N}$ when $0 < |x-a| < \delta$

$\Leftrightarrow$ for every $\mathrm{M} < 0$ there is a $\delta > 0$ such that $-f(x) < \mathrm{M}$ when $0 < |x-a| < \delta$

$\Leftrightarrow$ $\lim_{x\to a} -f(x) = -\infty$

Supplementary Exercises 1.7

1. Prove that $\lim_{x\to 2} \dfrac{3}{(x-2)^2} = +\infty$ by using Definition 1.7.1.

▹ We must show that for any $\mathrm{N} > 0$ there exists a $\delta > 0$ such that if $0 < |x-2| < \delta$ then $\dfrac{3}{(x-2)^2} > \mathrm{N}$

or, equivalently, because $(x-4)^2 > 0$ and $\mathrm{N} > 0$ if $0 < |x-2| < \delta$ then $(x-2)^2 < \dfrac{3}{\mathrm{N}}$

or, equivalently, if $0 < |x-2| < \delta$ then $|x-3| < \sqrt{\dfrac{3}{\mathrm{N}}}$

The above statement holds if $\delta = \sqrt{\dfrac{3}{\mathrm{N}}}$.

2. Prove that $\lim_{x\to 4} \dfrac{-2}{(x-4)^2} = -\infty$ by using Definition 1.7.2.

▹ We must show that for any $\mathrm{N} < 0$ there exists a $\delta > 0$ such that if $0 < |x-4| < \delta$ then $\dfrac{-2}{(x-4)^2} < \mathrm{N}$

or, equivalently, because $(x-4)^2 > 0$ and $\mathrm{N} < 0$ if $0 < |x-4| < \delta$ then $(x-4)^2 < -\dfrac{2}{\mathrm{N}}$

or, equivalently, if $0 < |x-4| < \delta$ then $|x-4| < \sqrt{-\dfrac{2}{\mathrm{N}}}$

The above statement holds if $\delta = \sqrt{-\dfrac{2}{\mathrm{N}}}$.

3. Prove Limit Theorem 11(ii): If r is any positive integer, then $\lim_{x\to 0^-} \dfrac{1}{x^r} = \begin{cases} -\infty & \text{if } r \text{ is odd} \\ +\infty & \text{if } r \text{ is even} \end{cases}$

If r is an odd positive integer we must show that for any $\mathrm{N} < 0$ there is a $\delta > 0$ such that

if $-\delta < x < 0$ then $\dfrac{1}{x^r} < \mathrm{N}$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{x^r} > -\mathrm{N}$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{(-x)^r} > -\mathrm{N}$ (because r is an odd integer)

$\Leftrightarrow$ if $0 < -x < \delta$ then $(-x)^r < \dfrac{1}{-\mathrm{N}}$ (because $-x > 0$ and $-\mathrm{N} > 0$)

$\Leftrightarrow$ if $0 < -x < \delta$ then $-x < \left(-\dfrac{1}{\mathrm{N}}\right)^{1/r}$ (because r is positive)

The last statement holds if $\delta = \left(-\dfrac{1}{\mathrm{N}}\right)^{1/r}$.

If r is an even positive integer we must show that for any $\mathrm{N} > 0$ there is a $\delta > 0$ such that

if $-\delta < x < 0$ then $\dfrac{1}{x^r} > \mathrm{N}$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{(-x)^r} > \mathrm{N}$ (because r is an even integer)

$\Leftrightarrow$ if $0 < -x < \delta$ then $(-x)^r < \dfrac{1}{\mathrm{N}}$ (because $-x > 0$ and $\mathrm{N} > 0$)

$\Leftrightarrow$ if $0 < -x < \delta$ then $-x < \left(\dfrac{1}{\mathrm{N}}\right)^{1/r}$ (because r is positive)

The last statement holds if $\delta = \left(\dfrac{1}{\mathrm{N}}\right)^{1/r}$.

4. Prove Theorem 1.7.4(ii).

▸ We want to prove that if $\lim_{x\to a} f(x) = 0$ with $f(x) \to 0$ through negative values of $f(x)$, and $\lim_{x\to a} g(x) = c$ with $c > 0$, then $\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$. Now $\lim_{x\to a} -f(x) = 0$ with $-f(x) \to 0$ through positive values, and so, by case (i) $\lim_{x\to a} \frac{g(x)}{-f(x)} = +\infty$. Hence by Theorem B, $\lim_{x\to a} -\left(\frac{g(x)}{-f(x)}\right) = -\infty$, or equivalently, $\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$.

5. Prove Theorem 1.7.4(iii).

▸ We want to prove that if $\lim_{x\to a} f(x) = 0$ with $f(x) \to 0$ through positive values of $f(x)$, and $\lim_{x\to a} g(x) = c$ with $c < 0$, then $\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$. Now $\lim_{x\to a} -g(x) = -c$ with $-c > 0$, and so, by case (i), $\lim_{x\to a} \frac{-g(x)}{f(x)} = +\infty$. Hence by Theorem B, $\lim_{x\to a} -\left(\frac{-g(x)}{f(x)}\right) = -\infty$, or equivalently, $\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$.

6. Prove Theorem 1.7.4(iv).

▸ We want to prove that if $\lim_{x\to a} f(x) = 0$ with $f(x) \to 0$ through negative values of $f(x)$, and $\lim_{x\to a} g(x) = c$ with $c < 0$, then $\lim_{x\to a} \frac{g(x)}{f(x)} = \infty$. Now $\lim_{x\to a} -f(x) = 0$ with $-f(x) \to 0$ through positive values, and $\lim_{x\to a} -g(x) = -c$ with $-c > 0$, and so, by case (i), $\lim_{x\to a} \frac{-g(x)}{-f(x)} = +\infty$, or equivalently, $\lim_{x\to a} \frac{g(x)}{f(x)} = \infty$.

7. Prove Theorem 1.7.5: If c is any constant and
(i) if $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} f(x) = c$ then $\lim_{x\to a} [f(x) + g(x)] = +\infty$
(ii) if $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = c$ then $\lim_{x\to a} [f(x) + g(x)] = -\infty$

Because $\lim_{x\to a} g(x) = c$, then for any $\epsilon > 0$ there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $|g(x) - c| < \epsilon$
Now $|g(x)| - |c| \leq |g(x) - c|$. Hence, if $\epsilon = |c| + 1$ there is a δ_1 such that
 if $0 < |x - a| < \delta_1$ then $|g(x) - c| < |c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta_1$ then $|g(x)| < 2|c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta_1$ then $-2|c| - 1 < g(x) < 2|c| + 1$ (1)

To prove part (i) we must show that for any N > 0 there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $f(x) + g(x) > N$
Because $\lim_{x\to a} f(x) = +\infty$, then for
$N + 2|c| + 1 > 0$ there is a $\delta_2 > 0$ such that
 if $0 < |x - a| < \delta_2$ then $f(x) > N + 2|c| + 1$
If $\delta = \min(\delta_1, \delta_2)$, by statement (1) we get
 if $0 < |x - a| < \delta$
 then $f(x) + g(x) > N + 2|c| + 1 - 2|c| - 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta$ then $f(x) + g(x) > N$
Hence part (i) is proved.

To prove part (ii) we must show that for any N < 0 there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $f(x) + g(x) < N$
Because $\lim_{x\to a} f(x) = -\infty$, then for
$N - 2|c| - 1 < 0$ there is a $\delta_3 > 0$ such that
 if $0 < |x - a| < \delta_3$ then $f(x) < N - 2|c| - 1$
If $\delta = \min(\delta_1, \delta_2)$, by statement (1) we get
 if $0 < |x - a| < \delta$
 then $f(x) + g(x) < N - 2|c| - 1 + 2|c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta$ then $f(x) + g(x) < N$
Hence part (ii) is proved.

8. Prove: Theorem 1.7.6:
If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = c$, where c is any constant except 0, then
(i) if $c > 0$, $\lim_{x\to a} f(x) \cdot g(x) = +\infty$ (ii) if $c < 0$, $\lim_{x\to a} f(x) \cdot g(x) = -\infty$

▸ By Theorem B, $\lim_{x\to a} -f(x) = -\infty$. By Theorem 1.7.7 proved below
if $c > 0$, $\lim_{x\to a} -f(x) \cdot g(x) = -\infty$ if $c < 0$, $\lim_{x\to a} -f(x) \cdot g(x) = +\infty$
Applying Theorem B again to the limit of the products, we obtain (i) and (ii).

9. Prove: Theorem 1.7.7:
If $\lim_{x \to a} f(x) = -\infty$ and $\lim_{x \to a} g(x) = c$, where c is any constant except 0, then
(i) if $c > 0$, $\lim_{x \to a} f(x) \cdot g(x) = -\infty$
(ii) if $c < 0$, $\lim_{x \to a} f(x) \cdot g(x) = +\infty$

▸ To prove part (i) we must show that for any $N < 0$ there is a $\delta > 0$ such that

if $0 < |x - a| < \delta$ then $f(x) \cdot g(x) < N$ (1)

Because $\lim_{x \to a} g(x) = c$ and $c > 0$ then for $\epsilon = \frac{1}{2}c > 0$ there is a $\delta_1 > 0$ such that

if $0 < |x - a| < \delta_1$ then $|g(x) - c| < \frac{1}{2}c$

$\Rightarrow$ if $0 < |x - a| < \delta_1$ then $g(x) - c > -\frac{1}{2}c$

$\Leftrightarrow$ if $0 < |x - a| < \delta_1$ then $g(x) > \frac{1}{2}c > 0$ (2)

Because $\lim_{x \to a} f(x) = -\infty$ then for $\frac{2N}{c} < 0$ there is a $\delta_2 > 0$ such that

if $0 < |x - a| < \delta_2$ then $-f(x) > -\frac{2N}{c} > 0$ (3)

Let $\delta = \min(\delta_1, \delta_2)$. It follows from statements (2) and (3) that

if $0 < |x - a| < \delta$ then $-f(x) \cdot g(x) > -\frac{2N}{c} \cdot \frac{c}{2} = -N$

This is statement (1) proving part (i).

To prove part (ii) we must show that for any $N > 0$ there is a $\delta > 0$ such that

if $0 < |x - a| < \delta$ then $f(x) \cdot g(x) > N$ (4)

Because $\lim_{x \to a} g(x) = c$ and $c < 0$ then for $\epsilon = -\frac{1}{2}c > 0$ there is a $\delta_3 > 0$ such that

if $0 < |x - a| < \delta_3$ then $|g(x) - c| < -\frac{1}{2}c$

$\Rightarrow$ if $0 < |x - a| < \delta_3$ then $-g(x) + c > \frac{1}{2}c$

$\Leftrightarrow$ if $0 < |x - a| < \delta_3$ then $-g(x) > -\frac{1}{2}c > 0$ (5)

Because $\lim_{x \to a} f(x) = -\infty$ then for $\frac{2N}{c} < 0$ there is a $\delta_4 > 0$ such that

if $0 < |x - a| < \delta_4$ then $-f(x) > -\frac{2N}{c} > 0$ (6)

Let $\delta = \min(\delta_3, \delta_4)$. It follows from statements (5) and (6) that

if $0 < |x - a| < \delta$ then $f(x) \cdot g(x) > -\frac{2N}{c} \cdot -\frac{c}{2} = N$

This is statement (4) proving part (ii).

10. Use Definition 1.7.1 to prove that $\lim_{x \to -3} \left| \frac{5 - x}{3 + x} \right| = +\infty$.

▸ Choose $\delta \leq 1$ so that $|x + 3| < \delta \Rightarrow -1 < x + 3 < 1 \Rightarrow -9 < x - 5 < -7 \Rightarrow 7 < |5 - x| < 9$. Then $\left| \frac{5 - x}{3 + x} \right| > \frac{7}{\delta} > N$ whenever $\delta < \frac{7}{N}$. Choose $\delta = \min(1, 7/N)$.

1.8 CONTINUITY OF A FUNCTION AT A NUMBER

There are many theorems having a hypothesis that includes the condition that a function be continuous at a number a. Hence we must be able to determine if a function is continuous or discontinuous at a. If there is a break in the graph of f at the point where $x = a$, then f is discontinuous at a. This situation is included in the following analytic definition.

1.8.1 Definition The function f is said to be *continuous* at the number a if and only if the following three conditions are satisfied.

(i) $f(a)$ exists
(ii) $\lim_{x \to a} f(x)$ exists
(iii) $\lim_{x \to a} f(x) = f(a)$

If one or more of these three conditions fails to hold at a, the function f is said to be *discontinuous* at a.

We note that we may have to consider one-sided limits and use Theorem 1.6.3 to determine whether or not condition (ii) in Definition 1.8.1 is satisfied. This is illustrated in Exercise 7. If f is a function that is discontinuous at the number a, but for which $\lim_{x \to a} f(x)$ exists and is the real number L, then either $f(a) \neq L$ or else $f(a)$ does not exist. Such a discontinuity is called a *removable discontinuity* because if we define $f(a) = L$ the new function is continuous at a. A removable discontinuity will not show as a break when the graph is plotted on a graphics calculator unless the x-range is carefully chosen. If $\lim_{x \to a} f(x)$ is not a real number, then f has an *essential discontinuity* at a, and the discontinuity cannot be removed.

The following theorems concerning the continuity of a function follow from Definition 1.8.1. Often they can be used to determine if a function is continuous at a number.

1.8.2 Theorem If f and g are two functions that are continuous at the number a, then

(i) $f+g$ is continuous at a
(ii) $f-g$ is continuous at a
(iii) $f\cdot g$ is continuous at a
(iv) f/g is continuous at a, provided that $g(a)\neq 0$

1.8.3 Theorem A polynomial function is continuous at every number.

1.8.4 Theorem A rational function is continuous at every number in its domain.

1.8.5 Theorem If n is a positive integer and $f(x)=\sqrt[n]{x}$, then
(i) if n is odd, f is continuous at every number
(ii) if n is even, f is continuous at every positive number

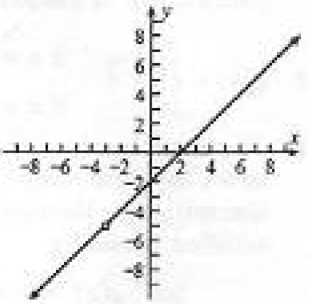
Theorem The absolute-value function is continuous at every number.

Exercises 1.8

In Exercises 1–14, sketch the graph of the function. By observing where there is a break in the graph, determine the number at which the function is discontinuous; and show why Definition 1.8.1 is not satisfied at this number.

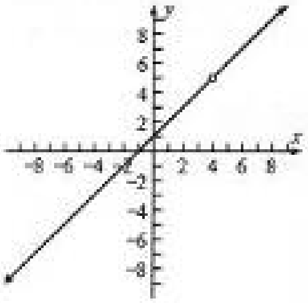
1. $f(x)=\dfrac{x^2+x-6}{x+3}=\dfrac{(x+3)(x-2)}{x+3}=x-2$ if $x\neq -3$.

▸ There is a break in the graph at -3.
$f(-3)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at -3.



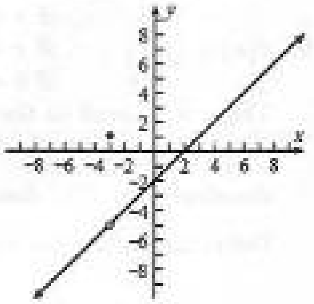
2. $F(x)=\dfrac{x^2-3x-4}{x-4}=\dfrac{(x-4)(x+1)}{x-4}=x+1$ if $x\neq 4$.

▸ There is a break in the graph at 4.
$f(4)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at 4.



3. $g(x)=\begin{cases}\dfrac{x^2+x-6}{x+3} & \text{if } x\neq -3\\ 1 & \text{if } x=-3\end{cases}=\begin{cases}x-2 & \text{if } x\neq -3\\ 1 & \text{if } x=-3\end{cases}$

▸ There is a break in the graph at -3.
(i) $g(-3)=1$; (ii) $\lim\limits_{x\to -3} g(x)=-5$;
(iii) $\lim\limits_{x\to -3} g(x)\neq g(-3)$.
Thus condition (iii) of Definition 1.8.1 fails at -3.
Hence, g is discontinuous at -3.



4. $G(x)=\begin{cases}\dfrac{x^2-3x-4}{x-4} & \text{if } x\neq 4\\ 2 & \text{if } x=4\end{cases}$

▸ Because

$$\frac{x^2-3x-4}{x-4}=\frac{(x-4)(x+1)}{x-4}=x+1 \quad \text{if } x\neq 4$$

then

$$G(x)=\begin{cases}x+1 & \text{if } x\neq 4\\ 2 & \text{if } x=4\end{cases}$$

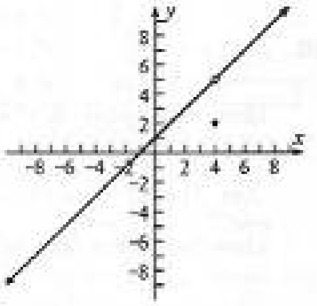
There is a "hole" in the line at the point (4, 5) because $G(x)\neq x+1$ if $x=4$, and thus G is discontinuous at 4. We show how Definition 1.8.1 is not satisfied. Because

$$\lim_{x\to 4} G(x)=\lim_{x\to 4}(x+1)=5 \quad \text{and} \quad G(4)=2$$

then

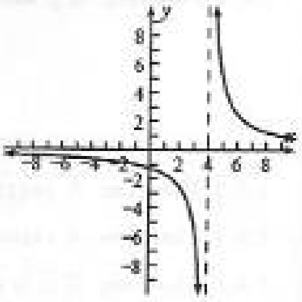
$$\lim_{x\to 4} G(x)\neq G(4)$$

and thus condition (iii) of Definition 1.8.1 is not satisfied.



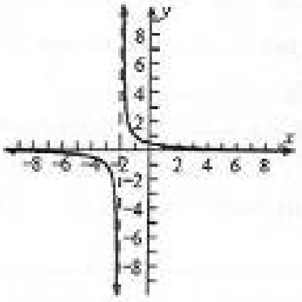
5. $h(x) = \dfrac{5}{x-4}$

▷ There is a break in the graph at 4.
$h(4)$ does not exist. Thus, condition (i) of Definition 1.8.1 fails at 4.
Therefore, f is discontinuous at 4.



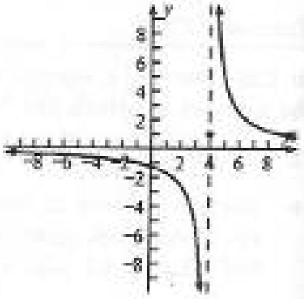
6. $H(x) = \dfrac{1}{x+2}$

▷ There is a break in the graph at -2.
$h(-2)$ does not exist. Thus, condition (i) of Definition 1.8.1 fails at -2.
Therefore, f is discontinuous at -2.



7. $f(x) = \begin{cases} \dfrac{5}{x-4} & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases}$

There is a break in the graph at 4.
$\lim\limits_{x \to 4^+} f(x) = +\infty$ and $\lim\limits_{x \to 4^-} f(x) = -\infty$. Hence, condition (ii) of Definition 1.8.1 fails at 4.
Therefore, f is discontinuous at 4.

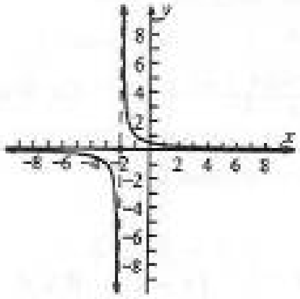


8. $g(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 0 & \text{if } x = -2 \end{cases}$

▷ There is a break in the graph at $x = -2$;
so we use Definition 1.8.1 at $a = -2$ to show there is a discontinuity. Because $g(-2) = 0$, condition (i) is satisfied. However

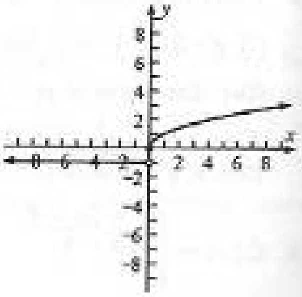
$$\lim_{x \to -2^+} g(x) = +\infty \quad \text{and} \quad \lim_{x \to -2^-} g(x) = -\infty$$

so condition (ii) is not satisfied. Therefore, g is discontinuous at -2.



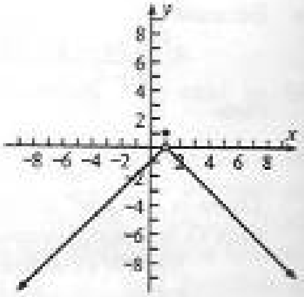
9. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x} & \text{if } 0 < x \end{cases}$

There is a break in the graph at 0.
$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-1) = -1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sqrt{x} = 0$; therefore $\lim\limits_{x \to 0} f(x)$ does not exist. Thus condition (ii) of Definition 1.8.1 fails at 0. Hence f is discontinuous at 0.

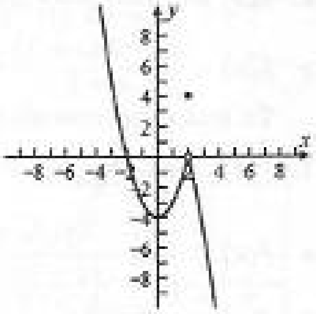


10. $f(x) = \begin{cases} x-1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 1-x & \text{if } x > 1 \end{cases}$

There is a break in the graph at 1.
(i) $f(1) = 1$; (ii) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x-1) = 0$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (1-x) = 0$. Thus $\lim\limits_{x \to 1} f(x) = 0$. (iii) $\lim\limits_{x \to 1} f(x) \neq f(1)$
Thus condition (iii) fails at 1. Hence f is discontinuous at 1.

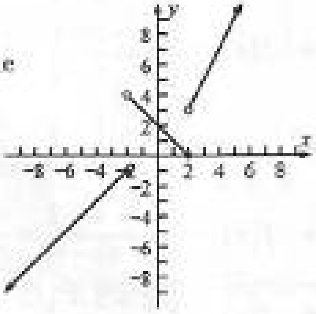


11. $g(t)=\begin{cases} t^2-4 & \text{if } t<2 \\ 4 & \text{if } t=2 \\ 4-t^2 & \text{if } 2<t \end{cases}$



▸ There is a break in the graph at 2.
(i) $g(2)=4$; (ii) $\lim\limits_{t\to 2^-} g(t)=\lim\limits_{t\to 2^-}(t^2-4)=0$ and
$\lim\limits_{t\to 2^+} g(t)=\lim\limits_{t\to 2^+}(4-t^2)=0$.
Therefore $\lim\limits_{t\to 0} g(t)=0$. (iii) $\lim\limits_{t\to 2} g(t)\neq g(2)$.
Thus condition (iii) of Definition 1.8.1 fails at 2.
Hence g is discontinuous at 2.

12. $H(x)=\begin{cases} 1+x & \text{if } x\le -2 \\ 2-x & \text{if } -2<x\le 2 \\ 2x-1 & \text{if } 2<x \end{cases}$



▸ There are breaks in the graph at the points where $x=-2$ and $x=2$. We consider the points separately. Since $H(-2)=1+2=-1$, condition (i) of Definition 1.8.1 is satisfied when $a=-2$. Because

$$\lim_{x\to -2^+} H(x)=\lim_{x\to -2^+}(2-x)=4$$

and

$$\lim_{x\to -2^-} H(x)=\lim_{x\to -2^-}(1+x)=-1$$

$\lim\limits_{x\to -2} H(x)$ does not exist. Therefore condition (ii) of Definition 1.8.1 is not satisfied when $a=-2$, and thus H is discontinuous at -2.
Since $H(2)=2-2=0$, condition (i) of Definition 1.8.1 is satisfied when $a=2$. Because

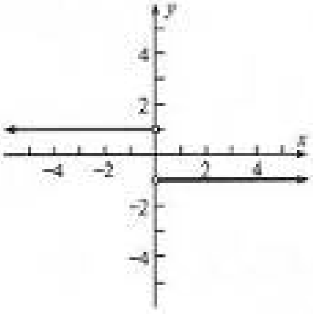
$$\lim_{x\to 2^+} H(x)=\lim_{x\to 2^+}(2x-1)=3$$

and

$$\lim_{x\to 2^-} H(x)=\lim_{x\to 2^-}(2-x)=0$$

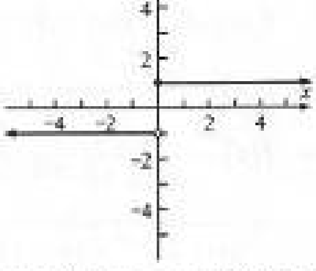
$\lim\limits_{x\to 2} H(x)$ does not exist. Thus condition (ii) in Definition 1.8.1 is not satisfied when $a=2$, and thus H is discontinuous at 2.

13. $f(x)=-\dfrac{|x|}{x}$, $x\neq 0$.



▸ There is a break in the graph at 0.
$f(0)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at 0.
Therefore, f is discontinuous at 0.

14. $g(x)=\begin{cases} |x|/x & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$



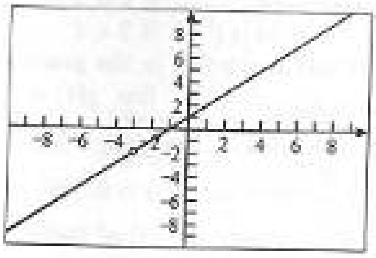
▸ There is a break in the graph at 0. (i) $f(0)=1$.
(ii) $\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-} -x/x=-1$ and
$\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+} x/x=1$. Thus $\lim\limits_{x\to 0} g(x)$ does not exist.
Condition (ii) in Definition 1.8.1 is not satisfied at 0, and so g is discontinuous at 0.

In Exercises 15–28, f is discontinuous at a. (a) Plot the graph of f and look for a break at $x=a$. Does the discontinuity appear to be removable? If so, how should f be redefined to remove it? (b) Confirm analytically.

15. $f(x)=\dfrac{x^2-4}{x-2}$; $a=2$. ▸ $f(x)=\dfrac{(x-2)(x+2)}{x-2}=x+2$ if $x\neq 2$. Define $f(2)=2+2=4$

16. $f(x)=\dfrac{x^2+4x+3}{x+3}$; $a=-3$

▷ $f(x)=\dfrac{(x+3)(x+1)}{x+3}=x+1$ if $x\neq -3$.
To make f continuous, we should define $f(-3)=-3+1=-2$.



17. $f(x)=\dfrac{x-9}{\sqrt{x}-3}$; $a=9$

▷ $f(x)=\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}=\sqrt{x}+3$ if $x\neq 9$. Define $f(9)=\sqrt{9}+3=6$

18. $f(x)=\dfrac{x-5}{\sqrt{x-1}-2}$; $a=5$

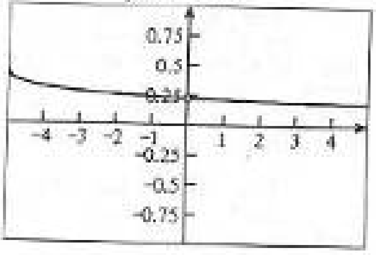
▷ $f(x)=\dfrac{(x-1)-4}{\sqrt{x-1}-2}=\dfrac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}-2}=\sqrt{x-1}+2$ if $x\neq 5$. Define $f(5)=\sqrt{5-1}+2=4$.

19. $f(x)=\dfrac{\sqrt{x+4}-3}{x-5}$; $a=5$

▷ $f(x)=\dfrac{\sqrt{x+4}-3}{(x+4)-9}=\dfrac{\sqrt{x+4}-3}{(\sqrt{x+4}-3)(\sqrt{x+4}+3)}=\dfrac{1}{\sqrt{x+4}+3}$ if $x\neq 5$. Define $f(5)=\dfrac{1}{\sqrt{5+4}+3}=\dfrac{1}{6}$.

20. $f(x)=\dfrac{\sqrt{x+5}-\sqrt{5}}{x}$; $a=0$

▷ $f(x)=\dfrac{\sqrt{x+5}-\sqrt{5}}{(x+5)-5}=\dfrac{\sqrt{x+5}-\sqrt{5}}{(\sqrt{x+5}-\sqrt{5})(\sqrt{x+5}+\sqrt{5})}=\dfrac{1}{\sqrt{x+5}+\sqrt{5}}$
if $x\neq 0$. To make f continuous, we should define
$f(0)=\dfrac{1}{\sqrt{0+5}+\sqrt{5}}=\dfrac{1}{2\sqrt{5}}$.



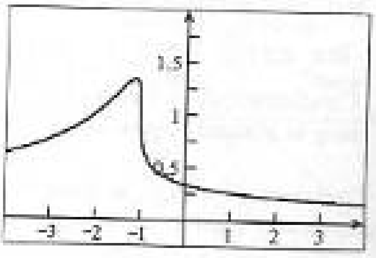
21. $f(x)=\dfrac{\sqrt{2}-\sqrt{x+2}}{x}$; $a=0$ ▷ $f(x)=\dfrac{\sqrt{2}-\sqrt{x+2}}{(x+2)-2}=\dfrac{\sqrt{2}-\sqrt{x+2}}{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}=\dfrac{-1}{\sqrt{x+2}+\sqrt{2}}$
if $x\neq 0$. Define $f(0)=\dfrac{-1}{\sqrt{0+2}+\sqrt{2}}=\dfrac{-1}{2\sqrt{2}}=-\frac{1}{4}\sqrt{2}$

22. $f(x)=\dfrac{2-\sqrt{x+1}}{x-3}$; $a=3$ ▷ $f(x)=\dfrac{2-\sqrt{x+1}}{(x+1)-4}=\dfrac{2-\sqrt{x+1}}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}=\dfrac{-1}{\sqrt{x+1}+2}$ if $x\neq 3$.
Define $f(3)=\dfrac{-1}{\sqrt{3+1}+2}=-\dfrac{1}{4}$.

23. $f(x)=\dfrac{\sqrt[3]{x}-2}{x-8}$; $a=8$ ▷ $f(x)=\dfrac{x^{1/3}-2}{(x^{1/3})^3-2^3}=\dfrac{x^{1/3}-2}{(x^{1/3}-2)(x^{2/3}+2x^{1/3}+4)}=\dfrac{1}{x^{2/3}+2x^{1/3}+4}$
if $x\neq 8$. Define $f(8)=\dfrac{1}{8^{2/3}+2\cdot 8^{1/3}+4}=\dfrac{1}{12}$.

24. $f(x)=\dfrac{\sqrt[3]{x+1}-1}{x}$; $a=0$

▷ $f(x)=\dfrac{(x+1)^{1/3}-1}{[(x+1)^{1/3}]^3-1^3}=\dfrac{(x+1)^{1/3}-1}{[(x+1)^{1/3}-1][(x+1)^{2/3}+(x+1)^{1/3}+1]}$
$=\dfrac{1}{(x+1)^{2/3}+(x+1)^{1/3}+1}$ if $x\neq 0$. To make f continuous,
we should define $f(0)=\dfrac{1}{(0+1)^{2/3}+(0+1)^{1/3}+1}=\dfrac{1}{3}$.



25. $f(x)=\dfrac{x+3}{3-|x|}$; $a=-3$

▷ $f(x)=\dfrac{x+3}{3-|x|}\cdot\dfrac{3+|x|}{3+|x|}=\dfrac{(x+3)(3+|x|)}{(3-x)(3+x)}=\dfrac{3+|x|}{3-x}$ if $x\neq -3$. Define $f(-3)=\dfrac{3+3}{3+3}=1$.

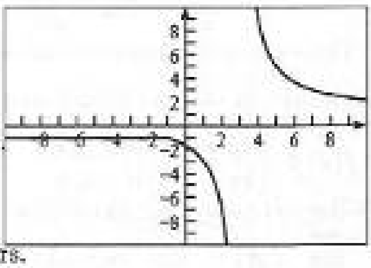
26. $f(x) = \dfrac{x+5}{|x+1|-4}$; $a = -5$ ▸ $f(x) = \dfrac{x+5}{|x+1|-4} \cdot \dfrac{|x+1|+4}{|x+1|+4} = \dfrac{(x+5)(|x+1|+4)}{[(x+1)-4][(x+1)+4]} = \dfrac{|x+1|+4}{x-3}$

if $x \neq -5$. Define $f(-5) = \dfrac{|-5+1|+4}{-5-3} = -1$

27. $f(x) = \dfrac{x+3}{3-|x|}$; $a = 3$ ▸ $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \dfrac{x+3}{3-x} = \dfrac{6}{0^+} = +\infty$. Essential discontinuity.

28. $f(x) = \dfrac{x+5}{|x+1|-4}$; $a = 3$

▸ $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3} \dfrac{x+5}{(x+1)+4}$ does not exist because the denominator is approaching 0 and the numerator is not. Hence the discontinuity is essential.



In Exercises 29–40, determine the numbers at which the function is continuous and state the reason.

29. $f(x) = x^2(x+3)^2$ is a polynomial. Hence, f is continuous for all real numbers.

30. $f(x) = (x-5)^3(x^2+4)^5$ is a polynomial. Hence, f is continuous for all real numbers.

31. $g(x) = \dfrac{x}{x-3}$ is a rational function. Hence f is continuous on its domain: all real numbers except 3.

32. $h(x) = \dfrac{x+1}{2x+5}$

▸ Because h is a rational function, by Theorem 1.8.4, h is continuous at every number in its domain. Therefore, h is continuous at every number for which $2x+5 \neq 0$ or equivalently, $x \neq -\frac{5}{2}$.

33. $F(x) = \dfrac{x^3+7}{x^2-4}$ is a rational function. Hence F is continuous on its domain: all real numbers except 2 and −2.

34. $G(x) = \dfrac{x-2}{x^2+2x-8} = \dfrac{x-2}{(x+4)(x-2)}$ is a rational function. Hence G is continuous on its domain: all real numbers except 2 and −4. The discontinuity at 2 is removable.

35. $f(x) = \begin{cases} 3x-1 & \text{if } x < 2 \\ 4-x^2 & \text{if } 2 \le x \end{cases}$. If $x \neq 2$, $f(x)$ is a polynomial. Hence f is continuous at numbers other than 2.

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (3x-1) = 5$ and $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (4-x^2) = 0$; thus $\lim\limits_{x\to 2} f(x)$ does not exist. Therefore, f is discontinuous at 2.

36. $f(x) = \begin{cases} (x+2)^2 & \text{if } x \le 0 \\ x^2+2 & \text{if } 0 < x \end{cases}$

▸ Because f is a polynomial function for $x < 0$ and for $x > 0$, by Theorem 1.8.3, f is continuous at every $x \neq 0$. Furthermore,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+2)^2 = 4 \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+2) = 2$$

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist, so f is discontinuous at 0.

37. $f(x) = \begin{cases} 1/(x+1) \text{ if } x \le 1 \\ 1/(3-x) \text{ if } x > 1 \end{cases}$. If $x \neq 1$, $f(x)$ is a rational function. Hence f is continuous except at −1 and 3. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 1/(x+1) = \frac{1}{2} = f(1)$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 1/(3-x) = \frac{1}{2}$. Hence f is continuous at 1.

38. $f(x) = \begin{cases} 1/x & \text{if } x < 3 \\ 2/(9-x) & \text{if } x \ge 3 \end{cases}$. If $x \neq 3$, $f(x)$ is a rational function. Hence f is continuous if $x \neq 0$ or 9. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (1/x) = \frac{1}{3}$ and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 2/(9-x) = \frac{1}{3} = f(3)$. Hence f is continuous at 3.

39. $h(x) = \begin{cases} x + \sqrt[3]{x} & \text{if } x < 0 \\ x - \sqrt{x} & \text{if } x \ge 0 \end{cases}$. If $x \neq 0$, f is continuous by Theorem 1.8.5.

$\lim\limits_{x\to 0^-} h(x) = \lim\limits_{x\to 0} (x + \sqrt[3]{x}) = 0$ and $\lim\limits_{x\to 0^+} h(x) = \lim\limits_{x\to 0^+} (x + \sqrt{x}) = 0$. Hence h is continuous for all real numbers.

40. $g(x)=\begin{cases}2x-\sqrt[3]{x} & \text{if } x\le 1\\ x\sqrt{x} & \text{if } x>1\end{cases}$

▸ The radicals are continuous by Theorem 1.8.5. $2x$ and x are polynomials and so are continuous. Hence if $x\ne 1$, g is continuous by Theorem 1.8.2 (ii) and (iii). Furthermore,

$$\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-}(2x-\sqrt[3]{x})=2-1=1=g(1) \text{ and } \lim_{x\to 1^+} g(x)=\lim_{x\to 1^+} x\sqrt{x}=1\cdot 1=1$$

Therefore g is continuous for all real numbers.

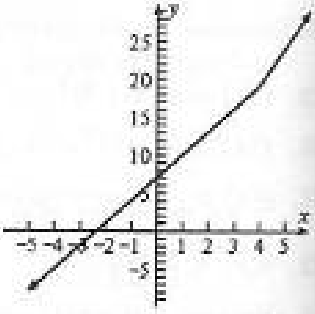
In Exercises 41–44: (a) Find c and k that make f continuous everywhere. (b) Sketch the graph of f.

41. $f(x)=\begin{cases}3x+7 & \text{if } x\le 4\\ kx-1 & \text{if } 4<x\end{cases}$

$\lim\limits_{x\to 4^-} f(x)=\lim\limits_{x\to 4^-}(3x+7)=19=f(4)$;

$\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4^+}(kx-1)=4k-1$

Hence f is continuous at 4 if and only if $4k-1=19$; $k=5$.

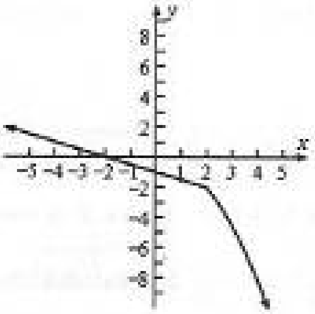


42. $f(x)=\begin{cases}kx-1 & \text{if } x<2\\ kx^2 & \text{if } x\ge 2\end{cases}$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+} kx^2=4k=f(2)$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-} kx-1=2k-1$

Hence f is continuous at 2 if and only if $4k=2k-1$; $2k=-1$; $k=-\frac{1}{2}$



43. $f(x)=\begin{cases}x & \text{if } x\le 1\\ cx+k & \text{if } 1<x<4\\ -2x & \text{if } 4\le x\end{cases}$

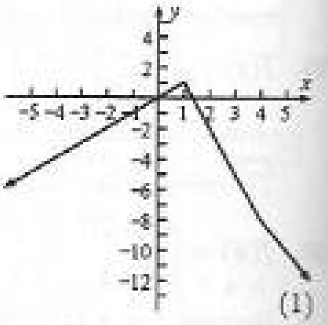
$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} x=1=f(1)$; $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(cx+k)=c+k$

Therefore f is continuous at 1 if and only if $c+k=1$.

$\lim\limits_{x\to 4^-} f(x)=\lim\limits_{x\to 4^-}(cx+k)=4c+k$; $\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4^+}(-2x)=-8=f(4)$

Therefore f is continuous at 4 if and only if $4c+k=-8$.

Solving $c+k=1$ and $4c+k=-8$ simultaneously, we get $c=-3$ and $k=4$.



44. $$f(x)=\begin{cases}x+2c & \text{if } x<-2\\ 3cx+k & \text{if } -2\le x\le 1\\ 3x-2k & \text{if } 1<x\end{cases} \qquad (1)$$

▸ For all values of c and k the function f is continuous at all x, except possibly at $x=-2$ and $x=1$.

If f is continuous at -2, then

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^+} f(x)$$
$$\lim_{x\to -2^-}(x+2c)=\lim_{x\to -2^+}(3cx+k)$$
$$-2+2c=-6c+k$$

If f is continuous at 1, then

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$$
$$\lim_{x\to 1^-}(3cx+k)=\lim_{x\to 1^+}(3x-2k)$$
$$3c+k=3-2k$$

Solving these equations simultaneously, we get $c=\frac{1}{3}$ and $k=\frac{2}{3}$.

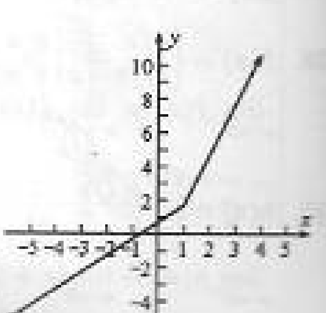
Substituting these values for c and k into Eq. (1), we have

$f(x)=\begin{cases}x+\frac{2}{3} & \text{if } x<-2\\ x+\frac{2}{3} & \text{if } -2\le x\le 1\\ 3x-\frac{4}{3} & \text{if } 1<x\end{cases}$ or, equivalently, $f(x)=\begin{cases}x+\frac{2}{3} & \text{if } x\le 1\\ 3x-\frac{4}{3} & \text{if } x>1\end{cases}$

Now, $f(1)=1+\frac{2}{3}=\frac{5}{3}$. Furthermore,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+\frac{2}{3})=\frac{5}{3}$ and $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(3x-\frac{4}{3})=\frac{5}{3}$

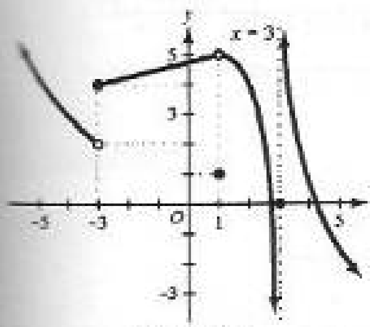
Thus, $\lim\limits_{x\to 1} f(x)=\frac{5}{3}=f(1)$. Therefore, f is continuous at 1. Hence, f is continuous on $(-\infty,+\infty)$.



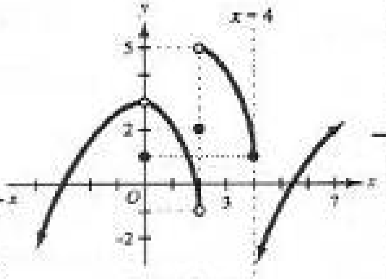
In Exercises 45–46, in (a)–(c) show that f in the figure is discontinuous by showing how Definition 1.8.1 fails.

45. (a) $x = -3$: $\lim\limits_{x \to -3^-} f(x) \neq \lim\limits_{x \to -3^+} f(x)$; essential (b) $x = 1$: $\lim\limits_{x \to 1} f(x) = 5 \neq f(1)$; removable: define $f(1) = 5$
(c) $x = 3$: $\lim\limits_{x \to 3^-} f(x)$ does not exist; essential

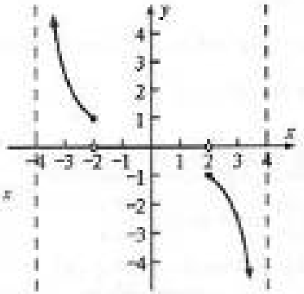
46. (a) $x = 0$: $\lim\limits_{x \to 0} f(x) = 3 \neq f(0)$; removable: define $f(0) = 3$ (b) $x = 2$: $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$; essential
(c) $x = 4$: $\lim\limits_{x \to 4^+} f(x)$ does not exist; essential



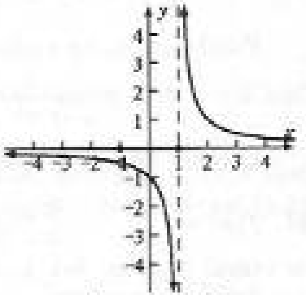
Exercise 45



Exercise 46



Exercise 47



Exercise 48

In Exercises 47 and 48, sketch a graph of some function f satisfying the given properties.

47. f is continuous in $(-4, -2)$, $(-2, 2)$ and $(2, 4)$; f is discontinuous at -2 and 2; $f(-2) = 0$ and $f(2) = 0$;

$$\lim_{x \to -4^+} f(x) = +\infty, \quad \lim_{x \to -2^-} f(x) = 0; \quad \lim_{x \to 2^+} f(x) = 0; \quad \lim_{x \to 4^-} f(x) = -\infty$$

48. f is continuous on $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$; f is discontinuous at -1 and 1; $f(-1) = 0$ and $f(1) = 0$; $\lim\limits_{x \to -1^-} f(x)$ and $\lim\limits_{x \to -1^+} f(x)$ both exist but neither is 0; neither $\lim\limits_{x \to 1^-} f(x)$ nor $\lim\limits_{x \to 1^+} f(x)$ exists.

In Exercises 49–52, state where the function is discontinuous by showing how Definition 1.8.1 fails.

49. $f(x)$ dollars is the cost of shipping x lb. $f(x) = \begin{cases} 2.2x & \text{if } 0 < x \leq 50 \\ 2.1x & \text{if } 50 < x \leq 200 \\ 2.05x & \text{if } x > 200 \end{cases}$.

▶ $x = 50$: $\lim\limits_{x \to 50^-} f(x) = 110 \neq 105 = \lim\limits_{x \to 50^+} f(x)$. $x = 200$: $\lim\limits_{x \to 200^-} f(x) = 420 \neq 410 = \lim\limits_{x \to 200^+} f(x)$

50. $\mathrm{F}(x)$ cents is the cost of mailing x ounces. $\mathrm{F}(x) = 6 - 23[\![-x]\!]$.

▶ If n is any positive integer, $\lim\limits_{x \to n^-} \mathrm{F}(x) \neq \lim\limits_{x \to n^+} \mathrm{F}(x)$; condition (ii) is not satisfied.

51. $g(x)$ cents is the cost of an x minute call. $g(x) = 10 - 30[\![-x]\!]$.

▶ If n is any positive integer, $\lim\limits_{x \to n^-} g(x) \neq \lim\limits_{x \to n^+} g(x)$; condition (ii) is not satisfied.

52. $\mathrm{G}(x)$ dollars is the admission for age x years. $\mathrm{G}(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \leq x < 60 \\ 5 & \text{if } 60 \leq x \end{cases}$.

▶ $x = 12$: $\lim\limits_{x \to 12^-} \mathrm{G}(x) = 4 \neq 7 = \lim\limits_{x \to 12^+} \mathrm{G}(x)$ $\qquad$ $x = 60$: $\lim\limits_{x \to 60^-} \mathrm{G}(x) = 7 \neq 5 = \lim\limits_{x \to 60^+} \mathrm{G}(x)$
Condition (ii) of Definition 1.8.1 is not satisfied at 12 and 60.

53. At t minutes, $r(t)$ meters is the radius. $r(t) = \begin{cases} 4t^2 + 20 & \text{if } 0 \leq t \leq 2 \\ 16t + 4 & \text{if } t > 2 \end{cases}$

▶ $\lim\limits_{t \to 2^-} r(t) = \lim\limits_{t \to 2^-} (4t^2 + 20) = 4(2)^2 + 20 = 36 = r(2)$ and $\lim\limits_{t \to 2^+} r(t) = \lim\limits_{t \to 2^+} (16t + 4) = 36$

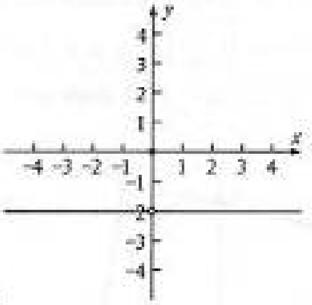
54. At t minutes, $\mathrm{A}(t)$ m^2 is the area. $\mathrm{A}(t) = \pi r^2 = \begin{cases} \pi(4t^2 + 20)^2 & \text{if } 0 \leq t \leq 2 \\ \pi(16t + 4)^2 & \text{if } t > 2 \end{cases}$

▶ $\lim\limits_{t \to 2^-} r(t) = \lim\limits_{t \to 2^-} \pi(4t^2 + 20)^2 = \pi(36)^2 = r(2)$ and $\lim\limits_{t \to 2^+} r(t) = \lim\limits_{t \to 2^+} \pi(16t + 4)^2 = \pi(36)^2$

55. If n is a positive integer, $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x^n - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x - 1)(x^{n-1} + x^{n-2} + \cdots + 1)}{x - 1}$
$= \lim\limits_{x \to 1} (x^{n-1} + x^{n-2} + \cdots + 1) = 1 + 1 + \cdots + 1 = n$. The discontinuity is removed by defining $f(1) = n$.

56. The function f is defined by $f(x) = \lim\limits_{n \to 0} \dfrac{2nx}{n^2 - nx}$.
Sketch the graph of f. At what values of x is f discontinuous?

▹ We divide the numerator and denominator by n to obtain the limit. Thus,



$$\lim_{n \to 0} \frac{2nx}{n^2 - nx} = \lim_{n \to 0} \frac{2x}{n - x} = \frac{2x}{-x} = -2 \quad \text{if } x \neq 0$$

Thus,

$$f(x) = -2 \quad \text{if} \quad x \neq 0$$

Furthermore, by replacing x with 0 in the definition of f,

$$f(0) = \lim_{n \to 0} \frac{0}{n^2} = 0$$

Because $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} -2 = -2$ and $f(0) = 0$, f is discontinuous at $x = 0$.

57. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$; $(f \cdot g)(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} = |x|$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 1 = 1$; hence $\lim f(x)$ does not exist.
Therefore, f is discontinuous at 0.

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} 1 = 1$ and $\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} x = 0$; thus $\lim g(x)$ does not exist.
Therefore, g is discontinuous at 0.

$(f \cdot g)(x) = |x|$ so $f \cdot g$ is continuous everywhere; in particular, $f \cdot g$ is continuous at 0.

58. Let $f(x) = x$ and $g(x) = \operatorname{sgn} x$. Then f is continuous at 0, g is discontinuous at 0 and $f(x)g(x) = |x|$ is continuous at 0.

59. Let $f(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } a \leq x \end{cases}$.
Then f and g are discontinuous at a because condition (ii) of Definition 1.8.1 fails at a for both functions. But $(f + g)(x) = 1$ for every x; therefore $f + g$ is continuous everywhere; in particular, $f + g$ is continuous at a.

1.9 CONTINUITY OF A COMPOSITE FUNCTION AND CONTINUITY ON AN INTERVAL

1.9.1 Theorem If $\lim\limits_{x \to a} g(x) = b$ and if the function f is continuous at b,

$$\lim_{x \to a} (f \circ g)(x) = f(b)$$

or, equivalently,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

1.9.2 Theorem If the function g is continuous at a and the function f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

1.9.3 Definition A function is said to be *continuous on an open interval* if and only if it is continuous at every number in the open interval.

1.9.4 Definition The function f is said to be *continuous from the right at the number* a if and only if the following three conditions are satisfied:

(i) $f(a)$ exists
(ii) $\lim\limits_{x \to a^+} f(x)$ exists
(iii) $\lim\limits_{x \to a^+} f(x) = f(a)$

1.9.5 Definition The function f is said to be *continuous from the left at the number* a if and only if the following three conditions are satisfied:

(i) $f(a)$ exists
(ii) $\lim\limits_{x \to a^-} f(x)$ exists
(iii) $\lim\limits_{x \to a^-} f(x) = f(a)$

A function f is continuous at a if and only if the function f is both continuous from the right at a and continuous from the left at a.

1.9.6 Definition A function whose domain includes the closed interval $[a,b]$ is said to be *continuous on* $[a,b]$ if and only if it is continuous on the open interval (a,b), as well as continuous from the right at a and continuous from the left at b.

A function f may be continuous on the closed interval $[a,b]$ with out being continuous at either a or b. For example, let f be the function defined as follows:

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } 0\le x\le 1\\ 1 & \text{if } x>1\end{cases}$$

Then f is continuous on the closed interval $[0,1]$, but f is discontinuous at both $x=0$ and $x=1$.

1.9.7 Definition

(i) A function whose domain includes the interval half-open on the right $[a,b)$ is said to be *continuous on* $[a,b)$ if and only if it is continuous on the open interval (a,b) and continuous from the right at a.

(ii) A function whose domain includes the interval half-open on the left $(a,b]$ is said to be *continuous on* $(a,b]$ if and only if it is continuous on the open interval (a,b) and continuous from the left at b.

If a function f is continuous on any interval I, then f is continuous on any subset of the interval I.

1.9.8 Intermediate-Value Theorem If the function f is continuous on the closed interval $[a,b]$ and if $f(a)\neq f(b)$, then for any number k (strictly) between $f(a)$ and $f(b)$ there exists a number c in the open interval (a,b) such that $f(c)=k$.

Exercises 1.9

In Exercises 1–6, define $f\circ g$ and determine the numbers at which $f\circ g$ is continuous.

1 and 17. (a) $f(x)=\sqrt{x}$; $g(x)=9-x^2$ ▹ $(f\circ g)(x)=\sqrt{9-x^2}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $-3<x<3$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-3,3)$. Because
$\lim\limits_{x\to -3^+}(f\circ g)(x)=0=(f\circ g)(-3)$ and $\lim\limits_{x\to 3^-}(f\circ g)(x)=0=(f\circ g)(3)$, $f\circ g$ is continuous on $[-3,3]$.

(b) $f(x)=\sqrt{x}$; $g(x)=x^2-16$ ▹ $(f\circ g)(x)=\sqrt{x^2-16}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $x<-4$ or $x>4$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-\infty,-4)\cup(4,+\infty)$.
Because $\lim\limits_{x\to -4^-}(f\circ g)(x)=0=(f\circ g)(-4)$ and $\lim\limits_{x\to 4^+}(f\circ g)(4)=0=(f\circ g)(4)$,
$f\circ g$ is continuous on $(-\infty,-4]\cup[4,+\infty)$.

2 and 18. (a) $f(x)=\sqrt{x}$; $g(x)=16-x^2$ ▹ $(f\circ g)(x)=\sqrt{16-x^2}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $-4<x<4$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-4,4)$. Because
$\lim\limits_{x\to -4^+}(f\circ g)(x)=0=(f\circ g)(-4)$ and $\lim\limits_{x\to 4^-}(f\circ g)(x)=0=(f\circ g)(4)$, $f\circ g$ is continuous on $[-4,4]$.

(b) $f(x)=\sqrt{x}$; $g(x)=x^2+4$ ▹ $(f\circ g)(x)=\sqrt{x^2+4}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ all x.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x, that is, on $(-\infty,\infty)$.

3 and 19. (a) $f(x)=\sqrt{x}$; $g(x)=\dfrac{1}{x-2}$ ▹ $(f\circ g)(x)=\sqrt{\dfrac{1}{x-2}}=\dfrac{1}{\sqrt{x-2}}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.4, g is continuous except at 2 and $g(x)>0$ for $x>2$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous in $(2,+\infty)$.
Because $\lim\limits_{x\to 2^+}(f\circ g)(x)$ does not exist, $f\circ g$ is continuous only on $(2,+\infty)$.

(b) $f(x) = \frac{1}{x-2}$; $g(x) = \sqrt{x}$ ▷ $(f \circ g)(x) = \frac{1}{\sqrt{x}-2}$

By Theorem 1.8.4, $f \circ g$ is continuous for $g(x) \neq 2$.
By Theorem 1.8.5(ii), g is continuous for $x > 0$ and $g(x) = 2$ for $x = 4$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous for all positive numbers except 4.
Because $\lim\limits_{x \to 0^+} (f \circ g)(x) = -\frac{1}{2} = (f \circ g)(0)$, $f \circ g$ is continuous on $[0,4) \cup (4,+\infty)$.

4 and 20. (a) $f(x) = \sqrt[3]{x}$; $g(x) = \sqrt{x+1}$ ▷ $(f \circ g)(x) = \sqrt[6]{x+1}$

By Theorem 1.8.5(i), $f \circ g$ is continuous for all $g(x)$.
By Theorem 1.8.5(ii), g is continuous for $x > -1$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous in $(-1,+\infty)$.
Because $\lim\limits_{x \to -1^+} (f \circ g)(x) = 0 = (f \circ g)(-1)$, $f \circ g$ is continuous on $[-1,+\infty)$.

(b) $f(x) = \sqrt{x+1}$; $g(x) = \sqrt[3]{x}$ ▷ $(f \circ g)(x) = \sqrt{\sqrt[3]{x}+1}$

By Theorem 1.8.5(ii), $f \circ g$ is continuous for $g(x) > -1$.
By Theorem 1.8.5(i), g is continuous for all x and $g(x) > -1$ for $x > -1$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous on $(-1,+\infty)$.
Because $\lim\limits_{x \to -1^+} (f \circ g)(x) = 0 = (f \circ g)(-1)$, $f \circ g$ is continuous on $[-0,+\infty)$.

5 and 21. $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x-1}}$; $g(x) = |x|$ ▷ $(f \circ g)(x) = \frac{\sqrt{4-x^2}}{\sqrt{|x|-1}}$

$\sqrt{4-x^2}$ is continuous for x in $(-2,2)$; $\sqrt{|x|-1}$ is continuous for x in $(-\infty,-1) \cup (1,+\infty)$.
By Theorem 1.8.2(iv), $f \circ g$ is continuous for x in $(-2,-1) \cup (1,2)$.
Because $\lim\limits_{x \to -2^+} (f \circ g)(x) = 0 = (f \circ g)(-2)$ and $\lim\limits_{x \to 2^-} (x) = 0 = (f \circ g)(2)$,
$f \circ g$ is continuous on $[-2,-1) \cup (1,2]$.

6 and 22. $f(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-x}}$; $g(x) = |x|$ ▷ $(f \circ g)(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-|x|}}$

$\sqrt{x^2-1}$ is continuous for x in $(-\infty,-1) \cup (1,+\infty)$; $\sqrt{4-|x|}$ is continuous for x in $(-4,4)$.
By Theorem 1.8.2(iv), $f \circ g$ is continuous for x in $(-4,-1) \cup (1,4)$.
Because $(f \circ g)(-4)$ and $(f \circ g)(4)$ do not exist, $f \circ g$ is continuous only on $(-4,-1) \cup (1,4)$.

In Exercises 7–16, find the domain of the function. Determine for each interval if it is continuous on that interval.

7. $f(x) = \frac{2}{x+5}$; the domain of f is all real numbers except -5.

f is continuous on$(3,7), (-5,+\infty), [-10,-5)$; f is discontinuous on$[-6,4], (-\infty,0), [-5,+\infty)$.

8. $g(x) = \frac{x}{x-2}$; $(-\infty,0]$, $[0,+\infty)$, $(0,2)$, $(0,2]$, $[2,+\infty)$, $(2,+\infty)$

▷ The domain is the set of all real numbers, except 2. Because a rational function is continuous on its domain, and g is a rational function, g is continuous on any interval that does not include 2. Therefore, g is continuous on $(-\infty,0]$, $(0,2)$, and $(2,+\infty)$. And g is discontinuous on $[0,+\infty)$, $(0,2]$, and $[2,+\infty)$.

9. $f(t) = \frac{1}{t^2-1}$; the domain of f is all real numbers except 1 and -1.

f is continuous on $(0,1), (-1,1), (-1,0], (1,+\infty)$; f is discontinuous on $[0,1]$, $(-\infty,-1]$.

10. $f(r) = \frac{r+3}{r^2-4}$; the domain of f is all real numbers except 2 and -2.

f is continuous on $(-2,2)$, $(2,+\infty)$ and discontinuous on $(0,4]$, $(-\infty,-2]$, $[-4,4]$, $(-2,2]$.

11. $g(x) = \sqrt{x^2-9}$; the domain of g is $\{x \mid x^2 - 9 \geq 0\} = (-\infty,-3] \cup [3,+\infty)$.
g is continuous on $(-\infty,-3)$, $(3,\infty)$; g is discontinuous on $(-3,3)$.

Because $\lim\limits_{x \to -3^-} g(x) = \lim\limits_{x \to -3^-} \sqrt{x^2-9} = 0 = g(-3)$ and $\lim\limits_{x \to 3^+} g(x) = \lim\limits_{x \to 3^+} \sqrt{x^2-9} = 0 = g(3)$,
g is also continuous on $(-\infty,-3]$ and $[3,+\infty)$.

12. $f(x)=[\![x]\!]$; $(-\frac{1}{2},\frac{1}{2})$, $(\frac{1}{4},\frac{1}{2})$, $(1,2)$, $[1,2)$, $(1,2]$

▸ The greatest integer function is defined for every real number, so the domain of f is the set of all real numbers. If n is any integer, then

$$[\![x]\!]=n-1 \text{ if } n-1\le x<n \text{ so that } \lim_{x\to n^-} f(x)=n-1$$

and

$$[\![x]\!]=n \text{ if } n\le x<n+1 \text{ so that } \lim_{x\to n^+} f(x)=n$$

Therefore $\lim\limits_{x\to n} f(x)$ does not exist and so f is discontinuous at each integer n. However, $f(n)=n$ so that f is continuous from the right at n because $\lim\limits_{x\to n^+} f(x)=f(n)$ and discontinuous from the left at n because $\lim\limits_{x\to n^-} f(x)\neq f(n)$. Therefore, f is continuous on the intervals $(\frac{1}{4},\frac{1}{2})$, $(1,2)$, and $[1,2)$, while f is discontinuous on the intervals $(-\frac{1}{2},\frac{1}{2})$ and $(1,2]$.

13. $f(t)=\dfrac{|t-1|}{t-1}$; the domain of f is all real numbers except 1.

f is continuous on $(-\infty,1)$ and $(1,\infty)$; f is discontinuous on $(-\infty,1]$, $[-1,1]$, $(-1,\infty)$.

14. $h(x)=\begin{cases}2x-3 & \text{if } x<-2\\ x-5 & \text{if } -2\le x\le 1\\ 3-x & \text{if } x>1\end{cases}$; the domain of h is all real numbers.

$\lim\limits_{x\to -2^-} h(x)=\lim\limits_{x\to -2^-}(2x-3)=-7$; $\lim\limits_{x\to -2^+} h(x)=\lim\limits_{x\to -2^+}(x-5)=-7=h(-2)$ and so h is continuous at -2. $\lim\limits_{x\to 1^-} h(x)=\lim\limits_{x\to 1^-}(x-5)=-4=h(1)$; $\lim\limits_{x\to 1^+} h(x)=\lim\limits_{x\to 1^+}(3-x)=2$ and so h is continuous from the left at 1. f is continuous on $(-\infty,1)$, $(-2,1)$, $[-2,1)$, $[-2,1]$; f is discontinuous on $(-2,+\infty)$.

15. $f(x)=\sqrt{4-x^2}$; the domain of f is $\{x \mid 4-x^2\ge 0\}=[-2,2]$.
f is continuous on $(-2,2)$; f is discontinuous on $(-\infty,-2]$ and $(2,+\infty)$.
Because $\lim\limits_{x\to -2^+} f(x)=\lim\limits_{x\to -2^+}\sqrt{4-x^2}=0=f(-2)$ and $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\sqrt{4-x^2}=0=f(2)$,
f is also continuous on $[-2,2]$, $(-2,2]$, and $[-2,2)$.

16. $F(y)=\dfrac{1}{3+2y-y^2}$; $(-1,3)$, $[-1,3]$, $[-1,3)$, $(-1,3]$

▸ $$\frac{1}{3+2y-y^2}=\frac{1}{(3-y)(1+y)}$$

The domain of F is the set of all real numbers except 3 and -1. Because F is a rational function, F is continuous on its domain. Thus, F is continuous on $(-1,3)$, and F is discontinuous on $[-1,3]$, $[-1,3)$, and $(-1,3]$.

17. See Ex.1 18. See Ex.2 19. See Ex.3 20. See Ex.4 21. See Ex.5 22. See Ex.6

23. $f(x)=\begin{cases}2 & \text{if } x<-2\\ \sqrt{4-x^2} & \text{if } -2\le x\le 2\\ -2 & \text{if } 2<x\end{cases}$ Continuous on $(-\infty,-2)\cup[-2,2]\cup(2,+\infty)$. See Ex.1.6.17 for fig. and limits.

24. Determine the largest interval (or union of intervals) on which the function is continuous:

$$f(x)=\begin{cases}x+5 & \text{if } x<-3\\ \sqrt{9-x^2} & \text{if } -3\le x\le 3\\ 3-x & \text{if } 3<x\end{cases}$$

▸ Because f is a polynomial on the intervals $(-\infty,-3)$ and $(3,+\infty)$, f is continuous on these intervals. Because $9-x^2$ is continuous for all x and positive on $(-3,3)$, f is continuous on $(-3,3)$ by Theorems 1.8.5 and 1.9.2. Because

$$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+}\sqrt{9-x^2}=0=f(-3) \quad\text{and}\quad \lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(x+5)=2\neq f(-3)$$

f is continuous from the right at -3 but not from the left. Because

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}\sqrt{9-x^2}=0=f(3) \quad\text{and}\quad \lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} 3-x=0=f(3)$$

f is continuous from the left and the right at 3 and hence is continuous there. Combining these facts, we see that f is continuous on $(-\infty,-3)\cup[-3,+\infty)$.

In Exercises 25–28, sketch the graph of a function f that satisfies the given properties.

25. f is continuous on $(-\infty, 2]$ and $(2, +\infty)$; $\lim_{x\to 0} f(x) = 4$; $\lim_{x\to 2^-} f(x) = -3$; $\lim_{x\to 2^+} f(x) = +\infty$; $\lim_{x\to 5} f(x) = 4$

26. f is continuous on $(-\infty, -2)$, $[-2, 4]$, and $(4, +\infty)$; $\lim_{x\to -5} f(x) = 0$; $\lim_{x\to -2^-} f(x) = -\infty$; $\lim_{x\to -2^+} f(x) = -3$; $\lim_{x\to 0} f(x) = -1$; $\lim_{x\to 4^-} f(x) = 2$; $\lim_{x\to 4^+} f(x) = 5$; $\lim_{x\to 6} f(x) = 0$

27. f is continuous on $(-\infty, -3]$, $(-3, 3)$, and $[3, +\infty)$; $\lim_{x\to -5} f(x) = 2$; $\lim_{x\to -3^-} f(x) = 0$; $\lim_{x\to -3^+} f(x) = 4$; $\lim_{x\to 0} f(x) = 1$; $\lim_{x\to 3^-} f(x) = 0$; $\lim_{x\to 3^+} f(x) = -5$; $\lim_{x\to 4} f(x) = 0$

28. f is continuous on $(-\infty, 0)$ and $[0, +\infty)$; $\lim_{x\to -4} f(x) = 0$; $\lim_{x\to 0^-} f(x) = 3$; $\lim_{x\to 0^+} f(x) = -3$; $\lim_{x\to 4} f(x) = 2$.

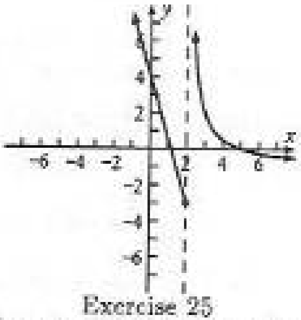
▷ Because f is continuous at -4, then $f(-4) = \lim_{x\to 4} f(x) = 0$, so the graph contains the point $(-4, 0)$.

Because f is continuous from the right at 0, then $f(0) = \lim_{x\to 0^+} f(x) = -3$, and the graph contains $(0, 3)$.

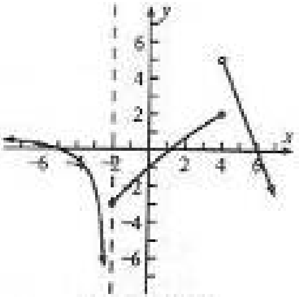
Because f is continuous at 4, and $\lim_{x\to 4} f(x) = 2$, then $f(4) = 2$.

Because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, there is a break in the graph at the point where $x = 0$.

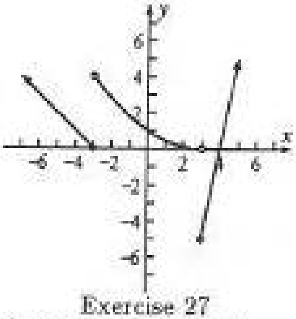
Because $\lim_{x\to 0^-} f(x) = 3$, from the left the graph approaches the point $(0, 3)$, but this point is not on the graph.



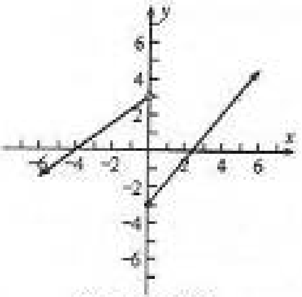
Exercise 25



Exercise 26



Exercise 27



Exercise 28

In Exercises 29–34, prove that the function obtained in Exercises 1.3 is continuous on its domain.

29. (a) Ex. 13. $a(x) = (120 - x)x = 120z - x^2$, $0 \le x \le 120$, is a polynomial.
(b) Ex. 15. $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$, is a polynomial.

30. (a) Ex. 14. $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$, is a polynomial.
(b) Ex. 16. $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$, is a polynomial.

31. (a) Ex. 17. $V(x) = (8 - 2x)(15 - 2x)x = 120z - 46x^2 + 4x^3$, $0 \le x \le 4$, is a polynomial.
(b) Ex. 19. $V(x) = (12 - 2x)(15 - 2x)x = 4x^3 - 54x^2 + 180x$, $0 \le x \le 6$, is a polynomial.

32. (a) Exercise 18. $V(x) = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x$, $0 \le x \le 6$.
(b) Exercise 20. $V(x) = (40 - 2x)(50 - 2x)x = 4x^3 - 180x^2 + 200x$, $0 \le x \le 20$.
Both of these are polynomials, continuous at every number by Theorem 1.8.3.

33. (a) Ex. 21. $C(r) = k(120/r + 4\pi r^2)$, $r > 0$, is a rational function, continuous on its domain by Th. 1.8.4.
(b) Ex. 23. $A(x) = (x + 2)(\frac{24}{x} + 3) = \dfrac{3x^2 + 30x + 48}{x}$, $x > 0$, is also a rational function.

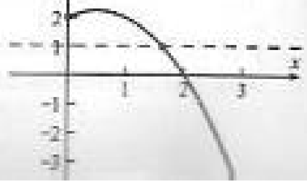
34. (a) Ex. 22. $C(r) = k(120/r + 2\pi r^2)$, $r > 0$, is a rational function, continuous on its domain by Th. 1.8.4.
(b) Ex. 24. $A(x) = (x + 30)\left(\dfrac{13200}{x} + 44\right)$, $x > 0$, is also a rational function.

In Exercises 35–42, does the intermediate-value theorem hold for the function f, interval $[a, b]$ and constant k? If so, solve $f(c) = k$ graphically and analytically to 4 decimals and sketch the graph showing the point (c, k).

35. $f(x) = 2 + x - x^2$; $[a, b] = [0, 3]$; $k = 1$

▷ 1 is between $f(0) = 2$ and $f(3) = -3$
and f is continuous on $[0, 3]$. Therefore,
the intermediate-value theorem holds and
there exists a number c between 0 and 3 such that $f(c) = 1$:

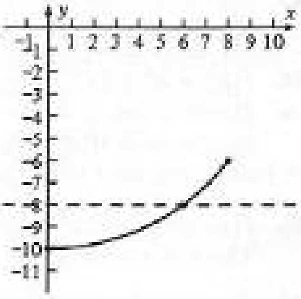
$2 + c - c^2 = 1$; $c^2 - c - 1 = 0$; $c = \frac{1}{2}(1 \pm \sqrt{5})$; and $\frac{1}{2}(1 + \sqrt{5}) \approx 1.6180$ is in $(0, 3)$.



36. $f(x) = -\sqrt{100 - x^2}$; $[a,b] = [0,8]$; $k = -8$

▷ Because $f(a) = f(0) = -10$ and $f(b) = f(8) = -6$, then $f(a) \neq f(b)$, and k is between $f(a)$ and $f(b)$. Furthermore, f is continuous on $[0,8]$. Thus, the hypothesis of the intermediate-value theorem is satisfied, and we can find a number c such that

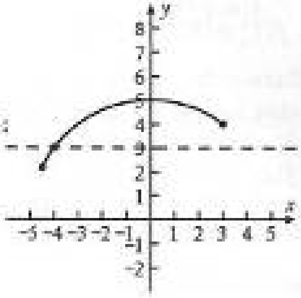
$$\begin{aligned} f(c) &= -8 \\ -\sqrt{100 - c^2} &= -8 \\ 100 - c^2 &= 64 \\ c &= \pm 6 \end{aligned}$$



Because -6 is not between 0 and 8, the only suitable choice of c is 6. The figure shows the graph of f and the line $y = -8$.

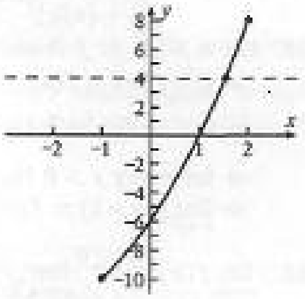
37. $f(x) = \sqrt{25 - x^2}$; $[a, b] = [-4.5, 3]$;

▷ 3 is between $f(-4.5) = \frac{1}{2}\sqrt{19} \approx 2.18$ and $f(3) = 4$ and f is continuous on $[-4.5, 3]$. Therefore, the intermediate-value theorem holds and there exists a number c between -4.5 and 3 such that $f(c) = 3$: $\sqrt{25 - c^2} = 3$; $25 - c^2 = 9$; $c^2 = 16$; $c = \pm 4$, and -4 is in $(-4.5, 3)$.



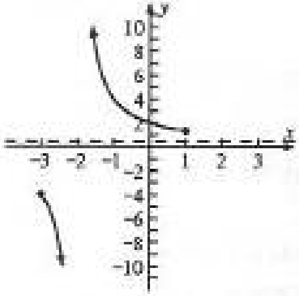
38. $f(x) = x^2 + 5x - 6$; $[a, b] = [-1, 2]$;

▷ 4 is between $f(-1) = -10$ and $f(2) = 8$ and f is continuous on $[-1, 2]$. Therefore, the intermediate-value theorem holds and there exists a number c between -1 and 2 such that $f(c) = 4$: $c^2 + 5c - 6 = 4$; $c^2 + 5c - 10 = 0$; $c = \frac{1}{2}(-5 \pm \sqrt{65})$ and $\frac{1}{2}(-5 + \sqrt{65}) \approx 1.5311$ is in $(-1, 2)$.



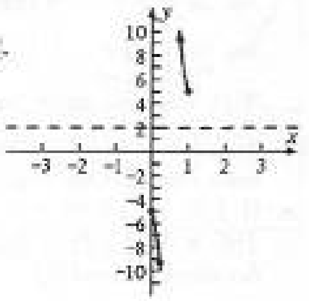
39. $f(x) = \frac{4}{x+2}$; $[a, b] = [-3, 1]$;

▷ $\frac{1}{2}$ is between $f(-3) = -4$ and $f(1) = \frac{4}{3}$ but f is discontinuous at -2, and -2 is in $[-3, 1]$. Therefore the intermediate-value theorem does not hold and there may not be a number c between -3 and 1 such that $f(c) = \frac{1}{2}$: $\frac{4}{c+2} = \frac{1}{2}$; $8 = c + 2$; $c = 6$. But 6 is not in $(-3, 1)$.



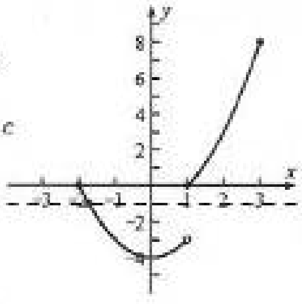
40. $f(x) = \frac{5}{2x-1}$; $[a, b] = [0, 1]$; $k = 2$

▷ Because f is discontinuous at $\frac{1}{2}$, then f is not continuous on the closed interval $[0, 1]$. Thus the hypothesis of the intermediate-value theorem is not satisfied and the theorem does not hold. The figure shows the graph of the function on $[0, 1]$ and the line $y = 2$. Because the line does not intersect the curve, there is no number c that satisfies the conclusion of the intermediate-value theorem.



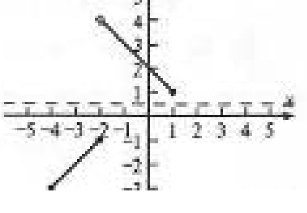
41. $f(x) = \begin{cases} x^2 - 4 \text{ if } -2 \le x < 1 \\ x^2 - 1 \text{ if } 1 \le x \le 3 \end{cases}$; $[a, b] = [-2, 3]$; $k = -1$

▷ -1 is not between $f(-2) = 0$ and $f(3) = 8$ and f is not continuous on $[-2, 3]$. Hence the intermediate-value theorem does not hold and there may not be a number c between -2 and 3 such that $f(c) = -1$: $c^2 - 4 = -1$; $c^2 = 3$; $c = \pm\sqrt{3}$; and $-\sqrt{3}$ is in $(-2, 3)$.



42. $f(x) = \begin{cases} 1 + x & \text{if } -4 \le x \le -2 \\ 2 - x & \text{if } -2 < x \le 1 \end{cases}$; $[a, b] = [-4, 1]$

▷ Because f is discontinuous at -2, and -2 is in $[-4, 1]$. Therefore the intermediate-value theorem does not hold and there may not be a number c between -4 and 1 such that $f(c) = \frac{1}{2}$. The graph shows there is no such number.

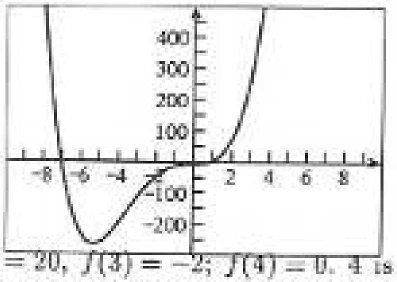


In Exercises 43–48, (a) apply the intermediate-zero theorem to show that f has the indicated number of zeros between a and b. (b) Approximate them to 2 decimal places.

43. $f(x) = x^3 - 6x + 3$. $f(-3) = -6$, $f(-2) = 7$; $f(0) = 3$, $f(1) = -2$; $f(2) = -1$, $f(3) = 12$. Thus there is a zero in each of the intervals: $(-3, -2)$, $(0, 1)$, $(1, 2)$. The zeros are $-2.669 \approx -2.67$, $0.524 \approx 0.52$, $2.1451 \approx 2.15$.

44. $f(x) = x^4 + 7x^3 + x - 8$; two zeros; $a = -10$; $b = 5$

▷ $f(-8) = 496$, $f(-7) = -15$; $f(0) = -8$, $f(1) = 1$
By the intermediate-zero theorem, there is a zero in each of the intervals $(-8, -7)$, $(0, 1)$. To two decimals, the zeros are -7.04 and 0.96.



45. $f(x) = 4x^4 - 3x^3 + 2x - 5$. $f(-1) = 0$; $f(1) = -2$, $f(2) = 39$. -1 is zero. There is a zero in the interval $(1, 2)$. The zero is $1.168 \approx 1.17$

46. $f(x) = 3x^4 - 21x^3 + 36x^2 + 2x - 8$. $f(-1) = 50$, $f(0) = -8$, $f(1) = 12$; $f(2) = 20$, $f(3) = -2$; $f(4) = 0$. 4 is a zero. There is one in each interval: $(-1, 0)$, $(0, 1)$, $(2, 3)$. They are $-0.440 \approx -0.44$, $0.518 \approx 0.52$, $2.922 \approx 2.92$.

47. $f(x) = x^3 - 4x^2 + x + 3$. $f(1) = 1$, $f(2) = -3$. To 3 decimals the root is 1.239.

48. Show that the intermediate-value theorem guarantees that $x^3 + x + 3 = 0$ has a root between -2 and 2 and use your graphics calculator to approximate the root to two decimal places.

▷ Let $f(x) = x^3 + x + 3$. $f(-2) = -7$ and $f(2) = 13$. Because 0 is between -7 and 13, there is a number c between -2 and 2 with $f(c) = 0$. Two 2 decimal places the root is -1.21.

49. $m(v) = \dfrac{m_0}{\sqrt{1 - (v/c)^2}}$ is continuous on $[0, c)$.

50. Let $x = a - t$. f is continuous at a

$\Rightarrow \lim_{x\to a} f(x) = f(a)$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(a - t) - f(a)| < \epsilon$ when $0 < |t| < \delta$
$\Rightarrow \lim_{t\to a} f(a - t) = f(a)$

51. $\lim_{x\to a} f(x) \overset{f(x)\geq 0}{=} \lim_{x\to a} \sqrt{[f(x)]^2} \overset{\text{L.T.10}}{=} \sqrt{\lim_{x\to a} [f(x)]^2}$, viewed as a one-sided limit if $\lim_{x\to a} f(x) = 0$.

52. Prove that if $\lim_{x\to a} f(x) = \mathrm{L}$, then $\lim_{x\to a} |f(x)| = |\mathrm{L}|$.

▷ By the triangle inequality, $|f(x) - \mathrm{L}| \geq \big||f(x)| - |\mathrm{L}|\big|$. Therefore
$\lim_{x\to a} f(x) = \mathrm{L}$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $\big||f(x)| - |\mathrm{L}|\big| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow \lim_{x\to a} |f(x)| = |\mathrm{L}|$.

Note that the converse is not true: $\lim_{x\to a} |\operatorname{sgn} x| = 1$ but $\lim_{x\to a} \operatorname{sgn} x$ does not exist.

53. Suppose f is a function for which $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$. Prove that if f is continuous on $[0, 1]$, there is at least one number c in $[0, 1]$ such that $f(c) = c$.

▷ If $f(0) = 0$, then $c = 0$ satisfies the conclusion. If $f(1) = 1$, then $c = 1$ satisfies the conclusion. Suppose that $f(0) \neq 0$ and $f(1) \neq 1$. Then because $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$, we have $f(0) > 0$ and $f(1) < 1$. Let g be the function defined by

$$g(x) = f(x) - x \tag{1}$$

Then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. Because g is continuous on $[0, 1]$, and 0 is between $g(0)$ and $g(1)$, by the intermediate-value theorem there exists a number c between 0 and 1 such that

$$g(c) = 0$$

By Eq. (1), we have

$$f(c) - c = 0$$
$$f(c) = c$$

54. $f(x) = [\![x^2 - 2]\!]$. We seek the largest value of k for which f is continuous on the interval $[3, 3 + k)$.
$f(3) = [\![9 - 2]\!] = [\![7]\!]$. If $x > 3$, then $f(x) \le 7$ if and only if $x^2 - 2 < 8$, that is $x < \sqrt{10}$.
Thus the largest value of k satisfies $3 + k = \sqrt{10}$; $k = \sqrt{10} - 3$.

55. Not equivalent: f continuous on the closed interval implies the existence of 1-sided limits at the endpoints; f continuous at every number in the closed interval implies the existence of 2-sided limits at the endpoints.

1.10 CONTINUITY OF THE TRIGONOMETRIC FUNCTIONS AND THE SQUEEZE THEOREM

1.10.1 Squeeze Theorem Suppose that the functions f, g, and h are defined on some open interval I containing a, except possibly at a itself, and that $f(x) \le g(x) \le h(x)$ for all x in I for which $x \ne a$. Also suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ both exist and are equal to L. Then $\lim_{x\to a} g(x)$ also exists and is equal to L.

The squeeze theorem is used to prove the following results.

1.10.2 Theorem $\displaystyle\lim_{t\to 0} \frac{\sin t}{t} = 1$ Corollary: $\displaystyle\lim_{t\to 0} \frac{t}{\sin t} = 1$

1.10.5 Theorem $\displaystyle\lim_{t\to 0} \frac{1 - \cos t}{t} = 0$

To verify Theorem 1.10.2 on a calculator, you must switch to radian mode. Note also that angle measure is based on arc length, which is not defined until Chapter **6**. The above theorems are used to prove:

1.10.6 Theorem The sine and cosine functions are continuous at every real number.

1.10.7 Theorem The tangent, cotangent, secant, and cosecant functions are continuous on their domains.

Therefore, the tangent and secant functions are continuous at every real number, except $\frac{1}{2}\pi + k\pi$, where k is any integer. The cotangent and cosecant functions are continuous at every real number except $k\pi$, where k is any integer.

Useful Identities $1 - \cos^2 x = \sin^2 x$ $\qquad 1 - \cos x = 2\sin^2 \frac{1}{2}x$

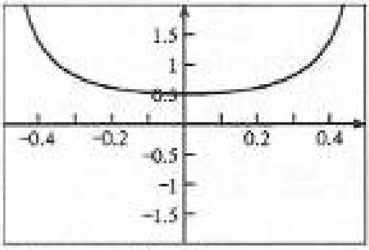
Exercises 1.10

In Exercises 1–26, (a) Estimate the limit by plotting the graph of the function. (b) Find the limit analytically.

1. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{x} = 4 \lim_{4x\to 0} \frac{\sin 4x}{4x} = 4 \cdot 1 = 4$

2. $\displaystyle\lim_{x\to 0} \frac{2x}{\sin 3x} = \frac{3}{2} \cdot \lim_{3x\to 0} \frac{3x}{\sin 3x} = \frac{3}{2} \cdot 1 = \frac{3}{2}$

3. $\displaystyle\lim_{x\to 0} \frac{\sin 9x}{\sin 7x} = \frac{9}{7} \cdot \frac{\lim_{9x\to 0} \frac{\sin 9x}{9x}}{\lim_{7x\to 0} \frac{\sin 7x}{7x}} = \frac{9}{7} \cdot \frac{1}{1} = \frac{9}{7}$



4. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 6x}$ ▹ The limit appears to be $\frac{1}{2}$.

To apply Theorem 1.10.2 we divide the numerator and denominator by $6x$.

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 6x} = \lim_{x\to 0} \frac{\frac{1}{2}\left(\frac{\sin 3x}{3x}\right)}{\frac{\sin 6x}{6x}} = \lim_{x\to 0} \frac{\frac{1}{2} \lim_{x\to 0} \frac{\sin 3x}{3x}}{\lim_{x\to 0} \frac{\sin 6x}{6x}} \qquad (1)$$

Furthermore, because both $3x$ and $6x$ approach 0 when x approaches 0, then by Theorem 1.10.2

$$\lim_{x\to 0} \frac{\sin 3x}{3x} = \lim_{3x\to 0} \frac{\sin 3x}{3x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin 6x}{6x} = \lim_{6x\to 0} \frac{\sin 6x}{6x} = 1$$

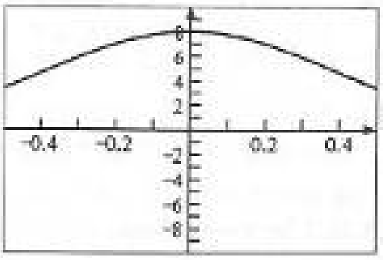
Substituting these into Eq. (1) we have $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 6x} = \frac{\frac{1}{2}(1)}{1} = \frac{1}{2}$

5. $\displaystyle\lim_{x\to 0} \frac{3x}{\sin 5x} = \frac{3}{5} \cdot \frac{1}{\lim_{5x\to 0} \frac{\sin 5x}{5x}} = \frac{3}{5} \cdot \frac{1}{1} = \frac{3}{5}$

6. $\displaystyle\lim_{x\to 0} \frac{\sin^3 x}{x^2} = \lim_{x\to 0} \sin x \cdot \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 0 \cdot 1 = 0$

7. $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin^2 3x} = \left(\lim_{x\to 0} \frac{x}{\sin 3x}\right)^2 = \left(\frac{1}{3} \cdot \lim_{3x\to 0} \frac{3x}{\sin 3x}\right)^2 = \left(\frac{1}{3} \cdot 1\right)^2 = \frac{1}{9}$

8. $\lim_{x\to 0}\dfrac{\sin^5 2x}{4x^5}$ ▹ The limit appears to be 8.



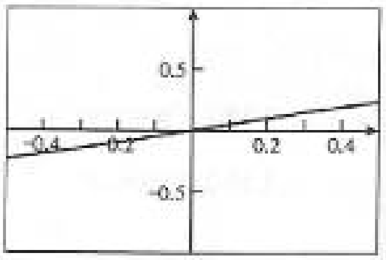
We apply Theorem 1.10.2. Because $2x \to 0$ when $x \to 0$, we have

$$\lim_{x\to 0}\frac{\sin^5 2x}{4x^5} = \lim_{x\to 0} 8\left(\frac{\sin 2x}{2x}\right)^5 = 8\left(\lim_{2x\to 0}\frac{\sin 2x}{2x}\right)^5 = 8\cdot 1^5 = 8$$

9. $\lim_{x\to 0}\dfrac{x}{\cos x} = \dfrac{\lim_{x\to 0} x}{\lim_{x\to 0}\cos x} = \dfrac{0}{1} = 0$ **10.** $\lim_{x\to 0}\dfrac{1-\cos x}{1+\sin x} = \dfrac{1-1}{1+0} = 0$

11. $\lim_{x\to 0}\dfrac{1-\cos 4x}{x} = 4\lim_{4x\to 0}\dfrac{1-\cos 4x}{4x} = 4\cdot 0 = 0$

12. $\lim_{z\to 0}\dfrac{1-\cos 2z}{4z}$ ▹ The limit appears to be 0.



Applying Theorem 1.10.5, we have

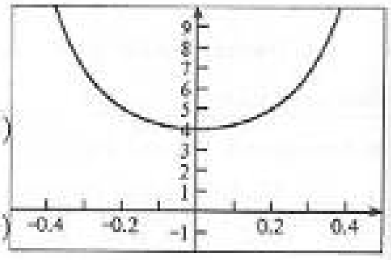
$$\lim_{z\to 0}\frac{1-\cos 2z}{4z} = \lim_{z\to 0}\frac{1}{2}\left(\frac{1-\cos 2z}{2z}\right) = \frac{1}{2}\lim_{2z\to 0}\frac{1-\cos 2z}{2z} = \frac{1}{2}\cdot 0 = 0$$

13. $\lim_{x\to 0}\dfrac{3x^2}{1-\cos^2\frac{1}{2}x} = \lim_{x\to 0}\dfrac{3x^2}{\sin^2\frac{1}{2}x} = \lim_{x\to 0}\dfrac{12(\frac{1}{2}x)^2}{\sin^2\frac{1}{2}x} = 12\left(\lim_{x/2\to 0}\dfrac{\frac{1}{2}x}{\sin\frac{1}{2}x}\right)^2 = 12(1)^2 = 12$

14. $\lim_{x\to 0}\dfrac{1-\cos^2 x}{2x^2} = \lim_{x\to 0}\dfrac{\sin^2 x}{2x^2} = \dfrac{1}{2}\left(\lim_{x\to 0}\dfrac{\sin x}{x}\right)^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$

15. $\lim_{x\to 0}\dfrac{\tan x}{2x} = \lim_{x\to 0}\dfrac{\frac{\sin x}{\cos x}}{2x} = \dfrac{1}{2}\cdot\lim_{x\to 0}\dfrac{\sin x}{x}\cdot\lim_{x\to 0}\dfrac{1}{\cos x} = \dfrac{1}{2}\cdot 1\cdot 1 = \dfrac{1}{2}$

16. $\lim_{x\to 0}\dfrac{\tan^4 2x}{4x^4}$ ▹ The limit appears to be 4.



$$\lim_{x\to 0}\frac{\tan^4 2x}{4x^4} = \lim_{x\to 0}\frac{\sin^4 2x}{4x^4\cos^4 2x} = \lim_{x\to 0}\frac{4}{\cos^4 2x}\left(\frac{\sin 2x}{2x}\right)^4$$
$$= \lim_{x\to 0}\frac{4}{\cos^4 2x}\cdot\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)^4 \qquad (1)$$

Because the cosine function is continuous at 0,

$$\lim_{x\to 0}\frac{4}{\cos^4 2x} = \frac{4}{\cos^4 0} = \frac{4}{1^4} = 4 \qquad (2)$$

By Theorem 1.10.2

$$\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)^4 = 1^4 = 1 \qquad (3)$$

Substituting from Equations (2) and (3) into Eq. (1), we obtain

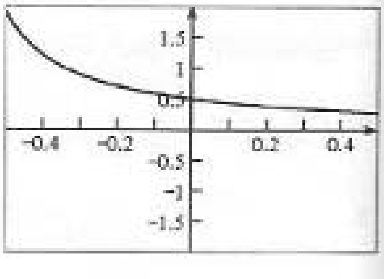
$$\lim_{x\to 0}\frac{\tan^4 2x}{4x^4} = 4\cdot 1 = 4$$

17. $\lim_{x\to 0}\dfrac{1-\cos 2x}{\sin 3x} = \lim_{x\to 0}\dfrac{1-\cos 2x}{x}\cdot\dfrac{x}{\sin 3x} = 2\lim_{2x\to 0}\dfrac{1-\cos 2x}{2x}\cdot\dfrac{1}{3}\cdot\dfrac{3x}{\sin 3x} = 2\cdot 0\cdot\dfrac{1}{3}\cdot 1 = 0$

18. $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim_{x\to 0}\dfrac{2\sin^2\frac{1}{2}x}{x^2} = \lim_{x\to 0}\dfrac{1}{2}\left(\dfrac{\sin\frac{1}{2}x}{\frac{1}{2}x}\right)^2 = \dfrac{1}{2}\left(\lim_{x/2\to 0}\dfrac{\sin\frac{1}{2}x}{\frac{1}{2}x}\right)^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$

19. $\lim_{x\to 0}\dfrac{x^2+3x}{\sin x} = \lim_{x\to 0}\dfrac{(x+3)x}{\sin x} = \lim_{x\to 0}(x+3)\cdot\lim_{x\to 0}\dfrac{x}{\sin x} = 3\cdot 1 = 3$

20. $\lim_{x\to 0}\dfrac{\sin x}{3x^2+2x}$ ▹ The limit appears to be $\frac{1}{2}$.



Applying Theorem 1.10.2, we have

$$\lim_{x\to 0}\frac{\sin x}{3x^2+2x} = \lim_{x\to 0}\frac{\sin x}{x(3x+2)} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{3x+2} = 1\cdot\frac{1}{2} = \frac{1}{2}$$

21. $\lim_{t\to 0^+}\dfrac{\sin t}{t^2} = \lim_{t\to 0^+}\dfrac{\sin t}{t}\cdot\lim_{t\to 0^+}\dfrac{1}{t} = +\infty$

22. $\lim\limits_{t\to 0^+} \dfrac{\sin 4t}{\cos 3t - 1} = \lim\limits_{t\to 0^+} -\dfrac{\sin 4t}{2\sin^2 \frac{3}{2}t} = \lim\limits_{t\to 0^+} -\dfrac{8}{9t} \cdot \dfrac{\sin 4t/4t}{(\sin \frac{3}{2}t/\frac{3}{2}t)^2} = -\infty$ by Limit Theorem 12

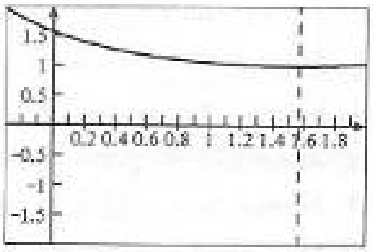
since $\lim\limits_{t\to 0^+} -\dfrac{8}{9t} = -\infty$ and $\lim\limits_{t\to 0^+} \dfrac{\sin 4t/4t}{(\sin \frac{3}{2}/\frac{3}{2}t)^2} = \dfrac{1}{(1)^2} = 1$

23. Let $t = \frac{1}{2}\pi - x$. Then $x = \frac{1}{2}\pi - t$. $\lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{\frac{1}{2}\pi - x} = \lim\limits_{t\to 0} \dfrac{1-\sin(\frac{1}{2}\pi - t)}{t} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{t} = 0$

24. $\lim\limits_{t\to\pi/2} \dfrac{\frac{1}{2}\pi - t}{\cos t}$ (*Hint:* Let $x = \frac{1}{2}\pi - t$.) ▹ The limit appears to be 1.

If $t = \frac{1}{2}x - x$, then $x = \frac{1}{2}\pi - t$. Thus

$$\lim_{t\to\pi/2} \frac{\frac{1}{2}\pi - t}{\cos t} = \lim_{x\to 0} \frac{x}{\cos(\frac{1}{2}\pi - x)} = \lim_{x\to 0} \frac{x}{\sin x} = 1$$



25. Let $t = x - \pi$. Then $x = \pi + t$.

$$\lim_{x\to\pi^+} \frac{\sin x}{x-\pi} = \lim_{t\to 0^+} \frac{\sin(\pi + t)}{t} = -\lim_{t\to 0^+} \frac{\sin t}{t} = -1$$

26. Let $t = x - \pi$. Then $x = t + \pi$. $\lim\limits_{x\to\pi^+} \dfrac{\tan x}{x-\pi} = \lim\limits_{t\to 0^+} \dfrac{\tan(t+\pi)}{t} = \lim\limits_{t\to 0^+} \dfrac{\tan t}{t} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t} \cdot \lim\limits_{t\to 0^+} \dfrac{1}{\cos t} = 1 \cdot 1 = 1$

27. Because $\sin x$ is continuous for all x and 2θ is continuous for all θ, the composition $\sin 2\theta$ is continuous for all θ, and so is the product $(V_0{}^2/g)\sin 2\theta$.

28. Let $F(\theta) = \dfrac{kW}{k\sin\theta + \cos\theta}$ where $0 < k < 1$. Prove that F is continuous on $[0, \frac{1}{2}\pi]$.

▹ Because $\sin\theta \geq 0$ and $\cos\theta \geq 0$ on $[0, \frac{1}{2}\pi]$ and both are not zero for the same θ, then $k\sin\theta + \cos\theta > 0$ on $[0, \frac{1}{2}\pi]$. Because $\sin\theta$, and hence $k\sin\theta$, and $\cos\theta$ are continuous for all θ, then by Theorem 1.8.2 (i) $k\sin\theta + \cos\theta$ is continuous for all θ. Hence by Theorem 1.8.2(iv) the quotient $F(\theta)$ is continuous on $[0, \frac{1}{2}\pi]$.

In Exercises 29–32, use the squeeze theorem to find the limit.

29. If $x \neq 0$ then $-1 \leq \cos\frac{1}{x} \leq 1$; so $-|x| \leq x\cos\frac{1}{x} \leq |x|$ (1)

Because $\lim\limits_{x\to 0}|x| = 0$, it follows from (1) and the squeeze theorem that $\lim\limits_{x\to 0} x\cos\frac{1}{x} = 0$.

30. Because $-1 \leq \sin x \leq 1$, then $-1 \leq \sin\dfrac{1}{\sqrt[3]{x}} \leq 1$ if $x \neq 0$.

Multiplying by x^2, we have $-x^2 \leq x^2 \sin\dfrac{1}{\sqrt[3]{x}} \leq x^2$ if $x \neq 0$.

Because $\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0}(-x^2) = 0$, we may apply the squeeze theorem to conclude that $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{\sqrt[3]{x}} = 0$

31. Because $0 \leq |g(x) + 4| < 2(3-x)^4$, then $-2(3-x)^4 < g(x) + 4 < 2(3-x)^4$ (1)

Because $\lim\limits_{x\to 3} 2(3-x)^4 = 0$, it follows from (1) and the squeeze theorem that

$\lim\limits_{x\to 3} g(x) + 4 = 0$. Therefore $\lim\limits_{x\to 3} g(x) = -4$.

32. $\lim\limits_{x\to -2} g(x)$, if $|g(x) - 3| < 5(x+2)^2$ for all x.

▹ Because $0 \leq |g(x) - 3| < 5(x+2)^2$, then $-5(x+2)^2 < g(x) - 3 < 5(x+2)^2$ (1)

Because $\lim\limits_{x\to -2} 5(x+2)^2 = 0$, it follows from (1) and the squeeze theorem that

$\lim\limits_{x\to -2} g(x) - 3 = 0$. Therefore $\lim\limits_{x\to -2} g(x) = 3$.

In Exercises 33 and 34, find the limit if it exists.

33. As x approaches 0 so does $\sin x$. Hence

$$\lim_{x\to 0} \frac{\sin(\sin x)}{x} = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} = \lim_{\sin x\to 0} \frac{\sin(\sin x)}{\sin x} \cdot \lim_{x\to 0} \frac{\sin x}{x} = 1 \cdot 1 = 1$$

34. If $x \neq 0$ then $-1 \leq \sin(1/x) \leq 1$; so $-|\sin x| \leq \sin x \sin(1/x) \leq |\sin x|$ (1)

Because $\lim\limits_{x\to 0}|\sin x| = 0$, it follows from (1) and the squeeze theorem that $\lim\limits_{x\to 0} \sin x \cos(1/x) = 0$.

35. Because $1 - \cos^2 x \le f(x) \le x^2$ for all x in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then

$$\sin^2 x \le f(x) \le x^2 \text{ for all } x \text{ in } (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \qquad (1)$$

Because $\lim_{x\to 0} \sin^2 x = 0$ and $\lim_{x\to 0} x^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 0} f(x) = 0$.

36. Given: $-\sin x \le f(x) \le 2 + \sin x$ for all x in the open interval $(-\pi, 0)$. Find $\lim_{x\to -\pi/2} f(x)$.

▹ Because $-\sin x \le f(x) \le 2 + \sin x$ for all x in $(-\pi, 0)$, then

$$-1 - \sin x \le f(x) - 1 \le 1 + \sin x \text{ for all } x \text{ in } (-\pi, 0) \qquad (1)$$

Because $\lim_{x\to -\pi/2} |1 + \sin x| = |1 - 1| = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to -\pi/2} f(x) - 1 = 0$ and so $\lim_{x\to -\pi/2} f(x) = 1$.

In Exercises 37–40, prove that the function is continuous on its domain.

37. Because $\tan x = \frac{\sin x}{\cos x}$, and both the sine and cosine functions are continuous,

the tangent function is continuous at all x for which $\cos x \ne 0$.
Therefore, the tangent function is continuous on its domain.

38. Because $\cot x = \frac{\cos x}{\sin x}$, and both the sine and cosine functions are continuous,

the cotangent function is continuous at all x for which $\sin x \ne 0$.
Therefore, the cotangent function is continuous on its domain.

39. Because $\sec x = \frac{1}{\cos x}$, and the cosine function is continuous,

the secant function is continuous at all values of x for which $\cos x \ne 0$.
Therefore, the secant function is continuous on its domain.

40. The cosecant function is defined by $\csc x = \frac{1}{\sin x}$.

The domain of the cosecant function is the set of all real numbers x for which $\sin x \ne 0$. Because a constant function and the sine function are continuous at every real number, and the cosecant function is the quotient of a constant function and the sine function, by Theorem 1.8.2(iv), the cosecant function is continuous at every real number x for which $\sin x \ne 0$. Thus the cosecant function is continuous on its domain.

41. If $|f(x)| \le M$ for all x, then $-M \le f(x) \le M$ and $-Mx^2 \le x^2 f(x) \le x^2 M$ (1)
Because $\lim_{x\to 0} Mx^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 0} x^2 f(x) = 0$.

42. If $|f(x)| \le M$ for all x, then $-M \le f(x) \le M$ and $-M|g(x)| \le f(x)g(x) \le M|g(x)|$ (1)
Because $\lim_{x\to a} M|g(x)| = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to a} f(x)g(x) = 0$.

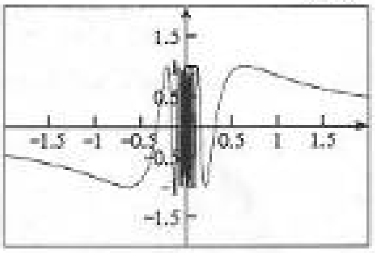
43. If $|f(x)| \le k|x - a|$ for all $x \ne a$, then $-k|x - a| \le f(x) \le k|x - a|$ (1)
Because $\lim_{x\to a} k|x - a| = 0$, it follows from (1) and the squeeze theorem that $\lim f(x) = 0$.

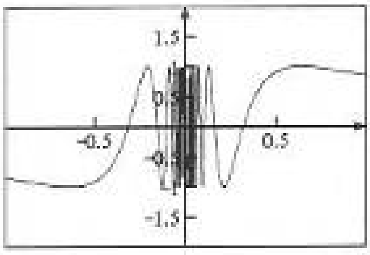
Remark: If $|f(x)| < g(x)$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} f(x) = 0$.

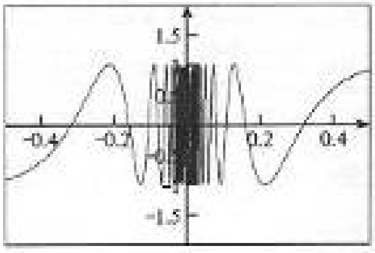
In fact $-g(x) < f(x) < g(x)$ and the remark follows from the squeeze theorem.
The results of Exercises 29, 31, 41 and 43 follow immediately from this remark.

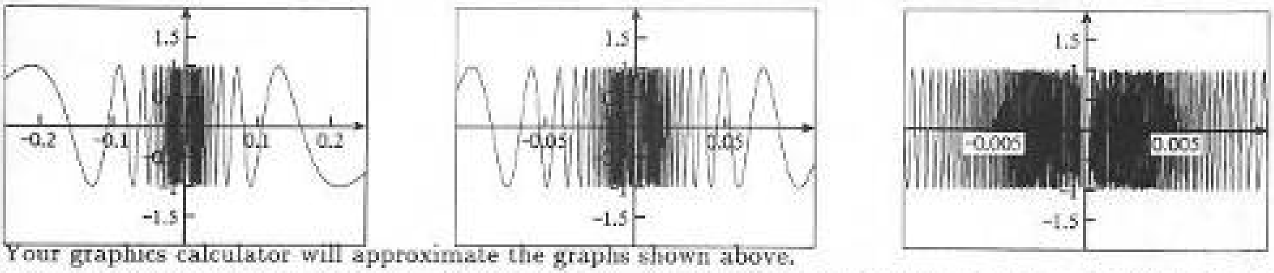
44. Plot the graph of $f(x) = \sin(1/x)$ in each of the following windows: (a) $[-2,2]\times[-2,2]$ (b) $[-1,1]\times[-2,2]$ (c) $[-0.5,0.5]\times[-2,2]$ (d) $[-0.25,0.25]\times[-2,2]$ (e) $[-0.1,0.1]\times[-2,2]$ (f) $[-0.01,0.01]\times[-2,2]$ (g) Make and verify a conjecture about $\lim_{x\to 0} f(x)$.

▹









Your graphics calculator will approximate the graphs shown above.

The limit does not exist. Suppose $\lim_{x\to 0}\sin(1/x) = L$. In the definition of limit, choose $\epsilon = 1$. Then for some $\delta > 0$ we must have $|\sin(1/x) - L| < 1$ whenever $-\delta < x < \delta$. Let k be a positive integer $> 1/(2\pi\delta)$ and let $x_1 = 1/(2k+\frac{1}{2})\pi$ and $x_2 = -1/(2k+\frac{1}{2})\pi$. Then $-\delta < x_1 < \delta$ and $-\delta < x_2 < \delta$. By the triangle inequality we get the following contradiction:

$$1 + 1 > |\sin(1/x_1) - L| + |\sin(1/x_2) - L| \geq |[\sin(1/x_1) - L] - [\sin(1/x_2) - L]| = |\sin(1/x_1) - \sin(1/x_2)|$$
$$= |\sin(2k+\tfrac{1}{2})\pi - \sin[-(2k+\tfrac{1}{2})\pi]| = |1 - (-1)| = 2$$

Thus there is no such number L and the limit does not exist.

45. $\lim_{x\to 0}\cos(1/x)$ does not exist. See Ex.44.

46. $\lim_{x\to 0}\tan(1/x)$ does not exist. See Ex.44.

Miscellaneous Exercises for Chapter 1

1. $f(x) = 4 - x^2$ ▹ (a) $f(1) = 4 - 1^2 = 3$ (b) $f(-2) = 4 - (-2)^2 = 0$ (c) $f(3) = 4 - 3^2 = -5$

(d) $f(x-1) = 4 - (x-1)^2 = -x^2 + 2x + 3$ (e) $f(x^2) = 4 - (x^2)^2 = 4 - x^4$

(f) If $h \neq 0$, $\dfrac{f(x+h) - f(x)}{h} = \dfrac{[4 - (x+h)^2] - [4 - x^2]}{h} = \dfrac{-2xh - h^2}{h} = -2x - h$

2. $g(x) = \sqrt{1-x}$ ▹ (a) $g(1) = \sqrt{1-1} = 0$ (b) $g(-3) = \sqrt{1-(-3)} = \sqrt{4} = 2$

(c) $g(x+1) = \sqrt{1-(x+1)} = \sqrt{-x}$ (d) $g(1-x^2) = \sqrt{1-(1-x^2)} = \sqrt{x^2} = |x|$

(e) If $h \neq 0$, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h} \cdot \dfrac{\sqrt{1-x-h} + \sqrt{1-x}}{\sqrt{1-x-h} + \sqrt{1-x}} = \dfrac{(1-x-h) - (1-x)}{h(\sqrt{1-x-h} + \sqrt{1-x})}$

$$= \frac{-h}{h(\sqrt{1-x-h} + \sqrt{1-x})} = \frac{-1}{\sqrt{1-x-h} + \sqrt{1-x}}$$

In *Exercises 3–6*, define the following functions and determine their domain D:
(a) $f+g$ (b) $f-g$ (c) $f\cdot g$ (d) f/g (e) g/f (f) $f\circ g$ (g) $g\circ f$

3. $f = \sqrt{x+2}$, $g = x^2 - 4$ (a) $f + g = \sqrt{x+2} + x^2 - 4$, D: $[-2, +\infty)$

(b) $f - g = \sqrt{x+2} - x^2 + 4$, D: $[-2, +\infty)$ (c) $f\cdot g = \sqrt{x+2}(x^2-4)$, D: $[-2, +\infty)$

(d) $f/g = \dfrac{\sqrt{x+2}}{x^2-4}$, D: $(-2, 2)\cup(2, +\infty)$ (e) $g/f = \dfrac{x^2-4}{\sqrt{x+2}}$, D: $(-2, +\infty)$

(f) $f(g(x)) = \sqrt{(x^2-4)+2} = \sqrt{x^2-2}$, D: $(-\infty, -\sqrt{2}]\cup[\sqrt{2}, +\infty)$

(g) $g(f(x)) = (\sqrt{x+2})^2 - 4 = (x+2) - 4 = x - 2$, D: $[-2, +\infty)$

4. $f = x^2 - 9$, $g = \sqrt{x+5}$ (a) $f + g = x^2 - 9 + \sqrt{x+5}$, D: $[-5, +\infty)$

(b) $f - g = x^2 - 9 - \sqrt{x+5}$, D: $[-5, +\infty)$ (c) $f\cdot g = (x^2-9)\sqrt{x+5}$, D: $[-5, +\infty)$

(d) $f/g = \dfrac{x^2-9}{\sqrt{x+5}}$, D: $(-5, +\infty)$ (e) $g/f = \dfrac{\sqrt{x+5}}{x^2-9}$, D: $[-5, -3)\cup(-3, 3)\cup(3, +\infty)$

(f) $f(g(x)) = (\sqrt{x+5})^2 - 9 = (x+5) - 9 = x - 4$, D: $[-5, +\infty)$

(g) $g(f(x)) = \sqrt{(x^2-9)+5} = \sqrt{x^2-4}$, D: $(-\infty, -2]\cup[2, +\infty)$

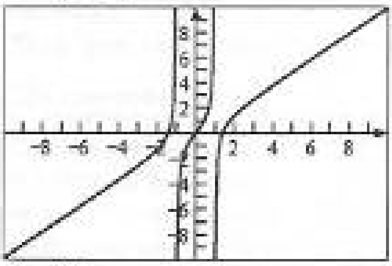
5. $f = \frac{1}{x^2}$, $g = \sqrt{x}$ (a) $f + g = \frac{1}{x^2} + \sqrt{x}$, D: $(0, +\infty)$ (b) $f - g = \frac{1}{x^2} - \sqrt{x}$, D: $(0, +\infty)$
(c) $f \cdot g = \frac{1}{x^2} \cdot \sqrt{x} = x^{-3/2}$, D: $(0, +\infty)$ (d) $\frac{f}{g} = \frac{1/x^2}{\sqrt{x}} = x^{-5/2}$, D: $(0, +\infty)$ (e) $\frac{g}{f} = \frac{\sqrt{x}}{1/x^2} = x^{5/2}$, D: $(0, +\infty)$
(f) $f(g(x)) = \frac{1}{(\sqrt{x})^2} = \frac{1}{x}$, D: $(0, +\infty)$ (g) $g(f(x)) = \sqrt{\frac{1}{x^2}} = \frac{1}{|x|}$, D: $x \neq 0$

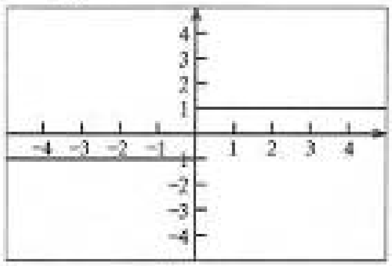
6. $f = \frac{x}{x-1}$, $g = \frac{1}{x+2}$ (a) $f + g = \frac{x}{x-1} \cdot \frac{x+2}{x+2} + \frac{1}{x+2} \cdot \frac{x-1}{x-1} = \frac{x^2+3x-1}{(x-1)(x+2)}$, D: $x \neq -2, 1$
(b) $f - g = \frac{x}{x-1} \cdot \frac{x+2}{x+2} - \frac{1}{x+2} \cdot \frac{x-1}{x-1} = \frac{x^2+x+3}{(x-1)(x+2)}$, D: $x \neq -2, 1$
(c) $f \cdot g = \frac{x}{x-1} \cdot \frac{1}{x+2} = \frac{x}{(x-1)(x+2)}$, D: $x \neq -2, 1$ (d) $f/g = \frac{x/(x-1)}{1/(x+2)} = \frac{x^2+2x}{x-1}$, D: $x \neq -2, 1$
(e) $g/f = \frac{1/(x+2)}{x/(x-1)} = \frac{x-1}{x(x+2)}$, D: $x \neq -2, 0, 1$
(f) $f(g(x)) = \frac{1/(x+2)}{1/(x+2)-1} = \frac{1}{1-(x+2)} = -\frac{1}{x+1}$, D: $x \neq -2, -1$
(g) $g(f(x)) = \frac{1}{x/(x-1)+2} = \frac{x-1}{x+2(x-1)} = \frac{x-1}{3x-2}$, D: $x \neq \frac{2}{3}, 1$

In *Exercises 7 and 8*, plot the graph and determine if the function is even, odd, or neither.

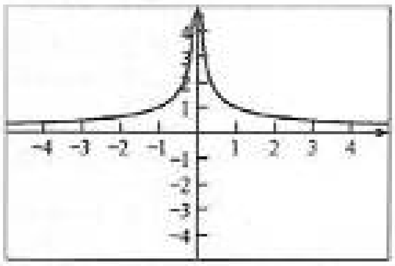
7. (a) $2x^3 - 3x$ is odd (odd − odd) (b) $5x^4 + 2x^2 - 1$ is even (sum of even)
(c) $3x^5 - 2x^3 + x^2 - x$ is neither (odd + even) (d) $(x^2+1)/(x^3-x)$ is odd (even ÷ odd)

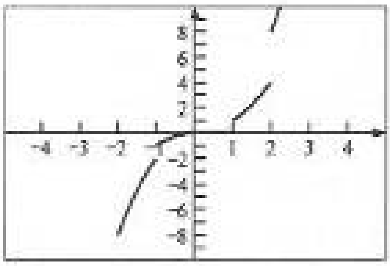
8. (a) $\frac{x^3-2x}{x^2-1}$ is odd (odd ÷ even) (b) $\frac{x}{|x|}$ is odd (odd ÷ even)





(c) $\sqrt[3]{x}/x$ is even (odd ÷ odd) (d) $F(x) = x^2 \cdot [\![x]\!]\, x$ is neither:
$F(0.5) = (0.5)^2[\![0.5]\!] = 0.25(0) = 0$ while $F(-0.5) = (-0.5)^2[\![-0.5]\!] = 0.25(1) = 0.25$





In *Exercises 9 and 10*, plot the graph of the function and determine its domain D and range R.

9. (a) $f(x) = 4 - 2x$
▹ D: $(-\infty, +\infty)$, R: $(-\infty, +\infty)$
(b) $g(x) = x^2 - 4$
D: $(-\infty, +\infty)$, R: $[-4, +\infty)$
(c) $h(x) = \sqrt{x^2 - 16}$
D: $(-\infty, -4] \cup [4, +\infty)$, R: $[0, +\infty)$
(d) $F(x) = \sqrt{16 - x^2}$
D: $[-4, 4]$, R: $[0, 4]$
(e) $f(x) = |5 - x|$
D: $(-\infty, +\infty)$, R: $[0, +\infty)$
(f) $g(x) = 5 - |x|$
D: $(-\infty, +\infty)$, R: $(-\infty, 5]$

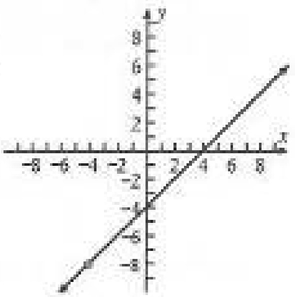
10. (a) $g(x) = 3x + 2$
▹ D: $(-\infty, +\infty)$, R: $(-\infty, +\infty)$
(b) $f(x) = 9 - x^2$
D: $(-\infty, +\infty)$, R: $(-\infty, 9]$
(c) $H(x) = \sqrt{1 - x^2}$
D: $[-1, 1]$, R: $[0, 1]$
(d) $G(x) = \sqrt{x^2 - 1}$
D: $(-\infty, -1] \cup [1, +\infty)$, R: $[0, +\infty)$
(e) $g(x) = |x + 4|$
D: $(-\infty, +\infty)$, R: $[0, +\infty)$
(f) $f(x) = |x| + 4$
D: $(-\infty, +\infty)$, R: $[4, +\infty)$

In Exercises 11–14, determine the domain D and range R of the function and sketch its graph.

11. (a) $g(x) = \dfrac{x^2 - 16}{x + 4}$

▷ $= \dfrac{(x-4)(x+4)}{x+4} = x - 4$

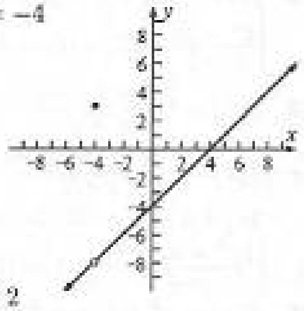
D: $x \neq -4$, R: $y \neq -8$



(b) $G(x) = \begin{cases} x - 4 & \text{if } x \neq -4 \\ 3 & \text{if } x = -4 \end{cases}$

▷ D: $(-\infty, +\infty)$

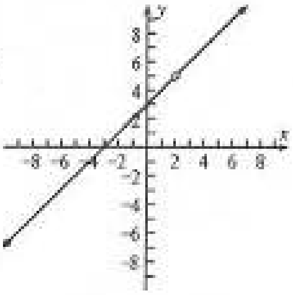
R: $y \neq -8$



12. (a) $f(x) = \dfrac{x^2 + x - 6}{x - 2}$

▷ $= \dfrac{(x-2)(x+3)}{x-2} = x + 3$

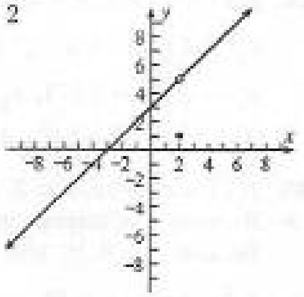
D: $x \neq 2$, R: $y \neq 5$



(b) $F(x) = \begin{cases} x + 3 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $y \neq 5$

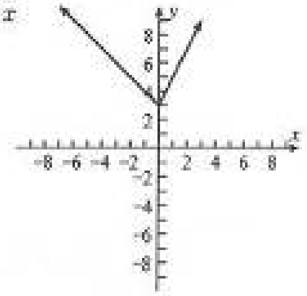


13. (a) $F(x) = \begin{cases} 3 - x & \text{if } x < 0 \\ 3 + 2x & \text{if } 0 \le x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(3, +\infty) \cup [3, +\infty)$

$= [3, +\infty)$

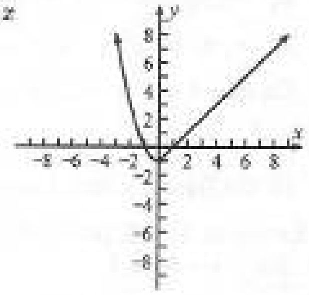


(b) $h(x) = \begin{cases} x^2 - 1 & \text{if } x \le 0 \\ x - 1 & \text{if } 0 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $[-1, +\infty) \cup (-1, +\infty)$

$= [-1, +\infty)$

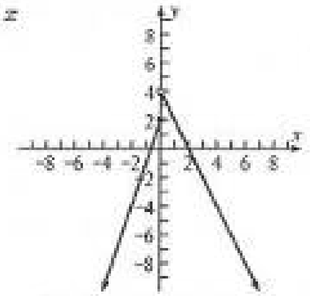


14. (a) $G(x) = \begin{cases} 3x + 2 & \text{if } x \le 0 \\ 4 - 2x & \text{if } 0 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(-\infty, 2] \cup (-\infty, 4)$

$= (-\infty, 4)$

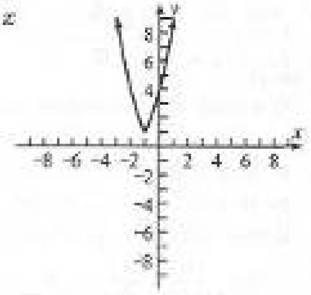


(b) $H(x) = \begin{cases} x^2 & \text{if } x < -1 \\ (x+2)^2 & \text{if } -1 \le x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(1, +\infty) \cup [1, +\infty)$

$= [1, +\infty)$



In Exercises 15–20, determine a $\delta > 0$ such that $\left| f(x) - L \right| < \epsilon$ when $0 < |x - a| < \delta$ by finding the smallest x_1 and largest x_2; and by using inequalities.

15. $f(x) = 2x - 5$, $a = 3$, $L = 1$, $\epsilon = .05$

▷ $2x_1 - 5 = 1 - .05$, $6 - 2x_1 = .05$, $3 - x_1 = .025$; $2x_2 - 5 = 1 + .05$, $2x_2 - 6 = .05$, $x_2 - 3 = .025$. $\delta = .025$.

$\left|(2x - 5) - 1\right| = |2x - 6| = 2|x - 3| < .05$ when $|x - 3| < .025 = \delta$

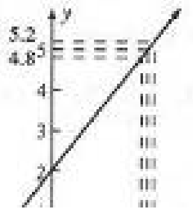
16. $f(x) = 3x + 2$, $a = 1$, $L = 5$, $\epsilon = 0.2$

▷ Because f is increasing, x_1 corresponds to the smaller value of f.

$3x_1 + 2 = 5 - 0.2$, $3 - 3x_1 = 0.2$, $1 - x_1 = \frac{1}{15}$

$3x_2 + 2 = 5 + 0.2$, $3x_2 - 3 = 0.2$, $x_2 - 1 = \frac{1}{15}$. $\delta = \frac{1}{15}$.

$\left|(3x + 2) - 5\right| = |3x - 3| = 3|x - 1| < 0.2$ when $|x - 1| < \frac{1}{15}$. $\delta = \frac{1}{15}$.



17. $f(x) = \frac{x^2 - 25}{x - 5}$, $a = 5$, $\mathrm{L} = 10$, $\epsilon = 0.1$

▹ $f(x) = x + 5$ if $x \neq 5$. $x_1 + 5 = 10 - 0.1$, $5 - x_1 = 0.1$. $x_2 + 5 = 10 + 0.1$, $x_2 - 5 = 0.1$. $\delta = 0.1$
$\left|\frac{x^2 - 25}{x - 5} - 10\right| = |(x + 5) - 10| = |x - 5| < 0.1$ when $0 < |x - 5| < 0.1$. $\delta = 0.1$

18. $f(x) = \frac{2x^2 + 9x + 10}{x + 2}$, $a = -2$, $\mathrm{L} = 1$, $\epsilon = .03$

▹ $f(x) = \frac{(2x + 5)(x + 2)}{x + 2} = 2x + 5$ if $x \neq -2$. $2x_1 + 5 = 1 - .03$, $-4 - 2x_1 = .03$, $-2 - x_1 = .015$
$2x_2 + 5 = 1 + .03$, $2x_2 + 4 = .03$, $x_2 - (-2) = .015$. $\delta = .015$
$\left|\frac{2x^2 + 9x + 10}{x + 2} - 1\right| = |(2x + 5) - 1| = |2x + 4| = 2|x + 2| < .03$ when $|x + 2| < .015$. $\delta = .015$

19. $f(x) = x^2 + 4$, $a = 2$, $\mathrm{L} = 8$, $\epsilon = 0.3$ ▹ Because $a > 0$, then $x_1, x_2 > 0$.
${x_1}^2 + 4 = 8 - 0.3$, ${x_1}^2 = 3.7$, $x_1 = \sqrt{3.7} = 1.924$, $2 - x_1 = .076$. ${x_2}^2 + 4 = 8 + 0.3$, ${x_2}^2 = 4.3$,
$x_2 = \sqrt{4.3} = 2.074$, $x_2 - 2 = .074$. $\delta = .074$. Choose $\delta < 1$ so $-1 < x - 2 < 1$, $3 < x + 2 < 5$.
$|(x^2 + 4) - 8| = |x^2 - 4| = |x + 2||x - 2| < 5|x - 2| < 0.3$ when $|x - 2| < .06 = \delta$.

20. $f(x) = x^2 - 3x$, $a = 3$, $\mathrm{L} = 0$, $\epsilon = .08$

▹ Because f is increasing at 3, x_1 corresponds to the smaller value of f.
Because $a > 0$, x_1 and $x_2 > 0$.

${x_1}^2 - 3x_1 = 0 - .08$, ${x_1}^2 - 3x_1 + .08 = 0$, $x_1 = \frac{1}{2}[3 + \sqrt{3^2 - 4(.08)}] = 2.973$.

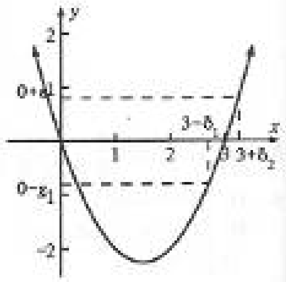
${x_2}^2 - 3x_2 = 0 + .08$, ${x_2}^2 - 3x_2 - .08 = 0$, $x_2 = \frac{1}{2}[3 + \sqrt{3^2 + 4(.08)}] = 3.026$

$3 - x_1 = .027$, $x_2 - 3 = .026$. $\delta = .026$

Choose $\delta < 1$ so $-1 < x - 3 < 1$, $2 < x < 4$.

$|(x^2 - 3x) - 0| = |x||x - 3| < 4|x - 3| < .08$ when $|x - 3| < .02 = \delta$

In the figure, ϵ and δ are exaggerated for clarity.



In Exercises 21–26, prove the limit by applying the definition (Definition 1.5.1).

21. $\lim_{x \to 3}(2x - 5) = 1$ ▹ $|(2x - 5) - 1| = |2x - 6| = 2|x - 3| < \epsilon$ when $|x - 3| = \frac{1}{2}\epsilon = \delta$

22. $\lim_{x \to -2}(8 - 3x) = 14$ ▹ $|(8 - 3x) - 14| = |-3x - 6| = 3|x + 2| < \epsilon$ when $|x + 2| < \frac{1}{3}\epsilon = \delta$

23. $\lim_{x \to -1}(3x + 8) = 5$ ▹ $|(3x + 8) - 5| = |3x + 3| = 3|x + 1| < \epsilon$ when $|x + 1| < \frac{1}{3}\epsilon = \delta$

24. $\lim_{x \to 5}(4x - 11) = 9$

▹ We wish to determine a $\delta > 0$ such that
if $0 < |x - 5| < \delta$ then $|(4x - 11) - 9| < \epsilon$
$\Leftrightarrow$ if $0 < |x - 5| < \delta$ then $4|x - 5| < \epsilon$
$\Leftrightarrow$ if $0 < |x - 5| < \delta$ then $|x - 5| < \frac{1}{4}\epsilon$
Hence take $\delta = \frac{1}{4}\epsilon$; then $0 < |x - 5| < \delta \Rightarrow 4|x - 5| < 4(\frac{1}{4}\epsilon) \Rightarrow |(4x - 11) - 9| < \epsilon$.

25. $\lim_{x \to -3/4} \frac{16x^2 - 9}{4x + 3} = -6$

▹ If $x \neq -\frac{3}{4}$, then $\left|\frac{16x^2 - 9}{4x + 3} - (-6)\right| = |(4x - 3) + 6| = |4x + 3| = 4\left|x + \frac{3}{4}\right| < \epsilon$ when $0 < \left|x + \frac{3}{4}\right| < \frac{1}{4}\epsilon = \delta$

26. $\lim_{x \to 1/3} \frac{1 - 9x^2}{1 - 3x} = 2$

▹ If $x \neq \frac{1}{3}$, then $\left|\frac{1 - 9x^2}{1 - 3x} - 2\right| = |(1 + 3x) - 2| = |3x - 1| = 3\left|x - \frac{1}{3}\right| < \epsilon$ when $0 < \left|x - \frac{1}{3}\right| < \frac{1}{3}\epsilon = \delta$

In Exercises 27–34, find the limit and, when appropriate, indicate the limit theorems used.

27. $\lim_{x \to 2}(3x^2 - 4x + 5) \overset{\text{L.T.4}}{=} \lim_{x \to 2} 3x^2 + \lim_{x \to 2}(-4x + 5) \overset{\substack{\text{L.T.7,L.T.8} \\ \text{L.T.1}}}{=} \lim_{x \to 2} 3(\lim_{x \to 2} x)^2 + (-3) \overset{\substack{\text{L.T.3} \\ \text{L.T.2}}}{=} 3(2)^2 - 3 = 9$

28. $\lim\limits_{x\to-2}\dfrac{x^2-x-6}{x^2-5x-14}$

▹
$$\begin{aligned}\lim_{x\to-2}\frac{x^2-x-6}{x^2-5x-14}&=\lim_{x\to-2}\frac{(x+2)(x-3)}{(x-2)(x-7)}\\&=\lim_{x\to-2}\frac{x-3}{x-7}\\&=\frac{\lim\limits_{x\to-2}(x-3)}{\lim\limits_{x\to-2}(x-7)}&&\text{(L.T. 9)}\\&=\frac{-5}{-9}=\frac{5}{9}&&\text{(L.T. 1)}\end{aligned}$$

29. $\lim\limits_{z\to-3}\dfrac{z^2-9}{z+3}=\lim\limits_{z\to-3}\dfrac{(z-3)(z+3)}{z+3}=\lim\limits_{z\to-3}(z-3)\overset{\text{L.T.1}}{=}-6$

30. $\lim\limits_{h\to1}\dfrac{h^2-4}{3h^3+6}\overset{\text{L.T.10}}{=}\dfrac{\lim\limits_{h\to1}(h^2-4)}{\lim\limits_{h\to1}(3h^3+6)}\overset{\text{L.T.4}}{=}\dfrac{\lim\limits_{h\to1}h^2-\lim\limits_{h\to1}4}{\lim\limits_{h\to1}3h^3+\lim\limits_{h\to1}6}\overset{\substack{\text{L.T.8}\\\text{L.T.7}}}{=}\dfrac{(\lim\limits_{h\to1}h)^2-\lim\limits_{h\to1}4}{\lim\limits_{h\to1}3(\lim\limits_{h\to1}h)^3+\lim\limits_{h\to1}6}\overset{\text{L.T.1}}{=}\dfrac{1^2-4}{3\cdot1^3+6}=\dfrac{-3}{9}=-\dfrac{1}{3}$

31. $\lim\limits_{x\to1/2}\sqrt[3]{\dfrac{4x^2+4x-3}{4x^2-1}}=\lim\limits_{x\to1/2}\sqrt[3]{\dfrac{(2x-1)(2x+3)}{(2x-1)(2x+1)}}=\lim\limits_{x\to1/2}\sqrt[3]{\dfrac{2x+3}{2x+1}}\overset{\text{L.T.10}}{=}\sqrt[3]{\lim\limits_{x\to1/2}\left(\dfrac{2x+3}{2x+1}\right)}$

$\overset{\text{L.T.9}}{=}\sqrt[3]{\dfrac{\lim\limits_{x\to1/2}(2x+3)}{\lim\limits_{x\to1/2}(2x+1)}}\overset{\text{L.T.1}}{=}\sqrt[3]{\dfrac{4}{2}}=\sqrt[3]{2}$

32. $\lim\limits_{t\to0}\dfrac{1-\sqrt{1+t}}{t}$

▹
$$\begin{aligned}\lim_{t\to0}\frac{1-\sqrt{1+t}}{t}&=\lim_{t\to0}\frac{1-\sqrt{1+t}}{t}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\\&=\lim_{t\to0}\frac{1-(1+t)}{t(1+\sqrt{1+t})}=\lim_{t\to0}\frac{-t}{t(1+\sqrt{1+t})}\\&=\lim_{t\to0}\frac{-1}{1+\sqrt{1+t}}\\&=\frac{-1}{\lim\limits_{t\to0}(1+\sqrt{1+t})}&&\text{(L.T. 9, 2)}\\&=\frac{-1}{1+\sqrt{1}}&&\text{(L.T. 5, 1, 10)}\\&=-\frac{1}{2}\end{aligned}$$

33. $\lim\limits_{t\to0}\dfrac{\sqrt{9-t}-3}{t}=\lim\limits_{t\to0}\dfrac{(\sqrt{9-t}-3)(\sqrt{9-t}+3)}{t(\sqrt{9-t}+3)}=\lim\limits_{t\to0}\dfrac{-t}{t(\sqrt{9-t}+3)}=\lim\limits_{t\to0}\dfrac{-1}{\sqrt{9-t}+3}\overset{\text{L.T.9}}{=}\dfrac{\lim\limits_{t\to0}(-1)}{\lim\limits_{t\to0}(\sqrt{9-t}+3)}$

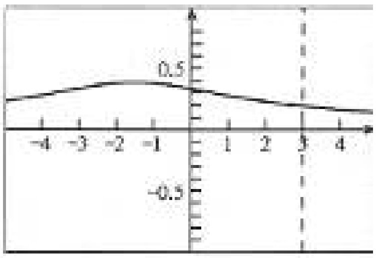
$\overset{\substack{\text{L.T.4}\\\text{L.T.2}}}{=}\dfrac{-1}{\lim\limits_{t\to0}\sqrt{9-t}+\lim\limits_{t\to0}3}\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\dfrac{-1}{\sqrt{\lim\limits_{t\to0}(9-t)}+3}\overset{\text{L.T.1}}{=}\dfrac{-1}{\sqrt{9}+3}=-\dfrac{1}{6}$

34. $\lim\limits_{y\to-4}\sqrt{\dfrac{5y+4}{y-5}}\overset{\text{L.T.10}}{=}\sqrt{\lim\limits_{y\to-4}\dfrac{5y+4}{y-5}}\overset{\text{L.T.9}}{=}\sqrt{\dfrac{\lim\limits_{y\to-4}(5y+4)}{\lim\limits_{y\to-4}(y-5)}}\overset{\text{L.T.1}}{=}\sqrt{\dfrac{5(-4)+4}{-4-5}}=\sqrt{\dfrac{-16}{-9}}=\sqrt{\dfrac{16}{9}}=\dfrac{4}{3}$

In Exercises 35–42, find the limit if it exists and support your answer by plotting the graph.

35. $\lim\limits_{x\to -1} \dfrac{2x^2-x-3}{3x^2+8x+5} = \lim\limits_{x\to -1} \dfrac{(2x-3)(x+1)}{(3x+5)(x+1)} = \lim\limits_{x\to -1} \dfrac{2x-3}{3x+5} = -\dfrac{5}{2}$

36. $\lim\limits_{y\to 3} \sqrt{\dfrac{y-3}{y^3-27}}$



▷
$$\begin{aligned}\lim_{y\to 3} \sqrt{\frac{y-3}{y^3-27}} &= \lim_{y\to 3} \sqrt{\frac{y-3}{(y-3)(y^2+3y+9)}}\\ &= \lim_{y\to 3} \sqrt{\frac{1}{y^2+3y+9}}\\ &= \sqrt{\frac{1}{27}} = \frac{1}{9}\sqrt{3} \approx 0.1925\end{aligned}$$

37. $\lim\limits_{x\to 9} \dfrac{2\sqrt{x}-6}{x-9} = 2\lim\limits_{x\to 9} \dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = 2\lim\limits_{x\to 9} \dfrac{x-9}{(x-9)(\sqrt{x}+3)} = 2\lim\limits_{x\to 9} \dfrac{1}{\sqrt{x}+3} = 2\cdot\dfrac{1}{6} = \dfrac{1}{3}$

38. Since $y \to 5^-$, then $y < 5$ and $5 - y > 0$. Thus,

$$\lim_{y\to 5^-} \frac{\sqrt{25-y^2}}{y-5} = \lim_{y\to 5^-} \frac{\sqrt{(5-y)(5+y)}}{-\sqrt{(5-y)^2}} = \lim_{y\to 5^-} \frac{\sqrt{5-y}\cdot\sqrt{5+y}}{-\sqrt{5-y}\sqrt{5-y}} = \lim_{y\to 5^-} \frac{\sqrt{5+y}}{-\sqrt{5-y}} = \frac{\sqrt{10}}{0^-} = -\infty$$

39. $\lim\limits_{s\to 7} \dfrac{5-\sqrt{4+3s}}{7-s} = \lim\limits_{s\to 7} \dfrac{5-\sqrt{4+3s}}{7-s}\cdot\dfrac{5+\sqrt{4+3s}}{5+\sqrt{4+3s}} = \lim\limits_{s\to 7} \dfrac{25-(4+3s)}{(7-s)(5+\sqrt{4+3s})} = \lim\limits_{s\to 7} \dfrac{3}{5+\sqrt{4+3s}}$
$= \dfrac{3}{5+\sqrt{4+3(7)}} = \dfrac{3}{10}$

40. $\lim\limits_{x\to 0^-} \dfrac{x^2-5}{2x^3-3x^2}$

▷ We apply Limit Theorem 12. Because $2x^3 - 3x^2 = x^2(2x-3)$, then $2x^3 - 3x^2 < 0$ when $x < \frac{3}{2}$. Thus,

$$\lim_{x\to 0^-} (2x^3 - 3x^2) = 0$$

and $2x^3 - 3x^2$ is approaching 0 through negative values. Furthermore,

$$\lim_{x\to 0^-} (x^2 - 5) = -5$$

Thus, by Limit Theorem 12(iv) we conclude that

$$\lim_{x\to 0^-} \frac{x^2-5}{2x^3-3x^2} = +\infty$$

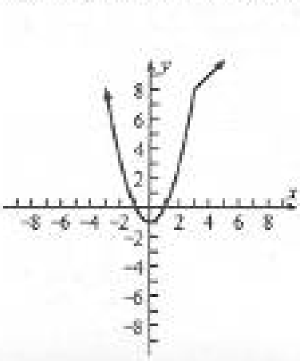
41. $\lim\limits_{x\to 2^+} ([\![x]\!] - 1) = 1$ and $\lim\limits_{x\to 2^+} ([\![x]\!] - x) = 2 - 2 = 0$ through negative values. Therefore $\lim\limits_{x\to 0} \dfrac{[\![x]\!]-1}{[\![x]\!]-x} = -\infty$

42. $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5} = \lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}\cdot\dfrac{\sqrt{x-1}+2}{\sqrt{x-1}+2} = \lim\limits_{x\to 5} \dfrac{(x-1)-4}{(x-5)(\sqrt{x-1}+2)} = \lim\limits_{x\to 5} \dfrac{1}{\sqrt{x-1}+2} = \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{4}$

In Exercises 43–48, sketch the graph and find the limit or state why it does not exist.

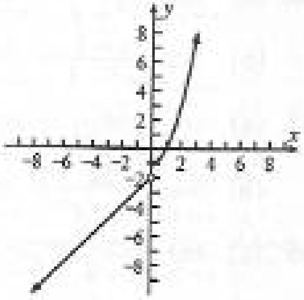
43. $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 3 \\ x + 5 & \text{if } 3 < x \end{cases}$

▷ $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2 - 1) = (3)^2 - 1 = 8;$
$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (x + 5) = 3 + 5 = 8.$
Hence, $\lim\limits_{x\to 3} f(x) = 8.$



44. $g(x) = \begin{cases} x-2 & \text{if } x \le 0 \\ x^2-1 & \text{if } x > 0 \end{cases}$

▸ $\lim_{x\to0^-} g(x) = \lim_{x\to0^-}(x-2) = -2$;
$\lim_{x\to0^+} g(x) = \lim_{x\to0^+} x^2-1 = -1$.
Because $\lim_{x\to0^-} g(x) \ne \lim_{x\to0^+} g(x)$,
then $\lim_{x\to0} g(x)$ does not exist.

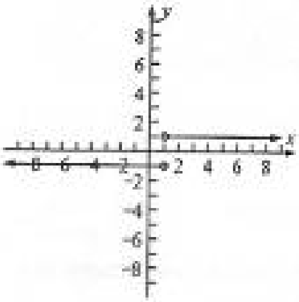


45. $h(t) = \dfrac{|t-1|}{t-1}$

▸ $\lim_{t\to1^-} h(t) = \lim_{t\to1^-} \dfrac{-(t-1)}{t-1} = \lim_{t\to1^-}(-1) = -1$;

$\lim_{t\to1^+} h(t) = \lim_{t\to1^+} \dfrac{t-1}{t-1} = \lim_{t\to1} 1 = 1$.

Hence, $\lim_{t\to1} h(t)$ does not exist.

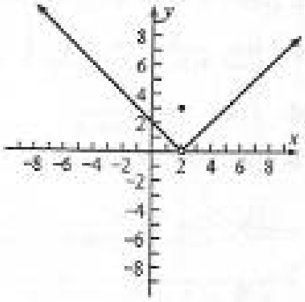


46. $f(r) = \begin{cases} |r-2| & \text{if } r \ne 2 \\ 3 & \text{if } r = 2 \end{cases}$

▸ $\lim_{r\to2^-} f(r) = \lim_{r\to2^-}(2-r) = 2-2 = 0$;

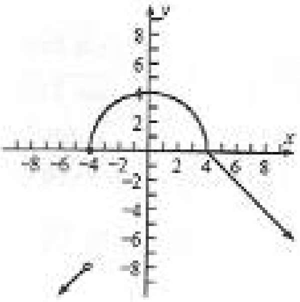
$\lim_{r\to2^+} f(r) = \lim_{r\to2^+}(r-2) = 2-2 = 0$.

Hence $\lim_{r\to2} f(r) = 0$



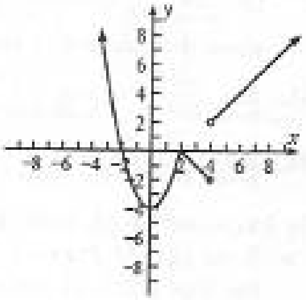
47. $g(x) = \begin{cases} x-4 & \text{if } x < -4 \\ \sqrt{16-x^2} & \text{if } -4 \le x \le 4 \\ 4-x & \text{if } x > 4 \end{cases}$

▸ $\lim_{x\to-4^-} g(x) = \lim_{x\to-4^-}(x-4) = -4-4 = -8$
$\lim_{x\to-4^+} g(x) = \lim_{x\to-4^+} \sqrt{16-x^2} = \sqrt{16-16} = 0$
Hence $\lim_{x\to-4} g(x)$ does not exist.
$\lim_{x\to4^-} g(x) = \lim_{x\to4^-} \sqrt{16-x^2} = \sqrt{16-16} = 0$
$\lim_{x\to4^+} g(x) = \lim_{x\to4^+}(4-x) = 4-4 = 0$
Hence $\lim_{x\to4} g(x) = 0$.



48. $h(x) = \begin{cases} x^2-4 & \text{if } x \le 2 \\ 2-x & \text{if } 2 < x \le 4 \\ x-2 & \text{if } x > 4 \end{cases}$

▸ (a) $\lim_{x\to2^-} h(x) = \lim_{x\to2^-}(x^2-4) = 4-4 = 0$
(b) $\lim_{x\to2^+} h(x) = \lim_{x\to2^+}(2-x) = 2-2 = 0$
(c) Because $\lim_{x\to2^-} h(x) = \lim_{x\to2^+} h(x) = 0$ then $\lim_{x\to2} h(x) = 0$.
(d) $\lim_{x\to4^-} h(x) = \lim_{x\to4^-}(2-x) = 2-4 = -2$
(e) $\lim_{x\to4^+} h(x) = \lim_{x\to4^+}(x-2) = 4-2 = 2$
(f) Because $\lim_{x\to4^-} h(x) \ne \lim_{x\to4^+} h(x)$, then $\lim_{x\to4} h(x)$ does not exist.



In Exercises 49–54, find the limit and support your answer graphically.

49. (a) $\lim_{x\to-4^-} \dfrac{2x}{16-x^2} = \lim_{x\to-4^-} \dfrac{2x}{(4-x)(4+x)} = \lim_{x\to-4^-} \dfrac{2x/(4-x)}{4+x} = \dfrac{-1}{0^-} = +\infty$

(b) $\lim_{x\to-4^+} \dfrac{2x}{16-x^2} = \lim_{x\to-4^+} \dfrac{2x}{(4-x)(4+x)} = \lim_{x\to-4^+} \dfrac{2x/(4-x)}{4+x} = \dfrac{-1}{0^+} = -\infty$

50. (a) $\lim\limits_{x\to -2^-}\frac{x-1}{x^2-4}=\lim\limits_{x\to -2^-}\frac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to -2^-}\frac{(x-1)/(x-2)}{x+2}=\frac{\frac{3}{4}}{0^-}=-\infty$

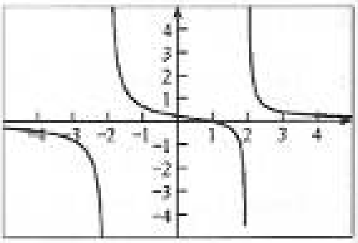
(b) $\lim\limits_{x\to -2^+}\frac{x-1}{x^2-4}=\lim\limits_{x\to -2^+}\frac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to -2^+}\frac{(x-1)/(x-2)}{x+2}=\frac{\frac{3}{4}}{0^+}=+\infty$

51. (a) $\lim\limits_{x\to 4^-}\frac{2x}{16-x^2}=\lim\limits_{x\to 4^-}\frac{2x}{(4-x)(4+x)}=\lim\limits_{x\to 4^-}\frac{2x/(4+x)}{4-x}=\frac{1}{0^+}=+\infty$

(b) $\lim\limits_{x\to 4^+}\frac{2x}{16-x^2}=\lim\limits_{x\to 4^+}\frac{2x}{(4-x)(4+x)}=\lim\limits_{x\to 4^+}\frac{2x/(4+x)}{4-x}=\frac{1}{0^-}=-\infty$

52. (a) $\lim\limits_{x\to 2^-}\frac{x-1}{x^2-4}=\lim\limits_{x\to 2^-}\frac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to 2^-}\frac{(x-1)/(x+2)}{x-2}=\frac{\frac{1}{4}}{0^-}=-\infty$

(b) $\lim\limits_{x\to 2^+}\frac{x-1}{x^2-4}=\lim\limits_{x\to 2^+}\frac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to 2^+}\frac{(x-1)/(x+2)}{x-2}=\frac{\frac{1}{4}}{0^+}=+\infty$



53. (a) $\lim\limits_{t\to 5}\frac{\sqrt{t-4}}{(t-5)^2}=\frac{1}{0^+}=+\infty$ (b) $\lim\limits_{t\to 5^+}\frac{4-t}{\sqrt{t-5}}=\frac{-1}{0^+}=-\infty$

54. (a) $\lim\limits_{x\to -2}\frac{\sqrt{3-x}}{(x+2)^2}=\frac{\sqrt{5}}{0^+}=+\infty$ (b) $\lim\limits_{x\to -2^+}\frac{x-3}{\sqrt{x+2}}=\frac{-5}{0^+}=-\infty$

In Exercises 55–62, (a) plot the graph, conjecture the limit as $x\to 0$ and (b) calculate it.

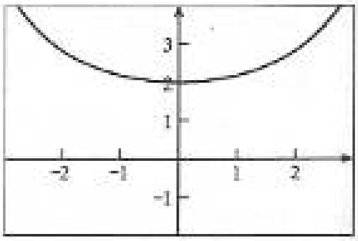
55. $\lim\limits_{x\to 0}\frac{x}{\sin 3x}=\lim\limits_{3x\to 0}\frac{1}{3}\cdot\frac{3x}{\sin 3x}=\frac{1}{3}$

56. $f(x)=\frac{x^2}{1-\cos x}$

▷ (a) See the figure. $f(x)$ appears to be approaching 2 as x approaches 0.

(b) $\lim\limits_{x\to 0}\frac{x^2}{1-\cos x}=\lim\limits_{x\to 0}\frac{x^2}{2\sin^2\frac{1}{2}x}=\lim\limits_{x/2\to 0}2\left(\frac{\frac{1}{2}x}{\sin\frac{1}{2}x}\right)^2=2\left(\lim\limits_{x/2\to 0}\frac{\frac{1}{2}x}{\sin\frac{1}{2}x}\right)^2$

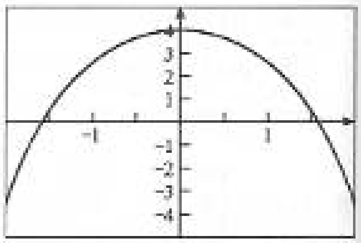
$=2\cdot 1=2$



57. $\lim\limits_{x\to 0}\frac{\sin 5x}{\sin 2x}=\lim\limits_{x\to 0}\frac{5}{2}\cdot\frac{\sin 5x/5x}{\sin 2x/2x}=\frac{5}{2}\cdot\frac{\lim\limits_{5x\to 0}(\sin 5x/5x)}{\lim\limits_{2x\to 0}(\sin 2x/2x)}=\frac{5}{2}\cdot\frac{1}{1}=\frac{5}{2}$

58. $\lim\limits_{x\to 0}\frac{1-\cos 3x}{\sin 3x}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{3}{2}x}{2\sin\frac{3}{2}x\cos\frac{3}{2}x}=\lim\limits_{x\to 0}\frac{\sin\frac{3}{2}x}{\cos\frac{3}{2}x}=\lim\limits_{x\to 0}\tan\frac{3}{2}x=\tan 0=0$

59. $\lim\limits_{x\to 0}\frac{1-\cos^2 x}{x}=\lim\limits_{x\to 0}\frac{\sin^2 x}{x}=\lim\limits_{x\to 0}\sin x\cdot\lim\limits_{x\to 0}\frac{\sin x}{x}=0\cdot 1=0$



60. $f(x)=\frac{4x}{\tan x}$

▷ (a) See the figure. $f(x)$ appears to be approaching 4 as x approaches 0.

(b) $\lim\limits_{x\to 0}\frac{4x}{\tan x}=\lim\limits_{x\to 0}\frac{4x}{\sin x/\cos x}=\lim\limits_{x\to 0}4\cos x\cdot\lim\limits_{x\to 0}\frac{x}{\sin x}=4\cos 0\cdot 1=4$ since the cosine function is continuous.

61. $\lim\limits_{x\to 0}\frac{\csc 3x}{\cot x}=\lim\limits_{x\to 0}\frac{1/\sin 3x}{\cos x/\sin x}=\lim\limits_{x\to 0}\frac{1}{3\cos x}\cdot\lim\limits_{3x\to 0}\frac{3x}{\sin 3x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{x}=\frac{1}{3}\cdot 1\cdot 1=\frac{1}{3}$

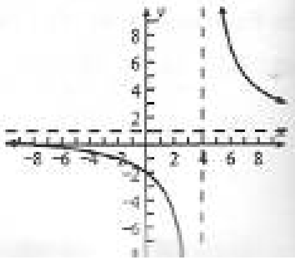
62. $\lim\limits_{x\to 0}\frac{2x^2-3x}{2\sin x}=\lim\limits_{x\to 0}\frac{x(2x-3)}{2\sin x}=\lim\limits_{x\to 0}\frac{2x-3}{2}\cdot\lim\limits_{x\to 0}\frac{x}{\sin x}=-\frac{3}{2}\cdot 1=-\frac{3}{2}$

In Exercises 63–68, find the vertical asymptotes and use them to sketch the graph of the function.

▷ Note that if $f(x)-L$ is arbitrarily close to 0 for all large values of x, then the graph of $y=f(x)$ approaches the line $y=L$ at the extreme left and right, *a horizontal asymptote* (see Section 3.7).

63. $f(x)=\frac{x+8}{x-4}$

▷ Because $\lim\limits_{x\to 4^-}f(x)=-\infty$ or $\lim\limits_{x\to 4^+}f(x)=+\infty$, $x=4$ is a vertical asymptote. Because $f(x)-1=\frac{x+8}{x-4}-1=\frac{12}{x-4}$ is small when x is large, $y=1$ is a horizontal asymptote.

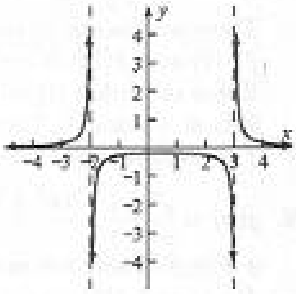


64. $f(x)=\frac{-2}{x^2-x-6}$

▸ To find the vertical asymptote, we factor the denominator.

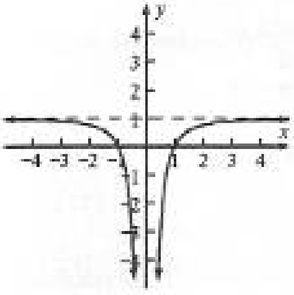
$$f(x)=\frac{-2}{(x+2)(x-3)}$$

Because $(x+2)(x-3)=0$ if $x=3$ or $x=-2$, we find the following limits. If $x>3$, then $(x+2)(x-3)>0$. Thus, by Limit Theorem 12(iii), we have $\lim_{x\to3^+} f(x)=-\infty$, and the line $x=3$ is a vertical asymptote of the graph of f. If $-2<x<3$, then $(x+2)(x-3)<0$, so by Limit Theorem 12(iv) $\lim_{x\to3^-} f(x)=+\infty$. Therefore, the curve approaches the asymptote $x=3$ as shown in the figure. Also, by Limit Theorem 12(iv) $\lim_{x\to-2^+} f(x)=+\infty$, so the line $x=-2$ is a vertical asymptote for the graph of f. Because $f(x)$ is small when x is large, the line $y=0$ is a horizontal asymptote.



65. $g(x)=1-\frac{1}{x^2}$

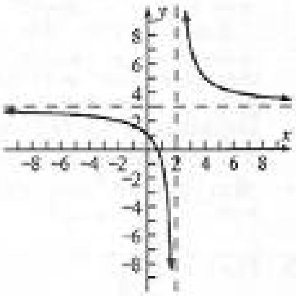
▸ Because $\lim_{x\to0^-} g(x)=-\infty$ or $\lim_{x\to0^+} g(x)=-\infty$, $x=0$ is a vertical asymptote. Because $g(x)$ is close to 1 when x is large, $y=1$ is a horizontal asymptote.



66. $f(x)=\frac{3x-2}{x-2}$

▸ Because $\lim_{x\to2^-} f(x)=-\infty$ or $\lim_{x\to2^+} f(x)=+\infty$, $x=2$ is a vertical asymptote.

Because $f(x)-3=\frac{3x-2}{x-2}-3=\frac{4}{x-2}$ is small when x is large, $y=3$ is a horizontal asymptote.

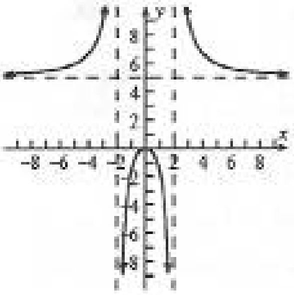


67. $f(x)=\frac{5x^2}{x^2-4}$

▸ Because $\lim_{x\to2^-} f(x)=-\infty$ or $\lim_{x\to2^+} f(x)=+\infty$, $x=2$ is a vertical asymptote.

Because $\lim_{x\to-2^-} f(x)=+\infty$ or $\lim_{x\to-2^+} f(x)=-\infty$, $x=-2$ is a vertical asymptote.

Because $f(x)-5=\frac{5x^2}{x^2-4}-5=\frac{20}{x^2-4}$ is small when x is large, $y=5$ is a horizontal asymptote.

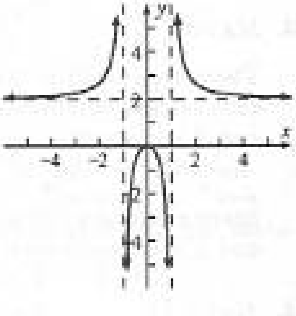


68. $h(x)=\frac{2x^2}{x^2-1}$

▸ Because $\lim_{x\to1^-} f(x)=-\infty$ or $\lim_{x\to1^+} f(x)=+\infty$, $x=1$ is a vertical asymptote.

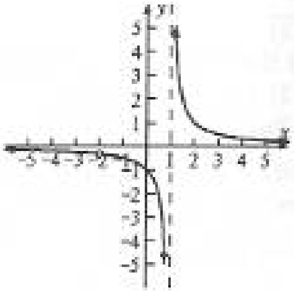
Because $\lim_{x\to-1^-} f(x)=+\infty$ or $\lim_{x\to-1^+} f(x)=-\infty$, $x=-1$ is a vertical asymptote.

Because $f(x)-2=\frac{2x^2}{x^2-1}-2=\frac{2}{x^2-1}$ is small when x is large, $y=2$ is a horizontal asymptote.

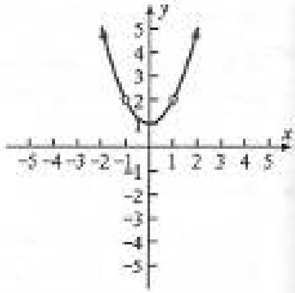


In Exercises 69–74, sketch the graph of the function; then by noting breaks in the graph, determine the values of at which the function is discontinuous, and show why Definition 1.8.1 is not satisfied at each discontinuity.

69. $f(x) = \dfrac{x+2}{x^2+x-2} = \dfrac{x+2}{(x+2)(x-1)} = \dfrac{1}{x-1}$ if $x \neq -2$
There are breaks in the graph at -2 and 1.
$f(-2)$ and $f(1)$ do not exist.
Hence condition (i) of Definition 2.6.1
fails at -2 and 1. Therefore,
f is discontinuous at -2 and 1.

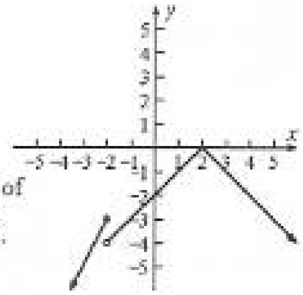


70. $g(x) = \dfrac{x^4-1}{x^2-1} = \dfrac{(x^2+1)(x+1)(x-1)}{(x+1)(x-1)} = x^2+1$ if $x \neq \pm 1$. There are breaks in the graph at -1 and 1.
$f(-1)$ and $f(1)$ do not exist.
Hence condition (i) of Definition 1.8.1
fails at -1 and 1. Therefore,
f is discontinuous at -1 and 1.



71. $g(x) = \begin{cases} 2x+1 & \text{if } x \leq -2 \\ x-2 & \text{if } -2 < x \leq 2 \\ 2-x & \text{if } 2 < x \end{cases}$

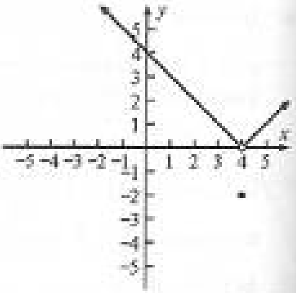
▹ There is a break in the graph at -2.
$\lim\limits_{x\to -2^-} g(x) = \lim\limits_{x\to -2^-}(2x+1) = -3;$
$\lim\limits_{x\to -2^+} g(x) = \lim\limits_{x\to -2^+}(x-2) = -4;$
therefore $\lim\limits_{x\to -2} g(x)$ does not exist. Thus condition (ii) of Definition 1.8.1 fails at -2, so g is discontinuous at -2.
g is continuous at 2 because
$\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^+} g(x) = 0 = g(2).$



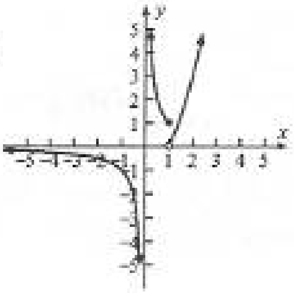
72. $F(x) = \begin{cases} |4-x| & \text{if } x \neq 4 \\ -2 & \text{if } x = 4 \end{cases}$

▹ There is a "hole" in the graph at the point $(4, 0)$ because
$F(x) \neq |4-x|$ if $x = 4$, and thus F is discontinuous at 4.
We show how Definition 1.8.1 is not satisfied.
Because the absolute-value function is continuous,

$\lim\limits_{x\to 4} F(x) = \lim\limits_{x\to 4} |4-x| = |0| = 0$ and $F(4) = -2$.
Thus, $\lim\limits_{x\to 4} F(x) \neq F(4)$ and so condition (iii) of Definition 1.8.1 is not satisfied.

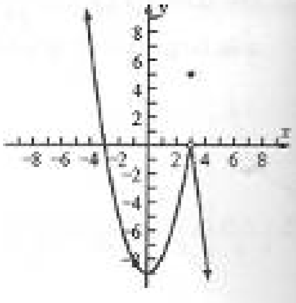


73. $h(x) = \begin{cases} 1/x & \text{if } x \leq 1 \\ x^2-1 & \text{if } 1 < x \end{cases}$
There are breaks in the graph at 0 and 1. $f(0)$ does not exist. Hence condition (i) of Definition 2.6.1 fails at 0 and f is discontinuous at 0.
$\lim\limits_{x\to 1^+} h(x) = \lim\limits_{x\to 1^+} x^2 - 1 = 0;\ h(1) = \frac{1}{1} = 1;$
hence condition (iii) of Definition 1.8.1 fails at 1 and h is discontinuous at 1.



74. $f(x) = \begin{cases} x^2-9 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ 9-x^2 & \text{if } x > 3 \end{cases}$

▹ There is a break in the graph at 3.
$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(x^2-9) = 0$ but $f(3) = 5$
Thus condition (ii) or condition (iii) of Definition 1.8.1 is not satisfied.



In Exercises 75–78, prove that f is discontinuous at a. If the discontinuity is removable, redefine $f(a)$ to remove it.

75. $f(x) = \dfrac{x^2+2x-8}{x^2+3x-4} = \dfrac{(x+4)(x-2)}{(x+4)(x-1)} = \dfrac{x-2}{x-1}$ if $x \neq -4$.

$f(-4)$ does not exist. Therefore f is discontinuous at -4.

$\lim\limits_{x\to-4} f(x) = \lim\limits_{x\to-4} \dfrac{x-2}{x-1} = \dfrac{-6}{-5} = \dfrac{6}{5}$. The discontinuity is removable by redefining $f(-4) = \frac{6}{5}$.

76. $f(x) = \begin{cases} 4-x^2 & \text{if } x < 1 \\ 2x+3 & \text{if } 1 \leq x \end{cases}$; $a = 1$

▷ $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (4-x^2) = 3$ $\qquad$ $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (2x+3) = 5$

Because $\lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$, then $\lim\limits_{x\to1} f(x)$ does not exist. Thus, there is an essential discontinuity at 1.

77. $f(x) = \begin{cases} 1/(x-2) & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

▷ $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} \dfrac{1}{x-2} = \dfrac{1}{0^-} = -\infty$. Thus, there is an essential discontinuity at 2.

78. $f(x) = \dfrac{|2x-6|}{2x-6}$

$f(3)$ does not exist. Therefore, f is discontinuous at 3.

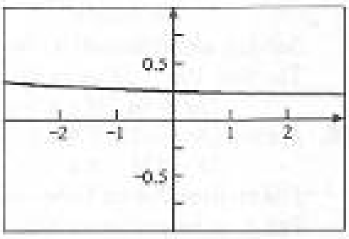
$\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^-} \dfrac{-(2x-6)}{2x-6} = \lim\limits_{x\to3^-} (-1) = -1$; $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \dfrac{2x-6}{2x-6} = \lim\limits_{x\to3^+} 1 = 1$

Because $\lim\limits_{x\to3^-} f(x) \neq \lim\limits_{x\to3^+} f(x)$, then $\lim\limits_{x\to3} f(x)$ does not exist. Therefore, the discontinuity is essential.

In Exercises 79–82, f is discontinuous at a. (a) Plot the graph of f and look for a break at $x = a$. Does the discontinuity appear to be removable? If so, how should f be redefined to remove it? (b) Confirm analytically.

79. $f(x) = \dfrac{|4-x|-3}{x-1}$; $a = 1$ $\qquad$ ▷ $\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} \dfrac{(4-x)-3}{x-1} = \lim\limits_{x\to1} \dfrac{1-x}{x-1} = \lim\limits_{x\to1} (-1) = -1$. Define $f(1) = -1$

80. $f(x) = \dfrac{2-\sqrt{x+4}}{x}$; $a = 0$ $\qquad$ ▷ We rationalize the numerator.



$f(x) = \dfrac{2-\sqrt{x+4}}{x} \cdot \dfrac{2+\sqrt{x+4}}{2+\sqrt{x+4}} = \dfrac{4-(x+4)}{x(2+\sqrt{x+4})} = \dfrac{-x}{x(2+\sqrt{x+4})}$

$= -\dfrac{1}{2+\sqrt{x+4}}$ if $x \neq 0$. To make f continuous, we should define

$f(0) = \dfrac{1}{2+\sqrt{0+4}} = \dfrac{1}{4}$.

81. $f(x) = \dfrac{x}{\sqrt{x+9}-3}$; $a = 0$ $\qquad$ ▷ $f(x) = \dfrac{x}{\sqrt{x+9}-3} \cdot \dfrac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$

$= \dfrac{x(\sqrt{x+9}+3)}{(x+9)-9} = \dfrac{x(\sqrt{x+9}+3)}{x} = \sqrt{x+9}+3$ if $x \neq 0$. Define $f(0) = \sqrt{9}+3 = 6$.

82. $f(x) = \dfrac{x-1}{\sqrt[3]{x}-1}$; $a = 1$

▷ $f(x) = \dfrac{(x^{1/3})^3-1}{x^{1/3}-1} = \dfrac{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)}{x^{1/3}-1} = x^{2/3}+x^{1/3}+1$ if $x \neq 1$. Define $f(1) = 1+1+1 = 3$.

In Exercises 83 and 84, (a) define $f \circ g$ and (b) find the numbers at which $f \circ g$ is continuous and state the reason.

83. (a) $f(x) = \sqrt{x}$; $g(x) = 25 - x^2$ $\qquad$ ▷ $(f \circ g)(x) = \sqrt{25-x^2}$

By Theorem 1.8.5(ii), $f \circ g$ is continuous for $g(x) > 0$.

By Theorem 1.8.3, g is continuous for all x and $g(x) > 0$ for $-5 < x < 5$.

Hence by Theorem 1.9.2, $f \circ g$ is continuous for all x in $(-5, 5)$.

(b) $f(x)=\dfrac{\sqrt{x^2-4}}{\sqrt{3-x}}$; $g(x)=|x|$ ▷ $(f\circ g)(x)=\dfrac{\sqrt{|x|^2-4}}{\sqrt{3-|x|}}=\dfrac{\sqrt{x^2-4}}{\sqrt{3-|x|}}$

By Theorems 1.8.2.4 and 1.8.5(ii), $f\circ g$ is continuous for $g(x)\in[(-\infty,-2)\cup(2,+\infty)]\cap(-\infty,3$ $=(-\infty,-2)\cup(2,3)=\text{I}$. The absolute-value function g is continuous for all x and $g(x)\in\text{I}$ fo $x\in(-3,-2)\cup(2,3)=\text{J}$. Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in J.

(c) $f(x)=\operatorname{sgn} x$; $g(x)=x^2-1$ ▷ $(f\circ g)(x)=\operatorname{sgn}(x^2-1)$

$f\circ g$ is continuous for $g(x)\neq 0$ and $g(x)\neq 0$ in $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)=\text{I}$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

84. (a) $f(x)=\sqrt{x}$; $g(x)=x^2-25$ ▷ $(f\circ g)(x)=\sqrt{x^2-25}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $(-\infty,-5)\cup(5,+\infty)=\text{I}$
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

(b) $f(x)=\sqrt{x+1}$; $g(x)=\dfrac{1}{x-3}$ ▷ $(f\circ g)(x)=\sqrt{\dfrac{1}{x-3}+1}=\sqrt{\dfrac{x-2}{x-3}}$

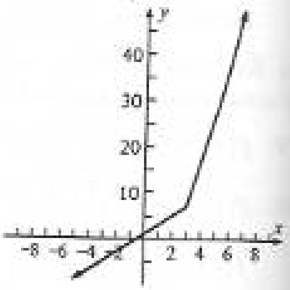
By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)+1>0$
By Theorem 1.8.4, g is continuous for all $x\neq 3$ and $g(x)+1>0$ when $\dfrac{x-2}{x-3}>0$; $x\in(-\infty,2)\cup(3,+\infty)$

(c) $f(x)=\operatorname{sgn} x$; $g(x)=x^2-x$ ▷ $(f\circ g)(x)=\operatorname{sgn}(x^2-x)$

$f\circ g$ is continuous for $g(x)\neq 0$. $g(x)=x(x-1)$ so $g(x)\neq 0$ in $(-\infty,0)\cup(0,1)\cup(1,+\infty)=\text{I}$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

In Exercises 85 and 86, find a and b that make f continuous at every number. Then sketch the graph of f.

85. $f(x)=\begin{cases}2x+1 & \text{if } x\le 3\\ ax+b & \text{if } 3<x<5\\ x^2+2 & \text{if } 5\le x\end{cases}$ ▷ $f(x)=\begin{cases}2x+1 & \text{if } x\le 3\\ 10x-23 & \text{if } 3<x<5\\ x^2+2 & \text{if } 5\le x\end{cases}$



f is continuous on $(-\infty,3)$, $(3,5)$ and $(5,+\infty)$.

For f to be continuous at 3,

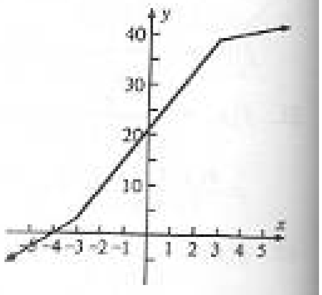
$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)$$
$$\lim_{x\to 3^-}(2x+1)=\lim_{x\to 3^+}(\text{a}x+\text{b})$$
$$7=3\text{a}+\text{b}$$

For f to be continuous at 5,

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)$$
$$\lim_{x\to 5^-}(\text{a}x+\text{b})=\lim_{x\to 5^+}(x^2+2)$$
$$5\text{a}+\text{b}=27$$

Solving simultaneously, we get $\text{a}=10$ and $\text{b}=-23$.
The line $10x-23$ joins the parabola x^2+2 smoothly at $(5,27)$.

86. $f(x)=\begin{cases}3x+6a & \text{if } x<-3\\ 3ax-7b & \text{if } -3\le x\le 3\\ x-12b & \text{if } x>3\end{cases}$ ▷ $f(x)=\begin{cases}3x+12 & \text{if } x<-3\\ 6x+21 & \text{if } -3\le x\le 3\\ x+36 & \text{if } 6\end{cases}$



f is continuous on $(-\infty,-3)$, $(-3,3)$ and $(3,+\infty)$.

For f to be continuous at -3,

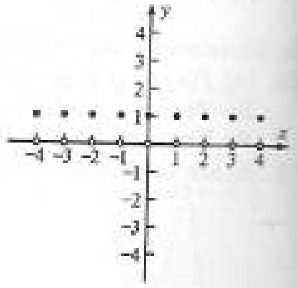
$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^+} f(x)$$
$$\lim_{x\to -3^-}(3x+6a)=\lim_{x\to -3^+}(3ax-7b)$$
$$-9+6a=-9a-7b$$
$$15a+7b=9$$

For f to be continuous at 3,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)$$
$$\lim_{x\to 3^-}(3ax-7b)=\lim_{x\to 3^+}(x-12b)$$
$$9a-7b=3-12b$$
$$9a+5b=3$$

Solving simultaneously, we get $\text{a}=2$ and $\text{b}=-3$.

87. $f(x)=\begin{cases}1 & \text{if } x \text{ is an integer}\\ 0 & \text{if } x \text{ is not an integer}\end{cases}$



(b) If a is an integer, then for all x in the open intervals $(\text{a}-1,\text{a})$, $(\text{a},\text{a}+1)$, $f(x)=0$. Thus $\lim\limits_{x\to a^-} f(x)=0$ and $\lim\limits_{x\to a^+} f(x)=0$.

Hence, $\lim\limits_{x\to a} f(x)=0$.

If a is not an integer, a is in the open interval $(\text{k},\text{k}+1)$ for some integer k.
Then for all x in the open intervals (k,a) and $(\text{a},\text{k}+1)$, $f(x)=0$.
Thus $\lim\limits_{x\to a^-} f(x)=0$ and $\lim\limits_{x\to a^+} f(x)=0$. Hence, $\lim\limits_{x\to a} f(x)=0$.
Therefore, $\lim\limits_{x\to a} f(x)$ exists for all values of a.

(c) f is discontinuous at every integer k because $\lim\limits_{x\to k} f(x)=0$, but $f(\text{k})=1$.

However for any number a that is not an integer, $\lim_{x\to a} f(x)=0$ and $f(a)=0$.

Therefore f is continuous at all non-integers.

88. Give an example of a function for which $\lim_{x\to 0}|f(x)|$ exists but $\lim_{x\to 0} f(x)$ does not exist.

▹ Any f for which $\lim_{x\to 0^-} f(x) = -\lim_{x\to 0^+} f(x)$ will do, for then $\lim_{x\to 0^-}|f(x)| = \lim_{x\to 0^+}|f(x)|$.

A simple example is $f(x)=\operatorname{sgn} x$: $\lim_{x\to 0^-}\operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1$ and $\lim_{x\to 0^+}\operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

In Exercises 89–92, find the largest interval or union of intervals on which the function is continuous.

89. (a) $f(x)=\sqrt{25-x^2}$ ▹ The domain of f is $\{x \mid 25-x^2 \ge 0\} = [-5,5]$.

f is continuous on $(-5,5)$. Because $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+}\sqrt{25-x^2} = 0 = f(-5)$ and $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}\sqrt{25-x^2} = 0 = f(5)$, f is also continuous on $[-5,5]$.

(b) $f(x)=\sqrt{x^2-25}$ ▹ The domain of f is $\{x \mid x^2-25 \ge 0\} = (-\infty,-5]\cup[5,+\infty) = \mathrm{I}$.

f is continuous on $(-\infty,-5)\cup(-5,+\infty)$. Because $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-}\sqrt{x^2-25} = 0 = f(-5)$ and $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\sqrt{x^2-25} = 0 = f(5)$, f is also continuous on I.

90. (a) $f(x)=\dfrac{|x|+1}{|x|-1}$ ▹ The domain of f is $\{x \mid |x|-1 \ne 0\} = (-\infty,-1)\cup(-1,1)\cup(1,+\infty) = \mathrm{I}$.

f is continuous on I. Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\dfrac{-x+1}{-x-1} = \dfrac{2}{0^+} = +\infty$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\dfrac{x+1}{x-1} = \dfrac{2}{0^+} = +\infty$, f is continuous only on I.

(b) $g(x)=\dfrac{\sqrt{9-x^2}}{x-2}$ ▹ The domain of g is $\{x \mid 9-x^2 \ge 0\} - \{2\} = [-3,2)\cup(2,3] = \mathrm{I}$.

g is continuous on $(-3,2)\cup(2,3)$. Because $\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+}\dfrac{\sqrt{9-x^2}}{x-2} = \dfrac{0}{-5} = 0 = g(-3)$,

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\dfrac{\sqrt{9-x^2}}{x-2} = \dfrac{\sqrt{5}}{0^+} = +\infty$, and $\lim_{x\to 3^-} g(x) = \dfrac{\sqrt{9-x^2}}{x-2} = \dfrac{0}{1} = 0 = g(3)$, g is continuous on I.

91. (a) $f(x)=\dfrac{|x-2|}{x-2}$ ▹ The domain of f is $\{x \mid x-2 \ne 0\} = (-\infty,2)\cup(2,+\infty) = \mathrm{I}$.

Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\dfrac{-(x-2)}{x-2} = \lim_{x\to 2^-}(-1) = -1$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\dfrac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$, f is continuous only on I.

(b) $g(x)=\dfrac{x}{x^2-4}$ ▹ The domain of g is $\{x \mid x^2-4 \ne 0\} = (-\infty,-2)\cup(-2,2)\cup(2,+\infty) = \mathrm{I}$.

Because $\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+}\dfrac{x}{x^2-4} = \dfrac{-2}{0^-} = +\infty$ and $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\dfrac{x}{x^2-4} = \dfrac{2}{0^+} = +\infty$, g is continuous only on I.

92. $\mathrm{F}(x) = \begin{cases} x+4 & \text{if } x < -4 \\ \sqrt{16-x^2} & \text{if } -4 \le x \le 4 \\ 2-x & \text{if } x > 4 \end{cases}$ ▹ The domain of F is all x. By Theorems 1.8.3 and 1.8.5(ii), F is continuous on $(-\infty,-4)\cup(-4,4)\cup(4,+\infty)$. Because

$\lim_{x\to -4^-}\mathrm{F}(x) = \lim_{x\to -4^-}(x+4) = 0$ and $\lim_{x\to -4^+}\mathrm{F}(x) = \lim_{x\to -4^+}\sqrt{16-x^2} = 0$

then F is continuous at -4. Because

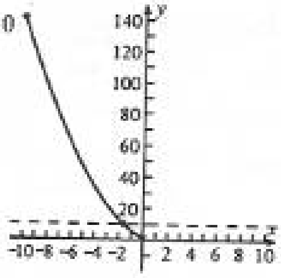
$\lim_{x\to 4^-}\mathrm{F}(x) = \lim_{x\to 4^-}\sqrt{16-x^2} = 0 = \mathrm{F}(4)$ and $\lim_{x\to 4^+}\mathrm{F}(x) = \lim_{x\to 4^+}(2-x) = -2$

then F is continuous only from the left at 4. Therefore F is continuous on $(-\infty,4]\cup(4,+\infty)$.

In Exercises 93–96, does the intermediate-value theorem hold for the function f, interval $[a, b]$ and constant k? so, solve $f(c) = k$ graphically and analytically to 4 decimals and sketch the graph showing the point (c, k).

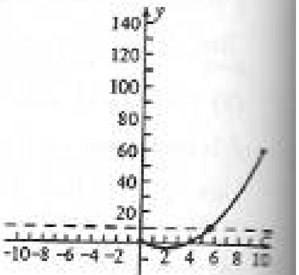
93. $f(x) = x^2 - 4x + 1$; $[a, b] = [-10, 0]$; $k = 10$

▸ The interval from $f(-10) = 141$ to $f(0) = 1$ contains 10 and f is continuous on $[-10, 0]$. Therefore, the intermediate-value theorem holds and there exists a number c between -10 and 0 such that $f(c) = 10$: $c^2 - 4c + 1 = 10$; $c^2 - 4c - 9 = 0$; $c = 2 \pm \sqrt{13}$ and $2 - \sqrt{13} \approx -1.6056$ is in $(-10, 0)$.



94. $f(x) = x^2 - 4x + 1$; $[a, b] = [0, 10]$; $k = 10$

▸ The interval from $f(0) = 1$ to $f(10) = 61$ contains 10 and f is continuous on $[0, 10]$. Therefore, the intermediate-value theorem holds and there exists a number c between 0 and 10 such that $f(c) = 10$: $c^2 - 4c + 1 = 10$; $c^2 - 4c - 9 = 0$; $c = \frac{1}{2}(4 \pm \sqrt{52})$ and $\frac{1}{2}(4 + \sqrt{52}) \approx 5.6056$ is in $(0, 10)$.

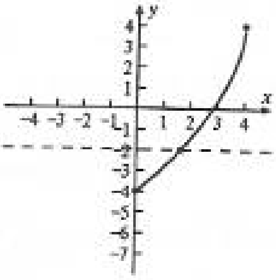


95. $f(x) = x - \sqrt{16 - x^2}$; $[a, b] = [0, 4]$; $k = -2$

▸ The interval from $f(0) = -4$ to $f(4) = 4$ contains -2 and f is continuous on $[0, 4]$. Therefore, the intermediate-value theorem holds and there exists a number c between 0 and 4 such that $f(c) = -2$:

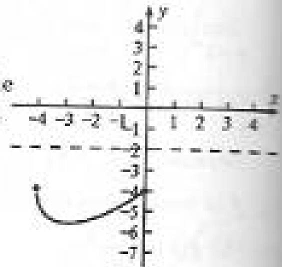
$c - \sqrt{16 - c^2} = -2$; $c + 2 = \sqrt{16 - c^2}$;

$c^2 + 4c + 4 = 16 - c^2$; $2c^2 + 4c - 12 = 0$; $c = -1 \pm \sqrt{7}$ and $-1 + \sqrt{7} \approx 1.6458$ is in $(0, 4)$.



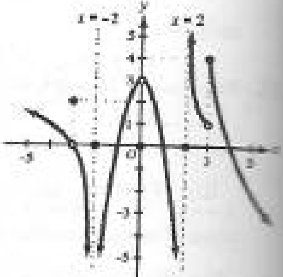
96. $f(x) = x - \sqrt{16 - x^2}$; $[a, b] = [-4, 0]$; $k = -2$

▸ The interval from $f(-4) = -4$ to $f(0) = 4$ does not contain -2. Thus the hypothesis of the intermediate-value theorem is not satisfied and the theorem does not hold. The figure shows the graph of the function on $[-4, 0]$ and the line $y = -2$. Because the line does not intersect the curve, there is no number c that satisfies the conclusion of the intermediate-value theorem. Try to solve $f(c) = -2$: $c - \sqrt{16 - c^2} = -2$; $c + 2 = \sqrt{16 - c^2}$; $c^2 + 4c + 4 = 16 - c^2$; $2c^2 + 4c - 12 = 0$; $c = -1 \pm \sqrt{7}$ and $-1 - \sqrt{7} \approx -3.6458$ is in $(-4, 0)$. However, $c + 2$ cannot be negative, and so the solution is extraneous.

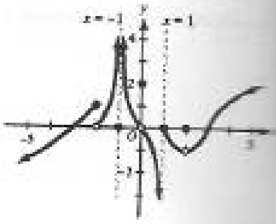


In Exercises 97 and 98, answer the questions from the graph of f in the figure.

97. (a) $\lim_{x \to -3} f(x) = 0$ (b) $\lim_{x \to -2} f(x) = -\infty$ (c) $\lim_{x \to 0} f(x) = 3$ (d) $\lim_{x \to 2^-} f(x) = -\infty$
(e) $\lim_{x \to 2^+} f(x) = +\infty$ (f) $\lim_{x \to 3^-} f(x) = 1$ (g) $\lim_{x \to 3^+} f(x) = 4$
(h) f is discontinuous at -3 (removable, define $f(-3) = 0$); -2 (essential); 0 (removable, define $f(0) = 3$); 2 (essential); 3 (essential)



98. (a) $\lim_{x \to -2^-} f(x) = 1$ (b) $\lim_{x \to -2^+} f(x) = 0$ (c) $\lim_{x \to -1} f(x) = +\infty$ (d) $\lim_{x \to 0} f(x) = 0$
(e) $\lim_{x \to 1^-} f(x) = -\infty$ (f) $\lim_{x \to 1^+} f(x) = 0$ (g) $\lim_{x \to 2} f(x) = -1$
(h) f is discontinuous at -2 (essential); -1 (essential); 0 (removable, define $f(0) = 0$); 1 (essential); 2 (removable, define $f(2) = -1$)

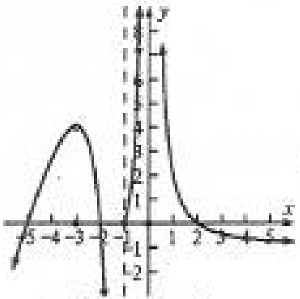


In Exercises 99–102, sketch the graph of a function satisfying the given conditions.

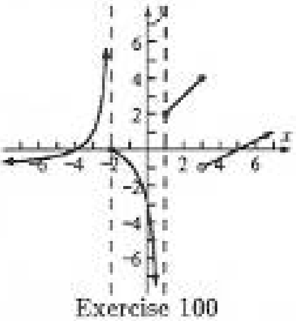
99. $-5, -3, -1, 0$ and 2 are the only zeros of f; $\lim_{x\to-3} f(x) = 4$; $\lim_{x\to-1^-} f(x) = +\infty$; $\lim_{x\to-1^+} f(x) = 0$; $\lim_{x\to0} f(x) = +\infty$; f is continuous on $(-\infty,-3)$, $(-3,-1)$, $(-1,0)$, $(0,+\infty)$

100. f is continuous on $(-\infty, -2)$, $[-2,1)$, $[1,3]$, and $(3,+\infty)$; $\lim_{x\to-4} f(x) = 0$; $\lim_{x\to-2^-} f(x) = +\infty$; $\lim_{x\to-2^+} f(x) = 0$; $\lim_{x\to0} f(x) = -3$; $\lim_{x\to1^-} f(x) = -\infty$; $\lim_{x\to1^+} f(x) = 2$; $\lim_{x\to3^-} f(x) = 4$; $\lim_{x\to3^+} f(x) = -1$; $\lim_{x\to5} f(x) = 0$.

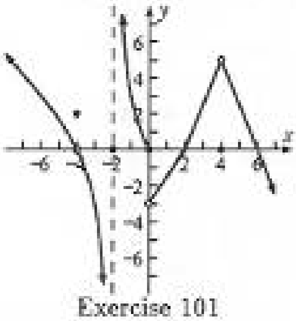
▹ Because f is continuous at -4 and $\lim_{x\to-4} f(x) = 0$ then $f(-4) = 0$, and the graph contains the point $(-4,0)$. Because $\lim_{x\to-2^-} f(x) = +\infty$, the line $x = -2$ is a vertical asymptote and the curve approaches the asymptote from the left in the upward direction. Because f is continuous from the right at -2 and $\lim_{x\to-2^+} f(x) = 0$, then $f(-2) = 0$, and the graph contains the point $(-2,0)$. The graph contains the point $(0,-3)$, because f is continuous at 0 and $\lim_{x\to0} f(x) = -3$. Because $\lim_{x\to1^-} f(x) = -\infty$, the line $x = 1$ is a vertical asymptote, and the curve approaches the asymptote from the left in the downward direction. Because f is continuous from the right at 1, and $\lim_{x\to1^+} f(x) = 2$, the graph contains the point $(1,2)$. Because f is continuous from the left at 3, and $\lim_{x\to3^-} f(x) = 4$, then the graph contains the point $(3,4)$. We are given that $\lim_{x\to3^+} f(x) = -1$. Because $\lim_{x\to3^-} f(x) = 4$, then f is discontinuous at 3, and there is a break in the graph at the point where $x = 3$. Because $\lim_{x\to3^+} f(x) = -1$, the point $(3,-1)$ is a limit point of the graph, but the point is not part of the graph; this is indicated by the open circle at the point $(3,-1)$. Finally, because $\lim_{x\to5} f(x) = 0$, and f is continuous at 5, the graph contains the point $(5,0)$.



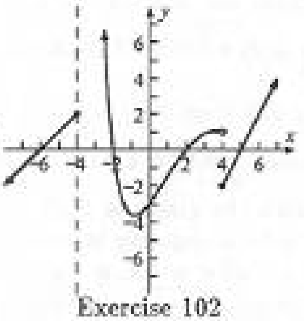
Exercise 99



Exercise 100



Exercise 101



Exercise 102

101. $f(-4) = 2$; $-2, 0, 2, 4$, and 5 are the only zeros of f; $\lim_{x\to-4} f(x) = 0$; $\lim_{x\to-2^-} f(x) = -\infty$; $\lim_{x\to-2^+} f(x) = +\infty$; $\lim_{x\to0^-} f(x) = 0$; $\lim_{x\to0^+} f(x) = -3$; $\lim_{x\to4} f(x) = 5$; f is continuous at all numbers except -4, -2, 0, and 4.

102. f is continuous on $(-\infty,-4]$, $(-4,4)$, and $[4,+\infty)$; $\lim_{x\to-6} f(x) = 0$; $\lim_{x\to-4^-} f(x) = 2$; $\lim_{x\to-4^+} f(x) = +\infty$; $\lim_{x\to-2} f(x) = 0$; $\lim_{x\to0} f(x) = -3$; $\lim_{x\to2} f(x) = 0$; $\lim_{x\to4^-} f(x) = 1$; $\lim_{x\to4^+} f(x) = -2$; $\lim_{x\to5} f(x) = 0$

103. x in. squares are cut from the corners of a 14 in × 18 in sheet and the sides turned up. (a) Find the volume $V(x)$ in^3. (b) Find dom(V). (c) Prove V is continuous on its domain. (d) Maximize the volume graphically.

▹ $V = \ell wh = (14-2x)(18-2x)x$, $0 \le x \le 7$. (c) V is a polynomial. (d) When $x \approx 2.6049$, $V_{max} = 292.86$ in^3.

104. An open box having a square base is to have a volume of 4000 in^3. (a) Find a mathematical model expressing the total surface area of the box as a function of the length of a side of the square base. (b) What is the domain of your function? (c) Prove that the function is continuous on its domain. (d) On your graphics calculator determine, to the nearest inch, the dimensions of the box that can be constructed with the least amount of material.

▹ (a) The base has side x in., total surface S in^2. Volume $= \ell wh = x^2h = 4000$, $h = \dfrac{4000}{x^2}$.

S = area of bottom + 4 · area of a side $= x^2 + 4xh = x^2 + 4x \cdot \dfrac{4000}{x^2} = x^2 + \dfrac{16{,}000}{x}$ (b) Dom S is $(0,+\infty)$

(c) A rational function is continuous on its domain. (d) Because $S = x^2 + \dfrac{8{,}000}{x} + \dfrac{8{,}000}{x}$ has a constant product

$8{,}000^2$, the sum S is least when the terms are equal: $x^2 = \frac{8{,}000}{x}$, $x^3 = 8{,}000$, $x = 20$. The dimensions a
20 in × 20 in × 10 in.

105. A sign with margins of 4 m at the top and bottom and 2 m at the sides is to contain 50 m^2 of print. (a) Fin the total area of the sign, $A(x)$ m^2, when the width of the printed region is x in. (b) Find the domain D of A (c) Prove that A is continuous on its domain (d) Determine to the nearest meter the size of the smallest sign.

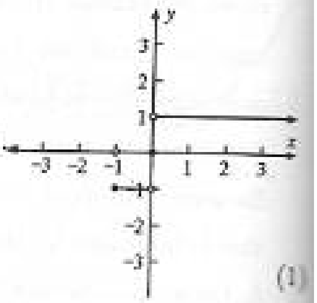
▹ (a) The length of the printed region is $\frac{50}{x}$ m. $A(x) = (x+4)(\frac{50}{x}+8) = 82 + (8x + \frac{200}{x})$ (product = 1600)
(b) D: $x > 0$ (c) A is a rational function (d) $A_{min} = 54$ when $8x = \frac{200}{x}$, $x^2 = 25$, $x = 5$. $x + 4 = 5 + 4 = 9$,
$\frac{50}{x} + 8 = \frac{50}{5} + 8 = 18$. The smallest sign is 9 m wide and 18 m long.

106. The growth rate f fish/week is jointly proportional to the number x of fish and the number $10{,}000 - x$ o capacity. $f(x) = kx(10{,}000 - x)$, $f(1000) = 90 = 1000k \cdot 9000$, $k = 1/100{,}000$.
$f(x) = x(10{,}000 - x)/100{,}000$, $0 \le x \le 10{,}000$. A polynomial is continuous everywhere.
$f(x) = k(-x^2 + 10{,}000x)$ is maximum when $x = -10{,}000/-2 = 5000$.

107. $F(x) = \operatorname{sgn} x \cdot U(x+1)$

$$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \cdot \begin{cases} 0 & \text{if } x+1 < 0 \\ 1 & \text{if } x+1 \ge 0 \end{cases} = \begin{cases} -1 \cdot 0 = 0 & \text{if } x < -1 \\ -1 \cdot 1 = -1 & \text{if } -1 \le x < 0 \\ 0 \cdot 1 = 0 & \text{if } x = 0 \\ 1 \cdot 1 = 1 & \text{if } x > 0 \end{cases}$$

F is discontinuous at −1 and 0 because the left- and right-hand limits disagree there.



In Exercises 108 and 109, use the squeeze theorem to find the limit.

108. $\lim_{x\to 4} g(x)$, if $|g(x) + 5| < 3(4 - x)^2$ for all x.

▹ Because $0 \le |g(x) + 5| < 3(4-x)^2$, then $-3(4-x)^2 < g(x) + 5 < 3(4-x)^2$ (1)
Because $\lim_{x\to 4} 3(4-x)^2 = 0$, it follows from (1) and the squeeze theorem that
$\lim_{x\to 4} g(x) + 5 = 0$. Therefore $\lim_{x\to 4} g(x) = -5$.

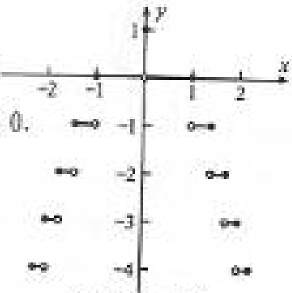
109. If $x \ne 1$ then $-1 \le \sin(1/\sqrt[3]{x-1}) \le 1$; so $-(x-1)^2 \le (x-1)^2 \sin(1/x) \le (x-1)^2$ (1)
Because $\lim_{x\to 1} (x-1)^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 1} (x-1)^2 \sin(1/\sqrt[3]{x-1}) = 0$.

110. Sketch the graph of $f(x) = [\![1 - x^2]\!]$ for x in $[-2, 2]$.

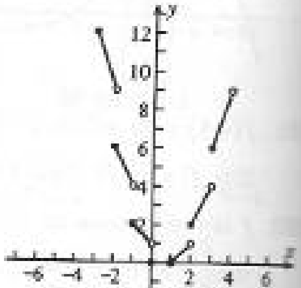
▹ Let n be a negative integer. $[\![1 - x^2]\!] = n$ when
$n \le 1 - x^2 < n + 1$, $n - 1 \le -x^2 < n$, $-(n-1) \ge x^2 > -n$
$x \in [-\sqrt{-(n-1)}, -\sqrt{-n}) \cup (\sqrt{-n}, \sqrt{-(n-1)}]$
$\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0 \ne f(0) = 1$ so f is not continuous at 0.

111. Sketch the graph $g(x) = (x-1)[\![x]\!]$ for x in $[-2, 2]$.

▹ $(x-1)[\![x]\!] = n(x-1)$ when $n \le x < n+1$.
$\lim_{x\to 1} g(x) = 0$ and g is continuous at 1.
$\lim_{x\to n} g(x)$ does not exist if n is any other integer.



Exercise 110



Exercise 111

112. Suppose $f(x) = g(x)$ for all values of x except a. (a) Prove that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if the limits exist.
(b) Prove that if $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)$ does not exist.

▹ (a) Suppose $\lim_{x\to a} f(x) = L$. Then
For every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.
Because $f(x) = g(x)$ when $x \ne a$, this is equivalent to
For every $\epsilon > 0$ there is a $\delta > 0$ such that $|g(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.
$\Leftrightarrow \lim_{x\to a} g(x) = L$

(b) The contrapositive of (b) is "If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} g(x)$" which is part of what was proved in (a).

113. (a) We are given that $\lim_{h\to 0} f(x+\mathrm{h}) = f(x)$. Let $\mathrm{h} = -\mathrm{k}$. Then $\lim_{-k\to 0} f(x-\mathrm{k}) = f(x)$.

But $-\mathrm{k}\to 0$ is equivalent to $\mathrm{k}\to 0$. Therefore, $\lim_{k\to 0} f(x-\mathrm{k}) = f(x)$.

(b) Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then $\lim_{h\to 0} f(0+\mathrm{h}) = \lim_{h\to 0} 1 = 1$; $\lim_{h\to 0} f(0-\mathrm{h}) = \lim_{h\to 0} 1 = 1$; but $f(0) = 0$. Thus $\lim_{h\to 0} f(0+\mathrm{h}) \neq f(0)$.

114. If t is any real number, then $f(t)$ exists by hypothesis. In order to prove that f is continuous at t we must show that $\lim_{x\to t} f(x) = f(t)$.

Because $f(\mathrm{a}+\mathrm{b}) = f(\mathrm{a}) + f(\mathrm{b})$, it follows that $f(x) = f[t + (x-t)] = f(t) + f(x-t)$. Thus, if the limits exist,

$$\lim_{x\to t} f(x) = \lim_{x\to t} f(t) + \lim_{x\to t} f(x-t)$$

$$\lim_{x\to t} f(x) = f(t) + \lim_{x\to t} f(x-t) \tag{1}$$

Because f is continuous at 0, it follows from Theorem 1.9.1 that

$$\lim_{x\to t} f(x-t) = f[\lim_{x\to t}(x-t)] = f(0)$$

substituting into (1) we obtain

$$\lim_{x\to t} f(x) = f(t) + f(0) \tag{2}$$

To find $f(0)$ we apply $f(\mathrm{a}+\mathrm{b}) = f(\mathrm{a}) + f(\mathrm{b})$ with $\mathrm{a} = \mathrm{b} = 0$. Then we have

$$f(0) = f(0) + f(0);\ f(0) = 0$$

Substituting into (2) we get $\lim_{x\to t} f(x) = f(t)$ which proves that f is continuous at t.

115. If the domain of f is the set of all real numbers, f is continuous at 0 and $f(a+b) = f(a)f(b)$ for all a and b, prove that f is continuous at every number.

▹ Because f is continuous at 0, $f(0) = \lim_{x\to 0} f(0+x) = \lim_{x\to 0} f(0)f(x) = f(0)f(0)$. Therefore, $f(0) = 0$ or $f(0) = 1$.

Suppose $f(0) = 0$. Then $f(x) = f(x+0) = f(x)f(0) = f(x)\cdot 0 = 0$. Thus f is a constant function and f is continuous.

Suppose $f(0) = 1$. Let a be any number. Then $\lim_{x\to 0} f(a+x) = \lim_{x\to 0}[f(a)f(x)] = f(a)f(0) = f(a)\cdot 1 = f(a)$.

Thus f is continuous at a.

116. Suppose the function f is defined on the open interval $(0,1)$ and $f(x) = \dfrac{\sin \pi x}{x(x-1)}$.

Define f at **0** and 1 so that f is continuous on the closed interval $[0,1]$.

▹ We must define $f(0)$ and $f(1)$ so that

$$\lim_{x\to 0^+} f(x) = f(0) \quad\text{and}\quad \lim_{x\to 0^-} f(x) = f(1)$$

Now

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin \pi x}{x}\cdot\frac{1}{x-1}\\ &= \pi\lim_{x\to 0^+}\frac{\sin \pi x}{\pi x}\cdot\lim_{x\to 0^+}\frac{1}{x-1}\\ &= \pi\lim_{\pi x\to 0}\frac{\sin \pi x}{\pi x}(-1)\\ &= -\pi\end{aligned}$$

(By Theorem 1.10.2)

Thus, we define $f(0) = -\pi$. Next, we let $t = x - 1$. Then

$$\begin{aligned}\lim_{x\to 1^-} f(t) &= \lim_{t\to 0^-}\frac{\sin(\pi t + \pi)}{(t+1)t}\\ &= \lim_{t\to 0^-}\frac{-\sin \pi t}{t}\cdot\lim_{t\to 0^-}\frac{1}{t+1}\\ &= -\pi\lim_{\pi t\to 0^-}\frac{\sin \pi t}{\pi t}\cdot 1\\ &= -\pi\cdot 1\end{aligned}$$

Thus we define $f(1) = -\pi$.

T W O

THE DERIVATIVE AND DIFFERENTIATION

2.1 THE TANGENT LINE AND THE DERIVATIVE

2.1.1 Definition Suppose the function f is continuous at x_1. The *tangent line* to the graph of f at the point $P(x_1, f(x_1))$ is

(i) the line through P having slope $m(x_1)$, given by

$$m(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{1}$$

if this limit exists

(ii) the line $x = x_1$ if

$$\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x} \text{ is } +\infty \text{ or } -\infty \text{ and } \lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x} \text{ is } +\infty \text{ or } -\infty$$

If neither (i) nor (ii) of Definition 2.1.1 holds, then there is no tangent line to the graph of f at the point $P(x_1, f(x_1))$.

If we want to find the slope of the tangent line to the curve at more than one point, we first find the limit and then make the indicated replacements for x_1. However, if we are interested in finding the tangent line at only one point and if the coordinates of the point of tangency are known, it is easier to first make the indicated replacement and then find the limit.

2.1.2 Definition The *normal line* to a graph at a given point is the line perpendicular to the tangent line at that point.

Formula (1) for the slope of the tangent line is a special case of the formula for the derivative of a function. Following is one of the most important definitions in the calculus.

2.1.3 Definition The *derivative* of the function f is that function, denoted by f', such that its value at a number x in the domain of f is given by the equivalent formulas

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \qquad \text{(3) and (3')}$$

if this limit exists, where $z = x + \Delta x$. (f' is read "f prime," and $f'(x)$ is read "f prime of x.") We also use the symbols $\frac{d}{dx} f(x)$ and $D_x f(x)$ to represent $f'(x)$. If $y = f(x)$, then the symbols $D_x y$, y', dy/dx are sometimes used to represent $f'(x)$.

If Δy is defined by $\Delta y = f(x + \Delta x) - f(x)$ then $D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

If x_1 is a particular number in the domain of f, then to find $f'(x_1)$ we may use either of the equivalent formulas,

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad (4) \quad \text{or} \quad f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} \tag{7}$$

To use formulas (3) and (4), note that
$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ and $(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

To use formulas (3′) and (7), note that
$\frac{x^2 - x_1^2}{x - x_1} = \frac{(x - x_1)(x + x_1)}{x - x_1} = x + x_1$ and $\frac{x^3 - x_1^3}{x - x_1} = \frac{(x - x_1)(x^2 + xx_1 + x_1^2)}{x - x_1} = x^2 + xx_1 + x_1^2$
The slope of the tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is precisely the derivative of f evaluated at x_1. Thus formulas (3), (3′), (4), and (7) are interchangeable.

The equation of a (tangent) line with slope m and passing through (a, b) is $y = m(x - a) + b$.
A normal line has slope $-1/f'(x_1)$.

Exercises 2.1

In Exercises 1–6, find an equation of the tangent line at the point. Sketch the graph and a segment of the tangent.

1. $y = 9 - x^2$; $(2, 5)$ ▷ Let $f(x) = 9 - x^2$.

$$m(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{(9-x^2)-(9-2^2)}{x-2} = \lim_{x\to 2}\frac{-(x^2-2^2)}{x-2} = \lim_{x\to 2} -(x+2) = -4.$$

An equation of the tangent line is $y = -4(x-2)+5$; $y = -4x+13$.

2. $y = x^2 + 4$; $(-1, 5)$ ▷ Let $f(x) = x^2 + 4$.

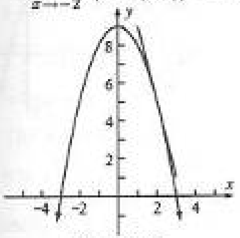
$$m(-1) = \lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1}\frac{x^2+4-[(-1)^2+4]}{x-(-1)} = \lim_{x\to -1}\frac{x^2-(-1)^2}{x-(-1)} = \lim_{x\to -1}(x+(-1)) = -2$$

An equation of the tangent line is $y = -2(x-(-1))+5$; $y = -2x+3$.

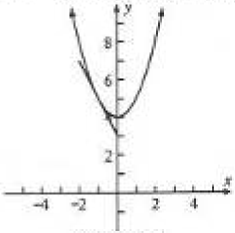
3. $y = 2x^2 + 4x$; $(-2, 0)$ ▷ Let $f(x) = 2x^2 + 4x$.

$$m(-2) = \lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to -2}\frac{2x^2+4x-[2(-2)^2+4(-2)]}{x-(-2)} = \lim_{x\to -2} 2\frac{x^2-(-2)^2}{x-(-2)} + 4\frac{x-(-2)}{x-(-2)}$$

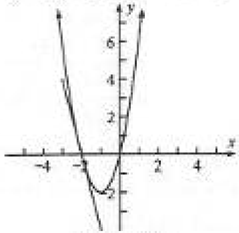
$= \lim_{x\to -2} 2(x+(-2)) + 4\cdot 1 = 2(-4)+4 = -4$. The tangent line is $y = -4(x-(-2))+0$; $y = -4x-8$.



Exercise 1



Exercise 2



Exercise 3

4. $y = x^2 - 6x + 9$; $(3, 0)$ ▷ Let $f(x) = x^2 - 6x + 9$. Use formula (4) with $x_1 = 3$.

$$m(3) = \lim_{\Delta x\to 0}\frac{f(3+\Delta x)-f(3)}{\Delta x} = \lim_{\Delta x\to 0}\frac{[(3+\Delta x)^2-6(3+\Delta x)+9]-0}{\Delta x} = \lim_{\Delta x\to 0}\frac{9+6\Delta x+(\Delta x)^2-18-6\Delta x+9}{\Delta x}$$

$= \lim_{\Delta x\to 0}\frac{(\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}\Delta x = 0$. An equation of the tangent line is $y = 0(x-3)+0$; $y = 0$.

5. $y = x^3 + 3$; $(1, 4)$ ▷ Let $f(x) = x^3 + 1$.

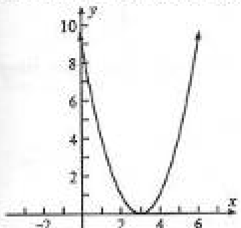
$$m(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(x^3+3)-(1^3+3)}{x-1} = \lim_{x\to 1}\frac{x^3-1^3}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$$

An equation of the tangent line is $y = 3(x-1)+4$; $y = 3x+1$.

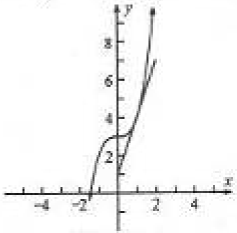
6. $y = 1 - x^3$; $(2, -7)$ ▷ Let $f(x) = 1 - x^3$.

$$m(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{1-x^3-(1-2^3)}{x-2} = \lim_{x\to 2}\frac{-(x^3-2^3)}{x-2} = \lim_{x\to 2} -(x^2+2x+4) = -12$$

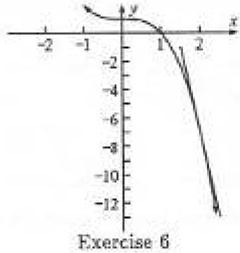
An equation of the tangent line is $y = -12(x-2)-7$; $y = -12x+17$



Exercise 4



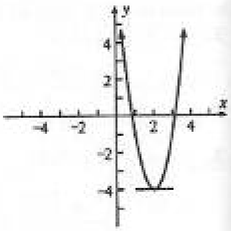
Exercise 5



Exercise 6

In Exercises 7–10, (a) find the slope of the tangent at $(x_1, f(x_1))$. (b) Find where the tangent is horizontal. Sketch.

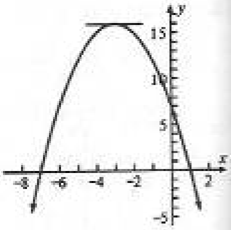
7. $f(x) = 3x^2 - 12x + 8$.



▹ (a) $$m(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[3(x_1 + \Delta x)^2 - 12(x_1 + \Delta x) + 8] - (3{x_1}^2 - 12x_1 + 8)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{6x_1 \Delta x + 3(\Delta x)^2 - 12\Delta x}{\Delta x} = 6x_1 - 12 = 6(x_1 - 2)$$

(b) $m(x_1) = 0$ when $x_1 = 2$ and $f(2) = -4$;
so the graph has a horizontal tangent at $(2, -4)$.

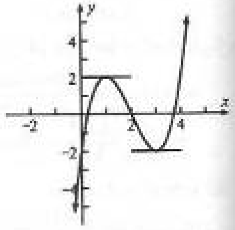
8. $f(x) = 7 - 6x - x^2$



▹ (a) Applying formula (1) we have
$$m(x_1) = \lim_{\Delta x \to 0} \frac{[7 - 6(x_1 + \Delta x) - (x_1 + \Delta x)^2] - [7 - 6x_1 - {x_1}^2]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7 - 6x_1 - 6\Delta x - {x_1}^2 - 2x_1\Delta x - (\Delta x)^2 - 7 + 6x_1 + {x_1}^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-6\Delta x - 2x_1\Delta x - (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (-6 - 2x_1 - \Delta x)$$
$$= -6 - 2x_1 = -2(x_1 + 3)$$

(b) $m(x_1) = 0$ when $x_1 = -3$ and $f(-3) = 16$ so the graph has a horizontal tangent at $(-3, 16)$. Other points on the graph are $(-7, 0)$, $(-5, 12)$, $(-3, 16)$, $(-1, 12)$ and $(1, 0)$.

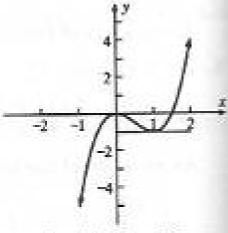
9. $f(x) = x^3 - 6x^2 + 9x - 2$



▹ (a) $$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1} \frac{(x^3 - 6x^2 + 9x - 2) - ({x_1}^3 - 6{x_1}^2 + 9x_1 - 2)}{x - x_1}$$
$$= \lim_{x \to x_1} \left(\frac{x^3 - {x_1}^3}{x - x_1} - 6\frac{x^2 - {x_1}^2}{x - x_1} + 9\frac{x - x_1}{x - x_1} \right) = \lim_{x \to x_1} [(x^2 + x_1 x + {x_1}^2) - 6(x + x_1) + 9]$$
$$= 3{x_1}^2 - 12x_1 + 9 = 3({x_1}^2 - 4x_1 + 3) = 3(x_1 - 1)(x_1 - 3)$$

(b) $m(x_1) = 0$ when $x_1 = 1$, $f(1) = 2$ and $x_1 = 3$, $f(3) = -2$
so the graph has horizontal tangents at $(1, 2)$ and $(3, -2)$.

10. $f(x) = 2x^3 - 3x^2$



▹ $$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1} \frac{(2x^3 - 3x^2) - (2{x_1}^3 - 3{x_1}^2)}{x - x_1}$$
$$= \lim_{x \to x_1} \left(2\frac{x^3 - {x_1}^3}{x - x_1} - 3\frac{x^2 - {x_1}^2}{x - x_1} \right) = \lim_{x \to x_1} [2(x^2 + x_1 x + {x_1}^2) - 3(x + x_1)]$$
$$= 6{x_1}^2 - 6x_1 = 6x_1(x_1 - 1)$$

(b) $m(x_1) = 0$ when $x_1 = 0$, $f(0) = 0$ and $x_1 = 1$, $f(1) = -1$
so the graph has horizontal tangents at $(0, 0)$ and $(1, -1)$.

In Exercises 11–16, find equations of the tangent and normal lines at the point. Plot the graph with these lines.

11. $y = \sqrt{x+1}$; $(3, 2)$

▹ $$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{(x+1) - 4} = \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{(\sqrt{x+1})^2 - 2^2} = \lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4}$$

The tangent line has equation $y = \frac{1}{4}(x - 3) + 2 = \frac{1}{4}x + \frac{5}{4}$.

The normal line has slope -4 and equation $y = -4(x - 3) + 2 = -4x + 14$.

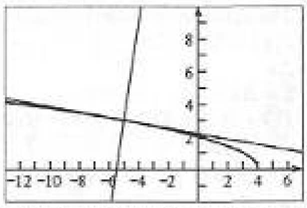
12. $y = \sqrt{4-x}$; $(-5, 3)$

▹ To find the slope of the tangent line to the curve at $(-5,3)$, use formula (7) with $f(x) = \sqrt{4-x}$ and $x_1 = -5$.

$$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

$$m(-5) = \lim_{x \to -5} \frac{\sqrt{4-x} - 3}{x - (-5)} = \lim_{x \to -5} - \frac{\sqrt{4-x} - 3}{(4-x) - 9} = \lim_{x \to -5} - \frac{\sqrt{4-x} - 3}{(\sqrt{4-x})^2 - 3^2}$$

$$= \lim_{x \to -5} - \frac{1}{\sqrt{4-x+3}} = -\frac{1}{6}$$



Use the point-slope form with $m = -\frac{1}{6}$ to find an equation of the tangent line to the curve at the point $(-5, 3)$.

$$y = -\tfrac{1}{6}(x - (-5)) + 3; \; y = -\tfrac{1}{6}x + \tfrac{13}{6}$$

Use the point-slope form with $m = 6$ to find an equation of the normal line to the curve at the point $(-5, 3)$.

$$y = 6(x - (-5)) + 3; \; y = 6x + 33$$

13. $y = 2x - x^3$; $(-2, 4)$ ▹ Let $f(x) = 2x - x^3$

$$m = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} \frac{2x - x^3 - (2(-2) - (-2)^3)}{x+2} = \lim_{x \to -2} \left(2\frac{x+2}{x+2} - \frac{x^3 + 2^3}{x+2} \right)$$

$= \lim_{x \to -2} [2 - (x^2 - 2x + 4)] = -25$. The tangent line has equation $y = -10(x+2) + 4 = -10x + 16$

The normal line has slope $\frac{1}{10}$ and equation $y = \frac{1}{10}(x+2) + 4 = \frac{1}{10}x + \frac{21}{5}$.

14. $y = x^4 - 4x$; $(0, 0)$ ▹ Let $f(x) = x^4 - 4x$.

$$m(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^4 - 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0} [(\Delta x)^3 - 4] = -4.$$

The tangent line has equation $y = -4x$; $4x + y = 0$. The normal line has equation $y = \frac{1}{4}x$; $x - 4y = 0$.

15. $y = 4/x^2$; $(2, 1)$ ▹ Let $f(x) = 4/x^2$.

$$m(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{4}{x^2} - \frac{4}{2^2}}{x - 2} \cdot \frac{-2^2 x^2}{-2^2 x^2} = \lim_{x \to 2} \frac{4}{-2^2 x^2} \cdot \frac{x^2 - 2^2}{x - 2} = \lim_{x \to 2} \frac{-1}{x^2} \cdot (x+2) = -1$$

The tangent line has equation $y = -(x-2) + 1 = -x + 3$.

The normal line has slope 1 and equation $y = 1(x-2) + 1 = x - 1$.

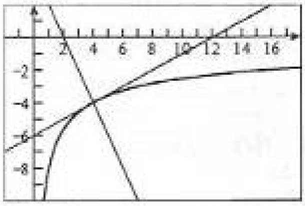
16. $y = -\dfrac{8}{\sqrt{x}}$; $(4, 4)$

▹ To find the slope of the tangent line to the curve at $(4, -4)$, use formula (7) with $f(x) = -8/\sqrt{x}$ and $x_1 = 4$.

$$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

$$m(4) = \lim_{x \to 4} \frac{-\frac{8}{\sqrt{x}} - (-4)}{x - 4} = \lim_{x \to 4} \frac{4\left(1 - \frac{2}{\sqrt{x}}\right)\sqrt{x}}{(x-4)\sqrt{x}} = \lim_{x \to 4} \frac{4(\sqrt{x} - 2)}{(x-4)\sqrt{x}} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x \to 4} \frac{4(x-4)}{(x-4)\sqrt{x}(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{4}{\sqrt{x}(\sqrt{x} + 2)} = \frac{4}{\sqrt{4}(\sqrt{4} + 2)} = \frac{1}{2}$$



Use the point-slope form with $m = \frac{1}{2}$ to find an equation of the tangent line to the curve at the point $(4, -4)$.

$$y = \tfrac{1}{2}(x - 4) + (-4); \; y = \tfrac{1}{2}x - 6$$

Use the point-slope form with $m = -2$ to find an equation of the normal line to the curve at the point $(4, -4)$.

$$y = -2(x - 4) + (-4); \; y = -2x + 4$$

In Exercises 17–20, (a) Tabulate $[f(2+\Delta x)-f(2)]/\Delta x$ when $\Delta x = 0.10$ to 0.01 step −0.01 and −0.10 to −0.01 step 0.01 and guess the limit. (b) Find $f'(2)$ using formula (4). (c) Tabulate $[f(x)-f(2)]/(x-2)$ when $x = 2.10$ to 2.01 step −0.01 and 1.90 to 1.99 step 0.01 and guess the limit. (d) Find $f'(2)$ using formula (7).

▹ The symmetric difference quotient of §2.3 is included for comparison. It is exact for quadratic functions.

17. $f(x) = 3x^2 - 7x$

▹ Tables for (a) and (c). The limit appears to be 5.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	5.30	5.27	5.24	5.21	5.18	5.15	5.12	5.09	5.06	5.03
$[f(2+\Delta x)-f(2-\Delta x)]/2\Delta x$	5	5	5	5	5	5	5	5	5	5

Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	4.70	4.73	4.76	4.79	4.82	4.85	4.88	4.91	4.94	4.97
$[f(2+\Delta x)-f(2-\Delta x)]/2\Delta x$	5	5	5	5	5	5	5	5	5	5

(b) $$\lim_{\Delta x\to 0}\frac{[3(2+\Delta x)^2-7(2+\Delta x)]-(3\cdot 2^2-7\cdot 2)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{3(2^2+4\Delta x+\Delta x^2)-7(2+\Delta x)-(3\cdot 2^2-7\cdot 2)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{12\Delta x+3\Delta x^2-7\Delta x}{\Delta x}=\lim_{\Delta x\to 0}(5-3\Delta x^2)=5$$

(d) $$\lim_{x\to 2}\frac{(3x^2-7x)-(3\cdot 2^2-7\cdot 2)}{x-2}$$
$$=\lim_{x\to 2}\left(3\frac{x^2-2^2}{x-2}-7\frac{x-2}{x-2}\right)$$
$$=\lim_{x\to 2}[3(x+2)-7]=5$$

18. $f(x) = x^3$

▹ Tables for (a) and (c). The limit appears to be 12.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	12.61	12.55	12.49	12.45	12.36	12.30	12.24	12.18	12.12	12.06
$\frac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	12.0100	12.0081	12.0064	12.0049	12.0036	12.0025	12.0016	12.0009	12.0004	12.0001

Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	11.41	11.47	11.53	11.58	11.64	11.70	11.76	11.82	11.88	11.94
$\frac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	12.0100	12.0081	12.0064	12.0049	12.0036	12.0025	12.0016	12.0009	12.0004	12.0001

(b) $$\lim_{\Delta x\to 0}\frac{(2+\Delta x)^3-2^3}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{2^3+3\cdot 2^2\Delta x+3\cdot 2\Delta x^2+\Delta x^3-2^2}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{12\Delta x+6\Delta x^2+\Delta x^3}{\Delta x}=\lim_{\Delta x\to 0}(12+6\Delta x+\Delta x^2)=12$$

(d) $$\lim_{x\to 2}\frac{x^3-2^3}{x-2}$$
$$=\lim_{x\to 2}(x^2+2x+2^2)=12$$

19. $f(x) = \sqrt{6-x}$

▹ Tables for (a) and (c). The limit appears to be $-\frac{1}{4}$.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	−.2516	−.2514	−.2513	−.2511	−.2509	−.2508	−.2506	−.2505	−.2503	−.2502
$\frac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	−.250019	−.250016	−.250012	−.250009	−.250006	−.250005	−.250004	−.250002	−.250001	−.250000

Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{f(x)-f(2)}{x-2}$	−.2485	−.2486	−.2488	−.2489	−.2491	−.2492	−.2494	−.2495	−.2497	−.2498
$\frac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	−.250019	−.250016	−.250012	−.250009	−.250006	−.250005	−.250004	−.250002	−.250001	−.250000

(b) $\lim\limits_{\Delta x\to 0}\dfrac{\sqrt{6-(2+\Delta x)}-\sqrt{6-2}}{\Delta x}$
$= \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{4-\Delta x}-2}{\Delta x}\cdot\dfrac{\sqrt{4+\Delta x}+2}{\sqrt{4-\Delta x}+2}$
$= \lim\limits_{\Delta x\to 0}\dfrac{(4-\Delta x)-4}{\Delta x(\sqrt{4-\Delta x}+2)} = \lim\limits_{\Delta x\to 0}\dfrac{-1}{\sqrt{4-\Delta x}+2} = -\frac{1}{4}$

(d) $\lim\limits_{x\to 2}\dfrac{\sqrt{6-x}-\sqrt{6-2}}{x-2}$
$= \lim\limits_{x\to 2} -\dfrac{\sqrt{6-x}-\sqrt{4}}{(6-x)-4}$
$= \lim\limits_{x\to 2}\dfrac{-1}{\sqrt{6-x}+2} = -\frac{1}{4}$

20. $f(x) = 1/(4-x)$ ▹ Tables for (a) and (c). The limit appears to be $\frac{1}{4}$.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	.2632	.2618	.2604	.2591	.2577	.2564	.2551	.2538	.2525	.2513
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	.250627	.250507	.250401	.250307	.250225	.250156	.250100	.250056	.250025	.250006

Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	.2381	.2392	.2404	.2415	.2427	.2439	.2451	.2463	.2475	.2488
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	.250627	.250507	.250401	.250307	.250225	.250156	.250100	.250056	.250025	.250006

(b) $\lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\left(\dfrac{1}{4-(2+\Delta x)}-\dfrac{1}{4-2}\right)$
$= \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\left(\dfrac{1}{2+\Delta x}-\dfrac{1}{2}\right)$
$= \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\cdot\dfrac{2-(2+\Delta x)}{(2+\Delta x)2} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\cdot\dfrac{\Delta x}{(2+\Delta x)2}$
$= \lim\limits_{\Delta x\to 0}\dfrac{1}{(2+\Delta x)2} = \dfrac{1}{4}$

(d) $\lim\limits_{x\to 2}\dfrac{1}{x-2}\left(\dfrac{1}{4-x}-\dfrac{1}{4-2}\right)$
$= \lim\limits_{x\to 2}\dfrac{1}{x-2}\left(\dfrac{1}{4-x}-\dfrac{1}{2}\right.$
$= \lim\limits_{x\to 2}\dfrac{1}{x-2}\cdot\dfrac{2-(4-x)}{(4-x)2} = \lim\limits_{x\to 2}\dfrac{1}{x-2}\cdot\dfrac{x-2}{(4-x)2}$
$= \lim\limits_{x\to 2}\dfrac{1}{(4-x)2} = \dfrac{1}{4}$

In Exercises 21–30, find $f'(x)$ (a) using formula 7 and (b) using formula (4).

21. $f(x) = \dfrac{8}{x-2}$; $x_1 = 6$

▹ (a) $f'(6) = \lim\limits_{x\to 6}\dfrac{1}{x-6}\left(\dfrac{8}{x-2}-\dfrac{8}{6-2}\right) = \lim\limits_{x\to 6}\dfrac{8}{x-6}\left(\dfrac{1}{x-2}-\dfrac{1}{4}\right) = \lim\limits_{x\to 6}\dfrac{8}{x-6}\cdot\dfrac{6-x}{(x-2)4} = \lim\limits_{x\to 6}\dfrac{-2}{x-2} = -\dfrac{1}{2}$

(b) $f'(6) = \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\left(\dfrac{8}{6+\Delta x-2}-\dfrac{8}{6-2}\right) = \lim\limits_{\Delta x\to 0}\dfrac{8}{\Delta x}\left(\dfrac{1}{4+\Delta x}-\dfrac{1}{4}\right) = \lim\limits_{\Delta x\to 0}\dfrac{8}{\Delta x}\cdot\dfrac{-\Delta x}{(4+\Delta x)4} = \lim\limits_{\Delta x\to 0}\dfrac{-2}{4+\Delta x} = -\dfrac{1}{2}$

22. $f(x) = \dfrac{2}{\sqrt{x}}-1$; $x_1 = 4$ ▹ (a) $f'(4) = \lim\limits_{x\to 4}\dfrac{1}{x-4}\left[\dfrac{2}{\sqrt{x}}-1-\left(\dfrac{2}{\sqrt{4}}-1\right)\right]$

$= \lim\limits_{x\to 4}\dfrac{2}{x-4}\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{4}}\right) = \lim\limits_{x\to 4}\dfrac{-2}{x-4}\cdot\dfrac{\sqrt{x}-\sqrt{4}}{\sqrt{x}\sqrt{4}} = \lim\limits_{x\to 4}\dfrac{-2}{(\sqrt{x}+\sqrt{4})\sqrt{x}\sqrt{4}} = \dfrac{-2}{16} = -\dfrac{1}{8}$

(b) $f'(4) = \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\left[\dfrac{2}{\sqrt{4+\Delta x}}-1-\left(\dfrac{2}{\sqrt{4}}-1\right)\right] = \lim\limits_{\Delta x\to 0}\dfrac{2}{\Delta x}\left(\dfrac{1}{\sqrt{4+\Delta x}}-\dfrac{1}{\sqrt{4}}\right) = \lim\limits_{\Delta x\to 0}\dfrac{2}{\Delta x}\cdot\dfrac{\sqrt{4}-\sqrt{4+\Delta x}}{\sqrt{x}\sqrt{4}}$

23. $f(x) = \sin x$; $x_1 = 0$ ▹ We use Theorem 1.10.2.

(a) $f'(0) = \lim\limits_{x\to 0}\dfrac{\sin x-\sin 0}{x-0} = \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ (b) $f'(0) = \lim\limits_{\Delta x\to 0}\dfrac{\sin(0+\Delta x)-\sin 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sin\Delta x}{\Delta x} = 1$

24. $f(x) = \cos x$; $x_1 = 0$ ▹ We use Theorem 1.10.5.

(a) $f'(0) = \lim\limits_{x\to 0}\dfrac{\cos x-\cos 0}{x-0} = \lim\limits_{x\to 0}\dfrac{\cos x-1}{x} = 0$ (b) $f'(0) = \lim\limits_{\Delta x\to 0}\dfrac{\cos(0+\Delta x)-\cos 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\cos\Delta x-1}{\Delta x} = 0$

25. $f(x) = \sin x$; $x_1 = \frac{1}{2}\pi$ ▹ In (a), let $x-\frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x\to\pi/2}\dfrac{\sin x-\sin\frac{1}{2}\pi}{x-\frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\dfrac{\sin(\frac{1}{2}\pi+\Delta x)-\sin\frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\cos\Delta x-1}{\Delta x} = 0$

26. $f(x) = \cos x$; $x_1 = \frac{1}{2}\pi$ ▹ In (a), let $x-\frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}x) = \lim\limits_{x\to\pi/2}\dfrac{\cos x-\cos\frac{1}{2}\pi}{x-\frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\dfrac{\cos(\frac{1}{2}\pi+\Delta x)-\cos\frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-\sin\Delta x}{\Delta x} = -1$

27. $f(x) = \sec x;\ x_1 = 0$ ▹ We use Theorem 1.10.5.

(a) $f'(0) = \lim\limits_{x\to 0}\dfrac{\sec x - \sec 0}{x - 0} = \lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{\cos x} - \dfrac{1}{\cos 0}\right) = \lim\limits_{x\to 0}\dfrac{1-\cos x}{x\cos x} = \lim\limits_{x\to 0}\dfrac{1}{\cos x}\cdot\lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 1\cdot 0 = 0$

(b) $f'(0) = \lim\limits_{\Delta x\to 0}\dfrac{\sec(0+\Delta x) - \sec 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}\left(\dfrac{1}{\cos(0+\Delta x)} - \dfrac{1}{\cos 0}\right) = \lim\limits_{\Delta x\to 0}\dfrac{1-\cos\Delta x}{\Delta x\cos\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\dfrac{1-\cos\Delta x}{\Delta x} = 1\cdot 0 = 0$

28. $f(x) = \tan x;\ x_1 = 0$ ▹ We use Theorem 1.10.2.

(a) $f'(0) = \lim\limits_{x\to 0}\dfrac{\tan x - \tan 0}{x - 0} = \lim\limits_{x\to 0}\dfrac{\sin x/\cos x}{x} = \lim\limits_{x\to 0}\dfrac{1}{\cos x}\cdot\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\cdot 1 = 1$

(b) $f'(0) = \lim\limits_{\Delta x\to 0}\dfrac{\tan(0+\Delta x) - \tan 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sin\Delta x/\cos\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\dfrac{\sin\Delta x}{\Delta x} = 1\cdot 1 = 1$

29. $f(x) = \cot x;\ x_1 = \frac{1}{2}\pi$ ▹ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x\to\pi/2}\dfrac{\cot x - \cot\frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\dfrac{\cot(\frac{1}{2}\pi + \Delta x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-\tan\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\dfrac{\sin\Delta x}{\Delta x} = -1\cdot 1 = -1$

30. $f(x) = \csc x;\ x_1 = \frac{1}{2}\pi$ ▹ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x\to\pi/2}\dfrac{\csc x - \csc\frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\dfrac{\csc(\frac{1}{2}\pi + \Delta x) - \csc\frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1/\sin(\frac{1}{2}\pi + \Delta x) - 1}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1/\cos\Delta x - 1}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\dfrac{1-\cos\Delta x}{\Delta x} = 1\cdot 0 = 0$

In Exercises 31–36, find $f'(x)$ by applying formula (3) or (3′).

31. $f(x) = -4$ ▹ $f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-4-(-4)}{\Delta x} = 0$

32. $f(x) = 10$ ▹ $f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{10-10}{\Delta x} = 0$

33. $f(x) = 7x + 3$ ▹ $f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{[7(x+\Delta x)+3] - (7x+3)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{7\Delta x}{\Delta x} = 7$

34. $f(x) = 8 - 5x$ ▹ $f'(x) = \lim\limits_{z\to x}\dfrac{f(z) - f(x)}{z - x} = \lim\limits_{z\to x}\dfrac{(8-5z) - (8-5x)}{z - x} = \lim\limits_{z\to x} -5\dfrac{z-x}{z-x} = -5$

35. $f(x) = 4 + 5x - 2x^2$ ▹ $f'(x) = \lim\limits_{z\to x}\dfrac{f(z) - f(x)}{z - x}$

$= \lim\limits_{z\to x}\dfrac{(4+5z-2z^2) - (4+5x-2x^2)}{z - x} = \lim\limits_{z\to x}\left(5\dfrac{z-x}{z-x} - 2\dfrac{z^2-x^2}{z-x}\right) = \lim\limits_{z\to x}[5 - 2(z+x)] = 5 - 4x$

36. $f(x) = 3x^2 - 2x + 1$ ▹ $f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{[3(x+\Delta x)^2 - 2(x+\Delta x)+1] - (3x^2 - 2x + 1)}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{[3(x^2 + 2x\Delta x + \Delta x^2) - 2(x+\Delta x)] - (3x^2 - 2x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\cancel{3x^2} + 6x\Delta x + 3\Delta x^2 - \cancel{2x} - 2\Delta x - \cancel{3x^2} + \cancel{2x}}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}(6x + 3\Delta x - 2) = 6x - 2$

In Exercises 37 – 40, find the derivative.

37. $\dfrac{d}{dx}(8 - x^3) = \lim\limits_{\Delta x\to 0}\dfrac{[8 - (x+\Delta x)^3] - (8 - x^3)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{[8 - x^3 - 3x^2\Delta x - 3x(\Delta x)^2 - (\Delta x)^3] - (8 - x^3)}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{-3x^2\Delta x - 3x(\Delta x)^2 - (\Delta x)^3}{\Delta x} = \lim\limits_{\Delta x\to 0}[-3x^2 - 3x\Delta x - (\Delta x)^2] = -3x^2$

38. $\dfrac{d}{dt}(t^3 + t) = \lim\limits_{z\to t}\dfrac{(z^3 + z) - (t^3 + t)}{z - t} = \lim\limits_{z\to t}\left(\dfrac{z^3 - t^3}{z - t} - \dfrac{z - t}{z - t}\right) = \lim\limits_{z\to t}(z^2 + zt + t^3 - 1) = 3t^2 - 1$

39. $D_r\left(\dfrac{2r+3}{3r-2}\right) = \lim\limits_{\Delta r\to 0}\dfrac{1}{\Delta r}\left[\dfrac{2(r+\Delta r)+3}{3(r+\Delta r)-2} - \dfrac{2r+3}{3r-2}\right] = \lim\limits_{\Delta r\to 0}\dfrac{[(2r+3) + 2\Delta Dr](3r-2) - (2r+3)[(3r-2) + 3\Delta r]}{\Delta r[3(r+\Delta r)-2](3r-2)}$

$= \lim\limits_{\Delta r\to 0}\dfrac{2\Delta r(3r-2) - 3\Delta r(2r+3)}{\Delta r[3(r+\Delta r)-2](3r-2)} = \lim\limits_{\Delta r\to 0}\dfrac{-13\Delta r}{\Delta r[3(r+\Delta r)-2](3r-2)} = \lim\limits_{\Delta r\to 0}\dfrac{-13}{[3(r+\Delta r)-2](3r-2)} = \dfrac{-13}{(3r-2)^2}$

40. $$D_x\left[\frac{1}{x^2}-x\right]=\lim_{\Delta x\to 0}\frac{\left[\frac{1}{(x+\Delta x)^2}-(x+\Delta x)\right]-\left(\frac{1}{x^2}-x\right)}{\Delta x}=\lim_{\Delta x\to 0}\left[-\Delta x+\frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2}\right]\frac{1}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[-\Delta x+\frac{-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2}\right]\frac{1}{\Delta x}=\lim_{\Delta x\to 0}\left[-1-\frac{2x+\Delta x}{(x+\Delta x)^2x^2}\right]=-1-\frac{2x}{x^4}=-1-\frac{2}{x^3}$$

In Exercises 41–44, find $\frac{dy}{dx}$.

41. $y=3x+\frac{6}{x^2}$ ▹ Let $f(x)=3x+\frac{6}{x^2}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{1}{z-x}\left[\left(3z+\frac{6}{z^2}\right)-\left(3x+\frac{6}{x^2}\right)\right]=\lim_{z\to x}\left[3\frac{z-x}{z-x}+\frac{6}{z-x}\left(\frac{1}{z^2}-\frac{1}{x^2}\right)\right]$$
$$=\lim_{z\to x}\left[3+\frac{6}{z-x}\cdot\frac{-(z^2-x^2)}{z^2x^2}\right]=\lim_{z\to x}\left[3-\frac{6(z+x)}{z^2x^2}\right]=3-\frac{12x}{x^4}=3-\frac{12}{x^3}$$

42. $y=\sqrt[3]{x}$. ▹ Let $f(x)=x^{1/3}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{z-x}=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{(z^{1/3})^3-(x^{1/3})^3}$$
$$=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{(z^{1/3}-x^{1/3})[(z^{1/3})^2+(z^{1/3})(x^{1/3})+(x^{1/3})^2}=\lim_{z\to x}\frac{1}{(z^{1/3})^2+(z^{1/3})(x^{1/3})+(x^{1/3})^2}$$
$=\frac{1}{x^{2/3}+x^{2/3}+x^{2/3}}=\frac{1}{3x^{2/3}}=\frac{3}{2}x^{-2/3}$. Note that while y is defined at 0, $\frac{dy}{dx}$ is not.

43. $y=\frac{1}{\sqrt{x-1}}$ ▹ Let $f(x)=\frac{1}{\sqrt{x-1}}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{1}{z-x}\left(\frac{1}{\sqrt{z-1}}-\frac{1}{\sqrt{x-1}}\right)=\lim_{z\to x}\frac{-(\sqrt{z-1}-\sqrt{x-1})}{[(z-1)-(x-1)]\sqrt{z-1}\sqrt{x-1}}$$
$$=\lim_{z\to x}\frac{-1}{(\sqrt{z-1}+\sqrt{x-1})\sqrt{z-1}\sqrt{x-1}}=\frac{-1}{2(x-1)^{3/2}}=-\tfrac{1}{2}(x-1)^{-3/2}$$

44. $y=\frac{4}{2x-5}$ ▹ Let $f(x)=\frac{4}{2x-5}$.
$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{4}{2(x+\Delta x)-5}-\frac{4}{2x-5}}{\Delta x}\cdot\frac{(2x-5)[2(x+\Delta x)-5]}{(2x-5)[2(x+\Delta x)-5]}$$
$$=4\lim_{\Delta x\to 0}\frac{(2x-5)-[2(x+\Delta x)-5]}{(2x-5)[2(x+\Delta x)-5]\Delta x}=4\lim_{\Delta x\to 0}\frac{-2\Delta x}{(2x-5)[2(x+\Delta x)-5]\Delta x}$$
$$=4\lim_{\Delta x\to 0}\frac{-2}{(2x-5)[2(x+\Delta x)-5]}=4\left[\frac{-2}{(2x-5)(2x-5)}\right]=\frac{-8}{(2x-5)^2}$$

45. Find an equation of the tangent line to the curve $y=2x^2+3$ that is parallel to the line $8x-y+3=0$.
▹ $8x-y+3=0$, or $y=8x+3$, has slope 8. Let $f(x)=2x^2+3$. We wish to find an x_1 for which $m(x_1)=8$.
$$m(x_1)=\lim_{\Delta x\to 0}\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{[2(x_1+\Delta x)^2+3]-(2{x_1}^2+3)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{4x_1\Delta x+2(\Delta x)^2}{\Delta x}=\lim_{\Delta x\to 0}(4x_1+2\Delta x)=4x_1$$
Therefore, we have $4x_1=8$; $x_1=2$. So an equation of the tangent line at $(2,11)$ is $y=8(x-2)+11=8x-5$.

46. Find an equation of the tangent line to the curve $y=3x^2-4$ that is parallel to the line $3x+y=4$.
▹ $y=-3x+4$ has slope -3. Let $f(x)=3x^2-4$. We wish to find an x_1 for which $m(x_1)=-3$.
$$m(x_1)=\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}=\lim_{x\to x_1}\frac{(3x^2-4)-(3{x_1}^2-4)}{x-x_1}=\lim_{x\to x_1}3\frac{x^2-{x_1}^2}{x-x_1}=\lim_{x\to x_1}3(x+x_1)=6x_1$$
Hence $6x_1=-3$; $x_1=-\frac{1}{2}$ and $f(x_1)=3(-\frac{1}{2})^2-4=-\frac{13}{4}$. The tangent line is $y=-3(x+\frac{1}{2})-\frac{13}{4}=-3x-\frac{19}{4}$.

47. Find an equation of the normal line to the curve $y = 2 - \frac{1}{3}x^2$ that is parallel to the line $x - y = 0$.

▹ The line $x - y = 0$, or $y = x$, has slope 1 so a line perpendicular to it has slope -1.
Let $f(x) = 2 - \frac{1}{3}x^2$. We wish to find an x_1 for which $m(x_1) = -1$.

$$m(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2 - \frac{1}{3}(x_1 + \Delta x)^2] - (2 - \frac{1}{3}{x_1}^2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2x_1 \Delta x - (\Delta x)^2}{3\Delta x} = \lim_{\Delta x \to 0} \frac{-2x_1 - \Delta x}{3} = -\frac{2}{3}x_1$$

We have $-\frac{2}{3}x_1 = -1$ when $x_1 = \frac{3}{2}$ and $f(\frac{3}{2}) = \frac{5}{4}$. Thus the required tangent line is at $(\frac{3}{2}, \frac{5}{4})$ and an equation is $y - \frac{5}{4} = -(x - \frac{3}{2})$, or $4x + 4y - 11 = 0$.

48. Find an equation of each normal line to the curve $y = x^3 - 3x$ that is parallel to the line $2x + 18y - 9 = 0$.

▹ We first find the slope of the tangent line to the given curve. Let $f(x) = x^3 - 3x$.

$$m(x_1) = \lim_{x \to x_1} \frac{(x^3 - 3x) - ({x_1}^3 - 3x_1)}{x - x_1} = \lim_{x \to x_1} \frac{(x - x_1)(x^2 + xx_1 + {x_1}^2) - 3(x - x_1)}{x - x_1} = \lim_{x \to x_1} (x^2 + xx_1 + {x_1}^2 - 3)$$
$$= 3{x_1}^2 - 3$$

Hence the slope of the normal line is $-\dfrac{1}{3{x_1}^2 - 3}$.

We find the slope of the given line by writing the given equation in slope-intercept form.

$$2x + 18y - 9 = 0; \qquad 18y = -2x + 9; \qquad y = -\tfrac{1}{9}x + \tfrac{1}{2}$$

Because $-\frac{1}{9}$ is the slope of the given line and the normal line is parallel to the given line, we have

$$-\frac{1}{9} = -\frac{1}{3{x_1}^2 - 3}; \qquad 3{x_1}^2 - 3 = 9; \qquad {x_1}^2 = 4; \qquad x_1 = \pm 2$$

We have found the x coordinate of each point on the given curve where the normal line is parallel to the given line. If we substitute $x = 2$ in the equation of the curve, we obtain $y = 2^3 - 3(2) = 2$. Thus, $(2, 2)$ is the point where the normal line intersects the given curve. Because the slope of the normal line is $-\frac{1}{9}$, we may write the point-slope form of the equation.

$$y - 2 = -\tfrac{1}{9}(x - 2); \qquad x + 9y - 20 = 0$$

Similarly, if $x = -2$, we have $y = (-2)^3 - 3(-2) = -2$, and

$$y + 2 = -\tfrac{1}{9}(x + 2); \qquad x + 9y + 20 = 0$$

49. Prove that there is no line through the point $(1,5)$ that is tangent to the the parabola $y = 4x^2$.

▹ At any point (x_1, y_1) on the curve we have $y_1 = 4{x_1}^2$. Since a tangent line at $(x_1, 4{x_1}^2)$ has slope $8x_1$ while the line through the points $(x_1, 4{x_1}^2)$ and $(1,5)$ has slope $\dfrac{4{x_1}^2 - 5}{x_1 - 1}$, we must have
$\dfrac{4{x_1}^2 - 5}{x_1 - 1} = 8x_1$; $4{x_1}^2 - 5 = 8{x_1}^2 - 8x_1$; $4{x_1}^2 - 8x_1 + 4 = -1$; $4(x_1 - 1)^2 = -1$
Since the last equation has no solution there can be no line through $(1,5)$ tangent to the curve $y = 4x^2$.

50. Prove that there is no line through the point $(1,2)$ that is tangent to the the parabola $y = 4 - x^2$.

▹ At any point (x_1, y_1) on the curve we have $y_1 = 4 - {x_1}^2$. Since a tangent line at $(x_1, 4 - {x_1}^2)$ has slope $-2x_1$ while the line through the points $(x_1, 4 - {x_1}^2)$ and $(1,2)$ has slope $\dfrac{4 - {x_1}^2 - 2}{x_1 - 1}$, we must have
$\dfrac{2 - {x_1}^2}{x_1 - 1} = -2x_1$; $2 - {x_1}^2 = -2{x_1}^2 + 2x_1$; ${x_1}^2 - 2x_1 + 1 = -1$; $(x_1 - 1)^2 = -1$
Since the last equation has no solution there can be no line through $(1,2)$ tangent to the curve $y = 4 - x^2$.

51. If g is continuous at a and $f(x) = (x - a)g(x)$, find $f'(a)$.

▹ $f'(a) = \displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x - a)g(x) - (a - a)g(a)}{x - a} = \lim_{x \to a} g(x) = g(a)$, since g is continuous at a.

52. If g is continuous at a and $f(x) = (x^2 - a^2)g(x)$, find $f'(a)$.

▹ $f'(a) = \displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x^2 - a^2)g(x) - (a^2 - a^2)g(a)}{x - a} = \lim_{x \to a} (x + a)g(x) = 2ag(a)$ (g is continuous at a

In Exercises 53 and 54, use the formula $f''(x) = \lim\limits_{\Delta x \to 0} \dfrac{f'(x+\Delta x) - f'(x)}{\Delta x} = \lim\limits_{z \to x} \dfrac{f'(z) - f(z)}{z - x}$.

53. $f(x) = \mathrm{a}x^2 + \mathrm{b}x$. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{[\mathrm{a}(x+\Delta x)^2 + \mathrm{b}(x+\Delta x)] - (ax^2 + \mathrm{b}x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\mathrm{a}(2x\Delta x + \Delta x^2) + \mathrm{b}\Delta x}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \mathrm{a}(2x + \Delta x) + \mathrm{b} = 2ax + \mathrm{b}$. $f''(x) = \lim\limits_{\Delta x \to 0} \dfrac{[2a(x+\Delta x) + \mathrm{b}] - (2ax + \mathrm{b})}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{2a\Delta x}{\Delta x} = \lim\limits_{\Delta x \to 0} 2a = 2a$

54. $f(x) = \frac{a}{x}$. $f'(x) = \lim\limits_{z \to x} \frac{1}{z-x}\left(\frac{a}{z} - \frac{a}{x}\right) = \lim\limits_{z \to x} \frac{a}{z-x} \cdot \frac{-(z-x)}{zx} = \lim\limits_{z \to x} \frac{-a}{zx} = -\frac{a}{x^2}$

$f''(x) = \lim\limits_{z \to x} \frac{1}{z-x}\left(\frac{-a}{z^2} - \frac{-a}{x^2}\right) = \lim\limits_{z \to x} \frac{a}{z-x} \cdot \frac{z^2 - x^2}{z^2x^2} = \lim\limits_{z \to x} \frac{a(z+x)}{z^2x^2} = \frac{2ax}{x^4} = \frac{2a}{x^3}$

55. If $f'(a)$ exists, prove that $f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a-\Delta x)}{2\Delta x}$.

▸ $\lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a-\Delta x)}{2\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a) + f(a) - f(a-\Delta x)}{2\Delta x}$

$= \lim\limits_{\Delta x \to 0} \left[\dfrac{f(a+\Delta x) - f(a)}{2} + \dfrac{f(a) - f(a-\Delta x)}{2}\right] = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{2\Delta x} + \lim\limits_{\Delta x \to 0} \dfrac{f(a) - f(a-\Delta x)}{2\Delta x}$

$= \frac{1}{2} \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{2\Delta x} + \frac{1}{2} \lim\limits_{\Delta x \to 0} \dfrac{f(a-\Delta x) - f(a)}{2\Delta x} = -f'(a) + \frac{1}{2} \lim\limits_{(-\Delta x) \to 0} \dfrac{f(a+(-\Delta x)) - f(a)}{2}$

$= \frac{1}{2}f'(a) + \frac{1}{2}f'(a) = f'(a)$, as desired.

56. Let f be a function whose domain is R and (i) $f(a+b) = f(a) \cdot f(b)$ for all a and b. Furthermore, suppose that (ii) $f(0) = 1$ and (iii) $f'(0)$ exists. Prove that $f'(x)$ exist for all x and that $f'(x) = f'(0) \cdot f(x)$.

▸ Let x be any real number. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \overset{\text{i}}{=} \lim\limits_{\Delta x \to 0} \dfrac{f(x)f(\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} f(x) \cdot \dfrac{f(\Delta x) - 1}{\Delta x}$

$\overset{\text{L.T.6}}{=} f(x) \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x) - 1}{\Delta x} \overset{\text{ii}}{=} f(x) \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x) - f(0)}{\Delta x} \overset{\text{iii}}{=} f(x)f'(0)$

57. Plot the parabola $y = \frac{1}{4}x^2$ and its tangent line y_t at $(2,1)$ and explain what happens as you zoom in.

▸ $y'(2) = \lim\limits_{x \to 2} \dfrac{\frac{1}{4}x^2 - \frac{1}{4}2^2}{x-2} = \lim\limits_{x \to 2} \frac{1}{4} \cdot \dfrac{x^2 - 2^2}{x-2} = \lim\limits_{x \to 2} \frac{1}{4}(x+2) = 1$. $y_t = 1 + 1(x-2) = x - 1$

The tangent line and the curve become indistinguishable, a characteristic of the tangent line.

58. Plot the parabola $y = \sqrt{x}$ and its tangent line y_t at $(1,1)$ and explain what happens as you zoom in.

▸ $y'(1) = \lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \frac{1}{2}$. $y_t = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$. See Exercise 57.

2.2 DIFFERENTIABILITY AND CONTINUITY

Differentiable A function is said to be *differentiable* at x_1 if $f'(x_1)$ exists.

2.2.1 Theorem If a function is differentiable at x_1, then f is continuous at x_1.

The theorem implies that if $f'(x_1)$ exists, then there must be no break in the graph of f at the point where $x = x_1$. The converse of Theorem 2.2.1 is not true. That is, a function that is continuous at x_1 may not be differentiable at x_1. For example, the absolute-value function defined by $f(x) = |x|$ is not differentiable at $x = 0$ although f is continuous there. Note that the graph of $|x|$ has a corner at $(0,0)$. If a function is differentiable at a point, then the graph of the function must be smooth at that point. Furthermore, if a function is differentiable at a point, then the tangent line to the graph of the function at that point must not be vertical. That is, if the tangent line to the graph of f at x_1 is vertical, then f is not differentiable at x_1.

A function f defined on an open interval containing x_1 is differentiable if and only if both one-sided derivatives exist and are equal. One-sided derivatives are defined as follows.

2.2.2 Definition If the function f is defined at x_1, then the *derivative from the right* of f at x_1, denoted by $f'_+(x_1)$, is defined by

$$f'_+(x_1) = \lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad \Leftrightarrow \quad f'_+(x_1) = \lim_{x \to x_1^+} \frac{f(x) - f(x_1)}{x - x_1}$$

if the limit exists.

2.2.3 Definition If the function f is defined at x_1, then the *derivative from the left* of f at x_1, denoted by $f'_-(x_1)$, is defined by

$$f'_-(x_1) = \lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad \Leftrightarrow \quad f'_-(x_1) = \lim_{x \to x_1^-} \frac{f(x) - f(x_1)}{x - x_1}$$

if the limit exists.

If f is continuous from the right and $\lim_{x \to x_1^+} f'(x)$ exists, then $\lim_{x \to x_1^+} f'(x) = f'_+(x_1)$.

If f is continuous from the left and $\lim_{x \to x_1^-} f'(x)$ exists, then $\lim_{x \to x_1^-} f'(x) = f'_-(x_1)$.

Exercises 2.2

In Exercises 1–20, do the following: (a) Sketch the graph of the function f. (b) Determine if f is continuous at x_1. (c) Find $f'_-(x_1)$ and $f'_+(x_1)$ if they exist. (d) Determine if f is differentiable at x_1.

1. $f(x) = \begin{cases} x + 2 & \text{if } x \le -4 \\ -x - 6 & \text{if } -4 < x \end{cases}$ $f(-4) = -4 + 2 = -2$.

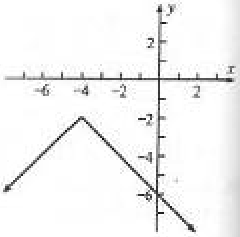
▹ (b) $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} (-x - 6) = -2$; $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} (x + 2) = -2$

Therefore $\lim_{x \to -4} f(x) = -2 = f(-4)$. Thus, f is continuous at -4.

(c) $f'_-(-4) = \lim_{x \to -4^-} \dfrac{f(x) - f(-4)}{x - (-4)} = \lim_{x \to -4^-} \dfrac{(x + 2) - (-2)}{x + 4} = 1$

$f'_+(-4) = \lim_{x \to -4^+} \dfrac{f(x) - f(-4)}{x - (-4)} = \lim_{x \to -4^+} \dfrac{(-x - 6) - (-2)}{x + 4} = -1$

(d) Since $f'_-(-4) \ne f'_+(-4)$, $f'(-4)$ does not exist; so f is not differentiable at -4.



2. $f(x) = \begin{cases} 3 - 2x & \text{if } x < 2 \\ 3x - 7 & \text{if } x \ge 2 \end{cases}$ $f(2) = 3 \cdot 2 - 7 = -1$.

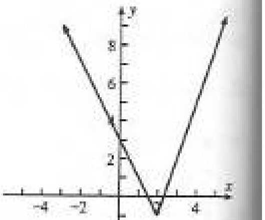
▹ (b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - 2x) = -1$; $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 7) = -1$

Therefore $\lim_{x \to 2} f(x) = -1 = f(2)$. Thus, f is continuous at 2.

(c) $f'_-(2) = \lim_{x \to 2^-} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \dfrac{(3 - 2x) - (-1)}{x - 2} = \lim_{x \to 2^-} \dfrac{4 - 2x}{x - 2} = -2$

$f'_+(2) = \lim_{x \to 2^+} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \dfrac{(3x - 7) - (-1)}{x - 2} = \lim_{x \to 2^+} \dfrac{3x - 6}{x - 2} = 3$

(d) Since $f'_-(2) \ne f'_+(2)$, $f'(2)$ does not exist; so f is not differentiable at 2.



3. $f(x) = |x - 3|$. $f(3) = 0$.

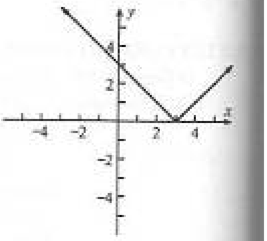
▹ (b) $\lim_{x \to 3^-} f(x) = \lim_{x \to 0} [-(x - 3)] = 0$; $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3) = 0$

Therefore, $\lim_{x \to 3} f(x) = 0 = f(3)$. Thus, f is continuous at 3.

(c) $f'_-(3) = \lim_{x \to 3^-} \dfrac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^-} \dfrac{-(x - 3) - 0}{x - 3} = \lim_{x \to 3^-} (-1) = -1$

$f'_+(3) = \lim_{x \to 3^+} \dfrac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \dfrac{(x - 3) - 0}{x - 3} = \lim_{x \to 3^+} 1 = 1$

(d) Since $f'_-(3) \ne f'_+(3)$, $f'(3)$ does not exist; so f is not differentiable at 3.



4. $f(x) = 1 + |x + 2|$; $x_1 = -2$

▹ $$f(x) = \begin{cases} 1 - (x + 2) & \text{if } x < -2 \\ 1 + (x + 2) & \text{if } x \ge -2 \end{cases} = \begin{cases} -x - 1 & \text{if } x < -2 \\ x + 3 & \text{if } x \ge -2 \end{cases}$$

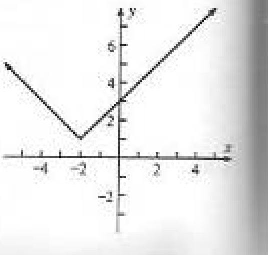
(a) A sketch of the graph is shown at the right.

(b) Because $f(-2) = 1$, and

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x - 1) = 1; \qquad \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x + 3) = 1$$

then f is continuous at -2.

(c) By Definition 2.2.3,



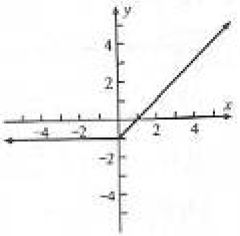
$$f'_-(-2) = \lim_{x\to-2^-} \frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to-2^-} \frac{(-x-1)-1}{x+2} = \lim_{x\to-2^-} \frac{-(x+2)}{x+2} = -1$$

and by Definition 2.2.2

$$f'_+(-2) = \lim_{x\to-2^+} \frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to-2^+} \frac{(x+3)-1}{x+2} = \lim_{x\to-2^+} \frac{x+2}{x+2} = 1$$

(d) Because $f'_-(-2) \neq f'_+(-2)$, then $f'(2)$ does not exist. Thus, f is not differentiable at -2. In the figure, note the corner at $x = -2$.

5. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x-1 & \text{if } 0 \le x \end{cases}$. $f(0) = 0 - 1 = -1$.



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (-1) = -1$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (x-1) = -1$

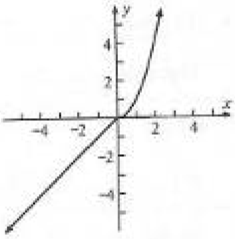
Therefore $\lim\limits_{x\to0} f(x) = -1 = f(0)$. Thus, f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{-1-(-1)}{x} = \lim\limits_{x\to0^-} \dfrac{0}{x} = 0$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{(x-1)-(-1)}{x} = \lim\limits_{x\to0^+} - = 1$

(d) Since $f'_-(0) \neq f'_+(0)$, $f'(0)$ does not exist; so f is not differentiable at 0.

6. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$. $f(0) = 0$



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} x = 0$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} x^2 = 0$

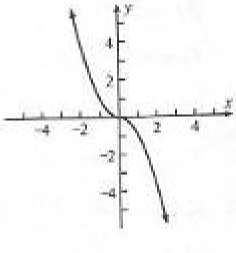
Therefore $\lim\limits_{x\to0} f(x) = 0 = f(0)$. Thus, f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{x-0}{x-0} = 1$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{x^2-0}{x-0} = \lim\limits_{x\to0^+} x = 0$

(d) Since $f'_-(0) \neq f'_+(0)$, $f'(0)$ does not exist; so f is not differentiable at 0.

7. $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ -x^2 & \text{if } 0 < x \end{cases}$. $f(0) = 0^2 = 0$



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} x^2 = 0$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (-x^2) = 0$

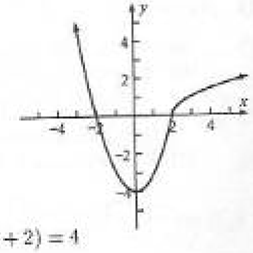
Therefore $\lim\limits_{x\to0} f(x) = 0 = f(0)$. Thus f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{x^2-0}{x} = \lim\limits_{x\to0^-} x = 0$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{-x^2}{x} = \lim\limits_{x\to0^+} (-x) = 0$

(d) Since $f'_+(0) = f'_-(0) = 0$, $f'(0) = 0$; so f is differentiable at 0.

8. $f(x) = \begin{cases} x^2-4 & \text{if } x < 2 \\ \sqrt{x-2} & \text{if } x \ge 2 \end{cases}$; $x_1 = 2$



▸ (a) A sketch of the graph is shown at the right.

(b) Because $f(2) = 0$ and

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (x_2 - 4) = 0; \qquad \lim_{x\to2^+} f(x) = \lim_{x\to2^+} \sqrt{x-2} = 0$$

then f is continuous at 2.

(c) By Definition 2.2.3,

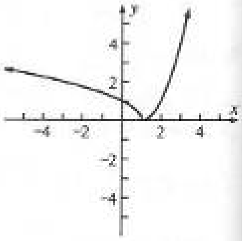
$$f'_-(2) = \lim_{x\to2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to2^-} \frac{(x^2-4)-0}{x-2} = \lim_{x\to2^-} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to2^-} (x+2) = 4$$

By Definition 2.2.2,

$$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{\sqrt{x-2}-0}{x-2} = \lim_{x\to 2^+} \frac{1}{\sqrt{x-2}} = +\infty$$

(d) Because $f'_+(2) = +\infty$, $f'(2)$ does not exist, and f is not differentiable at 2. In the figure, note the corner at $x = 2$.

9. $f(x) = \begin{cases} \sqrt{1-x} & \text{if } x < 1 \\ (1-x)^2 & \text{if } x \geq 1 \end{cases}$. $f(1) = (1-1)^2 = 0$.



▷ (b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = 0$; $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x)^2 = 0$

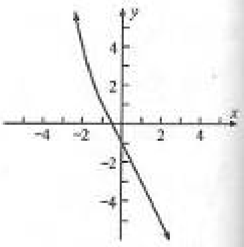
Therefore $\lim_{x\to 1} f(x) = 0 = f(1)$. Thus f is continuous at 0.

(c) $f'_-(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{\sqrt{1-x}-0}{-(1-x)} = \lim_{x\to 1^-} \frac{-1}{\sqrt{1-x}} = -\infty$

$f'_+(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(1-x)^2-0}{x-1} = \lim_{x\to 1^+} (x-1) = 0$

(d) Since $f'_-(1) \neq f'_+(1)$, $f'(1)$ does not exist; so f is not differentiable at 1.

10. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ -1-2x & \text{if } x \geq -1 \end{cases}$. $f(-1) = -1 - 2(-1) = 1$



▷ (b) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 = 1$; $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-1-2x) = 1$

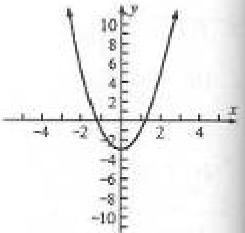
Therefore $\lim_{x\to -1} f(x) = 1 = f(-1)$. Thus f is continuous at -1.

(c) $f'_-(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^-} \frac{x^2-1}{x+1} = \lim_{x\to -1^-} (x-1) = -2$

$f'_+(-1) = \lim_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{(-1-2x)-1}{x+1} = \lim_{x\to -1^+} \frac{-2x-2}{x+1} = -2$

(d) Since $f'_-(-1) = f'_+(-1) = -2$, $f'(-1) = -2$; so f is differentiable at -1.

11. $f(x) = \begin{cases} 2x^2-3 & \text{if } x \leq 2 \\ 8x-11 & \text{if } 2 < x \end{cases}$. $f(2) = 2(2)^2 - 3 = 5$.



(b) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x^2-3) = 5$; $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (8x-1) = 5$

Therefore $\lim_{x\to 2} f(x) = 5 = f(2)$. Thus f is continuous at 2.

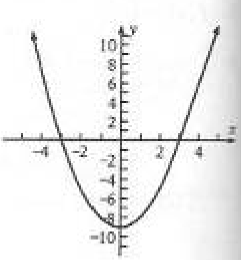
(c) $f'_-(2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{(2x^2-3)-5}{x-2} = \lim_{x\to 2^-} \frac{2(x-2)(x+2)}{x-2}$

$= \lim_{x\to 2^-} 2(x+2) = 8$

$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(8x-11)-5}{x-2} = \lim_{x\to 2^+} \frac{8(x-2)}{x-2} = 8$

(d) Since $f'_-(2) = f'_+(2) = 8$, $f'(2) = 8$; so f is differentiable at 2.

12. $f(x) = \begin{cases} x^2-9 & \text{if } x < 3 \\ 6x-18 & \text{if } 3 \leq x \end{cases}$; $x_1 = 3$



▷ (a) A sketch of the graph is shown at the right.

(b) Because $f(3) = 6(3) - 18 = 0$, and

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-9) = 0$; $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x-18) = 0$

then f is continuous at 3.

(c) By Definition 2.2.3 and by Definition 2.2.2

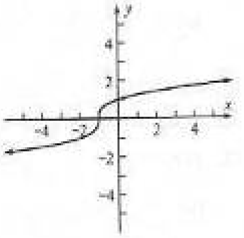
$f'_-(3) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$ and $f'_+(3) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$

$$= \lim_{x\to 3^-} \frac{(x^2-9)-0}{x-3} \qquad\qquad = \lim_{x\to 3^+} \frac{(6x-18)-0}{x-3}$$

$$= \lim_{x\to 3^-} (x+3) = 6 \qquad\qquad = \lim_{x\to 3^+} 6 = 6$$

(d) Because $f'_-(3) = f'_+(3)$, then $f'(3)$ exists, and f is differentiable at 3. In the figure, note that the curve is smooth at $x = 3$.

13. $f(x) = (x+1)^{1/3}$



(b) $\lim_{x\to -1} f(x) = \lim_{x\to -1} (x+1)^{1/3} = 0 = f(-1)$; so f is continuous at -1.

(c) $f'_-(-1) = f'_+(-1)$

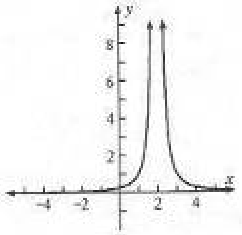
$$= \lim_{x\to -1} \frac{f(x)-f(-1)}{x+1} = \lim_{x\to -1} \frac{(x+1)^{1/3}-0}{x+1} = \lim_{x\to -1} \frac{1}{(x+1)^{2/3}} = +\infty$$

(d) $f'_-(-1)$, $f'_+(-1)$ and hence $f'(-1)$, do not exist;

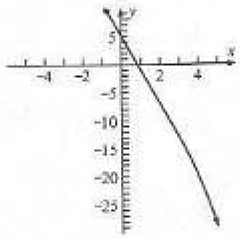
so f is not differentiable at -1. Note that there is a vertical tangent at 0.

14. $f(x) = (x-2)^{-2}$

▹ f is not defined at 2 so f is not continuous, neither derivative exists and f is not differentiable at 2.



15. $f(x) = \begin{cases} 5-6x & \text{if } x \le 3 \\ -4-x^2 & \text{if } 3 < x \end{cases}$. $f(3) = 5 - 6(3) = -13$.



(b) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (5-6x) = -13$; $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-4-x^2) = -13$

Therefore $\lim_{x\to 3} f(x) = -13 = f(3)$. Thus, f is continuous at 3.

(c) $f'_-(3) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^-} \frac{(5-6x)-(-13)}{x-3} = \lim_{x\to 3^-} \frac{-6x+18}{x-3}$

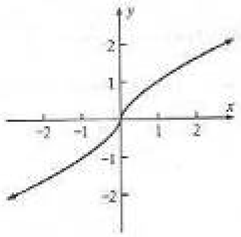
$$= \lim_{x\to 3^-} \frac{-6(x-3)}{x-3} = \lim_{x\to 0} (-6) = -6$$

$$f'_+(3) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^+} \frac{(-4-x^2)-(-13)}{x-3} = \lim_{x\to 3^+} \frac{-x^2+9}{x-3}$$

$$= \lim_{x\to 3^+} \frac{-(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} [-(x+3)] = -6$$

(d) Since $f'_-(3) = f'_+(3) = -6$, $f'(3) = -6$; so f is differentiable at 3.

16. $f(x) = \begin{cases} -x^{2/3} & \text{if } x \le 0 \\ x^{2/3} & \text{if } x > 0 \end{cases} \qquad x_1 = 0$



▹ (a) A sketch of the graph is shown at the right.

(b) Because $f(0) = 0$, and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x^{2/3}) = 0; \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{2/3} = 0$$

then f is continuous at 0.

(c) By Definitions 2.2.3 and 2.2.2

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \quad\text{and}\quad f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0^-} \frac{-x^{2/3}}{x} \qquad\qquad = \lim_{x\to 0^+} \frac{x^{2/3}}{x}$$

$$= \lim_{x\to 0^-} \frac{-1}{x^{1/3}} \qquad\qquad = \lim_{x\to 0^+} \frac{1}{x^{1/3}}$$

$$= +\infty \qquad\qquad = +\infty$$

(d) Because $f'_-(0) = +\infty$, then $f'(0)$ does not exist, and f is not differentiable at 0. Note that there is a vertical tangent at the point where $x = 0$.

17. $f(x) = \begin{cases} x-2 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \end{cases}$. $f(0) = 0^2 = 0$

(b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x-2) = -2$; $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0$

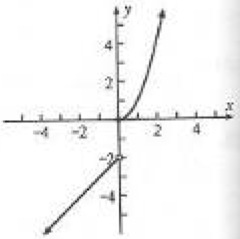
Since $\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist;

so f is not continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{(x-2)-0}{x} = -\infty$

$f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{x^2-0}{x} = \lim\limits_{x\to 0^+} x = 0$

(d) f is not continuous at 0, so f is not differentiable at 0.



18. $f(x) = \begin{cases} x^3 & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$. $f(1) = 1^3 = 1$

(b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^3 = 1$; $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x+1) = 2$

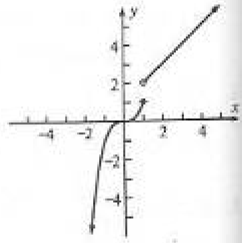
Since $\lim\limits_{x\to 1^-} f(x) \ne \lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$ does not exist;

so f is not continuous at 1.

(c) $f'_-(1) = \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1^-} (x^2+x+1) = 3$

$f'_+(1) = \lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{(x+1)-1}{x-1} = +\infty$

(d) f is not continuous at 1, so f is not differentiable at 1.



19. $f(x) = \begin{cases} 3x^2 & \text{if } x \le 2 \\ x^3 & \text{if } 2 < x \end{cases}$. $f(2) = 3(2)^2 = 12$

(b) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 3x^2 = 12$; $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} x^3 = 8$

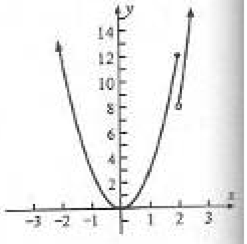
Since $\lim\limits_{x\to 2^-} f(x) \ne \lim\limits_{x\to 2^+} f(x)$, f is not continuous at 2.

(c) $f'_-(2) = \lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^-} \dfrac{3x^2-12}{x-2} = \lim\limits_{x\to 2^-} \dfrac{3(x-2)(x+2)}{x-2}$

$= \lim\limits_{x\to 2^-} 3(x+2) = 12$

$f'_+(2) = \lim\limits_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^+} \dfrac{x^3-12}{x-2} = -\infty$

(d) f is not continuous at 2, so f is not differentiable at 2.



20. $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 1-x^2 & \text{if } -1 \le x \end{cases}$ $\qquad x_1 = -1$

▹ (a) A sketch of the graph is shown at the right.

(b) Because

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (x^2+1) = 2 \qquad \lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (1-x^2) = 0$

then $\lim\limits_{x\to -1} f(x)$ does not exist, so f is discontinuous at -1.

Note that there is a break in the graph at the point where $x = -1$.

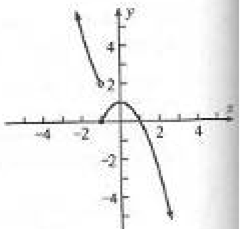
(c) Because $f(-1) = 1-(-1)^2 = 0$, by Definitions 2.2.3 and 2.2.2

$f'_-(-1) = \lim\limits_{x\to -1^-} \dfrac{f(x)-f(-1)}{x-(-1)}$ and $f'_+(-1) = \lim\limits_{x\to -1^+} \dfrac{f(x)-f(-1)}{x-(-1)}$

$= \lim\limits_{x\to -1^-} \dfrac{(x^2+1)-0}{x+1} \qquad = \lim\limits_{x\to -1^+} \dfrac{(1-x^2)-0}{x+1}$

$= -\infty \qquad = \lim\limits_{x\to -1^+} (1-x) = 2$

(d) Because f is discontinuous at -1 or because $f'(-1)$ does not exist, then f is not differentiable at -1.



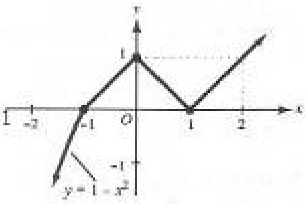
In Exercises 21–26, (a) Define piecewise the continuous function shown in the figure; assume that each part of the graph that appears to be a line segment is a line segment. Find (b) $f'_-(-1)$, (c) $f'_+(-1)$, (d) $f'_-(0)$, (e) $f'_+(0)$, (f) $f'_-(1)$, and (g) $f'_+(1)$. (h) At what numbers is f not differentiable?

▸ The derivative at either end of a segment is its slope.

21. (a) $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le -1 \\ x + 1 & \text{if } -1 < x \le 0 \\ -x + 1 & \text{if } 0 < x \le 1 \\ x - 1 & \text{if } x \ge 1 \end{cases}$

(b) $f'_-(-1) = \lim\limits_{x \to -1^-} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim\limits_{x \to -1^-} \dfrac{1 - x^2 - 0}{x + 1} = \lim\limits_{x \to -1^-} (1 - x) = 2$

(c) and (d) $f'_+(-1) = f'_-(0) = 1$ (e) and (f) $f'_+(0) = f'_-(1) = -1$ (g) $f'_+(1) = 1$



(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$;

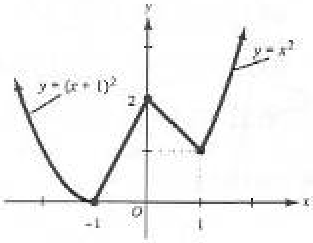
at 0 because $f'_-(0) \ne f'_+(0)$; at 1 because $f'_-(1) \ne f'_+(1)$

22. (a) $f(x) = \begin{cases} (x + 1)^2 & \text{if } x \le -1 \\ 2x + 2 & \text{if } -1 < x \le 0 \\ -x + 2 & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

(b) $f'_-(-1) = \lim\limits_{x \to -1^-} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim\limits_{x \to -1^-} \dfrac{(x + 1)^2 - 0}{x + 1} = \lim\limits_{x \to -1^-} (x + 1) = 0$

(c) and (d) $f'_+(-1) = f'_-(0) = 2$ (e) and (f) $f'_+(0) = f'_-(1) = -1$

(g) $f'_+(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1^+} (x + 1) = 2$



(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$;

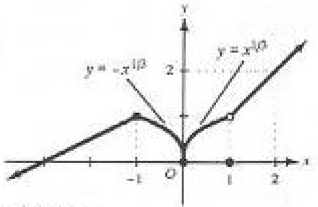
at 0 because $f'_-(0) \ne f'_+(0)$; at 1 because $f'_-(1) \ne f'_+(1)$

23. (a) If $x \le -1$, $m = \dfrac{1 - 0}{-1 - (-2)} = \frac{1}{2}$. $f(x) = \begin{cases} \frac{1}{2}(x + 3) & \text{if } x \le -1 \\ -x^{1/3} & \text{if } -1 < x \le 0 \\ x^{1/3} & \text{if } 0 < x \le 1 \\ x & \text{if } x > 1 \end{cases}$ (b) $f'_-(-1) = \frac{1}{2}$ (c) $f'_+(-1)$

$= \lim\limits_{x \to -1^+} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim\limits_{x \to -1^+} \dfrac{-x^{1/3} - 1}{x + 1} = \lim\limits_{x \to -1^+} \dfrac{-1}{(x^{2/3} - x^{1/3} + 1)} = -\dfrac{1}{3}$

(d) $f'_-(0) = \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{-x^{1/3} - 0}{x} = \lim\limits_{x \to 0^-} \dfrac{-1}{x^{2/3}} = -\infty$

(e) $f'_+(0) = \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{x^{1/3} - 0}{x} = \lim\limits_{x \to 0^+} \dfrac{1}{x^{2/3}} = +\infty$



(f) $f'_-(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{x^{1/3} - 1}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{1}{x^{2/3} + x^{1/3} + 1} = \dfrac{1}{3}$ (g) $f'_+(1) = 1$

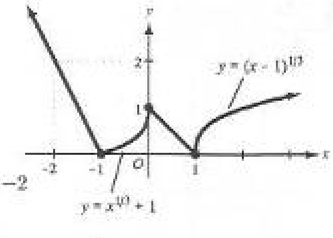
(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$; at 0 because $f'_-(0)$ and $f'_+(0)$ do not exist (there is a vertical tangent); at 1 because $f'_-(1) \ne f'_+(1)$.

24. (a) For $x \le -1$, slope $= \dfrac{0 - 2}{-1 - (-2)} = \dfrac{-2}{1} = -2$. Applying the point-slope formula at $(-1, 0)$, $y = -2(x - (-1)) + 0 = -2x - 2$.

For $0 < x \le 1$, slope $= \frac{0 - 1}{1 - 0} = -1$. Applying the point-slope formula at $(0, 1)$, $y = -1(x - 0) + 1 = -x + 1$. Therefore,

$$f(x) = \begin{cases} -2x - 2 & \text{if } x \le -1 \\ x^{1/3} + 1 & \text{if } -1 < x \le 0 \\ -x + 1 & \text{if } 0 < x \le 1 \\ (x - 1)^{1/3} & \text{if } x > 1 \end{cases}$$

(b) $f'_-(-1) = \text{slope}(-2x - 2) = -2$



(c) $f'_+(-1) = \lim \dfrac{f(x) - f(-1)}{} = \lim \dfrac{x^{1/3} + 1 - 0}{} = \lim \dfrac{1}{} = \dfrac{1}{}$

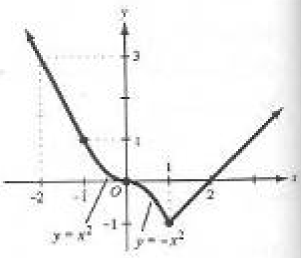
(d) $f'_-(0) = \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-}\dfrac{(x^{1/3}+1)-1}{x} = \lim\limits_{x\to 0^-}\dfrac{1}{x^{2/3}} = +\infty$

(e) and (f) $f'_-(0) = f'_+(1) = \text{slope}(-x+1) = -1$

(g) $f'_+(1) = \lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+}\dfrac{(x-1)^{1/3}-0}{x-1} = \lim\limits_{x\to 1^+}\dfrac{1}{(x-1)^{2/3}} = +\infty$

(h) f is not differentiable at -1 because $f'_-(-1) \neq f'_+(-1)$;
at 0 because $f'_-(0)$ does not exist; at 1 because $f'_+(1)$ does not exist.

25. (a) $f(x) = \begin{cases} -2x-1 & \text{if } x \le -1 \\ x^2 & \text{if } -1 < x \le 0 \\ -x^2 & \text{if } 0 < x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$ (b) $f'_-(-1) = -2$



(c) $f'_+(-1) = \lim\limits_{x\to -1^-}\dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^+}\dfrac{x^2-1}{x+1} = \lim\limits_{x\to -1^+}(x-1) = -2$

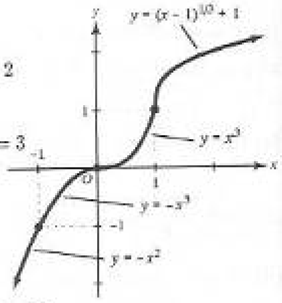
(d) $f'_-(0) = \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-}\dfrac{x^2-0}{x} = \lim\limits_{x\to 0^-} x = 0$

(e) $f'_+(0) = \lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+}\dfrac{-x^2-0}{x} = \lim\limits_{x\to 0^+}(-x) = 0$

(f) $f'_-(1) = \lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-}\dfrac{-x^2-(-1)}{x-1} = \lim\limits_{x\to 1^-} -(x+1) = -2$

(g) $f'_+(1) = 1$ (h) f is not differentiable at 1 because $f'_-(1) \neq f'_+(1)$.

26. (a) $f(x) = \begin{cases} -x^2 & \text{if } x \le -1 \\ x^3 & \text{if } -1 < x \le 1 \\ (x-1)^{1/3}+1 & \text{if } x > 1 \end{cases}$



(b) $f'_+(-1) = \lim\limits_{x\to -1^-}\dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^+}\dfrac{-x^2-(-1)}{x+1} = \lim\limits_{x\to -1^+} -(x-1) = 2$

(c) $f'_+(-1) = \lim\limits_{x\to -1^-}\dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^+}\dfrac{x^3-(-1)}{x+1} = \lim\limits_{x\to -1^+}(x^2-x+1) = 3$

(d) and (e) $f'_-(0) = f'_+(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^3-0}{x} = \lim\limits_{x\to 0} x^2 = 0$

(f) $f'_-(1) = \lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-}\dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1^-}(x^2+x+1) = 3$

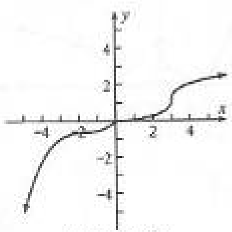
(g) $f'_+(1) = \lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+}\dfrac{[(x-1)^{1/3}+1]-1}{x-1} = \lim\limits_{x\to 1^+}\dfrac{1}{(x-1)^{2/3}} = +\infty$

(h) f is not differentiable at -1 because $f'_-(-1) \neq f'_+(-1)$; at 1 because $f'_+(1)$ does not exist.

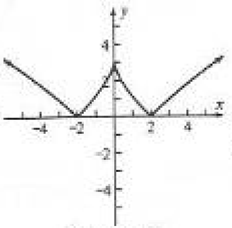
In Exercises 27–30, sketch the graph of a continuous function f defined on R and having the given properties.

27. The range of f is $(-\infty, +\infty)$, and f is differentiable at every number except 0 and 3; $f(-3) = -1$; $f(0) = 0$; $f(3) = 1$; $f'_-(0) = 1$; $f'_+(0) = 0$; $\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3} = +\infty$

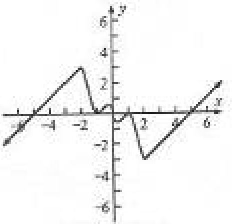
28. The range of f is $[0, +\infty)$; f is differentiable at every number except $-2, 0, 2$; $f(-2) = 0$; $f(0) = 3$; $f(2) = 0$; $f'_-(-2) = -1$; $f'_+(-2) = 1$; $f'_-(2) = -1$; $f'_+(2) = 1$; $\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x} = +\infty$; $\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x} = -\infty$



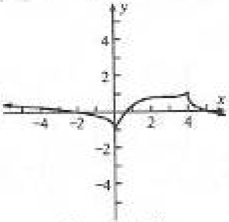
Exercise 27



Exercise 28



Exercise 29



Exercise 30

The range of f is R, f' exists except at $-2, 0, 2$; $f(-2)=0$; $f(-1)=0$; $f(0)=0$; $f(1)=0$; $f(2)=-3$; $f'_-(-2)=1$; $f'_+(-2)=-1$; $f'_-(2)=-1$; $f'_+(2)=1$; $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=-\infty$; $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=-\infty$

The range of f is $(-\infty,+\infty)$; f is differentiable at every number except 0 and 4; $f(-2)=0$; $f(0)=-1$; $f(4)=1$; $f(5)=0$; $f'_+(0)=2$; $f'_-(4)=\frac{1}{2}$; $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=-\infty$; $\lim_{x\to 4^+}\frac{f(x)-f(4)}{x-4}=-\infty$

Is the oil spill of Ex. 1.8.53 differentiable at 2? At t min, $r(t)$ m is the radius. $r(t)=\begin{cases}4t^2+20 & \text{if } 0\le t\le 2\\ 16t+4 & \text{if } t>2\end{cases}$

$r'_-(2)=\lim_{t\to 2^-}\frac{f(t)-f(2)}{t-2}=\lim_{t\to 2^-}\frac{(4t^2+20)-36}{t-2}=\lim_{t\to 2^-}\frac{4t^2-16}{t-2}=\lim_{t\to 2^-}4(t+2)=16$

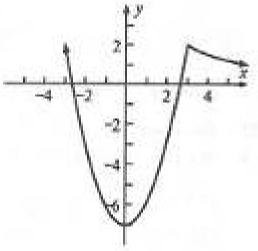
$r'_+(2)=\lim_{t\to 2^+}\frac{f(t)-f(2)}{t-2}=\lim_{t\to 2^+}\frac{(16t+4)-36}{t-2}=\lim_{t\to 2^+}16=16$. Since $r'_-(2)=r'_+(2)$, r is differentiable at 2.

$f(x)=x^2-1$ if $x<-1$ and $1-x^2$ if $-1\le x\le 1$. $\lim_{x\to -1^-}f(x)=\lim_{x\to -1}(x^2-1)=0=\lim_{x\to -1^+}f(x)=\lim_{x\to -1}1-x^2$ $=f(-1)\Rightarrow$continuous. $f'_-(-1)=\lim_{x\to -1^-}\frac{(x^2-1)-0}{x+1}=\lim_{x\to -1}(x-1)=-2$ but $f'_+(-1)=\lim_{x\to -1^+}\frac{(1-x^2)-0}{x+1}=2$

$f(x)=\begin{cases}x^2-7 & \text{if } 0<x\le b\\ 6/x & \text{if } b<x\end{cases}$ (a) Find b so f is continuous at b. (b) Sketch. (c) Is f differentiable at b?

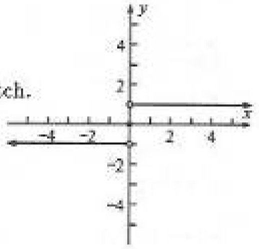
(a) $f(b)=b^2-7$; $\lim_{x\to b^-}f(x)=\lim_{x\to b^-}(x^2-7)=b^2-7$; $\lim_{x\to b^+}f(x)=\lim_{x\to b^+}\frac{6}{x}=\frac{6}{b}$

f will be continuous at b if $6/b=b^2-7$; $b^3-7b-6=0$; $(b-3)(b+1)(b+2)=0$. Thus f is continuous at b if $b=3$, -1, or -2; since $b>0$, only $b=3$ is admissible.



(c) $f'_-(3)=\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3^-}\frac{x^2-7-2}{x-3}=\lim_{x\to 3^-}\frac{(x-3)(x+3)}{x-3}$ $=\lim_{x\to 3^-}(x+3)=6$. $f'_+(3)=\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3^+}\frac{6/x-2}{x-3}=\lim_{x\to 3^+}\frac{6-2x}{x(x-3)}$ $=\lim_{x\to 3^+}\frac{-2}{x}=-\frac{2}{3}$. Hence $f'(3)$ does not exist, and f is not differentiable at 3.

$f(x)=\operatorname{sgn}x=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ +1 & \text{if } x>0\end{cases}$. Prove (a) $f'_-(0)$ and $f'_+(0)$ do not exist. (b) $\lim_{x\to 0^-}f'(x)=0$ and $\lim_{x\to 0^+}f'(x)=0$. (c) Sketch.



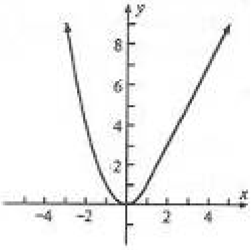
(a) $f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{-1-0}{x}=+\infty$;

$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{1-0}{x}=+\infty$

(b) $\lim_{x\to 0^-}f'(x)=\lim_{x\to 0^-}\left[\frac{d}{dx}(-1)\right]=\lim_{x\to 0^-}\left[\lim_{\Delta x\to 0}\frac{(-1)-(-1)}{\Delta x}\right]=\lim_{x\to 0^-}0=0$

$\lim_{x\to 0^+}f'(x)=\lim_{x\to 0^+}\left[\frac{d}{dx}(1)\right]=\lim_{x\to 0^+}\left[\lim_{\Delta x\to 0}\frac{1-1}{\Delta x}\right]=\lim_{x\to 0^+}\ =0$

$f(x)=\begin{cases}x^2 & \text{if } x<1\\ ax+b & \text{if } x\ge 1\end{cases}$. Find a and b such that f is differentiable at 1. Sketch.



For continuity we need $1^2=a\cdot 1+b$.

$f'_-(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x^2-1}{x-1}=\lim_{x\to 1^-}(x+1)=2$;

$f'_+(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\frac{(ax+b)-(a+b)}{x-1}=a$

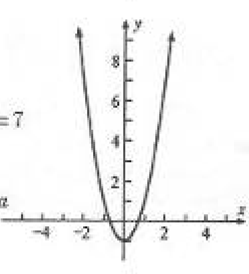
f will be differentiable at 1 if $a=2$. Since $a+b=1$, then $b=-1$

$f(x)=\begin{cases}ax+b & \text{if } x<2\\ 2x^2-1 & \text{if } 2\le x\end{cases}$. Find a and b such that f is differentiable at 2. Sketch.

$f(2)=2(2)^2-1=7$; $\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(ax+b)=2a+b$; $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(2x^2-1)=7$

For f to be differentiable at 2, f must be continuous at 2 so $2a+b=7$.

$f'_-(2)=\lim_{x\to 2^-}\frac{f(x)}{x-2}=\lim_{x\to 2^-}\frac{(ax+b)-7}{x-2}=\lim_{x\to 2^-}\frac{(ax+b)-(2a+b)}{x-2}=\lim_{x\to 2^-}\frac{a(x-2)}{x-2}=a$



$$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(2x^2-1)-7}{x-2} = \lim_{x\to 2^+} \frac{2(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} 2(x+2) = 8$$

f will be differentiable at 2 if $f'_-(2) = f'_+(2)$; so $a = 8$. Since $2a + b = 7$, $b = -9$.

In Exercises 37–40, treat the variable representing a nonnegative integer as if it was a nonnegative real number.

37. A trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.05 per student for each student over 150. (a) Express the income as function f of the number of students. (b) Prove that f is continuous on its domain. (c) If f differentiable at 150?

▷ (a) Let $f(x)$ dollars be the gross income if x students make the trip.

$$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ [15 - .05(x-150)]x & \text{if } 150 < x \le 250 \end{cases} = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 22.5x - .05x^2 & \text{if } 150 < x \le 250 \end{cases}$$

(b) $f(150) = 15(150) = 2250$; $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 150^-} 15x = 15(150) = 2250$

$$\lim_{x\to 150^+} f(x) = \lim_{x\to 150^+} [15 - 0.05(x-150)]x = 15(150) = 2250$$

Therefore, $\lim\limits_{x\to 150} f(x) = 2250 = f(150)$. Thus f is continuous at 150.

$$f'_-(150) = \lim_{x\to 150^-} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^-} \frac{15x-2250}{x-150} = \lim_{x\to 150^-} \frac{15(x-150)}{x-150} = \lim_{x\to 150^-} 15 = 15$$

$$f'_+(150) = \lim_{x\to 150^+} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^+} \frac{22.5x - .05x^2 - 2250}{x-150} = \lim_{x\to 150^+} \frac{-.05(x^2 - 450x + 45000)}{x-150}$$

$$= \lim_{x\to 150^+} \frac{-.05(x-150)(x-300)}{x-150} = \lim_{x\to 150^+} -.05(x-300) = 7.5$$

Since $f'_-(150) \ne f'_+(150)$, $f'(150)$ does not exist.

38. Do Exercise 37 if the reduction per student in excess of 150 is \$0.07.

▷ (a) Let $f(x)$ dollars be the gross income if x students make the trip.

$$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ [15 - .07(x-150)]x & \text{if } 150 < x \le 250 \end{cases} = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 25.5x - .07x^2 & \text{if } 150 < x \le 250 \end{cases}$$

(b) $f(150) = 15(150) = 2250$; $\lim\limits_{x\to 150^-} f(x) = \lim\limits_{x\to 150^-} 15x = 15(150) = 2250$

$$\lim_{x\to 150^+} f(x) = \lim_{x\to 150^+} [15 - 0.05(x-150)]x = 15(150) = 2250$$

Therefore, $\lim\limits_{x\to 150} f(x) = 2250 = f(150)$. Thus f is continuous at 150.

$$f'_-(150) = \lim_{x\to 150^-} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^-} \frac{15x-2250}{x-150} = \lim_{x\to 150^-} \frac{15(x-150)}{x-150} = \lim_{x\to 150^-} 15 = 15$$

$$f'_+(150) = \lim_{x\to 150^+} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^+} \frac{25.5x - .07x^2 - 2250}{x-150} \overset{\text{long division}}{=} \lim_{x\to 150^+} (-.07x + 15) = 4.5$$

Since $f'_-(150) \ne f'_+(150)$, $f'(150)$ does not exist.

39. Each tree produces 600 oranges, at up to 20 trees per acre. For each additional tree per acre, yield per tree decreases by 15 oranges. (a) Express the number of oranges as a function $f(x)$ of the number x of trees per acre. (b) Prove f is continuous. (c) Is f differentiable at 20?

▷ (a) $f(x) = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ [600 - 15(x-20)]x & \text{if } x > 20 \end{cases} = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ 900x - 15x^2 & \text{if } 20 < x \le 60 \end{cases}$

(b) $f(20) = 600(20) = 12000$; $\lim\limits_{x\to 20^-} f(x) = \lim\limits_{x\to 20^-} 600x = 12000$; $\lim\limits_{x\to 20^+} f(x) = \lim\limits_{x\to 20^+} (900x - 15x^2) = 12000$

Therefore, $\lim\limits_{x\to 20} f(x) = 12000 = f(20)$. Thus f is continuous at 100.

$$f'_-(20) = \lim_{x\to 20^-} \frac{f(x)-f(20)}{x-20} = \lim_{x\to 20^-} \frac{600x-12000}{x-20} = \lim_{x\to 20^-} \frac{600(x-20)}{x-20} = \lim_{x\to 20^-} 600 = 600$$

$$f'_+(20) = \lim_{x\to 20^+} \frac{f(x)-f(20)}{x-20} = \lim_{x\to 20^+} \frac{900x - 15x^2 - 12000}{x-20} \overset{\text{long division}}{=} \lim_{x\to 20^+} (-15x + 600) = 300$$

Since $f'_-(20) \ne f'_+(20)$, $f'(20)$ does not exist.

40. A club's annual dues is \$100 per member, less \$0.50 for each member over 600 and plus \$0.50 for each member less than 600. (a) Find a mathematical model expressing the club's revenue $f(x)$ as a function of the number x of members. (b) Prove that f is continuous on its domain. (c) Is f differentiable at 600?

» (a) $f(x) = x[100 - \frac{1}{2}(x - 600)] = x(400 - \frac{1}{2}x) = 400x - \frac{1}{2}x^2,\ 0 \le x \le 800$

(b) and (c) Because f is a polynomial it is continuous and differentiable on $(0, 800)$.

Let $f(x) = |x|$. Find $f'(x)$, $x \ne 0$.

» If $x \ne 0$, then since $|x| = \sqrt{x^2}$ we have $|x|^2 = x^2$ so

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} = \lim_{\Delta x \to 0} \frac{(|x+\Delta x| - |x|)(|x+\Delta x| + |x|)}{\Delta x(|x+\Delta x| + |x|)}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x(|x+\Delta x| + |x|)} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x(|x+\Delta x| + |x|)} = \lim_{\Delta x \to 0} \frac{2x + \Delta x}{|x+\Delta x| + |x|} = \frac{2x}{2|x|} = \frac{|x|}{x}$$

Given $f(x) = [\![x]\!]$, find $f'(x_1)$ if x_1 is not an integer. Prove that $f'(x_1)$ does not exist if x_1 is an integer. If x_1 is an integer, what can you say about $f'_-(x_1)$ and $f'_+(x_1)$?

» If x_1 is not an integer, then there is an integer N such that $\mathrm{N} < x_1 < \mathrm{N} + 1$. Hence, by definition, $f(x_1) = \mathrm{N}$. Furthermore, for all x such that $\mathrm{N} < x < \mathrm{N} + 1$, $f(x) = \mathrm{N}$. Thus, whenever $\mathrm{N} < x < \mathrm{N} + 1$, $f(x) - f(x_1) = 0$.

Therefore, $f'(x_1) = \lim\limits_{x \to x_1} \dfrac{f(x) - f(x_1)}{x - x_1} = \lim\limits_{x \to x_1} \dfrac{0}{x - x_1} = 0$

Thus, if x_1 is not an integer, $f'(x_1) = 0$. Furthermore, if x_1 is an integer, then

$$\lim_{x \to x_1^+} f(x) = x_1 \quad \text{and} \quad \lim_{x \to x_1^-} f(x) = x_1 - 1$$

Hence $\lim\limits_{x \to x_1} f(x)$ does not exist, and thus f is discontinuous at x_1. By Theorem 2.2.1, this proves that $f'(x_1)$ does not exist if x_1 is an integer. However, if x_1 is an integer, then

$$f'_-(x_1) = \lim_{x \to x_1^-} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1^-} \frac{(x_1 - 1) - x_1}{x - x_1} = \lim_{x \to x_1^-} \frac{-1}{x - x_1} = +\infty$$

$$f'_+(x_1) = \lim_{x \to x_1^+} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1^+} \frac{x_1 - x_1}{x - x_1} = \lim_{x \to x_1^+} \frac{0}{x - x_1} = 0$$

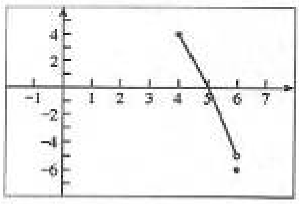
$f(x) = (x - 1)[\![x]\!]$

» (a) $f'_-(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{(x-1)[\![x]\!] - 0}{x - 1} = \lim\limits_{x \to 1^-} [\![x]\!] = 0$

(b) $f'_+(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} [\![x]\!] = 1$ (c) $f'_-(1) \ne f'_+(1)$; so $f'(1)$ does not exist.

$f(x) = (5 - x)[\![x]\!]$. Plot the graph of f for x in $[4, 6]$. Find, if they exist: (a) $f'_-(5)$; (b) $f'_+(5)$; (c) $f'(5)$.

» $f(x) = \begin{cases} (5-x)4 & \text{if } 4 \le x < 5 \\ (5-x)5 & \text{if } 5 \le x < 6 \\ -6 & \text{if } x = 6 \end{cases} \qquad f(5) = 0.$



(a) $f'_-(5) = \lim\limits_{x \to 5^-} \dfrac{f(x) - f(5)}{x - 5} = \lim\limits_{x \to 5^-} \dfrac{(5-x)4 - 0}{x - 5} = \lim\limits_{x \to 5^-} (-4) = -4$

(b) $f'_+(5) = \lim\limits_{x \to 5^+} \dfrac{f(x) - f(5)}{x - 5} = \lim\limits_{x \to 5^+} \dfrac{(5-x)5 - 0}{x - 5} = \lim\limits_{x \to 5^+} (-5) = -5$

Because $f'_-(5) \ne f'_+(5)$, then $f'(5)$ does not exist.

$f(x) = (x - \mathrm{a})[\![x]\!]$; a is any integer.

$f'_-(a) = \lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^-} \dfrac{(x - \mathrm{a})[\![x]\!] - 0}{x - a} = \lim\limits_{x \to a^-} [\![x]\!] = \mathrm{a} - 1$

$f'_+(a) = \lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^+} [\![x]\!] = \mathrm{a}$; so $f'_-(a) + 1 = f'_+(a)$.

Let the f be defined by $f(x) = \begin{cases} \dfrac{g(x) - g(a)}{x - a} & \text{if } x \ne a \text{ (i)} \\ g'(a) & \text{if } x = a \text{ (ii)} \end{cases}$. Prove that if $g'(a)$ exists (iii), f is continuous at a.

$\lim\limits_{x \to a} f(x) \overset{\text{i}}{=} \lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a} \overset{\text{iii}}{=} g'(a) \overset{\text{ii}}{=} f(a)$. Therefore, by Definition 1.8.1, f is continuous at a.

(a) If $f(x) = |x|$ and $g(x) = -|x|$ then $(f + g)(x) = 0$ for all real numbers.
(b) If $\mathrm{F}(x) = x^{-1}$ and $\mathrm{G}(x) = -x^{-1}$ then $(\mathrm{F} + \mathrm{G})(x) = 0$ for $x \ne 0$.

2.3 THE NUMERICAL DERIVATIVE

Symmetric Difference Quotient of f at a is $[f(a+\Delta x)-f(a-\Delta x)/(2\Delta x)$. As $\Delta x \to 0$, this approaches $f'(a)$, usually much faster than the ordinary difference quotient used in the definition, and sometimes approaches a limit when $f'(a)$ does not exist. The value of Δx is called the tolerance. With any tolerance, the result is exact for linear and quadratic functions. We shall used a fixed tolerance of 0.001. Thus

$$\mathbf{NDER}(f(x),x) = \frac{f(x+0.001)-f(x-0.001)}{0.002} \tag{4}$$

Note that some graphics calculators can plot the tangent line to a graph at any point and give the value of NDER in a single operation. If your graphics calculator can use rules of differentiation to get (a decimal expression for) the exact value DER1 (DER2 is the second derivative) of the derivative, you should use that capability (except in Exercises 27–30). The FRAC capability may be used to get the answer as a fraction.

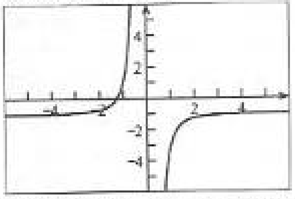
Exercises 2.3

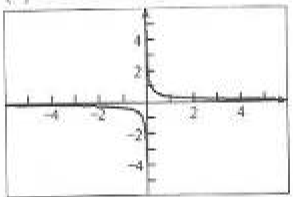
In Exercises 1–4, compare the symmetric difference quotient with standard one for the given Exercise of §2.1.

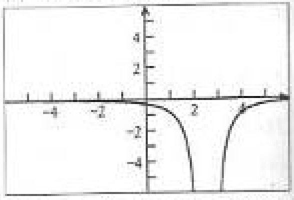
1. See Exercise 2.1.17 **2.** See Exercises 2.1.18 **3.** See Exercise 2.1.19 **4.** See Exercise 2.1.20

In Exercises 5–8, plot the graph of the numerical derivative at x to support the value found in the given Exercise.

5. (a) Exercise 2.1.33 (b) Exercise 2.1.35 (c) Exercise 2.1.37
6. (a) Exercise 2.1.34 (b) Exercise 2.1.36 (c) Exercise 2.1.38
7. (a) Exercise 2.1.39 (b) Exercise 2.1.41 (c) Exercise 2.1.43
8. (a) Exercise 2.1.40 (b) Exercise 2.1.42 (c) Exercise 2.1.44







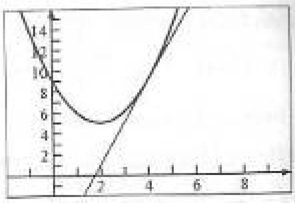
In Exercises 9–20, (a) use NDER to find the slope m of the tangent line to f at (x_1, y_1) where $y_1 = f(x_1)$; (b) write the equation of the tangent line T; (c) plot the tangent line and the graph of f.

9. $f(x)=(x-1)^2$; $x_1=2$ ▹ $y_1=f(2)=1$. $m=f'(2)=2$. T: $y=2(x-2)+1=2x-3$

10. $f(x)=2+2x-x^2$; $x_1=-1$ ▹ $y_1=f(-1)=-1$. $m=f'(-1)=4$. T: $y=4(x+1)-1=4x+3$

11. $f(x)=x^2-2x-4$; $x_1=3$ ▹ $y_1=f(3)=-1$. $m=f'(3)=4$. T: $y=4(x-3)-1=4x-13$

12. $f(x)=(2-x)^2+5$; $x_1=4$
▹ $y_1=f(4)=(2-4)^2+5=9$. Using NDER we find $m=f'(4)=4$.
Therefore the equation of the tangent line is
$y=4(x-4)+9$, $y=4x-7$



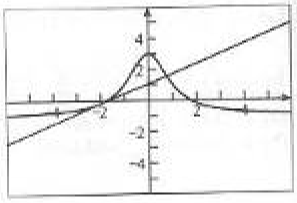
13. $f(x)=\sqrt{x^2-16}$; $x_1=-5$
▹ $y_1=f(-5)=3$. $m=f'(-5)=-\frac{5}{3}\approx -1.66\ldots$
T: $y=3-\frac{5}{3}(x+5)=-\frac{5}{3}x-\frac{16}{3}\approx -1.66\ldots x-5.33\ldots$

14. $f(x)=\sqrt{25-x^2}$; $x_1=3$ ▹ $y_1=f(3)=4$. $m=f'(3)=-.75$. T: $y=-.75(x-3)+4=-.75x+6.25$

15. $f(x)=\dfrac{x^2-1}{x^2+4}$; $x_1=1$ ▹ $y_1=f(1)=0$. $m=f'(1)=0.4$. T: $y=0.4(x-1)+0=0.4x-0.4$

16. $f(x)=\dfrac{3-x^2}{1+x^2}$; $x_1=-2$
▹ $y_1=f(-2)=\dfrac{3-(-2)^2}{1+(-2)^2}=-\frac{1}{5}=-0.2$.
$m=f'(-2)=0.64$ (using DER1) ≈ 0.6400001536 (using NDER)

Therefore the equation of the tangent line is
$y=0.64(x+2)-0.2$, $y=0.64x+1.08$

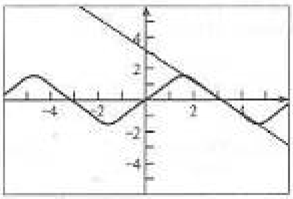


17. $f(x) = x \sin x;\ x_1 = 1$
▷ $y_1 = \sin 1 \approx 0.8415$. $m = f'(1) \approx 1.3818$. T: $y \approx 1.3818(x-1) + 0.8415 = 1.3818x - 0.5403$

18. $f(x) = x^2 \cos x;\ x_1 = 2$
▷ $y_1 = 4\cos 2 \approx -1.6646$. $m = f'(2) \approx -5.3018$. T: $y \approx -5.3018(x-2) - 1.6646 = -5.3018x + 8.9390$

19. $f(x) = \sin(\cos x);\ x_1 = 2$
▷ $y_1 = \sin(\cos 2) \approx -0.4042$. $m = f'(2) \approx -0.8317$. T: $y \approx -0.8317(x-2) - 0.4042 = -0.8317x + 1.2591$

20. $f(x) = \tan(\sin x);\ x_1 = 3$



▷ $y_1 = \tan(\sin 3) \approx 0.1421$

$m = f'(3) = -1.009972795$ (using DER1) $= -1.009972957$ (using NDER)

Therefore the equation of the tangent line is approximately

$y = -1.0100(x-3) + 0.1421$, $y = -1.0100x + 3.1720$

21 and 22. Prove that if f is a linear or quadratic function then NDER$(f(x),x)$ is exactly $f'(x)$.

▷ Let $f(x) = ax^2 + bx + c$ (a may be 0). NDER$(f(x),x) = \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

$$= \frac{[a(x+\Delta x)^2 + b(x+\Delta x) + c] - [a(x-\Delta x)^2 + b(x-\Delta x) + c]}{2\Delta x}$$

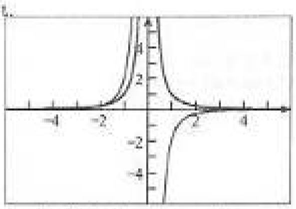
$$= \frac{[ax^2 + 2ax\Delta x + a\Delta x^2 + bx + b\Delta x] - [ax^2 - 2ax\Delta x + a\Delta x^2 + bx - b\Delta x]}{2\Delta x} = \frac{4ax\Delta x + 2b\Delta x}{2\Delta x} = 2ax + b = f'(x)$$

In Exercises 23–26, (a) plot f and NDER$(f(x),x)$. For what values of of x is (b) NDER > 0 and (c) NDER < 0? For what values of x does $f(x)$ appear to be (d) increasing and (e) decreasing. (f) Compare your answers in parts (b) and (d) and in parts (c) and (e).

▷ (f) The answers are identical: a function increases if its derivative f' is positive and decreases if f' is negative.

23. $f(x) = x^2 + 2$ ▷ (b) and (d) $x > 0$; (c) and (e) $x < 0$

24. $f(x) = 1/x^2$ ▷ The figure at the right show f dark and f' light.



▷ (b) and (d): NDER > 0 and $f(x)$ is increasing if $x < 0$

(c) and (e): NDER < 0 and $f(x)$ is decreasing if $x > 0$

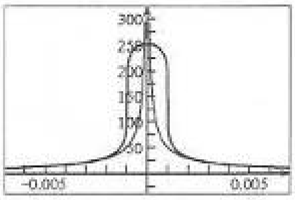
25. $f(x) = \sqrt{4 - x^2}$ ▷ (b) and (d) $x < 0$; (c) and (e) $x > 0$

26. $f(x) = \sqrt{x^2 - 4}$ ▷ (b) and (d) $x > 2$; (c) and (e) $x < 2$

27. Let $f(x) = x^{1/3}$. Compute NDER$(f(x),0)$ and explain why it exists even though $f'(0)$ does not exist.
▷ NDER $= (.001^{1/3} - (-.001)^{1/3})/.002 = (.1 + .1)/.002 = 100$. NDER always exists since we don't divide by 0.

In Exercises 28 and 29, (a) show that $f'(0)$ does not exist; compute NDER$(f(x),0)$ (b) by formula (4) and (c) by calculator. (d) Explain why NDER exists. (e) Plot the graph of NDER$(f(x),x)$. (f) Compare its behavior at 0 with the result of (a).

28. $f(x) = x^{1/5}$



▷ (a) $f'(0) = \lim\limits_{x\to 0} \dfrac{x^{1/5} - 0}{x - 0} = \lim\limits_{x\to 0} \dfrac{1}{x^{4/5}} = +\infty$.

(b) and (c) NDER$(x^{1/5},0) = \dfrac{.001^{1/5} - (-.001)^{1/5}}{.002} = 251.19$.

(d) NDER always exists since we don't divide by 0

(f) The figure at the right shows NDER dark and f' light.

29. $f(x) = x^{2/3}$. $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{x^{2/3} - 0}{x - 0} = \lim\limits_{x\to 0^-} \dfrac{1}{x^{1/3}} = -\infty$; $f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{1}{x^{1/3}} = +\infty$

Because f is an even function, NDER$(f(x),0) = \dfrac{f(\Delta x) - f(-\Delta x)}{2\Delta x} = \dfrac{0}{2\Delta x} = 0$.

30. The inconsistency in Exercise 29 is because the graph has a cusp at the origin.

2.4 THEOREMS ON DIFFERENTIATION OF ALGEBRAIC FUNCTIONS AND HIGHER ORDER-DERIVATIVES

The differentiation formulas that are proved in this section may be used to differentiate any rational function and should be memorized. Algebraic functions are discussed in Section 2.9.

2.4.1 Theorem Constant Rule If c is a constant,

$$\mathrm{D}_x(c) = 0$$

The derivative of a constant is zero.

2.4.3 Theorem Constant Multiple If c is a constant and if $\mathrm{D}_x f(x)$ exists,

$$\mathrm{D}_x[c \cdot f(x)] = c \cdot \mathrm{D}_x f(x)$$

The derivative of a constant times a function is the constant times the derivative of the function, if this derivative exists.

2.4.4 Theorem Sum rule If $\mathrm{D}_x f(x)$ and $\mathrm{D}_x g(x)$ exist,

$$\mathrm{D}_x[f(x) + g(x)] = \mathrm{D}_x f(x) + \mathrm{D}_x g(x)$$

The derivative of the sum of functions is the sum of their derivatives, if these derivatives exist. This rule holds for any number of terms.

2.4.6 Theorem Product rule If $\mathrm{D}_x f(x)$ and $\mathrm{D}_x g(x)$ exist,

$$\mathrm{D}_x[f(x) \cdot g(x)] = f(x) \cdot \mathrm{D}_x g(x) + g(x) \cdot \mathrm{D}_x f(x) = \mathrm{D}_x f(x) \cdot g(x) + f(x) \cdot \mathrm{D}_x g(x)$$

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function, if these derivatives exist. Warning: In the first form, the product and quotient rules have their terms interchanged; in the second form the product and quotient rules are in the same order.

General Product Rule (Exercise 53.) If $f(x)$, $g(x)$, $h(x)$ are differentiable,
$[f(x)g(x)h(x)]' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$
In the first term, we differentiate the first factor, in the second term we differentiate the second factor, etc. This rule holds for any number of factors.

2.4.7 Theorem Quotient rule If $g(x) \neq 0$ and $\mathrm{D}_x f(x)$ and $\mathrm{D}_x g(x)$ exist, then

$$\mathrm{D}_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \mathrm{D}_x f(x) - f(x) \cdot \mathrm{D}_x g(x)}{[g(x)]^2} = \frac{\mathrm{D}_x f(x) \cdot g(x) - f(x) \cdot \mathrm{D}_x g(x)}{[g(x)]^2}$$

The derivative of the quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of denominator, all divided by the square of the denominator, if the derivatives exist.

2.4.2 Theorem Power rule If r is a positive integer or r is a negative integer and $x \neq 0$, then

$$\mathrm{D}_x(cx^r) = crx^{r-1}$$

A power in the denominator is moved to the numerator, changing the sign of its exponent. The formula for the derivative of the power function was proved only for the case in which the exponent r is an integer. In Section 2.9 we show that the formula holds when r is any rational number, and in Section 5.5 we show that it is valid when r is any real number.

When convenient, we replace a given product by an equivalent sum before differentiating, because the formula for the derivative of a sum is easier to apply than the formula for the derivative of a product, as illustrated in Exercise 20. Also, if possible, we replace a given quotient by an equivalent sum before differentiating, as illustrated in Exercise 32.

Higher Order Derivatives The first derivative of the function f is denoted by f'; the second derivative of f is the first derivative of f' and is denoted by f''; the third derivative of f is the first derivative of f'' and is denoted by f'''; and so forth, provided these derivatives exist. If n is an integer greater than 1, the nth derivative of f is denoted by $f^{(n)}$, and $f^{(n)}$ is the first derivative of $f^{(n-1)}$.

Furthermore if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x)$, $\frac{d^3y}{dx^3} = f'''(x)$, $\frac{d^4y}{dx^4} = f^{(4)}(x)$, and so on.

Leibniz Rule (Exercise 58.) If all derivatives exist, following the pattern of the binomial theorem, $(fg)'' = f''g + 2f'g' + fg''$, $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$, etc

Exercises 2.4

In Exercises 1–24, differentiate the function by applying the theorems of this section.

1. $f'(x) = \mathrm{D}_x(7x - 5) = \mathrm{D}_x(7x) - \mathrm{D}_x(5) = 7$

2. $g'(x) = \mathrm{D}_x(8 - 3x) = \mathrm{D}_x(8) - \mathrm{D}_x(3x) = -3$

3. $g'(x) = \mathrm{D}_x(1 - 2x - x^2) = \mathrm{D}_x(1) + \mathrm{D}_x(-2x) + \mathrm{D}_x(-x^2) = -2 - 2x$

4. $f(x) = 4x^2 + 4x + 1$
▸ $f'(x) = \mathrm{D}_x(4x^2) + \mathrm{D}_x(4x) + \mathrm{D}_x(1) = 4(2x^{2-1}) + 4(1x^{1-1}) + 0 = 8x + 4$

5. $f'(x) = \mathrm{D}_x(x^3 - 3x^2 + 5x - 2) = \mathrm{D}_x(x^3) + \mathrm{D}_x(-3x^2) + \mathrm{D}_x(5x) + \mathrm{D}_x(-2) = 3x^2 - 6x + 5$

6. $f'(x) = \mathrm{D}_x(3x^4 - 5x^2 + 1) = \mathrm{D}_x(3x^4) - \mathrm{D}_x(5x^2) + \mathrm{D}_x(1) = 12x^3 - 10x$

7. $f'(x) = \mathrm{D}_x(\frac{1}{8}x^8 - x^4) = \mathrm{D}_x(\frac{1}{8}x^8) + \mathrm{D}_x(-x^4) = x^7 - 4x^3$

8. $g(x) = x^7 - 2x^5 + 5x^3 - 7x$
▸ $g'(x) = 7x^{7-1} - 2(5x^{5-1}) + 5(3x^{3-1}) - 7(x^{1-1}) = 7x^6 - 10x^4 + 15x^2 - 7$

9. $\mathrm{F}'(t) = \mathrm{D}_t(\frac{1}{4}t^4 - \frac{1}{2}t^2) = \mathrm{D}_t(\frac{1}{4}t^4) + \mathrm{D}_t(-\frac{1}{2}t^2) = t^3 - t$

10. $\mathrm{H}'(x) = \mathrm{D}_x(\frac{1}{3}x^3 - x + 2) = \mathrm{D}_x(\frac{1}{3}x^3) - \mathrm{D}_x(x) + \mathrm{D}_x(2) = x^2 - 1$

11. $\mathrm{V}'(r) = \mathrm{D}_r(\frac{4}{3}\pi r^3) = 4\pi r^2$

12. $\mathrm{G}(y) = y^{10} + 7y^5 - y^3 + 1$
▸ $\mathrm{G}'(y) = 10y^{10-1} + 7(5y^{5-1}) - 3y^{3-1} + 0 = 10y^9 + 35y^4 - 3y^2$

13. $\mathrm{F}'(x) = \mathrm{D}_x\left(x^2 + 3x + \frac{1}{x^2}\right) = \mathrm{D}_x(x^2) + \mathrm{D}_x(3x) + \mathrm{D}_x(x^{-2}) = 2x + 3 - 2x^{-3} = 2x + 3 - \frac{2}{x^3}$

14. $f'(x) = \mathrm{D}_x\left(\frac{x^3}{3} + \frac{3}{x^3}\right) = \mathrm{D}_x(\frac{1}{3}x^3) + \mathrm{D}_x(3x^{-3}) = x^2 - 9x^{-4} = x^2 - \frac{9}{x^4}$

15. $g'(x) = \mathrm{D}_x\left(4x^4 - \frac{1}{4x^4}\right) = \mathrm{D}_x(4x^4) + \mathrm{D}_x(-\frac{1}{4}x^{-4}) = 16x^3 + x^{-5} = 16x^3 + \frac{1}{x^5}$

16. $f(x) = x^4 - 5 + x^{-2} + 4x^{-4}$
▸ $f'(x) = 4x^{4-1} - 0 + (-2)x^{-2-1} + 4(-4)x^{-4-1} = 4x^3 - 2x^{-3} - 16x^{-5}$

17. $g'(x) = \mathrm{D}_x\left(\frac{3}{x^2} + \frac{5}{x^4}\right) = \mathrm{D}_x(3x^{-2}) + \mathrm{D}_x(5x^{-4}) = -6x^{-3} - 20x^{-4} = -\frac{6}{x^3} - \frac{20}{x^5}$

18. $\mathrm{H}'(x) = \mathrm{D}_x\left(\frac{5}{6x^5}\right) = \mathrm{D}_x(\frac{5}{6}x^{-5}) = -\frac{25}{6}x^{-6} = -\frac{25}{6x^6}$

19. $f'(s) = \mathrm{D}_s[\sqrt{3}(s^3 - s^2)] = \sqrt{3}\mathrm{D}_s(s^3 - s^2) = \sqrt{3}(3s^2 - 2s)$

20. $g(x) = (2x^2 + 5)(4x - 1)$
▸ We express $g(x)$ as a sum and use the sum rule.
$g(x) = 8x^3 - 2x^2 + 20x - 5,$ $\qquad g'(x) = 24x^2 - 4x + 20$

ALTERNATE SOLUTION: We use the product rule.

$$g'(x) = (2x^2 + 5)\mathrm{D}_x(4x - 1) + (4x - 1)\mathrm{D}_x(2x^2 + 5) = (2x^2 + 5)(4) + (4x - 1)(4x) = 8x^2 + 20 + 16x^2 - 4x$$
$$= 24x^2 - 4x + 20$$

21. $f'(x) = \mathrm{D}_x[(2x^4 - 1)(5x^3 + 6x)] = (2x^4 - 1)\mathrm{D}_x(5x^3 + 6x) + (5x^3 + 6x)\mathrm{D}_x(2x^4 - 1)$

$= (2x^4 - 1)(15x^2 + 6) + (5x^3 + 6x)(8x^3) = 30x^6 + 12x^4 - 15x^2 - 6 + 40x^6 + 48x^4 = 70x^6 + 60x^4 - 15x^2 - 6$

22. $f'(x) = \mathrm{D}_x[(4x^2 + 3)^2] = \mathrm{D}_x[(4x^2 + 3)(4x^2 + 3)] = \mathrm{D}_x(4x^2 + 3) \cdot (4x^2 + 3) + (4x^2 + 3)\mathrm{D}_x(4x^2 + 3)$

$= 2(8x)(4x^2 + 3) = 64x^3 + 48x$

23. $G'(y) = D_y[(7-3y^3)^2] = D_y[(7-3y^3)(7-3y^3)] = (7-3y^3)D_y(7-3y^3) + (7-3y^3)D_y(7-3y^3)$

$= (7-3y^3)(-9y^2) + (7-3y^3)(-9y^2) = -18y^2(7-3y^3)$

24. $F(t) = (t^3-2t+1)(2t^2+3t)$ ▹ We use the product rule.

$F'(t) = (t^3-2t+1)D_t(2t^2+3t) + (2t^2+3t)D_t(t^3-2t+1) = (t^3-2t+1)(4t+3) + (2t^2+2t)(3t^2-2)$

$= 4t^4 - 8t^2 + 4t + 3t^3 - 6t + 3 + 6t^4 + 9t^3 - 4t^2 - 6t = 10t^4 + 12t^3 - 12t^2 - 8t + 3$

ALTERNATE SOLUTION: First express $F(t)$ as a sum and then differentiate.

$F(t) = 2t^5 + 3t^4 - 4t^3 - 4t^2 + 3t$ $\qquad F'(t) = 10t^4 + 12t^3 - 12t^2 - 8t + 3$

In Exercises 25–36, compute the derivative. In Exercises 25–30, check by plotting your answer and NDER.

25. $D_x[(x^2-3x+2)(2x^3+1)] = (x^2-3x+2)D_x(2x^3+1) + (2x^3+1)D_x(x^2-3x+2)$

$= (x^2-3x+2)(6x^2) + (2x^3+1)(2x-3) = 6x^4 - 18x^3 + 12x^2 + 4x^4 - 6x^3 + 2x - 3$

$= 10x^4 - 24x^3 + 12x^2 + 2x - 3$

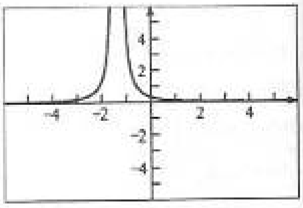
26. $D_x\left(\frac{2x}{x+3}\right) = \frac{D_x(2x)\cdot(x+3) - 2xD_x(x+3)}{(x+3)^2} = \frac{2(x+3)-2x}{(x+3)^2} = \frac{3}{(x+3)^2}$

27. $D_x\left(\frac{x}{x-1}\right) = \frac{(x-1)D_x x - xD_x(x-1)}{(x-1)^2} = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

28. $D_y\left(\frac{2y+1}{3y+4}\right)$ ▹ We use the quotient rule.

$$D_y\left(\frac{2y+1}{3y+4}\right) = \frac{(3y+4)\cdot D_y(2y+1) - (2y+1)\cdot D_y(3y+4)}{(3y+4)^2}$$

$= \frac{(3y+4)(2) - (2y+1)(3)}{(3y+4)^2} = \frac{5}{(3y+4)^2}$



29. $\frac{d}{dx}\left(\frac{x^2+2x+1}{x^2-2x+1}\right) = \frac{(x^2-2x+1)D_x(x^2+2x+1) - (x^2+2x+1)D_x(x^2-2x+1)}{(x^2-2x+1)^2}$

$= \frac{(x^2-2x+1)(2x+2) - (x^2+2x+1)(2x-2)}{(x^2-2x+1)^2} = \frac{2(x-1)^2(x+1) - 2(x+1)^2(x-1)}{[(x-1)^2]^2}$

$= \frac{2(x-1)(x+1)[(x-1)-(x+1)]}{(x-1)^4} = \frac{2(x+1)(-2)}{(x-1)^3} = -\frac{4(x+1)}{(x-1)^3}$

30. $\frac{d}{dx}\left(\frac{4-3x-x^2}{x-2}\right) = \frac{D_x(4-3x-x^2)\cdot(x-2) - (4-3x-x^2)D_x(x-2)}{(x-2)^2} = \frac{(-3-2x)(x-2) - (4-3x-x^2)}{(x-2)^2}$

$= \frac{-x^2-4x-2}{(x-2)^2}$

31. $\frac{d}{dt}\left(\frac{5t}{1+2t^2}\right) = \frac{(1+2t^2)D_t(5t) - (5t)D_t(1+2t^2)}{(1+2t^2)^2} = \frac{(1+2t^2)(5) - (5t)(4t)}{(1+2t^2)^2} = \frac{5+10t^2-20t^2}{(1+2t^2)^2} = \frac{5-10t^2}{(1+2t^2)^2}$

32. $\frac{d}{dx}\left(\frac{x^4-2x^2+5x+1}{x^4}\right)$ ▹ We divide and use the formula for the derivative of a sum.

$$\frac{d}{dx}\left(\frac{x^4-2x^2+5x+1}{x^4}\right) = \frac{d}{dx}(1 - 2x^{-2} + 5x^{-3} + x^{-4}) = 4x^{-3} - 15x^{-4} - 4x^{-5}$$

$$= x^{-5}(4x^2 - 15x^1 - 4) = \frac{4x^2-15x-4}{x^5}$$

33. $\frac{d}{dy}\left(\frac{y^3-8}{y^3+8}\right) = \frac{(y^3+8)(3y^2) - (y^3-8)(3y^2)}{(y^3+8)^2} = \frac{3y^2(y^3+8-y^3+8)}{(y^3+8)^2} = \frac{48y^2}{(y^3+8)^2}$

34. $\frac{d}{ds}\left(\frac{s^2-a^2}{s^2+a^2}\right) = \frac{D_s(s^2-a^2)\cdot(s^2+a^2) - (s^2-a^2)D_s(s^2+a^2)}{(s^2+a^2)^2} = \frac{2s(s^2+a^2) - (s^2-a^2)2s}{(s^2+a^2)^2} = \frac{4a^2s}{(s^2+a^2)^2}$

35. $D_x\left[\frac{2x+1}{x+5}(3x-1)\right] = \left(\frac{2x+1}{x+5}\right)D_x(3x-1) + (3x-1)D\left(\frac{2x+1}{x+5}\right)$

$= \left(\frac{2x+1}{x+5}\right)(3) + (3x-1)\frac{(x+5)2 - (2x+1)1}{(x+5)^2} = \frac{6x+3}{x+5} + (3x-1)\frac{2x+10-2x-1}{(x+5)^2}$

$= \frac{(6x+3)(x+5)}{(x+5)^2} + \frac{(3x-1)9}{(x+5)^2} = \frac{6x^2+33x+15+27x-9}{(x+5)^2} = \frac{6x^2+60x+6}{(x+5)^2} = \frac{6(x^2+10x+1)}{(x+5)^2}$

36. $\mathrm{D}_x\left[\dfrac{(x^3+1)(x^2-2x^{-1}+1)}{x^2+3}\right] = \mathrm{D}_x\left[\dfrac{x^5+x^3-x^2-2x^{-1}+1}{x^2+3}\cdot\dfrac{x}{x}\right] = \mathrm{D}_x\left[\dfrac{x^6+x^4-x^3+x-2}{x^3+3x}\right]$

$$= \frac{(x^3+3x)\mathrm{D}_x(x^6+x^4-x^3+x-2)-(x^6+x^4-x^3+x-2)\mathrm{D}_x(x^3+3x)}{(x+3x)^2}$$

$$= \frac{(x^3+3x)(6x^5+4x^3-3x^2+1)-(x^6+x^4-x^3+x-2)(3x^2+3)}{(x^3+3x)^2} = \frac{3x^8+16x^6+9x^4-8x^3+6x^2+6}{x^2(x^2+3)^2}$$

In Exercises 37 and 38, find all the derivatives of f.

37. $f(x) = 6x^5+3x^4-2x^3+5x^2-8x+9$, $f'(x) = 30x^4+12x^3-6x^2+10x-8$, $f''(x) = 120x^3+36x^2-12x+10$ $f'''(x) = 360x^2+72x-12$, $f^{(4)}(x) = 720x+72$, $f^{(5)} = 720$, $f^{(n)}(x) = 0$ if $n \geq 6$

38. $f(x) = 2x^7-x^5+5x^3-8x+4$, $f'(x) = 14x^6-5x^4+15x^2-8$, $f''(x) = 84x^5-20x^3+30x$, $f'''(x) = 420x^4-60x^2+30$, $f^{(4)}(x) = 1680x^3-120x$, $f^{(5)}(x) = 5040x^2-120$, $f^{(6)}(x) = 10080x$, $f^{(7)}(x) = 10080$, $f^{(n)}(x) = 0$ if $n \geq 8$

39. $\mathrm{D}_t^{\,3}\left(\dfrac{1}{6t^3}\right) = \mathrm{D}_t^{\,2}(\mathrm{D}_t\tfrac{1}{6}t^{-3}) = \mathrm{D}_t(\mathrm{D}_t-\tfrac{1}{2}t^{-4}) = \mathrm{D}_t(2t^{-5}) = -10t^{-6}$

40. Find $\dfrac{d^4}{dx^4}\left(x^5-\dfrac{1}{15x^5}\right)$ ▹ $= \mathrm{D}_x\mathrm{D}_x\mathrm{D}_x[\mathrm{D}_x(x^5-\tfrac{1}{15}x^{-5})] = \mathrm{D}_x\mathrm{D}_x\mathrm{D}_x(5x^4+\tfrac{1}{3}x^{-6})$
$= \mathrm{D}_x\mathrm{D}_x(20x^3-2x^{-7}) = \mathrm{D}_x(60x^2+14x^{-8}) = 120x-112x^{-9}$

In Exercises 41 and 42, find $\dfrac{d^2y}{dx^2}$. Check by plotting your answer and your calculator's second derivative.

41. $y = \dfrac{x^4+1}{x^2}$ ▹ $= x^2+x^{-2}$, $y' = 2x-2x^{-3}$, $y'' = 2+6x^{-4}$

42. $y = \dfrac{3}{x}-\dfrac{1}{3x^3}$ ▹ $= 3x^{-1}-\tfrac{1}{3}x^{-3}$, $y' = -3x^{-2}+x^{-4}$, $y'' = 6x^{-3}-4x^{-5}$

In Exercises 43–46, find an equation of the tangent line or normal line, at the point. Check by plotting the line and the curve.

43. $y = x^3-4$, $(2,4)$, tangent line ▹ Let $f(x) = x^3-4$. Then $f'(x) = 3x^2$, so $f'(2) = 12$. Therefore the tangent line has slope 12; so an equation of the tangent line is $y = 12(x-2)+4$, or $y = 12x-20$.

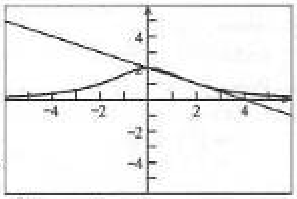
44. $y = 8/(x^2+4)$, $(2,1)$, tangent line.
▹ Because the slope of the tangent line is the value of the derivative,

$$m(x) = \mathrm{D}_x\left(\frac{8}{x^2+4}\right) = \frac{(x^2+4)\cdot 0-8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$$

Therefore $\quad m(2) = -\dfrac{16(2)}{(2^2+4)^2} = -\dfrac{1}{2}$

The line contains the point $(2,1)$ and has slope $-\frac{1}{2}$. An equation of the line is

$$y-1 = -\tfrac{1}{2}(x-2) \qquad x+2y-4 = 0$$



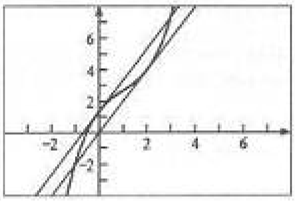
45. $y = \dfrac{10}{14-x^2}$, $(4,-5)$, normal line ▹ Let $f(x) = \dfrac{10}{14-x^2}$. Then $f'(x) = \dfrac{(14-x^2)0-10(-2x)}{(14-x^2)^2} = \dfrac{20x}{(14-x^2)^2}$; so $f'(4) = \dfrac{80}{(-2)^2}$. The normal line at $(4,-5)$ has slope $-\frac{1}{20}$; so its equation is $y = -\frac{1}{20}(x-4)-5$ or $y = -\frac{1}{20}x-\frac{24}{5}$.

46. $y = 4x^2-8x$, $(1,-4)$, normal line ▹ Let $f(x) = 4x^2-8x$. Then $f'(x) = 8x$; so $f'(1) = 8$. The normal line at $(1,-4)$ has slope $-\frac{1}{8}$; so its equation is $y = -\frac{1}{8}(x-1)-4$ or $y = -\frac{1}{8}x-\frac{31}{8}$.

47. Find an equation of the line tangent to $y = 3x^2-4x$ and parallel to the line $2x-y+3 = 0$.
▹ Let $f(x) = 3x^2-4x$. Then $f'(x) = 6x-4$. The given line has the equation $y = 2x+3$ and has slope 2. $f'(x) = 2$ when $6x-4 = 2$ so $x = 1$. Since $f(1) = -1$, the desired tangent line passes through $(1,-1)$ and has slope 2; so its equation is $y = 2(x-1)-1$ or $y = 2x-3$.

48. Find an equation of each of the tangent lines to $3y = x^3-3x^2+6x+4$ that is parallel to $2x-y+3 = 0$. Check by plotting the curve and the lines.
▹ Let L_1 be one of the required lines and let (x_1, y_1) be the point at which L_1 is tangent to the curve. The slope of L_1 is given by $m(x_1) = y'(x_1)$.

Because $\quad y = \tfrac{1}{3}x^3-x^2+2x+\tfrac{4}{3} \qquad (1)$
then $\quad y'(x) = x^2-2x+2$
and $\quad m(x_1) = x_1^{\,2}-2x_1+2 \qquad (2)$



Because the slope-intercept form of the equation of the given line is $y = 2x + 3$, the slope of this line is 2. Because L_1 is parallel to the given line, the slope of L_1 is also 2. Therefore, $m(x_1) = 2$, and by Eq. (2) we have

$$2 = {x_1}^2 - 2x_1 + 2, \quad x_1 = 0 \text{ or } x_1 = 2$$

If $x_1 = 0$, then by Eq. (1) $y = \frac{4}{3}$. Thus L_1 contains $(0, \frac{4}{3})$ and has slope 2. An equation of L_1 is

$$y = 2x + \tfrac{4}{3}$$

If $x_1 = 2$, then $y = \frac{1}{3}(2)^3 - (2)^2 + 2(2) + \frac{4}{3} = 4$. Therefore an equation of the second of the required lines is

$$y - 4 = 2(x - 2), \qquad y = 2x$$

49. Find an equation of each normal line to $y = x^3 - 4x$ that is parallel to the line $x + 8y - 8 = 0$.

▹ Let $f(x) = x^3 - 4x$. Then $f'(x) = 3x^2 - 4$. At the point $(x, f(x))$ on the curve, the slope of the normal line is $\dfrac{-1}{f'(x)} = \dfrac{-1}{3x^2 - 4}$. The given line has the equation $y = -\frac{1}{8}x + 1$ and hence has slope $-\frac{1}{8}$. Since the normal lines are parallel to the given line, we must have $\dfrac{-1}{3x^2 - 4} = -\dfrac{1}{8}$; $3x^2 - 4 = 8$; $x = \pm 2$

Now $f(2) = 0$ and $f(-2) = 0$; so the required normal lines are at $(2, 0)$ and $(-2, 0)$ and each line has slope $-\frac{1}{8}$. Hence, their equations are $y = -\frac{1}{8}(x - 2)$; $x + 8y - 2 = 0$ and $y = -\frac{1}{8}(x + 2)$; $x + 8y + 2 = 0$

50. Find an equation of the line tangent to $y = x^4 - 6x$ and perpendicular to the line $x - 2y + 6 = 0$.

▹ The given line has equation $y = \frac{1}{2}x + 3$ and slope $\frac{1}{2}$; a perpendicular has slope -2. Let $f(x) = x^4 - 6x$. Then $f'(x) = 4x^3 - 6$. At the point $(x, f(x))$ on the curve, the slope of the tangent line is $4x^3 - 6 = -2$ so $x = 1$. Because $f(1) = -5$, the required line is $y = -2(x - 1) - 5$ or $y = -2x - 3$.

51. Find an equation of each line through $(4, 13)$ that is tangent to the parabola $y = 2x^2 - 1$.

▹ Let $f(x) = 2x^2 - 1$. Then $f'(x) = 4x$. A line through the points $P(x_1, 2{x_1}^2 - 1)$ and $(4, 13)$ has slope $\dfrac{(2{x_1}^2 - 1) - 13}{x_1 - 4}$. This line will be tangent at P if its slope is $f'(x_1) = 4x_1$. Hence $\dfrac{(2{x_1}^2 - 1) - 13}{x_1 - 4} = 4x_1$; $2{x_1}^2 - 14 = 4x_1 - 16x_1$; $2x_1 - 16x_1 + 14 = 0$; $2(x_1 - 7)(x_1 - 1) = 0$ so $x_1 = 7$ or $x_1 = 1$. Now $f(7) = 97$ and $f'(7) = 28$ while $f(1) = 1$ and $f'(1) = 4$. Hence the equations of the tangent lines are
$y - 97 = 28(x - 7)$; $28x - y - 99 = 0$ and $y - 1 = 4(x - 1)$; $4x - y - 3 = 0$

52. Given $f(x) = \frac{1}{3}x^3 + 2x^2 + 5x + 5$, show that $f'(x) \geq 0$ for all values of x.

▹ $f'(x) = x^2 + 4x + 5 = (x + 2)^2 + 1 \geq 1$ for all x.

53. Prove the general product rule for three factors. ▹ Applying the product rule twice, we have

$$\begin{aligned}\{[f(x) \cdot g(x)] \cdot h(x)\}' &= [f(x) \cdot g(x)]' \cdot h(x) + [f(x) \cdot g(x)] \cdot h'(x)\\ &= [f'(x) \cdot g(x) + f(x) \cdot g'(x)] \cdot h(x) + [f(x) \cdot g(x)] \cdot h'(x)\\ &= f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)\end{aligned}$$

In Exercises 54–57, use the general product rule to find the derivative.

54. $[(x^2 + 3)(2x - 5)(3x + 2)]' = (x^2 + 3)'(2x - 5)(3x + 2) + (x^2 + 3)(2x - 5)'(3x + 2) + (x^2 + 3)(2x - 5)(3x + 2)'$
$= 2x(2x - 5)(3x + 2) + (x^2 + 3)(2)(3x + 2) + (x^2 + 3)(2x - 5)(3)$
$= (12x^3 - 22x^2 - 20x) + (6x^3 + 4x^2 + 18x + 12) + (6x^3 - 15x^2 + 18x - 45) = 24x^3 - 33x^2 + 16x - 33$

55. $[(3x + 2)^2(x^2 - 1)]' = [(3x + 2)(3x + 2)(x^2 - 1)]'$
$= (3x + 2)'(3x + 2)(x^2 - 1) + (3x + 2)(3x + 2)'(x^2 - 1) + (3x + 2)(3x + 2)(x^2 - 1)'$
$= 3(3x + 2)(x^2 - 1) + (3x + 2)(3)(x^2 - 1) + (3x + 2)(3x + 2)(2x)$
$= (3x + 2)(3x^2 - 3 + 3x^2 - 3 + 6x^2 + 4x) = (3x + 2)(12x^2 + 4x - 6) = 2(3x + 2)(6x^2 + 2x - 3)$

56. $[(3x^3 + x^{-3})(x + 3)(x^2 - 5)]'$
$= (3x^3 + x^{-3})'(x + 3)(x^2 - 5) + (3x^3 + x^{-3})(x + 3)'(x^2 - 5) + (3x^3 + x^{-3})(x + 3)(x^2 - 5)'$
$= (9x^2 - 3x^{-4})(x + 3)(x^2 - 5) + (3x^3 + x^{-3})(1)(x^2 - 5) + (3x^3 + x^{-3})(x + 3)(2x)$
$= (9x^5 + 27x^4 - 45x^3 - 135x^2 - 3x^{-1} - 9x^{-2} + 15x^{-3} + 45x^{-4}) + (3x^5 + x^{-1} - 5x^{-3}) + (6x^5 + 18x^4 + 2x^{-1} + 6x^{-2})$
$= 18x^5 + 45x^4 - 60x^3 - 135x^2 - 3x^{-2} + 10x^{-3} + 45x^{-4}$

57. $[(2x^2 + x + 1)^3]' = D[(2x^2 + x + 1)(2x^2 + x + 1)(2x^2 + x + 1)]$
$= (2x^2 + x + 1)'(2x^2 + x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(2x^2 + x + 1)'(2x^2 + x + 1)$
$+ (2x^2 + x + 1)(2x^2 + x + 1)(2x^2 + x + 1)'$
$= (4x + 1)(2x^2 + x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(4x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(2x^2 + x + 1)(4x + 1)$
$= 3(4x + 1)(2x^2 + x + 1)^2$

58. Prove the Leibniz rule for the second derivative.

▸ $(fg)'' = [(fg)']' = [f'g + fg']' = (f'g)' + (fg')' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$

59. If $y = x^n$, prove by mathematical induction that $\frac{d^ny}{dx^n} = n!$

▸ If $n = 1$, $y = x$ and $\frac{dy}{dx} = 1 = 1!$. Now assume the formula is true for $n = k$.
If $y = x^{k+1}$, then $\mathrm{D}^{k+1}y = \mathrm{D}^{k+1}(x^{k+1}) = \mathrm{D}[(k+1)x^k] = (k+1)\mathrm{D}_x(x^k) = (k+1)k! = (k+1)!$.
Thus the formula holds true for $n = k+1$ and hence for any integer n.

60. Use formula (7) of §2.1 to prove the power rule.

▸ $$\frac{d}{dx}x^n = \lim_{z\to x}\frac{z^n - x^n}{z - x} = \lim_{z\to x}\frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x}$$
$$= \lim_{z\to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) = x^{n-1} + \cdots + x^{n-1} \text{ (n terms)} = nx^{n-1}$$

61. Prove that if f and g are differentiable functions such that $f(0) = 0$ and $g(0) = 0$ then $f(x)g(x)$ cannot be x.

▸ Suppose $f(x)g(x) = x$. By the product rule: $f'(x)g(x) + f(x)g'(x) = 1$. At 0: $f'(0)\cdot 0 + 0\cdot g'(0) = 0 \neq 1$.

62. Show how three theorem on differentiation enable us to differentiate any polynomial.

$$(c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0)' \overset{\text{sum rule}}{=} (c_nx^n)' + (c_{n-1}x^{n-1})' + \cdots + (c_1x^1)' + (c_0x^0)'$$
$$\overset{\text{constant multiple}}{=} c_n(x^n)' + c_{n-1}(x^{n-1})' + \cdots + c_1(x^1)' + c_0(x^0)' \overset{\text{power}}{=} nc_nx^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + c_1$$

2.5 RECTILINEAR MOTION

2.5.1 Definition If f is a function given by the equation $s = f(t)$ and a particle is moving along a straight line so that s is the number of units in the directed distance of the particle from a fixed point on the line at t units of time, then the *instantaneous velocity* of the particle at t units of time is v units of velocity, where

$$v = f'(t) \quad\Leftrightarrow\quad v = \frac{ds}{dt}$$

if it exists. The instantaneous rate of change of the velocity is called the *instantaneous acceleration.*

Exercises 2.5

In Exercises 1–8, a particle is moving along a horizontal line according to the given equation, where s meters is the directed distance of the particle from a point O at t seconds. Find the instantaneous velocity $v(t)$ meters per second at t seconds, and then find $v(t_1)$ for the particular value of t_1.

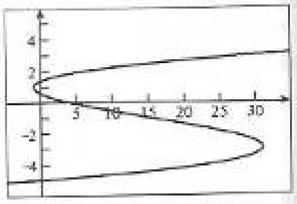
1. $s = 3t^2 + 1$; $v(t) = \frac{ds}{dt} = 6t$; $v(3) = 18$
2. $s = 8 - t^2$; $v(t) = \frac{ds}{dt} = -2t$; $v(5) = -10$
3. $s = \frac{1}{4}t^{-1}$; $v(t) = \frac{ds}{dt} = -\frac{1}{4}t^{-2}$; $v(\frac{1}{2}) = -1$
4. $s = \frac{3}{t^2}$; $t_1 = -2$

▸ $v(t) = \mathrm{D}_t(3t^{-2}) = -6t^{-3}$; $v(-2) = -6(-2)^{-3} = \frac{3}{4}$

5. $s = 2t^3 - t^2 + 5$; $v(t) = \frac{ds}{dt} = 6t^2 - 2t$; $v(-1) = 8$
6. $s = 4t^3 + 2t - 1$; $v(t) = \frac{ds}{dt} = 12t^2 + 2$; $v(\frac{1}{2}) = 5$
7. $s = \frac{2t}{4+t}$; $v(t) = \frac{ds}{dt} = \frac{2(4+t) - 2t}{(4+t)^2} = \frac{8}{(4+t)^2}$; $v(0) = \frac{1}{2}$
8. $s = \frac{1}{t} + \frac{3}{t^2}$; $t_1 = 2$

▸ $v(t) = \mathrm{D}_t(t^{-1} + 3t^{-2}) = -t^{-2} - 6t^{-3}$; $\quad v(2) = -2^{-2} - 6(2^{-3}) = -\frac{1}{4} - \frac{3}{4} = -1$

In Exercise 9–14, a particle is moving along a horizontal line according to the given equation where s meters is the directed distance of the particle from a point O at t seconds. The positive direction is to the right. Determine the intervals of time when the particle is moving to the right and when it is moving to the left. Also determine when it reverses its direction. Simulate the motion on your graphics calculator. In Exercises 15 and 16, plot the motion parametrically with $x = s$ and $y = t$.

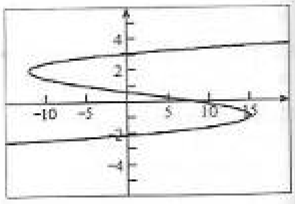


9 and 15. $s = t^3 + 3t^2 - 9t + 4$

▹ $v(t) = \frac{ds}{dt} = 3t^2 + 6t - 9 = s(t^2 + 2t - 3) = 3(t + 3)(t - 1).$

$v(t) = 0$ when $t = -3$ and $t = 1$.

	$t+3$	$t-1$	v	particle is
$t < -3$	$-$	$-$	$+$	moving to the right
$t = -3$	0	$-$	0	changing direction from right to left
$-3 < t < 1$	$+$	$-$	$-$	moving to the left
$t = 1$	$+$	0	0	changing direction from left to right
$1 < t$	$+$	$+$	$+$	moving to the right



10 and 16. $s = 2t^3 - 3t^2 - 12t + 8$

▹ $v(t) = \frac{ds}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t + 1)(t - 2).$

$v(t) = 0$ when $t = -1$ and $t = 2$

	$t+1$	$t-2$	v	particle is
$t < -1$	$-$	$-$	$+$	moving to the right
$t = -1$	0	$-$	0	changing direction from right to left
$-1 < t < 2$	$+$	$-$	$-$	moving to the left
$t = 2$	$+$	0	0	changing direction from left to right
$2 < t$	$+$	$+$	$+$	moving to the right

11. $s = \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t + 4$; $v(t) = \frac{ds}{dt} = 2t^2 + 3t - 2 = (t + 2)(2t - 1)$; $v(t) = 0$ when $t = -2$ and $t = \frac{1}{2}$

	$t+2$	$2t-1$	v	particle is
$t < -2$	$-$	$-$	$+$	moving to the right
$t = -2$	0	$-$	0	changing direction from right to left
$-2 < t < \frac{1}{2}$	$+$	$-$	$-$	moving to the left
$t = \frac{1}{2}$	$+$	0	0	changing direction from left to right
$\frac{1}{2} < t$	$+$	$+$	$+$	moving to the right

12. $s = \frac{t}{1 + t^2}$

▹ The instantaneous velocity is given by

$$v(t) = \frac{d}{dt}\left(\frac{t}{1 + t^2}\right) = \frac{(1 + t^2)(1) - t(2t)}{(1 + t^2)^2} = \frac{1 - t^2}{(1 + t^2)^2}$$

$$= \frac{(1 - t)(1 + t)}{(1 + t^2)^2} \tag{1}$$

Because $v(t) = 0$ when $t = 1$ and $t = -1$, the particle reverses direction at each of these times. Table 12a indicates the sign of each factor of v when t is in each interval. The sign of v is negative when v has one odd factor. The particle is moving to the right when $v > 0$ and is moving to the left when $v < 0$. The behavior of the motion is illustrated in the figure. Table 12b gives values of s and v for specific replacements of t, where we use the original equation of motion to calculate s and Eq. (1) to calculate v.

Table 12a

	$1-t$	$1+t$	$(1+t^2)^2$	v	particle is
$t < -1$	$+$	$-$	$+$	$-$	moving to the left
$t = -1$	$+$	0	$+$	0	reversing direction
$-1 < t < 1$	$+$	$+$	$+$	$+$	moving to the right
$t = 1$	0	$+$	$+$	0	reversing direction
$t > 1$	$-$	$+$	$+$	$-$	moving to the left

Table 12b

t	s	v
-2	$-\frac{2}{5}$	$-\frac{3}{25}$
-1	$-\frac{1}{2}$	0
0	0	1
1	$\frac{1}{2}$	0
2	$\frac{2}{5}$	$-\frac{3}{25}$

13. $s = \frac{1}{9+t^2}$; $v(t) = \frac{ds}{dt} = \frac{9+t^2-2t\cdot t}{(9+t^2)^2} = \frac{9-t^2}{(9+t^2)^2} = \frac{(3+t)(3-t)}{(9+t^2)^2}$; $v(t) = 0$ when $t = -3$ and $t = 3$

	3+t	3−t	$(9+t^2)^2$	v	particle is
$t < -3$	−	+	+	−	moving to the left
$t = -3$	0	+	+	0	changing direction from left to right
$-3 < t < 3$	+	+	+	+	moving to the right
$t = 3$	+	0	+	0	changing direction from right to left
$3 < t$	+	−	+	−	moving to the left

14. $s = \frac{t+1}{t^2+4}$; $v(t) = \frac{ds}{dt} = \frac{1\cdot(t^2+4)-(t+1)(2t)}{(t^2+4)^2} = \frac{-t^2-2t+4}{(t^2+4)^2}$; $v(t) = 0$ when $t = -1 \pm \sqrt{5}$

	$-t^2-2t+4$	$(t^2+4)^2$	v	particle is
$t < -1-\sqrt{5}$	−	+	−	moving to the left
$t = -1-\sqrt{5}$	0	+	0	changing direction from left to right
$-1-\sqrt{5} < t < -1+\sqrt{5}$	+	+	+	moving to the right
$t = -1+\sqrt{5}$	0	+	0	changing direction from right to left
$t > -1+\sqrt{5}$	−	+	−	moving to the left

In Exercises 17–21, an object moves in a vertical line such that $s = -16t^2 + v_0t + s_0$, where s_0 feet and v_0 ft/sec are the initial height and velocity and s ft is the height after t sec and the positive direction is upward.

17. A stone is dropped from a height of 256 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 2 sec. (c) When does it reach the ground? (d) How fast is going then?

▸ (a) $v_0 = 0$, $s_0 = 256$. $s = -16t^2 + 256$ (b) $v(t) = \frac{ds}{dt} = -32t$. $v(1) = -32$. Therefore, 1 sec after the stone is dropped its velocity is −32ft/sec. $v(2) = -64$. Thus, 2 sec after the stone is dropped its velocity is −64ft/sec. (c) The stone reaches the ground when $s = 0$. Then $-16t^2 + 256 = 0$; $t^2 = 16$; $t = 4$. Thus the stone reaches the ground in 4 sec. (d) $v(4) = -128$. Hence when the stone reaches the ground its velocity is −128ft/sec.

18. A chandelier is dropped from a height of 160 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 1.5 sec. (c) When does it reach the ground? (d) How fast is going then?

▸ (a) $v_0 = 0$, $s_0 = 160$. $s = -16t^2 + 160$ (b) $v(t) = \frac{ds}{dt} = -32t$. $v(1) = -32$. Therefore, 1 sec after it is dropped its velocity is −32ft/sec. $v(1.5) = -48$. Thus, 1.5 sec after it is dropped its velocity is −48 ft/sec. (c) It reaches the ground when $s = 0$. Then $-16t^2 + 160 = 0$; $t^2 = 10$; $t = \sqrt{10}$. Thus it reaches the ground in about 3.2 sec. (d) $v(\sqrt{10}) = -32\sqrt{10}$. Hence when it reaches the ground its velocity is about −101.2 ft/sec.

19. A chandelier is thrown down from a height of 160 ft at 48 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 1.5 sec. (c) When does it reach the ground? (d) How fast is going then?

▸ (a) $v_0 = -48$, $s_0 = 160$. $s = -16t^2 - 48t + 160$ (b) $v(t) = \frac{ds}{dt} = -32t - 48$. $v(1) = -80$. Hence, 1 sec after it is thrown its velocity is −80ft/sec. $v(1.5) = -96$. Thus, 1.5 sec after it is dropped its velocity is −96 ft/sec. (c) It reaches the ground when $s = 0$. Then $-16t^2 - 48t + 160 = 0$; $0 = t^2 + 3t - 10 = (t+5)(t-2)$; $t = 2$. Thus it reaches the ground in 2 sec. (d) $v(2) = -112$. Hence when it reaches the ground its velocity is −112 ft/sec.

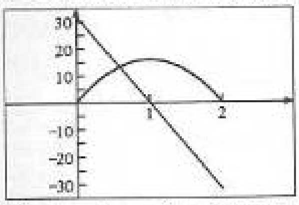
20. A ball is thrown up from the ground at 32 ft/sec. (a) Write its equation of motion. (b) Simulate its motion on your graphics calculator and estimate when and where its highest point will be. (c) Calculate the results. (d) Find its velocity and (e) its speed at 0.5 sec and 1.25 sec (f) Find its speed when it reaches the ground.

▸ (a) $v_0 = 32$, $s_0 = 0$. $s = -16t^2 + 32t$ (c) $v(t) = \frac{ds}{dt} = -32t + 32$. The ball reaches its highest point when $v = 0$; that is, when $t = 1$ and $s(1) = 16$. Therefore it takes the ball 1 sec to reach its maximum height of 16 ft. (d) $v(.75) = 8$. Hence, .75 sec after it is thrown its velocity is 8ft/sec. $v(1.25) = -8$. Thus, 1.25 sec after it is thrown its velocity is −8 ft/sec. (e) In both cases, its speed is 8 ft/sec. (f) It reaches the ground when $s = 0$. Then $-16t^2 + 32t = 0$; $0 = -16t(t-2)$; $t = 2$. $v(2) = 32$. Hence when it reaches the ground its speed is 32 ft/sec, the same as when it was thrown.

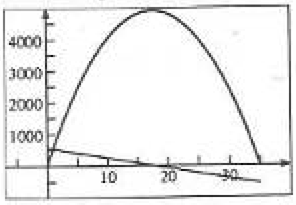
21. A rocket is fired up from the ground at 560 ft/sec. (a) Write its equation of motion. (b) Simulate its motion on your graphics calculator and estimate when and where its highest point will be. (c) Calculate the results. (d) Find its velocity and (e) its speed at 10 sec and 25 sec (f) Find its speed when it reaches the ground.

▹ (a) $v_0 = 560$, $s_0 = 0$. $s = -16t^2 + 560t$ (c) $v(t) = ds/dt = -32t + 560$. The rocket reaches its highest point when $v = 0$; that is, when $t = 17.5$ and $s(17.5) = 4900$. Therefore it takes the rocket 17.5 sec to reach its maximum height of 4900 ft. (d) $v(10) = 240$. Hence, 10 sec after it is fired its velocity is 240 ft/sec. $v(25) = -240$. Thus, 25 sec after it is fired its velocity is -240 ft/sec. (e) In both cases, its speed is 240 ft/sec. (f) It reaches the ground when $s = 0$. Then $-16t^2 + 560t = 0$; $0 = -16t(t - 35)$; $t = 35$ $v(35) = 560$. Hence when it reaches the ground its speed is 560 ft/sec, the same as when it was thrown.

22. Plot the path of the ball in Exercise 20 with $x = t$ and $y = s$, and also the velocity.



23. Plot the path of the rocket in Exercise 21 with $x = t$ and $y = s$, and also the velocity.



24. Simulate the motion of the particle of Ex. 3 on your graphics calculator. Explain why this supports the results.

▹ In Example 3 we have $s = 3t^2 - t^3$ and $v = \dfrac{ds}{dt} = 6t - 3t^2$. For visibility, we simulate the motion of the particle on the line $y = 2$. With our calculator in parametric mode, we let $x_1(t) = t^3 - 12t^2 + 36t - 24$ and $y_1(t) = 2$. In the window $[-1, 4] \times [-3, 5]$ we let $[t\text{Min}, t\text{Max}] = [0, 4]$ and t-step $= 0.05$. We now press the $\boxed{\text{TRACE}}$ key and then press the ◃ key and hold it down until the cursor is at $t = 0$. Notice the information at the bottom of the screen: $t = 0$, $x = 0$, $y = 2$. We press the ▹ key and hold it down. The cursor represents the particle moving along the line $y = 2$. Note that the particle is moving to the right until $t = 2$ and $x = 4$, when it stops and changes direction. The particle then moves to the left and disappears off the screen to the left.

In Exercises 25 and 26, a particle is moving along a line according the the equation, where s ft is the position at $t \geq 0$ sec. Find the time when the acceleration is 0 and the position and velocity at that time.

25. $s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 1$ ▹ $v = t^2 - 3t + 2$, $a = \dfrac{dv}{dt} = 2t - 3 = 0$ when $t = \frac{3}{2}$. $s(\frac{3}{2}) = \frac{7}{4}$. $v(\frac{3}{2}) = -\frac{1}{4}$

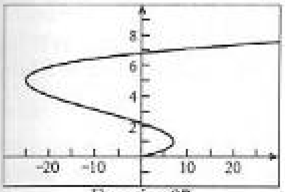
26. $s = 2t^3 - 6t^2 + 3t - 4$ ▹ $v = 6t^2 - 12t$, $a = \dfrac{dv}{dt} = 12t - 12 = 0$ when $t = 1$. $s(1) = -5$. $v(1) = -6$

In Exercises 27 and 28, make a table giving s, v, a. Include the intervals of time when the particle is moving to the left and right, and when the velocity or speed is increasing or decreasing. Plot with $x = s$ and $y = t$.

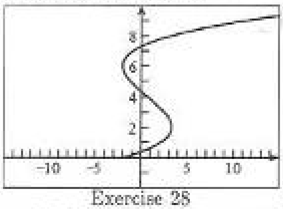
27. $s = t^3 - 9t^2 + 15t = t(t^2 - 9t + 15) = t(t - t_1)(t - t_2)$, where the roots of $t^2 - 9t + 15 = 0$ are
$t_1 = \frac{1}{2}(9 - \sqrt{81 - 4 \cdot 15}) = \frac{1}{2}(9 - \sqrt{21}) \approx 2.21$ and $t_2 = \frac{1}{2}(9 + \sqrt{21}) \approx 6.79$.
$v = D_t s = 3t^2 - 18t + 15 = 3(t^2 - 6t + 15) = 3(t - 1)(t - 5)$; $a = D_t s = 6t - 18 = 6(t - 3)$

	s	v	a	Conclusion
$t = 0$	0	+	−	Particle is at the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$0 < t < 1$	+	+	−	Particle is right of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = 1$	7	0	−	Particle is 7 m right of the origin and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$1 < t < t_1$	+	−	−	Particle is right of the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t = t_1$	0	−	−	Particle is at the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t_1 < t < 3$	−	−	−	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 3$	−	−	0	Particle is left of the origin, and it is moving to the left

				The velocity is not changing; so the speed is not changing.
$3 < t < 5$	$-$	$-$	$+$	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 5$	-25	0	$+$	Particle is 25 m left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is decreasing.
$5 < t < t_2$	$-$	$+$	$+$	Particle is left of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t = t_2$	0	$+$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_2 < t$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.



Exercise 27



Exercise 28

28. $s = \frac{1}{6}t^3 - 2t^2 + 6t - 2 = \frac{1}{6}(t - t_1)(t - t_2)(t - t_3)$, where $t_1 \approx 0.3799$, $t_2 \approx 4.3365$, $t_3 \approx 7.2836$

$v = \frac{ds}{dt} = \frac{1}{2}t^2 - 4t + 6 = \frac{1}{2}(t^2 - 8t + 12) = \frac{1}{2}(t - 2)(t - 6)$; $a = \frac{ds}{dt} = t - 4$

	s	v	a	Conclusion
$0 < t < t_1$	$-$	$+$	$-$	Particle is left of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = t_1$	0	$+$	$-$	Particle is at the origin, and it is moving to the left
$t_1 < t < 2$	$+$	$+$	$-$	Particle is right of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = 2$	$\frac{10}{3}$	0	$-$	Particle is $\frac{10}{3}$ m right of the origin and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$2 < t < 4$	$+$	$-$	$-$	Particle is right of the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t = 4$	$\frac{2}{3}$	$-$	0	Particle is right of the origin, and it is moving to the left The velocity is not changing; so the speed is not changing.
$4 < t < t_2$	$+$	$-$	$+$	Particle is right of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = t_2$	0	$-$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_2 < t < 6$	$-$	$-$	$+$	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 6$	-2	0	$+$	Particle is 2 m left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is decreasing.
$6 < t < t_3$	$-$	$+$	$+$	Particle is left of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t = t_3$	0	$+$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_3 < t$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.

29. Simulate the motion of the particle of Exercise 27 on your graphics calculator.

30. Simulate the motion of the particle of Exercise 28 on your graphics calculator and explain why this supports your results.

▷ We have $s = \frac{1}{6}t^3 - 2t^2 + 6t - 2$ and $v = \frac{ds}{dt} = 6t - 3t^2$. For visibility, we simulate the motion of the particle on the line $y = 2$. With our calculator in parametric mode, we let $x_1(t) = \frac{1}{6}t^3 - 2t^2 + 6t - 2$ and $y_1(t) = 2$. In the window $[-13, 4] \times [-3, 5]$ we let $[t\text{Min}, t\text{Max}] = [0, 8]$ and t-step $= 0.05$. We now press the $\boxed{\text{TRACE}}$ key and then press the ◁ key and hold it down until the cursor is at $t = 0$. Notice the information at the bottom of the screen: $t = 0$, $x = -2$, $y = 2$. We press the ▷ key and hold it down. The cursor represents the particle moving along the line $y = 2$. Observe that the particle is moving to the right until $t = 2$ and $x = 4$, when it stops and changes direction. The particle then moves to the left until $t = 6$ and $x = -12$ when it stops and changes direction. Then it moves to the right and disappears off the screen to the right.

In Exercises 31 and 32, the equation of motion is $s = \frac{1}{2}at^2 + v_0t + s_0$ where a the acceleration for the body. A stone dropped from a cliff hits the ground in T sec. (a) What is the height of the cliff? (b) With what velocity does it hit the ground? (c) What velocity is needed to throw it back to its original position?

31. On the moon, $a = -5.5$; T = 4. The stone starts at the origin. $s = -2.75t^2$. $s(4) = -44$; the cliff is 44 ft high. $v = -5.5t$, $v(4) = -22$. The stone hits the ground at 22 ft/sec and 22 ft/sec is the velocity needed to return it.

32. On Mars, $a = -12$; T = 3

▷ (a) The stone starts at the origin so $s_0 = 0$. The equation of motion is $s = -6t^2$.
Because $s(3) = -54$, the cliff is 54 ft high.
(b) $v(t) = \text{D}_ts = \text{D}_t(-6t^2) = -12t$ and $v(3) = -36$.
The stone hits the ground at 36 ft/sec and 36 ft/sec is the velocity needed to return it.

33. A sprinter is s meters from the finish t sec after the start, where $s = 100 - \frac{1}{4}(t^2 + 33t)$. Find his speed (a) at the start and (b) at the finish.

▷ $v = -\frac{ds}{dt} = \frac{1}{4}(2t + 33)$. (a) $v(0) = \frac{33}{4} = 8.25$. (b) $s = 0$ when $t^2 + 33t - 400 = 0$, $t = t_1 = \frac{1}{2}(-33 + \sqrt{2689}) \approx 9.42$
$v(t_1) = \frac{1}{4}\sqrt{2689} \approx 12.96$.

34. s ft is the distance of the ball from the starting point at t sec. $s = 24t + 10t^2$; $v(t) = \frac{ds}{dt} = 24 + 20t$

(a) $v(t_1) = 24 + 20t_1$. Therefore, at $t_1$sec the instantaneous velocity of the ball is $(24 + 20t_1)$ft/sec.
(b) $v(t) = 48$ when $24 + 20t = 48$; $20t = 24$; $t = \frac{6}{5}$. Hence it takes $\frac{6}{5}$ sec for the velocity to increase to 48 ft/sec.

35. s cm is the distance of the ball from its initial position at t sec. $s = 100t^2 + 100t$; $v(t) = \frac{ds}{dt} = 200t + 100$
The billiard ball hits the cushion when $s = 39$ so we have
$100t^2 + 100t = 39$; $100t^2 + 100t - 39 = 0$; $(10t - 3)(10t + 13) = 0$; $t = \frac{3}{10}$ or $t = -\frac{13}{10}$.
We reject the negative value of t. Therefore, the billiard ball hits the cushion in $\frac{3}{10}$ sec and $v(\frac{3}{10}) = 160$. Thus the velocity of the billiard ball is 160cm/sec when it hits the cushion.

36. Two particles, A and B, move to the right along a horizontal line. They start at point O, s meters is the directed distance of the particles from O at t seconds, and the equations of motion are

$$s = 4t^2 + 5t \qquad \text{(for particle A)} \qquad\qquad s = 7t^2 + 3t \qquad \text{(for particle B)}$$

If $t = 0$ at the start, for what values of t will the velocity of particle A exceed the velocity of particle B?

▷ The velocity of A is given by $\quad v_\text{A} = \text{D}_t(4t^2 + 5t) = 8t + 5$

The velocity of B is given by $\quad v_\text{B} = \text{D}_t(7t^2 + 3t) = 14t + 3$

We want to find when $v_\text{A} > v_\text{B}$, or equivalently, when

$$8t + 5 > 14t + 3$$
$$-6t > -2$$
$$t < \tfrac{1}{3}$$

Thus, the velocity of A exceeds the velocity of B when $0 \le t < \frac{1}{3}$.

2.6 THE DERIVATIVE AS A RATE OF CHANGE

2.6.1 Definition If $y = f(x)$, the *instantaneous rate of change of y per unit change in x* at x_1 is $f'(x_1)$ or, equivalently, the derivative of y with respect to x at x_1, if it exists. The *marginal cost* and *marginal revenue* are the derivatives of the cost and revenue functions.

Relative Rate of Change of y with respect to x at x_1 is given by $f'(x_1)/f(x_1) = (dy/dx)/y$ evaluated at $x = x_1$.

Exercises 2.6

1. A square of side x cm has area $A(x)$ cm^2. Find the average rate of change of A as x changes from (a) 4 to 4.6; (b) 4 to 4.3; (c) 4 to 4.1; (d) 4 to 4.05. (e) Find the instantaneous rate of change when $x = 4$.

▸ $A(x) = x^2$ (a) $\Delta A/\Delta x = (4.6^2 - 4^2)/(4.6 - 4) = 8.6$ (b) $\Delta A/\Delta x = (4.3^2 - 4^2)/(4.3 - 4) = 8.3$ (c) $\Delta A/\Delta x = (4.1^2 - 4^2)/(4.1 - 4) = 8.1$ (d) $(4.05^2 - 4^2)/(4.05 - .05) = 8.05$ (e) $A'(x) = 2x$, $A'(4) = 8$

2. A rectangle of width w in. and length $w + 4$ in. has area A(w) in^2. Find the average rate of change of A as w changes from (a) 3 to 3.2; (b) 3 to 3.1; (c) 3 to 3.01; (d) 3 to 3.001. (e) Find the instantaneous rate of change when $w = 3$.

▸ $A(w) = w(w + 4)$, $A(3) = 21$ (a) $\Delta A/\Delta w = (3.2 \cdot 7.2 - 21)/.2 = 10.2$ (b) $\Delta A/\Delta w = (3.1 \cdot 7.1 - 21)/.1 = 10.1$ (c) $\Delta A/\Delta w = (3.01 \cdot 7.01 - 21)/.01 = 10.01$ (d) $(3.001 \cdot 7.001 - 21)/.001 = 10.001 \approx 10.00$ (e) $A'(w) = 2w + 4$, $a'(3) = 10$

3. The measure of emission is $R = kT^4$, where T degrees is the Kelvin temperature. Find (a) the average rate of change as T increases from 200 to 300; (b) the instantaneous rate of change when $T = 200$.

▸ (a) $\Delta R/\Delta T = (k300^4 - k200^4)/(300 - 200) = 65{,}000{,}000k$ (b) $R'(T) = 4kT^3$, $R'(200) = 32{,}000{,}000k$

4. A circular cylinder of height 10 in. and base of r in. has volume V in^3. Find the average rate of change of V with respect to r as r changes from (a) 5.00 to 5.40; (b) 5.00 to 5.10; (c) 5.00 to 5.01. (d) Find the instantaneous rate of change of V with respect to r when r is 5.00.

▸ The volume of a circular cylinder is given by $V = \pi r^2 h = 10\pi r^2$. We simplify the calculation by factoring 10π.

(a) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi \cdot 5.4^2 - 10\pi \cdot 5^2}{5.4 - 5} = \dfrac{10\pi(5.4^2 - 5^2)}{0.4} = 104\pi$ (b) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi(5.1^2 - 5^2)}{5.1 - 5} = 101\pi$

(c) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi(5.01^2 - 5^2)}{5.01 - 5} = 100.1\pi$ (d) $V'(r) = 20\pi r$, $V'(5) = 100\pi$

5. A circular plate of radius r in. has area $A(r)$ in^2 and circumference $C(r)$ in. Find the instantaneous rate of change of (a) $A(r)$ and (b) $C(r)$.

▸ (a) $A = \pi r^2$, $A' = 2\pi r$ (b) $C = 2\pi r$, $C' = 2\pi$ (c) Note that $A' = C$.

6. A right circular cylinder whose length is twice its radius r is capped by two hemispheres and has volume $V(r)$ cubic units. Find the instantaneous rate of change of $V(r)$ with respect to r.

▸ The two hemispheres combine to form a sphere of volume $\frac{4}{3}\pi r^3$. The cylinder has volume $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$. Thus $V(r) = \frac{10}{3}\pi r^3$, $V'(r) = 10\pi r^2$.

7. Let x units be the total length of the solid of Exercise 6. Find the instantaneous rate of change of $V(x)$ with respect to x.

▸ $x = 4r$, $r = \frac{1}{4}x$. $V(x) = \frac{10}{3}\pi(\frac{1}{4}x)^3 = \frac{5}{96}\pi x^3$, $V'(x) = \frac{5}{32}\pi x^2$

8. Boyle's law for the expansion of a gas is $PV = C$, where P units is the pressure and V units is the volume, and C is a constant. (a) Show that V decreases at a rate proportional to the inverse square of P. (b) Find the instantaneous rate of change of V with respect to P when $P = 4$ and $V = 8$.

▸ (a) Solving for V, we find $V = CP^{-1}$ so that $V' = -CP^{-2} = -C/P^2$ which proves the assertion. (b) When $P = 4$ and $V = 8$, we find $C = 4 \cdot 8 = 32$. The instantaneous rate of change is $V' = -32/4^2 = -2$.

9. t days after a sickness starts, the temperature is $f(t)$ °F, where $f(t) = 98.6 + 1.2t - 0.12t^2$, $0 \le t \le 10$. (a) Find the rate of change of f. What is the temperature and its rate of change after (b) 3 days and (c) 8 days. (d) When does the maximum temperature occur and what is it?

▸ (a) $f'(t) = 1.2 - 0.24t$ (b) $f(3) = 101.12$, $f'(3) = 0.48$ (c) $f(8) = 100.52$, $f'(8) = -0.72$ (d) The temperature is increasing as long as $f'(t) > 0$, that is $0.24t < 1.2$, $t < 5$. The maximum temperature is at 5 days; it is 101.6°.

10. Find the rate of change of the volume of a spherical tumor with respect to the radius when the radius is (a) 0.5 cm and (b) 1 cm.

▷ The volume is V cm^3 when the radius is r cm, where $V = \frac{4}{3}\pi r^3$. $V'(r) = 4\pi r^2$. (a) $V'(0.5) = 4\pi(0.5)^2 = \pi$ cm^3/cm (b) $V'(1) = 4\pi(1)^2 = 4\pi$ cm^3/cm.

11. Find the rate of change of the volume of a spherical cell with respect to the radius when the radius is (a) 1.5 μm and (b) 2 μm.

▷ The volume is V μm^3 when the radius is r μm, where $V = \frac{4}{3}\pi r^3$. $V'(r) = 4\pi r^2$. (a) $V'(1.5) = 4\pi(1.5)^2 = 9\pi$ $\mu\text{m}^3/\mu\text{m}$ (b) $V'(2) = 4\pi(2)^2 = 16\pi$ $\mu\text{m}^3/\mu\text{m}$.

12. Find the rate of change of the surface of a spherical tumor with respect to the radius when the radius is (a) 0.5 cm and (b) 1 cm.

▷ The surface is S cm^2 when the radius is r cm, where $S = 4\pi r^2$. $S'(r) = 8\pi r$.
(a) $S'(0.5) = 8\pi(0.5) = 4\pi$. The surface is increasing at the rate of 4π cm^2/cm.
(b) $S'(1) = 8\pi$. The surface is increasing at the rate of 8π cm^2/cm.

13. Find the rate of change of the surface of a spherical tumor with respect to the radius when the radius is (a) 1.5 μm and (b) 2 μm.

▷ The surface is S μm^2 when the radius is r μm, where $S = 4\pi r^2$. $S'(r) = 8\pi r$.
(a) $S'(1.5) = 8\pi(1.5) = 12\pi$. The surface is increasing at the rate of 9π $\mu\text{m}^2/\mu\text{m}$
(b) $S'(2) = 16\pi$. The surface is increasing at the rate of 16π $\mu\text{m}^2/\mu\text{m}$.

14. The height of a cone is twice the radius. Find the rate of change of the volume with respect to the radius when the height is (a) 4m and (b) 8m.

▷ When the radius is rm, the height is $h = 2r$m and the volume is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$ m^3. $V' = 2\pi r^2$.
(a) When $h = 4$, $r = 2$. $V'(2) = 8\pi$ m^3/m (b) When $h = 8$, $r = 4$. $V'(4) = 32\pi$ m^3/m.

15. T degrees is the temperature t hours after midnight. $T = 0.1(400 - 40t + t^2)$, $0 \leq t \leq 12$

▷ (a) The average rate of change of T with respect to t between 5 a.m. and 6 a.m. is

$$\frac{0.1[400 - 40(6) + (6)^2] - 0.1[400 - 40(5) + (5)^2]}{6 - 5} = \frac{0.1(-204 + 175)}{1} = -2.9$$

Therefore between 5 a.m. and 6 a.m. the average rate of change of the temperature with respect to time is a decrease of 2.9 degrees per hour. (b) $T'(t) = 0.1(-40 + 2t)$; $T'(5) = 0.1[-40 + 2(5)] = -3$. Therefore the instantaneous rate of change of the temperature at 5 a.m. is a decrease of 3 degrees per hour.

16. A worker can paint y frames x hours after starting work at 8 A.M., and $y = 3x + 8x^2 - x^3$, $0 \leq x \leq 4$. (a) Find the rate at which he is painting at 10 A.M. (b) Find the number of frames he paints between 10 and 11 A.M.

▷ (a) $y' = 3 + 16x - 3x^2$. At 10 A.M., $x = 2$. $y'(2) = 3 + 16(2) - 3(2)^2 = 23$. He is painting at the rate of 23 frames per hour. (b) At 11 A.M., $x = 3$. $y(3) - y(2) = (3 \cdot 3 + 8 \cdot 3^2 - 3^3) - (3 \cdot 2 + 8 \cdot 2^2 - 2^3) = 24$. He paints 24 frames between 10 A.M. and 11 A.M.

17. V liters is the volume of water in the pool t minutes after the draining starts, where $V = 250(1600 - 80t + t^2)$.

▷ (a) The number of liters per minute in the average rate at which the volume of water in the pool is changing during the first 5 min is $\dfrac{250[1600 - 80(5) + (5)^2] - 250[1600 - 80(0) + 0^2]}{5 - 0} = \dfrac{250(-375)}{5} = -18{,}750$

Therefore, the average rate at which the water leaves the pool during the first 5 min is 18,750 liters per minute. (b) $V'(t) = 250(-80 + 2t)$, $V'(5) = 250[-80 + 2(5)] = -17{,}500$. Therefore, 5 min after the draining starts the water is leaving the pool at the rate of 17,500 liters per minute.

18. Find the rate of change of the area of a circular ripple when it radius is (a) 4 cm and (b) 7 cm.

▷ When the radius is r cm, the area is $A = \pi r^2$ cm^2. $A' = 2\pi r$. $A'(4) = 8\pi$ and $A'(7) = 14\pi$. The area is increasing at the rate of (a) 8π cm^2/cm and (b) 14π cm^2/cm.

19. $C(x)$ dollars is the total cost of manufacturing x watches: $C(x) = 1500 + 3x + x^2$

▷ (a) C' is the marginal cost function: $C'(x) = 3 + 2x$ (b) $C'(40) = 83$
(c) The number of dollars in the actual cost of manufacturing the forty-first watch is

$$C(41) - C(40) = [1500 + 3 \cdot 41 + (41)^2] - [1500 + 3 \cdot 40 + (40)^2] = 84$$

The total revenue received from the sale of x desks is $R(x)$ dollars, and $R(x) = 200x - \frac{1}{3}x^2$. Find (a) the marginal revenue function; (b) the marginal revenue when $x = 30$; (c) the actual revenue from the sale of the thirty-first desk.

▸ (a) The marginal revenue function is given by $R'(x) = 200 - \frac{2}{3}x$

(b) The marginal revenue when $x = 30$ is given by $R'(30) = 200 - \frac{2}{3}(30) = 180$
The marginal revenue is \$180 when $x = 30$.
(c) The actual revenue from the sale of the thirty-first desk is given by
$R(31) - R(30) = [200(31) - \frac{1}{3}(31)^2] - [200(30) - \frac{1}{3}(30)^2] = 5879.67 - 5700 = 179.67$
The actual revenue from the sale of the thirty-first desk is \$179.67.

$R(x)$ dollars is the total revenue from the sale of x television sets: $R(x) = 600x - \frac{1}{20}x^3$
▸ (a) R' is the marginal revenue function: $R'(x) = 600 - \frac{3}{20}x^2$.
(b) $R'(20) = 600 - \frac{3}{20}(20)^2 = 600 - 60 = 540$. Therefore the marginal revenue is \$540 when $x = 20$.
(c) The number of dollars in the actual revenue from the sale of the twenty-first television set is $R(21) - R(20)$. $R(21) - R(20) = \left|600(21) - \frac{1}{20}(21)^3\right| - \left|600(20) - \frac{1}{20}(20)^3\right| = 536.95$

$C(x)$ dollars is the total cost of making x paperweights: $C(x) = 200 + \frac{50}{x} + \frac{x^2}{5}$.
▸ (a) C' is the marginal cost function: $C'(x) = -\frac{50}{x^2} + \frac{2x}{5}$. (b) $C'(10) = -\frac{50}{10^2} + \frac{2(10)}{5} = 3.5$. The marginal cost is \$3.50 when $x = 10$. (c) The number of dollars in the actual cost of making the 11th paperweight is
$C(11) - C(10) = \left[200 + \frac{50}{11} + \frac{11^2}{5}\right] - \left[200 + \frac{50}{10} + \frac{10^2}{5}\right] = 3.745 \approx 3.75$

p dollars is the annual gross earnings of the company t years after January 1, 1994. $p(t) = 0.4t^2 + 2t + 10$.
▸ $p'(t) = 0.8t + 2$. (a) On January 1, 1996, $t = 2$ and $p'(2) = 0.8(2) + 2 = 3.6$. Hence on January 1, 1996 the gross earnings are growing at a rate of 3.6 million dollars per year.
(b) The relative rate of growth of the gross earnings on January 1, 1996 is $p'(2)/p(2)$.
$p(2) = 0.4(2)^2 + 2(2) + 10 = 15.6$; $p'(2)/p(2) = 3.6/15.6 = 0.231 = 23.1\%$
(c) On January 1, 2000, $t = 6$ and $p'(6) = 0.8(6) + 2 = 6.8$. Hence on January 1, 2000 the gross earnings are growing at a rate of 6.8 million dollars per year.
(d) The relative rate of growth of the gross earnings on January 1, 2000 is $p'(6)/p(6)$.
$p(6) = 0.4(6)^2 + 2(6) + 10 = 36.4$; $p'(2)/p(2) = 6.8/36.4 = 0.187 = 18.7\%$

24. p dollars is the annual gross earnings of a company t years after April 1, 1993, where $p = 50{,}000 + 18{,}000t + 600t^2$. Find (a) the rate at which the gross earnings were growing on April 1, 1995; (b) the relative rate of growth of the gross earnings on April 1, 1995 to the nearest 0.1%; (c) the rate at which the gross earnings should be growing on April 1, 2003; (d) the anticipated relative rate of growth of the gross earnings on April 1, 2003 to the nearest 0.1%.
▸ $p' = 18{,}000 + 1200t$ (a) On April 1, 1995, $t = 2$ and $p'(2) = 18{,}000 + 1200(2) = 20{,}400$. Hence on April 1, 1995 the gross earnings are growing at a rate of \$20,400 per year.
(b) The relative rate of growth of the gross earnings on April 1, 1996 is $p'(2)/p(2)$.
$p(2) = 50{,}000 + 18{,}000(2) + 600(2)^2 = 88{,}400$; $p'(2)/p(2) = 20{,}400/88{,}400 = 0.2308 \approx 23.1\%$
(c) On April 1, 2003, $t = 10$ and $p'(10) = 18{,}000 + 1200(10)) = 30{,}000$. Hence on April 1, 2003 the gross earnings are growing at a rate of \$30,000 per year.
(d) The relative rate of growth of the gross earnings on January 1, 2003 is $p'(10)/p(10)$.
$p(10) = 50{,}000 + 18{,}000(10) + 600(10)^2 = 290{,}000$; $p'(10)/p(10) = 30{,}000/290{,}000 = 0.1034 \approx 10.3\%$

5. $P(t)$ is the number of people in a population t years after January 1, 1995, where $P(t) = 40t^2 + 200t + 10{,}000$.
▸ $P'(t) = 80t + 200$. (a) On January 1, 2004, $t = 9$ and $P'(9) = 80 \cdot 9 + 200 = 920$. Therefore on January 1, 2004 the population will be growing at the rate of 920 people per year.
(b) The relative rate of growth of the population on January 1, 2004 is $P'(9)/P(9)$.
$P(9) = 40(9)^2 + 200(9) + 10{,}000 = 15{,}040$; $P'(9)/P(9) = 920/15{,}040 = 0.061 = 6.1\%$
(c) On January 1, 2010, $t = 15$ and $P'(15) = 80 \cdot 15 + 200 = 1400$. Hence on January 1, 2010 the population will be growing at the rate of 1400 people per year.
(d) The relative rate of growth of the population on January 1, 2010 is $P'(15)/P(15)$.
$P(15) = 40(15)^2 + 200(15) + 10{,}000 = 22{,}000$; $P'(15)/P(15) = 1{,}400/22{,}000 = 0.064 = 6.4\%$

26. Let r be the reciprocal of n. Find the instantaneous rate of change and the relative rate of change of r with respect to n when n is (a) 4 and (b) 10.

▷ $r = \frac{1}{n}$, $r' = -\frac{1}{n^2}$ (a) $r'(4) = -\frac{1}{4^2} = -\frac{1}{16}$, $\frac{r'(4)}{r(4)} = \frac{-1/16}{1/4} = -\frac{1}{4}$ (b) $r'(10) = -\frac{1}{10^2} = -\frac{1}{100}$, $\frac{r'(10)}{r(10)} = \frac{-1/100}{1/10} = -\frac{1}{10}$

27. The profit of a store is $100y$ dollars when x dollars are spent daily on advertising and $y = 2500 + 36x - 0.2x^2$.

▷ $y'(x) = 36 - 0.4x$. (a) $y'(60) = 36 - 0.4(60) = 2$. Therefore the rate of change of y with respect to x is positive when $x = 60$. Thus it is profitable to increase the daily advertising budget when $x = 60$.
(b) $y'(300) = 36 - 0.4(300) = -84$. Because the rate of change of y with respect to x is negative when $x = 300$, it is not profitable to increase the daily advertising budget when $x = 300$.
(c) The maximum value for x below which it is profitable to increase the advertising budget occurs when $y'(x) = 0$. Thus $36 - 0.4 = 0$; $0.4x = 36$; $x = 90$.

28. The supply equation for a shirt is $x = 3p^2 + 2p$, where p dollars is the price per shirt when $1000x$ shirts are supplied. (a) Find the average rate of change of the supply when the price is increased for \$10 to \$11. (b) Find the marginal rate change of the supply when the price is \$10.

▷ (a) $x(11) - x(10) = [3(11)^2 + 2(11)] - [3(10)^2 + 2(10)] = 65$. The supply increases by 65,000 shirts when the price increases fro \$10 to \$11. (b) $x' = 6p + 2$. $x'(10) = 6(10) + 2 = 62$. The marginal rate of change is 62,000 shirts per dollar.

29. Find the slope of the tangent line at each point of the graph of $y = x^4 + x^3 - 3x^2$ where the rate of change of the slope is zero.

▷ slope $= y' = 4x^3 + 3x^2 - 6x$. $y'' = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1) = 0$ when $x = -1, \frac{1}{2}$.
$y'(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) = 5$, $y'(\frac{1}{2}) = 4(\frac{1}{2})^3 + 3(\frac{1}{2})^2 - 6(\frac{1}{2}) = -\frac{7}{4}$

30. Find the instantaneous rate of change of the slope of the tangent line to $y = 2x^3 - 6x^2 - x + 1$ at $(3, -2)$.

▷ slope $= y' = 6x^2 - 12x - 1$. $y'' = 12x - 12$. $y''(3) = 12(3) - 12 = 24$

31. At t min $r(t)$ m is the radius of an oil spill. $r(t) = \begin{cases} 4t^2 + 20 & \text{if } 0 \le t \le 2 \\ 16t + 4 & \text{if } t > 2 \end{cases}$. In Ex. 2.2.31, it was shown that $r'(2) = 16$. Find the rate at which the the radius is changing at (a) 0.4 min; (b) 2 min; (c) 3.2 min.

▷ (a, b) If $r \le 2$, $r' = 8t$ so $r'(0.4) = 8(0.4) = 3.2$, $r'(2) = 8(2) = 16$ (c) If $r > 2$, $r' = 16$ so $r'(3.2) = 16$

32. Show that for any linear function f, the average rate of change of $f(x)$ as x changes from x_1 to $x_1 + k$ is the same as the instantaneous rate of change of $f(x)$ at x_1.

▷ Let $f(x) = mx + b$. The average rate of change is $\frac{f(x_1 + k) - f(x_1)}{(x_1 + k) - x_1} = \frac{m(x_1 + k) + b - (mx_1 + b)}{k} = \frac{mk}{k} = m$.
$f'(x) = m$ which the instantaneous rate of change at any point.

33. (a) The instantaneous rate of change of the area of a circle is $D_r(\pi r^2) = 2\pi r$, the circumference of the circle.
(b) The instantaneous rate of change of the volume of a sphere is $D_r(\frac{4}{3}\pi r^3) = 4\pi r^2$, the surface of the sphere.

2.7 DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

2.7.1 Theorem $D_x(\sin x) = \cos x$ **2.7.2 Theorem** $D_x(\cos x) = -\sin x$

2.7.3 Theorem $D_x(\tan x) = \sec^2 x$ **2.7.4 Theorem** $D_x(\cot x) = -\csc^2 x$

2.7.5 Theorem $D_x(\sec x) = \sec x \tan x$ **2.7.6 Theorem** $D_x(\csc x) = -\csc x \cot x$

Functions of rational multiples of π in the first quadrant requiring at most one radical (Ex. in Before Calculus):
$\tan \frac{1}{12}\pi = 2 - \sqrt{3}$ (9.2.5), $\sin \frac{1}{10}\pi = \cos \frac{2}{5}\pi = \frac{1}{4}(\sqrt{5} - 1)$ (10.6.22) $\tan \frac{1}{8}\pi = \sqrt{2} - 1$, $\sin \frac{1}{6}\pi = \cos \frac{1}{3}\pi = \frac{1}{2}$,
$\cos \frac{1}{6}\pi = \sin \frac{1}{3}\pi = \frac{1}{2}\sqrt{3}$, $\tan \frac{1}{6}\pi = \frac{1}{3}\sqrt{3}$, $\cos \frac{1}{5}\pi = \sin \frac{3}{10}\pi = \frac{1}{4}(\sqrt{5} + 1)$, $\sin \frac{1}{4}\pi = \cos \frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$, $\tan \frac{1}{4}\pi = 1$,
$\tan \frac{1}{3}\pi = \sqrt{3}$, $\tan \frac{3}{8}\pi = \sqrt{2} + 1$ (9.3.28), $\tan \frac{5}{12}\pi = 2 + \sqrt{3}$ (9.2.6)

Exercises 2.7

1. $D_x(\cot x) = D_x\frac{\cos x}{\sin x} = \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = \frac{(-\sin x)\sin x - \cos x(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$
$= -\csc x$

2. $D_x(\csc x) = D_x\frac{1}{\sin x} = \frac{(1)'\sin x - 1(\sin x)'}{\sin^2 x} = \frac{0 - \cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$

In Exercises 3–18, find the derivative of the function.

3. $f'(x) = \mathrm{D}_x(3 \sin x) = 3 \cos x$

4. $g(x) = \sin x + \cos x$ ▸ $g'(x) = \mathrm{D}_x(\sin x) + \mathrm{D}_x(\cos x) = \cos x - \sin x$

5. $g'(x) = \mathrm{D}_x(\tan x + \cot x) = \sec^2 x - \csc^2 x$

6. $f'(x) = \mathrm{D}_x(4 \sec x - 2 \csc x) = 4 \sec x \tan x + 2 \csc x \cot x$

7. $f'(t) = \mathrm{D}_t(2t \cos t) = 2 \cos t + 2t(-\sin t) = 2(\cos t - t \sin t)$

8. $f(x) = 4x^2 \cos x$

▸ We apply the product rule. $f'(x) = 4x^2\mathrm{D}_x(\cos x) + \cos x \cdot \mathrm{D}_x(4x^2) = -4x^2 \sin x + 8x \cos x$

9. $g'(x) = \mathrm{D}_x(x \sin x + \cos x) = (1 \sin x + x \cos x) - \sin x = x \cos x$

10. $g'(y) = \mathrm{D}_y(3 \sin y - y \cos y) = 3 \cos y - [1 \cdot \cos y + y(-\sin y)] = 2 \cos y + y \sin y$

11. $h'(x) = \mathrm{D}_x(4 \sin x \cos x) = 4[\sin x(-\sin x) + (\cos x)\cos x] = 4(-\sin^2 x + \cos^2 x) = 4 \cos 2x$

12. $f(x) = x^2 \sin x + 2x \cos x$

▸ $$\begin{aligned} f'(x) &= \mathrm{D}_x(x^2 \sin x) + \mathrm{D}_x(2x \cos x) = x^2 \cdot \mathrm{D}_x(\sin x) + \sin x \cdot \mathrm{D}_x(x^2) + 2x \cdot \mathrm{D}_x(\cos x) + \cos (x) \cdot \mathrm{D}_x(2x) \\ &= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x = x^2 \cos x + 2 \cos x \end{aligned}$$

13. $f'(x) = \mathrm{D}_x(x^2 \cos x - 2x \sin x - 2 \cos x) = [x^2(-\sin x) + 2x \cos x] - (2x \cos x + 2 \sin x) + 2\sin x = -x^2 \sin x$

14. $h'(y) = \mathrm{D}_y(y^3 - y^2 \cos y + 2y \sin y + 2 \cos y) = 3y^2 - [2y \cos y + y^2(-\sin y)] + (2 \sin y + 2y \cos y) - 2 \sin y$
$= 3y^2 + y^2 \sin y$

15. $f'(x) = \mathrm{D}_x(3 \sec x \tan x) = 3[(\sec x \tan x)\tan x + \sec x(\sec^2 x)] = 3 \sec x(\tan^2 x + \sec^2 x)$

16. $f(t) = \sin t \tan t$

▸ $f'(t) = \sin t\, \mathrm{D}_t(\tan t) + \tan t\, \mathrm{D}_t(\sin t) = \sin t \sec^2 t + \tan t \cos t = \sin t \sec^2 t + \sin t$

17. $f'(y) = \mathrm{D}_y(\cos y \cot y) = (\cos y)'\cot y + \cos y(\cot y)' = -\sin y \cot y - \cos y \csc^2 y$

18. $h'(x) = \mathrm{D}_x(\cot x \csc x) = \cot x(-\csc x \cot x) + \csc x(-\csc^2 x) = -\csc x(\cot^2 x + \csc^2 x)$

In Exercises 19–30, find the derivative.

19. $\mathrm{D}_z\left(\frac{2 \cos z}{z+1}\right) = \frac{(z+1)(-2 \sin z) - 2 \cos z \cdot 1}{(z+1)^2} = -2\frac{(z+1)\sin z + \cos z}{(z+1)^2}$

20. $\mathrm{D}_t\left(\frac{\sin t}{t}\right)$ ▸ We apply the quotient rule.

$$\mathrm{D}_t\left(\frac{\sin t}{t}\right) = \frac{t \cdot \mathrm{D}_t(\sin t) - \sin t \cdot \mathrm{D}_t t}{t^2} = \frac{t \cos t - \sin t}{t^2}$$

21. $\frac{d}{dx}\left(\frac{\sin x}{1 - \cos x}\right) = \frac{(1 - \cos x)\cos x - \sin x \cdot \sin x}{(1 - \cos x)^2} = \frac{\cos x - \cos^2 x - \sin^2 x}{(1 - \cos x)^2} = \frac{\cos x - 1}{(\cos x - 1)^2} = \frac{1}{\cos x - 1}$

22. $\frac{d}{dx}\left(\frac{x+4}{\cos x}\right) = \frac{1 \cdot \cos x - (x+4)(-\sin x)}{\cos^2 x} = \frac{\cos x + x \sin x + 4 \sin x}{\cos^2 x}$

23. $\frac{d}{dt}\left(\frac{\tan t}{\cos t - 4}\right) = \frac{(\cos t - 4)\sec^2 t - \tan t(-\sin t)}{(\cos t - 4)^2} = \frac{\sec t - \sec^2 t + \tan t \sin t}{(\cos t - 4)^2} = \frac{1 - 4 \sec t - \sin^2 t}{\cos t(\cos t - 4)^2}$

24. $\frac{d}{dy}\left(\frac{\cot y}{1 - \sin y}\right)$

▸ $$\begin{aligned} \frac{d}{dy}\left(\frac{\cot y}{1 - \sin y}\right) &= \frac{(1 - \sin y)\mathrm{D}_y(\cot y) - \cot y\, \mathrm{D}_y(1 - \sin y)}{(1 - \sin y)^2} = \frac{(1 - \sin y)(-\csc^2 y) - \cot y(-\cos y)}{(1 - \sin y)^2} \\ &= \frac{-\csc^2 y + \csc y + \cot y \cos y}{(1 - \sin y)^2} \end{aligned}$$

25. $\frac{d}{dy}\left(\frac{1 + \sin y}{1 - \sin y}\right) = \frac{(1 - \sin y)(\cos y) - (1 + \sin y)(-\cos y)}{(1 - \sin y)^2} = \frac{\cos y - \sin y \cos y + \cos y + \sin y \cos y}{(1 - \sin y)^2} = \frac{2 \cos y}{(1 - \sin y)^2}$

26. $\frac{d}{dx}\left(\frac{\sin x-1}{\cos x+1}\right)=\frac{\cos x(\cos x+1)-(\sin x-1)(-\sin x)}{(\cos x+1)^2}=\frac{\cos^2 x+\cos x+\sin^2 x-\sin x}{(\cos x+1)^2}=\frac{1+\cos x-\sin x}{(\cos x+1)^2}$

27. $\mathrm{D}[(x-\sin x)(x+\cos x)]=(1-\cos x)(x+\cos x)+(x-\sin x)(1-\sin x)$

28. $\mathrm{D}_z[(z^2+\cos z)(2z-\sin z)]$

▷ Applying the product rule, we obtain

$$\begin{aligned}\mathrm{D}_z[(z^2+\cos z)(2z-\sin z)]&=(z^2+\cos z)\mathrm{D}_z(2z-\sin z)+(2z-\sin z)\mathrm{D}_z(z^2+\cos z)\\&=(z^2+\cos z)(2-\cos z)+(2z-\sin z)(2z-\sin z)\\&=(z^2+\cos z)(2-\cos z)+(2z-\sin z)^2\end{aligned}$$

29. $\mathrm{D}_t\left(\frac{2\csc t-1}{\csc t+2}\right)=\frac{(\csc t+2)(-2\csc t\cot t)-(2\csc t-1)(-\csc t\cot t)}{(\csc t+2)^2}$

$=\frac{-2\csc^2 t\cot t-4\csc t\cot t+2\sec^2 t\cot t-\sec t\cot t}{(\csc t+2)^2}=\frac{-5\csc t\cot t}{(\csc t+2)^2}$

Alternatively, $\mathrm{D}_t\left(\frac{2\csc t-1}{\csc t+2}\right)=\mathrm{D}_t\left(\frac{2-\sin t}{1+2\sin t}\right)=\frac{(1+2\sin t)(-\cos t)-(2-\sin t)(2\cos t)}{(1+2\sin t)^2}$

$=\frac{-\cos t-2\sin t\cos t-4\cos t+2\sin t\cos t}{(1+2\sin t)^2}=\frac{-5\cos t}{(1+2\sin t)^2}$

30. $\mathrm{D}_y\left(\frac{\tan y+1}{\tan y-1}\right)=\frac{\sec^2 y(\tan y-1)-(\tan y+1)\sec^2 y}{(\tan y-1)^2}=\frac{-2\sec^2 y}{(\tan y-1)^2}$

In Exercises 31–42, compute $\mathrm{NDER}(f(x),a)$ on your calculator. Then compute the exact value of $f'(a)$.

31. $f(x)=x\cos x$; $f'(x)=\cos x-x\sin x$; so $f'(0)=\cos 0-0\sin 0=1$

32. $f(x)=x\sin x$; $\quad a=\frac{3}{2}\pi$

▷ NDER gives −0.9999995. Now we find $f'(x)$.

$$f'(x)=x\cdot\mathrm{D}_x\sin x+\sin x\cdot\mathrm{D}_x x=x\cos x+\sin x$$

Next, we replace x with $\frac{3}{2}\pi$. Thus,

$$f'(\tfrac{3}{2}\pi)=(\tfrac{3}{2}\pi)\cos(\tfrac{3}{2}\pi)+\sin(\tfrac{3}{2}\pi)=(\tfrac{3}{2}\pi)(0)+(-1)=-1$$

33. $f(x)=\frac{\cos x}{x}$; $f'(x)=\frac{x(-\sin x)-\cos x\cdot 1}{x^2}$; so $f'(\frac{1}{2}\pi)=\frac{\frac{1}{2}\pi(-1)-0}{(\frac{1}{2}\pi)^2}=\frac{-1}{\frac{1}{2}\pi}=-\frac{2}{\pi}$

34. $f(x)=\frac{\sec x}{x^2}$; $f'(x)=\frac{(\sec x\tan x)x^2-\sec x(2x)}{x^4}$; so $f'(\pi)=\frac{0+2\pi}{\pi^4}=\frac{2}{\pi^3}\approx 0.06450307$. NDER = 0.06450311

35. $f(x)=x^2\tan x$; $f'(x)=2x\tan x+x^2\sec^2 x$; so $f'(\pi)=2\pi\tan\pi+\pi^2\sec^2\pi=2\pi(0)+\pi^2(-1)^2=\pi^2$

36. $f(x)=x^2\cos x-\sin x$; $\quad a=0$

▷ NDER gives −0.9999998

$$f'(x)=x^2\cdot\mathrm{D}_x\cos x+\cos x\cdot\mathrm{D}_x x^2-\mathrm{D}_x\sin x=-x^2\sin x+2x\cos x-\cos x$$

Thus, with $x=0$, we have

$$f'(0)=-0^2(\sin 0)+2(0)\cos 0-\cos 0=-1$$

37. $f(x)=\sin x(\cos x-1)$; $f'(x)=\cos x(\cos x-1)+\sin x(-\sin x)$; so $f'(\pi)=(-1)(-2)+0(0)=2$

38. $f(x)=(\cos x+1)(x\sin x-1)$; $f'(x)=-\sin x(x\sin x-1)+(\cos x+1)(\sin x+x\cos x)$;

$f'(\frac{1}{2}\pi)=-(\frac{1}{2}\pi-1)+1=2-\frac{1}{2}\pi\approx 0.4292037$. NDER = 0.4292041

39. $f(x)=x\cos x+x\sin x$; $f'(x)=\cos x-x\sin x+\sin x+x\cos x$
so $f'(\frac{1}{4}\pi)=\frac{1}{2}\sqrt{2}-(\frac{1}{4}\pi)(\frac{1}{2}\sqrt{2})+\frac{1}{2}\sqrt{2}+(\frac{1}{4}\pi)(\frac{1}{2}\sqrt{2})=\sqrt{2}$

40. $f(x)=\tan x+\sec x$; $\quad a=\frac{1}{6}\pi$

▷ NDER gives 2.000002.

$$f'(x)=\sec^2 x+\sec x\tan x$$

$$f'(\tfrac{1}{6}\pi)=\left(\frac{2}{\sqrt{3}}\right)^2+\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}=\frac{4}{3}+\frac{2}{3}=2$$

41. $f(x)=2\cot x-\csc x$; $f'(x)=-2\csc^2 x+\csc x\cot x$; so $f'(\frac{2}{3}\pi)=-2\left(\frac{2}{\sqrt{3}}\right)^2+\frac{2}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{10}{3}$

42. $f(x) = \dfrac{1}{\cot x - 1}$; $f'(x) = \dfrac{0 - (-\csc^2 x)}{(\cot x - 1)^2}$; $f'(\frac{3}{4}\pi) = \frac{2}{4} = \frac{1}{2} = 0.5$. NDER $= 0.5000002$

43. (a)

h	1	0.5	0.1	0.01	0.001	-1	-0.5	-0.1	-0.01	$-.001$
$\dfrac{\sin(\frac{1}{3}\pi + h) - \sin\frac{1}{3}\pi}{h}$	.0226	.2674	.4559	.4956	.4995	.8188	.6915	.5424	.5043	.5006

The quotient appears to approach $\frac{1}{2}$. (b) $\displaystyle\lim_{h\to 0} \frac{\sin(\frac{1}{3}\pi + h) - \sin\frac{1}{3}\pi}{h} = \frac{d}{dx}\sin x\Big|_{x=\pi/3} = \cos\tfrac{1}{3}\pi = \tfrac{1}{2}$

44. (a) Use a calculator to tabulate to four decimal places values of $\dfrac{\cos(\frac{5}{6}\pi + h) - \cos\frac{5}{6}\pi}{h}$ when h is 1, 0.5, 0.1, 0.01, 0.001 and h is -1, -0.5, -0.1, -0.001. What does the quotient appear to be approaching as h approaches 0? (b) Find $\displaystyle\lim_{h\to 0} \frac{\cos(\frac{5}{6}\pi + h) - \cos\frac{5}{6}\pi}{h}$ by interpreting it as a derivative.

▸ (a) See the table. The quotient appears to be approaching -0.5 as h approaches 0.

h	1	0.5	0.1	0.01	0.001
$\dfrac{\cos(\frac{5}{6}\pi + h) - \cos\frac{5}{6}\pi}{h}$	-0.0226	-0.2674	-0.4559	-0.4957	-0.4996

h	-1	-0.5	-0.1	-0.01	-0.001
$\dfrac{\cos(\frac{5}{6}\pi + h) - \cos\frac{5}{6}\pi}{h}$	-0.8188	-0.6915	-0.5424	-0.5043	-0.5004

(b) By formula (4) of Section 2.1 and Theorem 2.7.2,

$$\lim_{h\to 0} \frac{\cos(\frac{5}{6}\pi + h) - \cos\frac{5}{6}\pi}{h} = \mathrm{D}_x\cos x\,|_{5\pi/6} = -\sin(\tfrac{5}{6}\pi) = -\tfrac{1}{2}$$

45.

h	0.1	0.01	0.001	0.0001	10^{-5}	-0.1	-0.01	-0.001	$-.0001$	-10^{-5}
$\dfrac{\tan(\frac{1}{4}\pi + h) - \tan\frac{1}{4}\pi}{h}$	2.2305	2.0203	2.0020	2.0002	2	1.8237	1.9803	1.9980	1.9998	2

The quotient appears to approach 2. (b) $\displaystyle\lim_{h\to 0} \frac{\tan(\frac{1}{4}\pi + h) - \tan\frac{1}{4}\pi}{h} = \frac{d}{dx}\tan x\Big|_{x=\pi/4} = \sec^2\tfrac{1}{4}\pi = 2$

46.

h	0.1	0.01	0.001	0.0001	10^{-5}	-0.1	-0.01	-0.001	$-.0001$	-10^{-5}
$\dfrac{\sec(\frac{1}{6}\pi + h) - \sec\frac{1}{6}\pi}{h}$	0.7716	0.6764	0.6676	0.6668	0.6667	0.5775	0.6571	0.6657	0.6666	0.6667

Quotient appears to approach $\frac{2}{3}$. (b) $\displaystyle\lim_{h\to 0} \frac{\sec(\frac{1}{6}\pi + h) - \sec\frac{1}{6}\pi}{h} = \frac{d}{dx}\sec x\Big|_{x=\pi/6} = \sec\tfrac{1}{6}\pi\tan\tfrac{1}{6}\pi = \frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{2}{3}$

47.

x	$\frac{3}{20}\pi$	$\frac{19}{120}\pi$	$\frac{33}{200}\pi$	$\frac{199}{1200}\pi$	$\frac{333}{2000}\pi$	$\frac{11}{60}\pi$	$\frac{7}{40}\pi$	$\frac{101}{600}\pi$	$\frac{67}{400}\pi$	$\frac{1001}{6000}\pi$
$x - \frac{1}{6}\pi$	$-\frac{1}{60}\pi$	$-\frac{1}{120}\pi$	$-\frac{1}{600}\pi$	$-\frac{1}{1200}\pi$	$-\frac{1}{6000}\pi$	$\frac{1}{60}\pi$	$\frac{1}{120}\pi$	$\frac{1}{600}\pi$	$\frac{1}{1200}\pi$	$\frac{1}{6000}\pi$
$\dfrac{\cos x - \cos\frac{1}{6}\pi}{x - \frac{1}{6}\pi}$	$-.4771$	$-.4886$	$-.4977$	$-.4989$	$-.4998$	$-.5224$	$-.5113$	$-.5023$	$-.5011$	$-.5002$

The quotient appears to approach $-\frac{1}{2}$. (b) $\displaystyle\lim_{x\to\pi/6} \frac{\cos x - \cos\frac{1}{6}\pi}{x - \frac{1}{6}\pi} = \frac{d}{dx}\cos x\Big|_{x=\pi/6} = \sin\tfrac{1}{6}\pi = -\tfrac{1}{2}$

48. (a) Use a calculator to tabulate to four decimal places values of $\dfrac{\sin x - \sin\frac{1}{3}\pi}{x - \frac{1}{3}\pi}$ when x is $\frac{3}{10}\pi$, $\frac{19}{60}\pi$, $\frac{33}{100}\pi$, $\frac{199}{600}\pi$, $\frac{333}{1000}\pi$ and x is $\frac{11}{30}\pi$, $\frac{7}{20}\pi$, $\frac{101}{300}\pi$, $\frac{67}{200}\pi$, $\frac{1001}{3000}\pi$. What does the quotient seem to be approaching as x approaches $\frac{1}{3}\pi$? (b) Find $\displaystyle\lim_{x\to\pi/3} \frac{\sin x - \sin\frac{1}{3}\pi}{x - \frac{1}{3}\pi}$ by interpreting it as a derivative.

▶ (a) See the table. The quotient appears to be approaching 0.5 as x approaches $\frac{1}{3}\pi$.

x	$\frac{3}{10}\pi$	$\frac{19}{60}\pi$	$\frac{33}{100}\pi$	$\frac{199}{600}\pi$	$\frac{333}{1000}\pi$
$x-\frac{1}{3}\pi$	$-\frac{1}{30}\pi$	$-\frac{1}{60}\pi$	$-\frac{1}{300}\pi$	$-\frac{1}{600}\pi$	$-\frac{1}{3000}\pi$
$\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}$	0.5444	0.5224	0.5045	0.5023	0.5005

x	$\frac{11}{30}\pi$	$\frac{7}{20}\pi$	$\frac{101}{300}\pi$	$\frac{67}{200}\pi$	$\frac{1001}{3000}\pi$
$x-\frac{1}{3}\pi$	$\frac{1}{30}\pi$	$\frac{1}{60}\pi$	$\frac{1}{300}\pi$	$\frac{1}{600}\pi$	$\frac{1}{3000}\pi$
$\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}$	0.4538	0.4771	0.4955	0.4977	0.4994

(b) $\lim\limits_{x\to\pi/3}\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}=D_x\sin x\,\big|_{\pi/3}=\cos\frac{1}{3}\pi=\frac{1}{2}$

49.

x	$\frac{3}{5}\pi$	$\frac{19}{30}\pi$	$\frac{33}{50}\pi$	$\frac{199}{300}\pi$	$\frac{333}{500}\pi$	$\frac{11}{15}\pi$	$\frac{7}{10}\pi$	$\frac{101}{150}\pi$	$\frac{167}{100}\pi$	$\frac{1001}{1500}\pi$
$x-\frac{2}{3}\pi$	$-\frac{1}{15}\pi$	$-\frac{1}{30}\pi$	$-\frac{1}{150}\pi$	$-\frac{1}{300}\pi$	$-\frac{1}{1500}\pi$	$\frac{1}{15}\pi$	$\frac{1}{30}\pi$	$\frac{1}{150}\pi$	$\frac{1}{300}\pi$	$\frac{1}{1500}\pi$
$\dfrac{\csc x-\csc\frac{2}{3}\pi}{x-\frac{2}{3}\pi}$	0.4929	0.5736	0.6468	0.6567	0.6647	0.9116	0.7770	0.6872	0.6768	0.6687

Quotient seems to approach $\frac{2}{3}$. (b) $\lim\limits_{x\to 2\pi/3}\dfrac{\csc x-\csc\frac{2}{3}\pi}{x-\frac{2}{3}\pi}=\dfrac{d}{dx}\csc x\Big|_{x=2\pi/3}=-\csc\frac{2}{3}\pi\cot\frac{2}{3}\pi=-\dfrac{2}{\sqrt{3}}\left(-\dfrac{1}{\sqrt{3}}\right)=\frac{2}{3}$

50.

x	$\frac{29}{40}\pi$	$\frac{59}{80}\pi$	$\frac{299}{400}\pi$	$\frac{599}{800}\pi$	$\frac{2999}{4000}\pi$	$\frac{31}{40}\pi$	$\frac{61}{80}\pi$	$\frac{301}{400}\pi$	$\frac{601}{800}\pi$	$\frac{3001}{4000}\pi$
$x-\frac{3}{4}\pi$	$-\frac{1}{40}\pi$	$-\frac{1}{80}\pi$	$-\frac{1}{400}\pi$	$-\frac{1}{800}\pi$	$-\frac{1}{4000}\pi$	$\frac{1}{40}\pi$	$\frac{1}{80}\pi$	$\frac{1}{400}\pi$	$\frac{1}{800}\pi$	$\frac{1}{4000}\pi$
$\dfrac{\cot x-\cot\frac{3}{4}\pi}{x-\frac{3}{4}\pi}$	-1.8579	-1.9254	-1.9845	-1.9922	-1.9984	-2.1753	-2.0289	-2.0159	-2.0079	-2.0289

The quotient appears to approach -2. (b) $\lim\limits_{x\to 3\pi/4}\dfrac{\cot x-\cot\frac{3}{4}\pi}{x-\frac{3}{4}\pi}=\dfrac{d}{dx}\cot x\Big|_{x=3\pi/4}=-\csc^2\frac{3}{4}\pi=-(\sqrt{2})^2=-2$

51. Let $f(x)=\sin x$. Then $f'(x)=\cos x$.
(a) If $x=0$, $y=0$, and the slope of the tangent line is $f'(0)=\cos 0=1$; thus an equation of the tangent line at $(0,0)$ is $y=x$.
(b) If $x=\frac{1}{3}\pi, y=\frac{1}{2}\sqrt{3}$, and the slope of the tangent line is $f'(\frac{1}{3}\pi)=\cos(\frac{1}{3}\pi)=\frac{1}{2}$; thus an equation of the tangent line at $(\frac{1}{3}\pi,\frac{1}{2}\sqrt{3})$ is $y-\frac{1}{2}\sqrt{3}=\frac{1}{2}(x-\frac{1}{3}\pi)$; $3x-6y-\pi+3\sqrt{3}=0$.
(c) If $x=\pi$, $y=0$, and the slope of the tangent line is $f'(\pi)=-1$; thus an equation of the tangent line at $(\pi,0)$ is $y=-1(x-\pi)$; $x+y-\pi=0$.

52. Find an equation of the tangent line to the graph of the cosine function at the point where (a) $x=\frac{1}{2}\pi$; (b) $x=-\frac{1}{2}\pi$; (c) $x=\frac{1}{6}\pi$.

▶ Let f be the cosine function. That is $f(x)=\cos x$. Then $f'(x)=-\sin x$

(a) Because $f(\frac{1}{2}\pi)=\cos(\frac{1}{2}\pi)=0$, $f'(\frac{1}{2}\pi)=-\sin(\frac{1}{2}\pi)=-1$

then $y=0$ when $x=\frac{1}{2}\pi$ and the slope of the tangent line is -1. By the point-slope formula, an equation of the tangent line is $\quad y-0=-(x-\frac{1}{2}\pi), \qquad y=-x+\frac{1}{2}\pi$

(b) Because $\quad f(-\frac{1}{2}\pi)=\cos(-\frac{1}{2}\pi)=0, \quad f'(-\frac{1}{2}\pi)=-\sin(-\frac{1}{2}\pi)=1$

then $y=0$ when $x=-\frac{1}{2}\pi$ and the slope of the tangent line is 1. Thus, an equation of the tangent line is
$y=x+\frac{1}{2}\pi$

(c) Because $f(\frac{1}{6}\pi) = \cos(\frac{1}{6}\pi) = \frac{1}{2}\sqrt{3}$, $f'(\frac{1}{6}\pi) = -\sin(\frac{1}{6}\pi) = -\frac{1}{2}$
then $y = \frac{1}{2}\sqrt{3}$ when $x = \frac{1}{6}\pi$ and the slope of the tangent line is $-\frac{1}{2}$. Thus, an equation of the tangent line is

$$y - \tfrac{1}{2}\sqrt{3} = -\tfrac{1}{2}(x - \tfrac{1}{6}\pi), \qquad y = -\tfrac{1}{2}x + \tfrac{1}{12}\pi + \tfrac{1}{2}\sqrt{3}$$

Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$.
(a) When $x = 0$, $y = 0$, and the slope of the tangent line is $f'(0) = \sec^2 0 = 1$; thus an equation of the tangent line at $(0, 0)$ is $y = x$.
(b) When $x = \frac{1}{4}\pi, y = 1$, and the slope of the tangent line is $f'(\frac{1}{4}\pi) = \sec^2(\frac{1}{4}\pi) = 2$; thus an equation of the tangent line at $(\frac{1}{4}\pi, 1)$ is $y - 1 = 2(x - \frac{1}{4}\pi)$; $4x - 2y + 2 - \pi = 0$.
(c) When $x = -\frac{1}{4}\pi$, $y = -1$, and the slope of the tangent line is $f'(-\frac{1}{4}\pi) = \sec^2(-\frac{1}{4}\pi) = 2$; thus an equation of the tangent line at $(-\frac{1}{4}\pi, -1)$ is $y + 1 = 2(x + \frac{1}{4}\pi)$; $4x - 2y - 2 + \pi = 0$.

Let $f(x) = \sec x$. Then $f'(x) = \sec x \tan x$.
(a) When $x = \frac{1}{4}\pi$, $y = \sqrt{2}$, and the slope of the tangent line is $f'(\frac{1}{4}) = \sec \frac{1}{4}\pi \tan \frac{1}{4}\pi = \sqrt{2}$; thus an equation of the tangent line at $(\frac{1}{4}\pi, \sqrt{2})$ is $y = \sqrt{2}(x - \frac{1}{4}\pi) + \sqrt{2}$.
(b) When $x = -\frac{1}{4}\pi, y = \sqrt{2}$, and the slope of the tangent line is $f'(-\frac{1}{4}\pi) = \sec(-\frac{1}{4}\pi)\tan(-\frac{1}{4}\pi) = -\sqrt{2}$; thus an equation of the tangent line at $(-\frac{1}{4}\pi, \sqrt{2})$ is $y = -\sqrt{2}(x + \frac{1}{4}\pi) + \sqrt{2}$.
(c) When $x = \frac{3}{4}\pi$, $y = -\sqrt{2}$, and the slope of the tangent line is $f'(\frac{3}{4}\pi) = \sec \frac{3}{4}\pi \tan \frac{3}{4}\pi = \sqrt{2}$; thus an equation of the tangent line at $(\frac{3}{4}\pi, -\sqrt{2})$ is $y = \sqrt{2}(x - \frac{3}{4}\pi) - \sqrt{2}$

Exercises 55–58, a particle is moving along a straight line according to the equation where s cm is the directed tance from the origin at t seconds. (a) What is the instantaneous velocity and acceleration of the particle at t_1 nds? (b) Find the instantaneous velocity and acceleration of the particle at t_1 seconds for each value of t_1.

$s = 4 \sin t$. (a) The instantaneous velocity and acceleration are given by $v(t) = \dfrac{ds}{dt} = 4 \cos t$ and $a(t) = \dfrac{dv}{dt} = -4 \sin t$ (b) $v(0) = 4 \cos 0 = 4$, $a(0) = -4 \sin 0 = 0$; $v(\frac{1}{3}\pi) = 4 \cos(\frac{1}{3}\pi) = 4(\frac{1}{2}) = 2$; $a(\frac{1}{3}\pi) = -4 \sin \frac{1}{3}\pi = -4 \cdot \frac{1}{2}\sqrt{3} = -2\sqrt{3}$; $v(\frac{1}{2}\pi) = 4 \cos(\frac{1}{2}\pi) = 4(0) = 0$, $a(\frac{1}{2}\pi) = -4 \sin \frac{1}{2}\pi = -4 \cdot 1 = -4$; $v(\frac{2}{3}\pi) = 4 \cos(\frac{2}{3}\pi) = 4(-\frac{1}{2}) = -2$; $a(\frac{2}{3}\pi) = -4 \sin \frac{2}{3}\pi = -4 \cdot \frac{1}{2}\sqrt{3} = -2\sqrt{3}$; $v(\pi) = 4 \cos \pi = 4(-1) = -4$, $a(\pi) = -4 \sin \pi = -4 \cdot 0 = 0$

$s = 6 \cos t$; t_1 is $0, \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi$ and π

(a) Because $v(t) = D_t(6 \cos t) = -6 \sin t$ and $a(t) = D_t(-6 \sin t) = -6 \cos t$
the instantaneous velocity and acceleration of the particle at t sec is $-6 \sin t$ cm/sec and $-6 \cos t$ cm/sec^2.
(b) The instantaneous velocity and acceleration for each value of t_1 is given in the table below.

t_1	v	velocity	a	acceleration
0	$-6 \sin 0 = -6(0) = 0$	0	$-6 \cos 0 = -6$	-6 cm/sec^2
$\frac{1}{6}\pi$	$-6 \sin(\frac{1}{6}\pi) = -6(\frac{1}{2}) = -3$	-3 cm/sec	$-6 \cos \frac{1}{6}\pi = -6 \cdot \frac{1}{2}\sqrt{3} = -3\sqrt{3}$	$-3\sqrt{3}$ cm/sec^2
$\frac{1}{2}\pi$	$-6 \sin(\frac{1}{2}\pi) = -6(1) = -6$	-6 cm/sec	$-6 \cos \frac{1}{2}\pi = -6 \cdot 0 = 0$	0
$\frac{5}{6}\pi$	$-6 \sin(\frac{5}{6}\pi) = -6(\frac{1}{2}) = -3$	-3 cm/sec	$-6 \cos \frac{5}{6}\pi = -6(-\frac{1}{2}\sqrt{3}) = 3\sqrt{3}$	$3\sqrt{3}$ cm/sec^2
π	$-6 \sin \pi = -6(0) = 0$	0	$-6 \cos \pi = -6(-1) = 6$	6 cm/sec^2

$s = -3 \cos t$ (a) The instantaneous velocity and acceleration are given by $v(t) = \dfrac{ds}{dt} = 3 \sin t$ and $a(t) = \dfrac{dv}{dt} = 3 \cos t$. (b) $v(0) = 3(0) = 0$, $a(0) = 3 \cos 0 = 3 \cdot 1 = 3$; $v(\frac{1}{6}\pi) = 3 \cdot \frac{1}{2} = \frac{3}{2}$, $a(\frac{1}{6}\pi) = 3 \cos \frac{1}{6}\pi = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$; $v(\frac{1}{3}\pi) = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$, $a(\frac{1}{3}\pi) = 3 \cos \frac{1}{3}\pi = 3 \cdot \frac{1}{2} = \frac{3}{2}$; $v(\frac{1}{2}\pi) = 3 \cdot 1 = 3$, $v(\frac{2}{3}\pi) = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\pi$, $a(\frac{2}{3}\pi) = 3 \cos \frac{2}{3}\pi = 3(-\frac{1}{2}) = -\frac{3}{2}$, $v(\frac{5}{6}\pi) = 3 \cdot \frac{1}{2} = \frac{3}{2}$, $a(\frac{5}{6}\pi) = 3 \cos \frac{5}{6}\pi = 3(\frac{1}{2}\sqrt{3}) = \frac{3}{2}\sqrt{3}$; $v(\pi) = 3 \cdot 0 = 0$, $a(\pi) = 3 \cos \pi = 3(-1) = -3$

$s = -\frac{1}{2} \sin t$ (a) The instantaneous velocity and acceleration are $v(t) = \dfrac{ds}{dt} = -\frac{1}{2} \cos t$ and $a(t) = \frac{1}{2} \sin t$.
(b) $v(0) = -\frac{1}{2} \cos 0 = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$, $a(0) = \frac{1}{2} \sin 0 = \frac{1}{2} \cdot 0 = 0$; $v(\frac{1}{6}\pi) = -\frac{1}{2} \cos \frac{1}{6}\pi = -\frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = -\frac{1}{4}\sqrt{3}$, $a(\frac{1}{6}\pi) = \frac{1}{2} \sin \frac{1}{6}\pi = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; $v(\frac{1}{3}\pi) = -\frac{1}{2} \cos \frac{1}{3}\pi = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$, $a(\frac{1}{3}\pi) = \frac{1}{2} \sin \frac{1}{3}\pi = \frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$;

$v(\frac{1}{2}\pi) = -\frac{1}{2}\cos\frac{1}{2}\pi = -\frac{1}{2}\cdot 0 = 0$, $a(\frac{1}{2}\pi) = \frac{1}{2}\sin\frac{1}{2}\pi = \frac{1}{2}\cdot 1 = \frac{1}{2}$; $v(\frac{2}{3}\pi) = -\frac{1}{2}\cos\frac{2}{3}\pi = -\frac{1}{2}(-\frac{1}{2}) = \frac{1}{4}$,
$a(\frac{2}{3}\pi) = \frac{1}{2}\sin\frac{2}{3}\pi = \frac{1}{2}\cdot\frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$; $v(\frac{5}{6}\pi) = -\frac{1}{2}\cos\frac{5}{6}\pi = -\frac{1}{2}(-\frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$, $a(\frac{5}{6}\pi) = \frac{1}{2}\sin\frac{5}{6}\pi = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$;
$v(\pi) = -\frac{1}{2}\cos\pi = -\frac{1}{2}(-1) = \frac{1}{2}$, $a(\pi) = \frac{1}{2}\sin\pi = \frac{1}{2}\cdot 0 = 0$

59. $F = \dfrac{hW}{k\sin\theta + \cos\theta} = \dfrac{\frac{1}{2}W}{\frac{1}{2}\sin\theta + \cos\theta}$

The instantaneous rate of change of F with respect to θ is $F'(\theta) = \dfrac{\frac{1}{2}W(\frac{1}{2}\cos\theta - \sin\theta)}{(\frac{1}{2}\sin\theta + \cos\theta)^2}$.

(a) $F'(\frac{1}{4}\pi) = \dfrac{-\frac{1}{2}W(\frac{1}{2}\cdot\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2})}{(\frac{1}{2}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2})^2} = \dfrac{-\frac{1}{2}W(-\frac{1}{4}\sqrt{2})}{(\frac{3}{4}\sqrt{2})^2} = \dfrac{\frac{1}{8}\sqrt{2}W}{\frac{9}{8}} = \frac{1}{9}\sqrt{2}W$ (b) $F'(\frac{1}{2}\pi) = \dfrac{-\frac{1}{2}W(\frac{1}{2}\cdot 0 - 1)}{(\frac{1}{2}\cdot 1 + 0)^2} = 2W$

60. A projectile is shot from a gun at an angle of elevation having radian measure $\frac{1}{2}\alpha$ and an initial velocity of v_0 ft/sec. If R feet is the range of the projectile, then $R = (v_0{}^2/g)\sin\alpha$, $0 \le \alpha \le \pi$ where g ft/sec^2 is the acceleration due to gravity. (a) If $v_0 = 480$, find the rate of change of R with respect to α when $\alpha = \frac{1}{2}\pi$ (That is, the angle of elevation has radian measure $\frac{1}{4}\pi$). Take $g = 32$. (b) Find the values of α for which $D_\alpha R > 0$.

▷ (a) Because $v_0 = 480$ and $g = 32$, we are given that $R(\alpha) = \dfrac{480^2}{32}\sin\alpha = 7200\sin\alpha$

Differentiating with respect to α, we have $R'(\alpha) = 7200\cos\alpha$, $R'(\frac{1}{2}\pi) = 7200\cos(\frac{1}{2}\pi) = 0$

Hence, the rate of change of R with respect to α is 0 when $\alpha = \frac{1}{2}\pi$.
(b) If $R'(\alpha) > 0$, we have $7200\cos\alpha > 0$, $\cos\alpha > 0$

If $\cos\alpha > 0$ and $0 \le \alpha \le \pi$ then $0 \le \alpha < \frac{1}{2}\pi$. We conclude that $D_\alpha R > 0$ when $0 \le \alpha < \frac{1}{2}\pi$.

61. We wish to prove by mathematical induction that if k is any positive integer then

$$D_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4k \\ \cos x & \text{if } n = 4k+1 \\ -\sin x & \text{if } n = 4k+2 \\ -\cos x & \text{if } n = 4k+3 \end{cases} \quad (1)$$

We prove that formula (1) holds when $k = 1$. $D_x(\sin x) = \cos x$;
$D^2(\sin x) = -\sin x$; $D^3(\sin x) = -\cos x$
Thus, by taking successive derivatives

$$D_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4 \\ \cos x & \text{if } n = 5 \\ -\sin x & \text{if } n = 6 \\ -\cos x & \text{if } n = 7 \end{cases} \quad (2)$$

Because (2) is (1) for $k = 1$, we have proved formula (1) for $k = 1$.

We now wish to show that the formula holds for $k = t + 1$ if it holds for $k = t$, that is, if

$$D_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4t \\ \cos x & \text{if } n = 4t+1 \\ -\sin x & \text{if } n = 4t+2 \\ -\cos x & \text{if } n = 4t+3 \end{cases} \quad (3)$$

When $k = t + 1$, then $4k = 4t + 4$, $4k + 1 = 4t + 5$, $4k + 2 = 4t + 6$, $4k + 3 = 4t + 7$. From (3) we have $D^n(\sin x) = -\cos x$ if $n = 4t + 3$.
Thus, by taking successive derivatives

$$D_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4t+4 \\ \cos x & \text{if } n = 4t+5 \\ -\sin x & \text{if } n = 4t+6 \\ -\cos x & \text{if } n = 4t+7 \end{cases} \quad (4)$$

Because (4) is (1) when $k = t + 1$, we have proved that if (1) holds for $k = t$ it also holds for $k = t + 1$. Hence (1) holds for every positive integer k.

62. The formula of Ex. 61 can be expressed as $D_x{}^n(\sin x) = \sin(x + \frac{1}{2}n\pi)$ for any positive integer n. We wish to prove by induction that if n is any positive integer, $D_x{}^n(\cos x) = \cos(x + \frac{1}{2}n\pi)$. (1) We prove that formula (1) holds when $n = 1$: $D_x(\cos x) = -\sin x$ and $\cos(x + \frac{1}{2}\pi) = -\sin x$. Suppose the formula is true for some integer k. Using this hypothesis and the formula for $\cos(A + B)$, we have

$$D_x{}^{k+1}(\cos x) = D[D_x{}^k(\cos x)] \overset{\text{hyp}}{=} D[\cos(x + k\tfrac{1}{2}\pi)] \overset{\text{add}}{=} D(\cos x\cos k\tfrac{1}{2}\pi - \sin x\sin k\tfrac{1}{2}\pi)$$
$$= -\sin x\cos k\tfrac{1}{2}\pi - \cos x\sin k\tfrac{1}{2}\pi = \cos(x + \tfrac{1}{2}\pi)\cos k\tfrac{1}{2}\pi - \sin(x + \tfrac{1}{2}\pi)\sin k\tfrac{1}{2}\pi \overset{\text{add}}{=} \cos[x + (k+1)\tfrac{1}{2}\pi]$$

We have proved that if (1) holds for $n = k$, it hold for $n = k + 1$. Hence (1) holds for every positive integer k.

THE DERIVATIVE OF A COMPOSITE FUNCTION AND THE CHAIN RULE

The Chain Rule If the function g is differentiable at x and the function f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The following differentiation formulas are special cases of the chain rule. In each formula we assume that u is a function differentiable at x. In the formulas involving the tangent, cotangent, secant, and cosecant functions we assume that the trigonometric function is defined at u.

$$\mathrm{D}_x u^n = n u^{n-1} \mathrm{D}_x u \quad \text{if } n \text{ is any integer}$$

$$\mathrm{D}_x \sin u = \cos u \, \mathrm{D}_x u \qquad \mathrm{D}_x \cos u = -\sin u \, \mathrm{D}_x u$$

$$\mathrm{D}_x \tan u = \sec^2 u \, \mathrm{D}_x u \qquad \mathrm{D}_x \cot u = -\csc^2 u \, \mathrm{D}_x u$$

$$\mathrm{D}_x \sec u = \sec u \tan u \, \mathrm{D}_x u \qquad \mathrm{D}_x \csc u = -\csc u \cot u \, \mathrm{D}_x u$$

When computing derivatives by the chain rule we don't actually write the functions f and g, but we bear them in mind. If a function is described as "power of …", "sine of …", or "product of …", then we first apply the power rule, sine rule, or product rule. In the final form of an answer, the simpler factors appear first.

Simple Harmonic Motion: An object moving on a line so that the measure of its acceleration is proportional to the measure of its displacement s from a fixed point on the line, and the acceleration and displacement are oppositely directed. It is a sum of terms of the form $\mathrm{A}\cos(kx + a)$ and $\mathrm{B}\sin(kx + b)$ of *amplitudes* $|\mathrm{A}|$ and $|\mathrm{B}|$, *period* $2\pi/|k|$ and *frequency* $|k|/2\pi$.

Exercises 2.8

Exercises 1–12, find the derivative of the function.

$f'(x) = \mathrm{D}_x(2x+1)^3 = 3(2x+1)^2\mathrm{D}_x(2x+1) = 3(2x+1)^2(2) = 6(2x+1)^2$

$f'(x) = \mathrm{D}_x(10-5x)^4 = 4(10-5x)^3\mathrm{D}_x(10-5x) = 4(10-5x)^3(-5) = -20(10-5x)^3$

$F'(x) = \mathrm{D}_x(x^2+4x-5)^4 = 4(x^2+4x-5)^3\mathrm{D}_x(x^2+4x-5) = 4(x^2+4x-5)^3(2x+4) = 8(x+2)(x^2+4x-5)^3$

$g(r) = (2r^4+8r^2+1)^5$

Because $g(r)$ is the fifth power of $2r^4+8r^2+1$, we use the power rule first.

$g'(r) = 5(2r^4+8r^2+1)^4 \cdot \mathrm{D}_r(2r^4+8r^2+1) = 5(2r^4+8r^2+1)^4(8r^2+16r) = 40r(r+2)(2r^4+8r^2+1)^4$

$f'(t) = \mathrm{D}_t(2t^4-7t^3+2t-1)^2 = 2(2t^4-7t^3+2t-1)\mathrm{D}_t(2t^4-7t^3+2t-1)$
$= 2(2t^4-7t^3+2t-1)(8t^3-21t^2+2)$

$H'(z) = \mathrm{D}_z(z^3-3z^2+1)^{-3} = -3(z^3-3z^2+1)^{-4}\mathrm{D}_z(z^3-3z^2+1) = -3(z^3-3z^2+1)^{-4}(3z^2-6z)$
$= -9z(z-2)(z^3-3z^2+1)^{-4}$

$f'(x) = \mathrm{D}[(x^2+4)^{-2}] = -2(x^2+4)^{-3}\mathrm{D}_x(x^2+4) = -2(x^2+4)^{-3}(2x) = -4x(x^2+4)^{-3}$

$g(x) = \sin x^2$

Because $g(x)$ is the sine of x^2, we use the sine rule first.

$g'(x) = \cos x^2 \, \mathrm{D}_x(x^2) = \cos x^2 \, (2x) = 2x \cos x^2$

$f'(x) = \mathrm{D}_x(4\cos 3x - 3\sin 4x) = -4(\sin 3x)(3) - 3(\cos 4x)(4) = -12(\sin 3x + \cos 4x)$

$G'(x) = \mathrm{D}_x(\sec^2 x) = 2\sec x \, \mathrm{D}_x(\sec x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

$h'(x) = \mathrm{D}_x(\frac{1}{3}\sec^3 2x - \sec 2x) = (\sec^2 2x - 1)\mathrm{D}_x(\sec 2x) = (\sec^2 2x - 1)(\sec 2x \tan 2x)(2)$
$= (\tan^2 2x)(2\sec 2x \tan 2x) = 2\sec 2x \tan^3 2x$

12. $f(x) = \cos(3x^2 + 1)$

▹ Applying the chain rule to the cosine rule, we have

$$f'(x) = -\sin(3x^2+1)\mathrm{D}_x(3x^2+1) = -\sin(3x^2+1)(6x) = -6x\sin(3x^2+1)$$

In Exercises 13–16, compute the derivative.

13. $\frac{d}{dx}(\sec^2 x \tan^2 x) \overset{\text{prod}}{=} \mathrm{D}_x(\sec^2 x)\tan^2 x + \sec^2 x\, \mathrm{D}_x(\tan^2 x) = 2\sec x(\sec x \tan x)\tan^2 x + \sec^2 x(2\tan x \sec^2 x)$

$= 2\sec^2 x \tan^3 x + 2\sec^2 x \tan x = 2\sec^2 x \tan x(\tan^2 x + \sec^2 x)$

14. $\frac{d}{dt}(2\sin^3 t \cos^2 t) \overset{\text{prod}}{=} 2[\mathrm{D}_t(\sin^3 t)\cdot\cos^2 t + \sin^3 t\cdot \mathrm{D}_t(\cos^2 t)] = 2[3\sin^2 t\cdot \mathrm{D}_t\sin t\cdot\cos^2 t + \sin^3 t\cdot 2\cos t\cdot \mathrm{D}_t\cos t]$

$= 6\sin^2 t\cos t - 4\sin^4 t\cos t$

15. $\frac{d}{dt}(\cot^4 t - \csc^4 t) = 4\cot^3 t(-\csc^2 t) - 4\csc^3 t(-\csc t\cot t) = -4\cot t\csc^2 t(\cot^2 t - \csc^2 t)$

$= -4\cot t\csc^2 t(-1) = 4\cot t\csc^2 t$. Alternatively,

$\frac{d}{dt}(\cot^4 t - \csc^4 t) = \frac{d}{dt}[(\cot^2 t - \csc^2 t)(\cot^2 t + \csc^2 t)] = \frac{d}{dt}[-(2\cot^2 t - 1)] = 4\cot t\csc^2 t$

16. $\frac{d}{dx}[(4x^2+7)^2(2x^3+1)^4]$

▹ Because the function is the product of $(4x^2+7)^2$ and $(2x^3+1)^4$, we use the product rule first, then the power rule. In the final step we remove the repeated factors. Note that we cannot describe the whole function as a power of something.

$$\begin{aligned}
&\frac{d}{dx}[(4x^2+7)^2(2x^3+1)^4]\\
&\quad= (4x^2+7)^2\cdot \mathrm{D}_x(2x^3+1)^4 + (2x^3+1)^4\cdot \mathrm{D}_x(4x^2+7)^2\\
&\quad= (4x^2+7)^2\cdot 4(2x^3+1)^3\mathrm{D}_x(2x^3+1) + (2x^3+1)^4\cdot 2(4x^2+7)\mathrm{D}_x(4x^2+7)\\
&\quad= (4x^2+7)^2\cdot 4(2x^3+1)^3(6x^2) + (2x^3+1)^4\cdot 2(4x^2+7)(8x)\\
&\quad= 24x^2(4x^2+7)^2(2x^3+1)^3 + 16x(4x^2+7)(2x^3+1)^4\\
&\quad= 8x(4x^2+7)(2x^3+1)^3[3x(4x^2+7) + 2(2x^3+1)]\\
&\quad= 8x(4x^2+7)(2x^3+1)^3(16x^3+21x+2)
\end{aligned}$$

In Exercises 17–24, find the derivative of the function and check by plotting your answer and NDER.

17. $\frac{d}{dx}\left(\frac{x-7}{x+2}\right)^2 = 2\left(\frac{x-7}{x+2}\right)^1\frac{d}{dx}\left(\frac{x-7}{x+2}\right) = 2\left(\frac{x-7}{x+2}\right)\frac{(x+2)1-(x-7)1}{(x+2)^2} = \frac{2(x-7)9}{(x+2)^2} = \frac{18(x-7)}{(x+2)^2}$

18. $\frac{d}{dt}\left[\left(\frac{2t^2+1}{3t^3+1}\right)^2\right] = 2\left(\frac{2t^2+1}{3t^3+1}\right)\cdot \mathrm{D}_x\left(\frac{2t^2+1}{3t^3+1}\right) = 2\cdot\frac{2t^2+1}{3t^3+1}\cdot\frac{(3t^3+1)(4t)-(2t^2+1)(9t^2)}{(3t^3+1)^2} = 2\cdot\frac{2t^2+1}{3t^3+1}\cdot\frac{-t^4-9t^2+4t}{(3t^3+1)^2}$

$= \frac{-2t(2t^2+1)(6t^3+9t-4)}{(3t^3+1)^3}$

19. $g'(t) = \mathrm{D}_t\sin^2(3t^2-1) = 2\sin(3t^2-1)\mathrm{D}_t\sin(3t^2-1) = 2\sin(3t^2-1)\cos(3t^2-1)(6t) = 6t\sin(6t^2-2)$

20. $g(x) = \tan^2 x^2$

▹ Because the function is the square of $\tan x^2$, we apply the power rule first.

$g'(x) = 2\tan x^2\cdot \mathrm{D}_x(\tan x^2) = 2\tan x^2\sec^2 x^2\cdot \mathrm{D}_x(x^2) = 4x\tan x^2\sec^2 x^2$

21. $f'(x) = \mathrm{D}_x(\tan^2 x - x^2)^3 = 3(\tan^2 x - x^2)^2\mathrm{D}_x(\tan^2 x - x^2) = 3(\tan^2 x - x^2)^2[2\tan x(\sec^2 x) - 2x]$

$= 6(\tan^2 x - x^2)^2(\tan x\sec^2 x - x)$

22. $\mathrm{G}'(x) = \mathrm{D}_x(2\sin x - 3\cos x)^3 = 3(2\sin x - 3\cos x)^2\mathrm{D}_x(2\sin x - 3\cos x)$

$= 3(2\sin x - 3\cos x)^2(2\cos x + 3\sin x)$

23. $\mathrm{F}'(x) = \mathrm{D}[4\cos(\sin 3x)] = -4\sin(\sin 3x)\mathrm{D}_x(\sin 3x) = -4\sin(\sin 3x)\cos 3x(3) = -12\cos 3x\sin(\sin 3x)$

24. $f(x) = \sin^2(\cos 2x)$

▹ Because $f(x)$ is the square of a sine, we use the power rule first, then the sine rule.

$$\begin{aligned}
f'(x) &= 2\sin(\cos 2x)\mathrm{D}_x\sin(\cos 2x) = 2\sin(\cos 2x)\cos(\cos 2x)\mathrm{D}_x\cos 2x\\
&= 2\sin(\cos 2x)\cos(\cos 2x)(-\sin 2x)\mathrm{D}_x(2x) = 2\sin(\cos 2x)\cos(\cos 2x)(-2\sin 2x)\\
&= -2\sin 2x\sin(2\cos 2x)
\end{aligned}$$

Exercises 25 and 26, find an equation of the tangent line at the point. Check by plotting the curve and tangent.

$y = (x^2-1)^2$ at $(2, 9)$

▸ $y'(x) = 2(x^2-1)D_x(x^2-1) = 2(x^2-1)(2x) = 4x(x^2-1)$. $y'(2) = 24$. $y = 24(x-2)+9$; $y = 24x - 39$

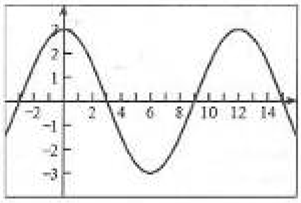
$y = 4 \tan 2x$ at $(\frac{1}{8}\pi, 4)$

▸ $y = (x) = 4\sec^2 2x\ D_x(2x) = 8\sec^2 2x$. $y'(\frac{1}{8}\pi) = 8\sec^2 \frac{1}{4}\pi = 8(\sqrt{2})^2 = 16$. $y = 16(x - \frac{1}{8}\pi) + 4$

Exercises 27–30, s cm is the distance of a weight from its central position at t seconds, and the positive ection is upward. (a) Find the velocity and acceleration. (b) Show that the motion is simple harmonic. (c) Find amplitude A, period p, and frequency f of the motion. (d) Simulate the motion on your calculator. (e) Plot graph.

$s = 6 \sin \frac{1}{4}\pi t$ (a) $v = \frac{ds}{dt} = \frac{3}{2}\pi \cos \frac{1}{4}\pi t$, $a = \frac{dv}{dt} = -\frac{3}{8}\pi^2 \sin \frac{1}{4}\pi t$ (b) Because $a = -(\frac{1}{4}\pi)^2 s$ and $-(\frac{1}{4}\pi)^2$ is negative, the motion is simple harmonic. (c) $A = 6$, $p = 2\pi/\frac{1}{4}\pi = 8$, $f = 1/p = \frac{1}{8}$

$s = 3 \cos \frac{1}{6}\pi t$ ▹ (a) The graph is at the right.



(a) $v = \frac{ds}{dt} = D_t(3 \cos \frac{1}{6}\pi t) = -\frac{1}{2}\pi \sin \frac{1}{6}\pi t$,

$a = \frac{dv}{dt} = D_t(-\frac{1}{2}\pi \sin \frac{1}{6}\pi t) = -\frac{1}{12}\pi^2 \cos \frac{1}{6}\pi t = -(\frac{1}{6}\pi)^2(3 \cos \frac{1}{6}\pi t)$

(b) Because $-(\frac{1}{6}\pi)^2$ is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because $-(-\frac{1}{6}\pi)^2$ is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic. (c) The amplitude is 3. The period is $2\pi/\frac{1}{6}\pi = 12$ and the frequency is $\frac{1}{12}$. (d) To simulate the motion, in parametric mode let $x_1(t) = 2$, $y_1(t) = 3 \cos \frac{1}{6}\pi t$. In the window $[0, 4] \times [-4, 4]$, we let $[t\text{Min}, t\text{Max}] = [0, 12]$, t-step $= .05$. Press the $\boxed{\text{TRACE}}$ key, press the ◃ key until the cursor is at 0, the press the ▹ key to observe the motion.

$s = 4 \cos \pi(2t - \frac{1}{3})$ (a) $v = \frac{ds}{dt} = -8\pi \sin \pi(2t - \frac{1}{3})$, $a = \frac{dv}{dt} = -16\pi^2 \cos \pi(2t - \frac{1}{3})$ (b) Because $a = -(2\pi)^2 s$ and $-(2\pi)^2$ is negative, the motion is simple harmonic. (c) $A = 4$, $p = 2\pi/2\pi = 1$, $f = 1/1 = 1$

$s = 8 \sin \pi(3t + \frac{1}{2})$ (a) $v = \frac{ds}{dt} = 24\pi \cos \pi(3t + \frac{1}{2})$, $a = \frac{dv}{dt} = -72\pi^2 \sin \pi(3t + \frac{1}{2})$ (b) Because $a = -(3\pi)^2 s$ and $-(3\pi)^2$ is negative, the motion is simple harmonic. (c) $A = 8$, $p = 2\pi/3\pi = \frac{2}{3}$, $f = 1/p = \frac{3}{2}$

Exercises 31 and 32, s m is the distance of a particle from the origin at t seconds. Find (a) the velocity and (b) acceleration. (c) Show that the motion is simple harmonic.

$s = b \cos(kt + c)$ (a) $\frac{ds}{dt} = -bk \sin(kt + c)$ (b) $\frac{d^2s}{dt^2} = -bk^2 \cos(kt + c)$ (c) Because $\frac{d^2s}{dt^2} = -k^2 s$, the measure of the acceleration is always proportional to the measure of the displacement and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic.

$s = A \sin 2\pi kt + B \cos 2\pi kt$, where A, B, and k are constants.

▸ We find the velocity and then the acceleration of the particle by differentiating with respect to t. Thus,

$$v = \frac{ds}{dt} = 2\pi k A \cos 2\pi kt - 2\pi k B \sin 2\pi kt$$
$$a = \frac{dv}{dt} = -4\pi^2 k^2 A \sin 2\pi kt - 4\pi^2 k^2 B \cos 2\pi kt = -4\pi^2 k^2 (A \sin 2\pi kt + B \cos 2\pi kt)$$

Substituting, we have $a = -4\pi^2 k^2 s$. Because $-4\pi^2 k^2$ is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because $-4\pi^2 k^2$ is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic.

Exercises 33–36, s ft is the distance of a particle from the origin at t seconds. (a) Find v and a. (b) Show that motion is simple harmonic. (c) Simulate the motion on your calculator.

$s = 5 \sin \pi t + 3 \cos \pi t$ (a) $v = \frac{ds}{dt} = 5\pi \cos \pi t - 3\pi \sin \pi t$, $a = \frac{dv}{dt} = -5\pi^2 \sin \pi t - 3\pi^2 \cos \pi t$

(b) Because $a = -\pi^2 s$ and $-\pi^2$ is negative, the motion is simple harmonic. The amplitude is $\sqrt{5^2 + 3^2} = \sqrt{34}$.

34. $s = \sin(6t - \frac{1}{3}\pi) + \sin(6t + \frac{1}{6}\pi)$; $v = \frac{ds}{dt} = 6\cos(6t - \frac{1}{3}\pi) + 6\cos(6t + \frac{1}{6}\pi)$
$a = \frac{dv}{dt} = -36\sin(6t - \frac{1}{3}\pi) - 36\sin(6t + \frac{1}{6}\pi) = -36[\sin(6t - \frac{1}{3}\pi) + \sin(6t + \frac{1}{6}\pi)] = -36s$
Because $a = -36s$, the measure of the acceleration is always proportional to the measure of the displacemen and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic. Th amplitude is $\sqrt{2}$.

35. $s = 5 - 10\sin^2 2t = 5 - 10 \cdot \frac{1}{2}[1 - \cos 2(2t)] = 5\cos 4t$; $v = \frac{ds}{dt} = -20\sin 4t$; $a = \frac{dv}{dt} = -80\cos 4t = -16s$
Because $a = -16s$, the measure of the acceleration is always proportional to the measure of the displacemen and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic.

36. $s = 8\cos^2 6t - 4$

▹ Because $s = 4(2\cos^2 6t - 1)$ we may apply the identity $\cos 2x = 2\cos^2 x - 1$ with x replaced by $6t$ to simplif the given equation. Thus, $s = 4\cos 12t$. We differentiate to find v and a. Thus,

(a) $v = \frac{ds}{dt} = 4(-\sin 12t)(12) = -48\sin 12t$
$a = \frac{dv}{dt} = -48(\cos 12t)(12) = -144(4\cos 12t) = -144s$

(b) Because the acceleration is proportional to the displacement, and a and s are oppositely directed, th motion is simple harmonic. (c) To simulate the motion, in parametric mode let $x_1(t) = 2$, $y_1(t) = 4\cos 12t$. In the window $[0, 4] \times [-5, 5]$, we let $[t\text{Min}, t\text{Max}] = [0, 2]$, t-step = .05. Press the $\boxed{\text{TRACE}}$ key, press the ◃ ke until the cursor is at 0, the press the ▹ key to observe the motion.

37. (a) $h(\theta) = 10 - 10\cos\theta = 10(1 - \cos\theta) = 10 \cdot 2\sin^2 \frac{1}{2}\theta = 20\sin^2 \frac{1}{2}\theta$. $h'(\theta) = 2 \cdot 20\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta \cdot \frac{1}{2} = 10\sin\theta$
(b) $h'(\frac{1}{6}\pi) = 10\sin\frac{1}{6}\pi = 10 \cdot \frac{1}{2} = 5$ (c) $h'(\frac{1}{3}\pi) = 10\sin\frac{1}{3}\pi = 10 \cdot \frac{1}{2}\sqrt{3} = 5\sqrt{3}$ (d) $h'(\frac{1}{2}\pi) = 10\sin\frac{1}{2}\pi = 10 \cdot 1 = 10$

38. $K = 25\sin\alpha$. $K'(\alpha) = 25\cos\alpha$ (a) $K'(\frac{1}{6}\pi) = 25\cos\frac{1}{6}\pi = 25 \cdot \frac{1}{2}\sqrt{3} = \frac{25}{2}\sqrt{3}$ (b) $K'(\frac{1}{4}\pi) = 25\cos\frac{1}{4}\pi = 25 \cdot \frac{1}{2}\sqrt{2}$
(c) $K'(\frac{1}{3}\pi) = 25\cos\frac{1}{3}\pi = 25 \cdot \frac{1}{2} = \frac{25}{2}$

39. $P = \left(\frac{40 + T}{140}\right)^5$, $T > 80$. Find the rate of change of P with respect to T when (a) $T = 100$; (b) $P = 32$.

▹ $P' = \frac{5}{140}\left(\frac{40 + T}{140}\right)^4$ (a) $P'(100) = \frac{5}{140} \cdot 1^4 = \frac{1}{28}$
(b) $P = 32 = \left(\frac{40 + T}{140}\right)^5$ when $2 = \frac{40 + T}{140}$, $T = 240$. $P'(240) = \frac{5}{140}\left(\frac{280}{140}\right)^4 = \frac{5 \cdot 16}{140} = \frac{4}{7}$

40. If $q = -\frac{A}{\omega}\cos(\omega t + \phi)$, find i, the rate of change of q with respect to t.

▹ $i = \frac{dq}{dt} = \frac{A}{\omega}\sin(\omega t + \phi) \cdot w = A\sin(\omega t + \phi)$

41. If $\theta = 0.2\cos\pi(t - 0.5)$ at t seconds, find, to the nearest tenth, how fast the angle θ is changing at 3.1 sec.

▹ $\theta'(t) = -0.2\pi\sin\pi(t - 0.5)$, $\theta'(3.1) = -0.2\pi\sin 2.6\pi = -0.598 \approx -0.60$ rad/sec

42. $V = \frac{1}{16}x(100 - x)^2$, $V' = \frac{1}{16}[(100 - x)^2 + x \cdot 2(100 - x)(-1)] = \frac{1}{16}(100 - x)(100 - 3x)$. $V'(32) = \frac{1}{16} \cdot 68 \cdot 4 = 17$,
$V'(33) = \frac{1}{16} \cdot 67 \cdot 1 = 4.1875 \approx 4.2$, $V'(34) = \frac{1}{16} \cdot 66(-2) = -8.25 \approx -8.3$

43. $E(t) = 50\sin 120\pi t$; $E'(t) = 50\cos 120\pi t(120\pi) = 6{,}000\pi\cos 120\pi t$
(a) $E'(0.02) = 6{,}000\pi\cos 2.4\pi = 6000\pi\cos .4\pi \approx 5824.8$. Hence, the instantaneous rate of change of $E(t)$ wit respect to t at $t = 0.02$ sec is 5824.8 volts/sec. (b) $E'(0.2) = 6000\pi\cos 24\pi = 6000\pi \approx 18{,}850$. Hence, th instantaneous rate of change of $E(t)$ with respect to t at $t = 0.2$sec is 18,850 volts/sec.

44. A wave produced by a simple sound has the equation $P(t) = 0.003\sin 1800\pi t$ where $P(t)$ dynes per squa centimeter is the difference between the atmospheric pressure and the air pressure at the eardrum at t second Find the instantaneous rate of change of $P(t)$ with respect to t at (a) $\frac{1}{9}$ sec; (b) $\frac{1}{8}$ sec; (c) $\frac{1}{7}$ sec.

▹ The instantaneous rate of change of $P(t)$ with respect to t is $P'(t)$.
$P'(t) = 0.003\cos 1800\pi t \cdot D_x(1800\pi t) = 0.003(1800\pi)\cos 1800\pi t = 5.4\pi\cos 1800\pi t$
$P'(\frac{1}{9}) = 5.4\pi\cos(1800\pi \cdot \frac{1}{9}) = 5.4\pi\cos 200\pi = 5.4\pi\cos 0 = 5.4\pi \approx 17.0$
$P'(\frac{1}{8}) = 5.4\pi\cos(1800\pi \cdot \frac{1}{8}) = 5.4\pi\cos 225\pi = 5.4\pi\cos\pi = -5.4\pi \approx -17.0$
$P'(\frac{1}{7}) = 5.4\pi\cos(\frac{1800}{7}\pi) \approx -15.3$
Thus the instantaneous rate of change of $P(t)$ with respect to t is

(a) 17.0 dynes/cm^2 per second when $t = \frac{1}{9}$; (b) -17.0 dynes/cm^2 per second when $t = \frac{1}{8}$;
(c) -15.3 dynes/cm^2 per second when $t = \frac{1}{7}$.

The demand equation for a particular toy is $p^2x = 5000$, where x toys are demanded per month when p dollars is the price per toy. It is expected that in t months, where $t \in [0, 6]$, the price of the toy will be p dollars, where $20p = t^2 + 7t + 100$. What is the anticipated rate of change of the demand with respect to time in 5 months? Do not express x in terms of t, but use the chain rule.

▸ We have $x = 5000p^{-2}$ and $p = \frac{1}{20}(t^2 + 7t + 100)$

By the chain rule $\quad \dfrac{dx}{dt} = \dfrac{dx}{dp} \cdot \dfrac{dp}{dt} = -10{,}000p^{-3} \cdot \frac{1}{20}(2t + 7) = -\dfrac{500(2t + 7)}{p^3} \qquad (1)$

When $t = 5$ then $\quad p = \frac{1}{20}[(5)^2 + 7(5) + 100] = 8$

Substituting for t and p in Eq. (1), we obtain $\quad \left.\dfrac{dx}{dt}\right|_{t=5} = -\dfrac{500(17)}{8^3} \approx -16.6$

that is, in 5 months the demand will be decreasing at the rate of 16.6 toys per month.

At t min, the area of an oil spill is $A(t) = \begin{cases} \pi(4t^2 + 20)^2 & \text{if } 0 \le t \le 2 \\ \pi(16t + 4)^2 & \text{if } t > 2 \end{cases}$ (a) Prove that A is differentiable at 2. (b) Define $A'(t)$; find the rate at which the area of the spill is changing at (c) 0.4 min, (d) 2 min, (e) 3.2 min.

▸ (a) $f'_-(2) = 2\pi(4t^2 + 20)8t\big|_{t=2} = 1152\pi$; $f'_+(2) = 2\pi(16t + 4)16\big|_{t=2} = 1152\pi$. Thus A is differentiable at 2 and (b) $A'(t) = \begin{cases} 16\pi t(4t^2 + 20) & \text{if } 0 < t \le 2 \\ 32\pi(16t + 4) & \text{if } t > 2 \end{cases}$ (c) $A'(0.4) = 16\pi(0.4)[4(0.4)^2 + 20] = 414.99 \approx 415$ m^2/min

(d) $A'(2) = 1152\pi \approx 3619$ m^2/min (e) $A'(3.2) = 32\pi[16(3.2) + 4] \approx 9571$ m^2/min.

$f(x) = x^3$; $g(x) = f(x^2)$

▸ (a) $f'(x) = 3x^2$, so $f'(x^2) = 3(x^2)^2 = 3x^4$ (b) $g'(x) = D[f(x^2)] = f'(x^2)(2x) = 3x^4(2x) = 6x^5$

Given $f(u) = u^2 + 5u + 5$ and $g(x) = (x + 1)/(x - 1)$. Find the derivative of $f \circ g$ in two ways: (a) by first finding $(f \circ g)(x)$; (b) by using the chain rule.

▸ (a) $D_x(f \circ g)(x) = D_x\left[\left(\dfrac{x+1}{x-1}\right)^2 + 5\left(\dfrac{x+1}{x-1}\right) + 5\right] = 2\left(\dfrac{x+1}{x-1}\right)D_x\left(\dfrac{x+1}{x-1}\right) + 5D_x\left(\dfrac{x+1}{x-1}\right)$

$= \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right]\dfrac{1(x-1) - (x+1)1}{(x-1)^2} = \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right] \cdot \dfrac{-2}{(x-1)^2}$

(b) $f'(u) = 2u + 5$ and $g'(x) = D_x\left(\dfrac{x+1}{x-1}\right) = \dfrac{-2}{(x-1)^2}$. $(f \circ g)'(x) = f'(g(x))g'(x) = \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right] \cdot \dfrac{-2}{(x-1)^2}$

$D_x(\cos x) = D[\sin(\frac{1}{2}\pi - x)] = \cos(\frac{1}{2}\pi - x)(-1) = -\cos(\frac{1}{2}\pi - x) = -\sin x$

Use the chain rule to prove that (a) the derivative of an even function is an odd function, and (b) the derivative of an odd function is an even function, provided that these derivatives exist.

▸ (a) If f is an even function, then $f(-x) = f(x)$.
Differentiating on both sides with respect to x and using the chain rule on the left side, we have
$f'(-x)D_x(-x) = f'(x); \quad f'(-x)(-1) = f'(x); \quad f'(-x) = -f'(x)$
Therefore f' is an odd function.
(b) If f is an odd function, then $f(-x) = -f(x)$.
Differentiating on both sides with respect to f we have
$f'(-x)D_x(-x) = -f'(x); \quad f'(-x)(-1) = -f'(x); \quad f'(-x) = f'(x)$
Therefore f' is an even function.

Use the result of Exercise 50(a) to prove that if f and g are differentiable, g is an even function and $h(x) = (f \circ g)(x)$, then $h'(0) = 0$.

▸ If g is an even function then by Ex.51(a), g$'$ is an odd function so $g(-x) = g(x)$ and $g'(-x) = -g'(x)$ for every x in the domain of g$'$. If $h(x) = (f \circ g)(x)$, then $h'(x) = f'(g(x)) \cdot g'(x)$, so
$h'(-x) = f'(g(-x)) \cdot g'(-x) = f'(g(x))(-g'(x)) = -h'(x)$.
Substituting $x = 0$ gives $h'(0) = -h'(0)$; $2h'(0) = 0$; $h'(0) = 0$.

Suppose that f and g are functions such that $f'(x) = 1/x$ (1) and $(f \circ g)(x) = x$. Prove that if $g'(x)$ exists, then $g'(x) = g(x)$.

▸ $D_x(f \circ g)(x) = D_x x$. By the chain rule, $f'(g(x))g'(x) = 1$. By (1), $[1/g(x)]g'(x) = 1$; $g'(x) = g(x)$

53. Let $f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(a) Prove that f is continuous at 0. (b) Compute $f'(x)$. (c) Prove that f' is discontinuous at 0.

▷ (a) Because $|\sin x| \leq 1$, $0 \leq \left|x^2\sin(1/x)\right| \leq x^2 \to 0$ as $x \to 0$. Hence by the squeeze theorem $\lim_{x\to 0} f(x) = 0 = f(0)$ and so f is continuous at 0. (b) $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ if $x \neq 0$ and $f'(0) = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = x \sin \frac{1}{x} = 0$. (c) As $x \to 0$, the first term of $f'(x)$ approaches 0 but the second term oscillates between ± 1 and so $\lim_{x\to 0} f'(x)$ does not exist.

54. If f'' and g'' exist, find $(f \circ g)''$.

▷ $(f \circ g)'' = \mathrm{D}_x(\mathrm{D}_x(f \circ g)) \overset{\text{chain}}{=} \mathrm{D}_x(f'(g)g') \overset{\text{product}}{=} \mathrm{D}_x(f'(g))g' + f'(g)\mathrm{D}_x g' \overset{\text{chain}}{=} f''(g)(g')^2 + f'(g)g''$

55. See Exercise 27. **56.** See Exercise 28.

57. Suppose that f and g are such that (i) $g'(x_1)$ and $f'(g(x_1))$ exist and (ii) for all $x \neq x_1$ in some open interval containing x_1, $g(x) - g(x_1) \neq 0$. Then $\dfrac{(f \circ g)(x) - (f \circ g)(x_1)}{x - x_1} = \dfrac{(f \circ g)(x) - (f \circ g)(x_1)}{g(x) - g(x_1)} \cdot \dfrac{g(x) - g(x_1)}{x - x_1}$.

(a) Prove that as $x \to x_1$, $g(x) \to (x_1)$ and hence that $(f \circ g)'(x_1) = f'(g(x_1))g'(x_1)$ thus simplifying the proof of the chain rule under the additional hypothesis (ii). (b) Explain why this proof applies at $x_1 = 0$ if $f(x) = x^2$ and $g(x) = x^3$, but not if $g(x) = \operatorname{sgn} x$.

▷ (a) Because $f'(g(x_1))$ exists, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\dfrac{f(g(x)) - f(g(x_1))}{g(x) - g(x_1)} - f'(g(x_1))\right| < \epsilon$ whenever $0 < \left|g(x) - g(x_1)\right| < \delta$. Because $g'(x_1)$ exists, $g(x)$ is continuous at x_1 and so $g(x) \to g(x_1)$ as $x \to x_1$. Hence there is a δ_1 such that $\left|g(x) - g(x_1)\right| < \delta$ whenever $0 < \left|g(x) - g(x_1)\right| < \delta_1$. By (ii), we have $0 < \left|g(x) - g(x_1)\right| < \delta$ whenever $0 < |x - x_1| < \delta_1$, which proves that the first factor has limit $f'(g(x_1))$. The second factor has limit $g'(x_1)$ by definition, which completes the proof. (b) The proof does not apply to $g(x) = \operatorname{sgn} x$ because $g'(0)$ does not exist.

2.9 THE DERIVATIVE OF THE POWER FUNCTION FOR RATIONAL EXPONENTS AND IMPLICIT DIFFERENTIATION

Two theorems proved in this section extend the formulas for the derivative of a power function and for the derivative of a composite power function to include those powers with rational number exponents. To apply them to radicals, rewrite using exponents.

2.9.1 Theorem If f is the power function defined by $f(x) = x^r$, where r is any rational number, then f is differentiable and

$$f'(x) = rx^{r-1}, \quad \text{where } r \geq 1 \text{ if } x = 0 \text{ and } r \text{ has an odd denominator if } x < 0.$$

2.9.2 Theorem If f and g are functions such that $f(x) = [g(x)]^r$, where r is any rational number, and if $g'(x)$ exists, then f is differentiable, and

$$f'(x) = r[g(x)]^{r-1}g'(x)$$

Algebraic Function A sum of terms of the form cx^ny^m is identically 0. We can find y' without solving for y.

Implicit Differentiation With implicit differentiation we must be careful to distinguish the independent variable from the dependent variable. If x is the independent variable, then we regard y as a function of x. Thus,

$$\mathrm{D}_x x = 1$$

but $\mathrm{D}_x y$ is not known. Similarly,

$$\mathrm{D}_x(x^r) = rx^{r-1}$$

but to find $\mathrm{D}_x(y^r)$ we must use the chain rule since y^r is a composite function of x. Thus,

$$\mathrm{D}_x(y^r) = ry^{r-1}\frac{dy}{dx}$$

If we regard y as the independent variable and differentiate implicitly with respect to y, then the roles of x and y are interchanged. Thus,

$$\mathrm{D}_y(x^r) = rx^{r-1}\frac{dx}{dy} \quad \text{and} \quad \mathrm{D}_y(y^r) = ry^{r-1}$$

Exercises 2.9

In Exercises 1–12, find the derivative of the function.

1. $f'(x) = \mathrm{D}_x(4x^{1/2} + 5x^{-1/2}) = \frac{1}{2}\cdot 4x^{-1/2} - \frac{1}{2}\cdot 5x^{-3/2} = 2x^{-1/2} - \frac{5}{2}x^{-3/2}$

2. $f'(x) = \mathrm{D}_x(3x^{2/3} - 6x^{1/3} + x^{-1/3}) = \frac{2}{3}\cdot 3x^{-1/3} - \frac{1}{3}\cdot 6x^{-2/3} - \frac{1}{3}x^{-4/3} = 2x^{-1/3} - 2x^{-2/3} - \frac{1}{3}x^{-4/3}$

3. $g'(x) = \mathrm{D}_x(\sqrt{1} + 4x^2) = \mathrm{D}_x(1+4x^2)^{1/2} = \frac{1}{2}(1+4x^2)^{-1/2}(8x) = 4x(1+4x^2)^{-1/2} = \dfrac{4x}{\sqrt{1+4x^2}}$

4. $f(s) = \sqrt{2-3s^2}$

▸ We replace the radical sign by a fractional exponent and use the chain power rule, Theorem 2.9.2.

$$f(s) = (2-3s^2)^{1/2} \qquad\qquad f'(s) = \tfrac{1}{2}(2-3s^2)^{-1/2}\cdot(-6s) = \frac{-3s}{\sqrt{2-3s^2}}$$

5. $f'(x) = \mathrm{D}_x(5-3x)^{2/3} = \frac{2}{3}(5-3x)^{-1/3}(-3) = -2(5-3x)^{-1/3}$

6. $g'(x) = \mathrm{D}_x(\sqrt[3]{4x^2-1}) = \mathrm{D}_x(4x^2-1)^{1/3} = \frac{1}{3}(4x^2-1)^{-2/3}\cdot \mathrm{D}_x(4x^2-1) = \frac{8}{3}x(4x^2-1)^{-2/3}$

7. $g'(y) = \mathrm{D}_y\,\dfrac{1}{\sqrt{25-y^2}} = \mathrm{D}_y(25-y^2)^{-1/2} = -\frac{1}{2}(25-y^2)^{-3/2}(-2y) = \dfrac{y}{(25-y^2)^{3/2}}$

8. $f(x) = (5-2x^2)^{-1/3}$

▸ We apply Theorem 2.9.2. Thus,

$$f'(x) = -\tfrac{1}{3}(5-2x^2)^{-(1/3)-1}\mathrm{D}_x(5-2x^2) = -\tfrac{1}{3}(5-2x^2)^{-4/3}(-4x) = \tfrac{4}{3}x(5-2x^2)^{-4/3} = \frac{4x}{3(5-2x^2)^{4/3}}$$

9. $h'(t) = \mathrm{D}_t(2\cos\sqrt{t}) = \mathrm{D}_t[2\cos(t^{1/2})] = -2\sin(t^{1/2})(\frac{1}{2}t^{-1/2}) = -t^{-1/2}\sin(t^{1/2}) = -\sin\sqrt{t}/\sqrt{t}$

10. $f'(x) = \mathrm{D}_x(4\sec\sqrt{x}) = 4\sec\sqrt{x}\tan\sqrt{x}\ \mathrm{D}_x(x^{1/2}) = 4\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2}x^{-1/2} = 2\sec\sqrt{x}\tan\sqrt{x}/\sqrt{x}$

11. $g'(r) = \mathrm{D}_r(\cot\sqrt{3r}) = \mathrm{D}_r[\csc^2(\sqrt{3}r^{1/2})] = -\csc^2(\sqrt{3}r^{1/2})(\frac{1}{2}\sqrt{3}r^{-1/2}) = -\sqrt{3}\csc^2\sqrt{3r}/2\sqrt{r}$

12. $g(x) = \sqrt{3\sin x}$

▸ We introduce a fractional exponent and then use Theorem 2.9.2.

$$g(x) = \sqrt{3}\sqrt{\sin x} = \sqrt{3}(\sin x)^{1/2}$$
$$g'(x) = \sqrt{3}(\tfrac{1}{2})(\sin x)^{-1/2}\mathrm{D}_x\sin x = \frac{\sqrt{3}\cos x}{2\sqrt{\sin x}}$$

In Exercise 13–16, compute the derivative and check by plotting NDER.

13. $\dfrac{d}{dt}\sqrt{\dfrac{\sin t}{1-\sin t}} = \dfrac{d}{dt}\Big(\dfrac{\sin t}{1-\sin t}\Big)^{1/2} = \dfrac{1}{2}\Big(\dfrac{\sin t}{1-\sin t}\Big)^{-1/2}\cdot\dfrac{\cos t(1-\sin t)-\sin t(-\cos t)}{(1-\sin t)^2} = \dfrac{\cos t}{\sqrt{\sin t}(1-\sin t)^{3/2}}$

14. $\dfrac{d}{dx}\sqrt{\dfrac{\cos x-1}{\sin x}} = \dfrac{d}{dx}\Big(\dfrac{\cos x-1}{\sin x}\Big)^{1/2} = \dfrac{1}{2}\Big(\dfrac{\cos x-1}{\sin x}\Big)^{-1/2}\dfrac{(-\sin x)\sin x-(\cos x-1)\cos x}{\sin^2 x} = \dfrac{1}{2}\Big(\dfrac{\sin x}{\cos x-1}\Big)^{1/2}\dfrac{\cos x-1}{\sin^2 x}$

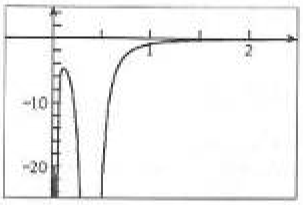
$= \dfrac{1}{2}\sqrt{\dfrac{\cos x-1}{\sin^3 x}}$. Note that $\dfrac{\sqrt{\cos x-1}}{2\sin^{3/2}x}$ is incorrect because $\sqrt{\cos x-1}$ does not exist.

15. $\mathrm{D}_x(\sqrt{9+\sqrt{9-x}}) = \mathrm{D}_x\{[9+(9-x)^{1/2}]^{1/2}\} = \frac{1}{2}[9+(9-x)^{1/2}]^{-1/2}\mathrm{D}[9+(9-x)^{1/2}]$

$= \frac{1}{2}[9+(9-x)^{1/2}]^{-1/2}[\frac{1}{2}(9-x)^{-1/2}(-1)] = \dfrac{-1}{4\sqrt{9+\sqrt{9-x}}\sqrt{9-x}}$

16. $\mathrm{D}_x\Big(\sqrt{x}\tan\sqrt{\frac{1}{x}}\Big)$

▸
$$\begin{aligned}
&= \mathrm{D}_x(x^{1/2}\tan x^{-1/2}) = x^{1/2}\ \mathrm{D}_x\tan x^{-1/2} + \tan x^{-1/2}\ \mathrm{D}_x(x^{1/2})\\
&= x^{1/2}\sec^2 x^{-1/2}(-\tfrac{1}{2}x^{-3/2}) + \tan x^{-1/2}(\tfrac{1}{2}x^{-1/2})\\
&= \tfrac{1}{2}x^{-1}(-\sec^2 x^{-1/2} + x^{1/2}\tan x^{-1/2})\\
&= \frac{-\sec^2\sqrt{\frac{1}{x}} + \sqrt{x}\tan\sqrt{\frac{1}{x}}}{2x}
\end{aligned}$$



The figure shows a plot of NDER$(\sqrt{x}\tan\sqrt{1/x})$.

In Exercises 17–32, find dy/dx by implicit differentiation.

17. $x^2 + y^2 = 16;\ 2x + 2y\frac{dy}{dx} = 0;\ \frac{dy}{dx} = -\frac{x}{y}$

18. $4x^2 - 9y^2 = 1;\ 8x - 18y\frac{dy}{dx} = 0;\ \frac{dy}{dx} = \frac{4x}{9y}$

19. $x^3 + y^3 = 8xy;\ 3x^2 + 3y^2\frac{dy}{dx} = 8y + 8x\frac{dy}{dx};\ \frac{dy}{dx}(3y^2 - 8x) = 8y - 3x^2;\ \frac{dy}{dx} = \frac{8y - 3x^2}{3y^2 - 8x}$

20. $x^2 + y^2 = 7xy$

We differentiate with respect to x on both sides.

$$D_x x^2 + D_x y^2 = D_x(7xy) \tag{1}$$

Then $D_x x^2 = 2x$. Because y is a function of x, we use the chain rule to find $D_x y^2$.

$$D_x y^2 = (2y)\frac{dy}{dx} \tag{2}$$

To find $D_x(7xy)$ we use the product rule

$$\begin{aligned} D_x(7xy) &= (7x)D_x y + y \cdot D_x(7x) \\ &= (7x)\frac{dy}{dx} + 7y \end{aligned} \tag{3}$$

Substituting from (2) and (3) into (1), we have

$$2x + 2y\frac{dy}{dx} = 7x\frac{dy}{dx} + 7y$$

Next, we solve algebraically for $\frac{dy}{dx}$.

$$\begin{aligned} 2y\frac{dy}{dx} - 7x\frac{dy}{dx} &= 7y - 2x \\ (2y - 7x)\frac{dy}{dx} &= 7y - 2x \\ \frac{dy}{dx} &= \frac{7y - 2x}{2y - 7x} \end{aligned} \tag{4}$$

ALTERNATE SOLUTION: We divide on both sides by x^2. Thus

$$1 + \left(\frac{y}{x}\right)^2 = 7\left(\frac{y}{x}\right)$$

Therefore

$$\frac{y}{x} = k \tag{5}$$

where the constant k is one of the two roots of the quadratic equation

$$1 + k^2 = 7k$$

It follows from Eq. (5) that

$$y = kx$$ (6

Thus the graph consists of two lines through the origin. Because y is not a function of x in any neighborhoo of $(0, 0)$, then dy/dx does not exist at $(0, 0)$. Differentiating both sides of Eq. (6) with respect to x and usin Eq. (5) we find

$$\frac{dy}{dx} = k = \frac{y}{x}$$ (7

Using the given relation, it is easy to show that Eq. (7) is equivalent to Eq. (4).

21. $\frac{1}{x} + \frac{1}{y} = 1;\ -x^{-2} - y^{-2}\frac{dy}{dx} = 0;\ \frac{dy}{dx} = -\frac{x^{-2}}{y^{-2}} = -\frac{y^2}{x^2}$

22. $\frac{3}{x} - \frac{3}{y} = 2x;\ 3x^{-2} - 3y^{-2}\frac{dy}{dx} = 2;\ \frac{dy}{dx} = \frac{3x^{-2} - 2}{3y^{-2}} = \frac{3y^2 - 2x^2y^2}{3x^2}$

23. $\sqrt{x} + \sqrt{y} = 4;\ \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0;\ \frac{dy}{dx} = \frac{x^{-1/2}}{y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$

24. $2x^3y + 3xy^3 = 5$

▹ We differentiate with respect to x, and have

$$D_x(2x^3y) + D_x(3xy^3) = 0$$

Because $2x^3y$ is the product of two functions of x, namely $2x^3$ and y, and because $3xy^3$ is the product of and y^3, we use the product rule. Thus,

$$\left(2x^3\frac{dy}{dx} + y\cdot 6x^2\right) + \left(3x\cdot 3y^2\frac{dy}{dx} + y^3\cdot 3\right) = 0$$
$$(2x^3 + 9xy^2)\frac{dy}{dx} = -6x^2y - 3y^3$$
$$\frac{dy}{dx} = \frac{-6x^2y - 3y^3}{2x^3 + 9xy^2}$$

$x^2y^2 = x^2 + y^2$; $2xy^2 + 2x^2y\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$; $\frac{dy}{dx}(2x^2y - 2y) = 2x - 2xy^2$; $\frac{dy}{dx} = \frac{2x - 2xy^2}{2x^2y - 2y} = \frac{x - xy^2}{x^2y - y}$

$(2x + 3)^4 = 3y^4$; $4(2x + 3)^3 = 12y^3\frac{dy}{dx}$; $\frac{dy}{dx} = \frac{(2x + 3)^3}{3y^3}$

$y = \cos(x - y)$; $\frac{dy}{dx} = -\sin(x - y)\left(1 - \frac{dy}{dx}\right) = -\sin(x - y) + \sin(x - y)\frac{dy}{dx}$
$\frac{dy}{dx}[1 - \sin(x - y)] = -\sin(x - y)$; $\frac{dy}{dx} = \frac{-\sin(x - y)}{1 - \sin(x - y)} = \frac{\sin(x - y)}{\sin(x - y) - 1}$

$x = \sin(x + y)$

▸ Differentiating with respect to x, we have

$$1 = \cos(x + y)\mathrm{D}_x(x + y)$$
$$1 = \cos(x + y)\left(1 + \frac{dy}{dx}\right)$$
$$1 = \cos(x + y) + \cos(x + y)\cdot\frac{dy}{dx}$$
$$1 - \cos(x + y) = \cos(x + y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1 - \cos(x + y)}{\cos(x + y)}$$

$\sec^2 x + \csc^2 y = 4$; $2\sec x(\sec x\tan x) + 2\csc y(-\csc y\cot y)\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{\sec^2 x\tan x}{\csc^2 y\cot y}$

$\cot xy + xy = 0$; $(-\csc^2 xy + 1)\left(y + x\frac{dy}{dx}\right) = 0$; $\frac{dy}{dx} = -\frac{y}{x}$
Alternatively, $xy = k$, where k is one of the roots of $\cot k + k = 0$; $y = \frac{k}{x}$, $\frac{dy}{dx} = -\frac{k}{x^2} = -\frac{y}{x}$

$x\sin y + y\cos x = 1$; $\sin y + x\cos y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = 0$; $\frac{dy}{dx}(x\cos y + \cos x) = y\sin x - \sin y$
$\frac{dy}{dx} = \frac{y\sin x - \sin y}{x\cos y + \cos x}$

$\cos(x + y) = y\sin x$

▸ We differentiate on both sides with respect to x.

$$-\sin(x + y)\mathrm{D}_x(x + y) = y\,\mathrm{D}_x(\sin x) + \sin x\,\mathrm{D}_x y$$
$$-\sin(x + y)\left(1 + \frac{dy}{dx}\right) = y\cos x + \sin x\frac{dy}{dx}$$
$$-\sin(x + y) - \sin(x + y)\frac{dy}{dx} = y\cos x + \sin x\frac{dy}{dx}$$
$$[\sin(x + y) + \sin x]\frac{dy}{dx} = y\cos x + \sin(x + y)$$
$$\frac{dy}{dx} = -\frac{y\cos x + \sin(x + y)}{\sin x + \sin(x + y)}$$

Exercises 33–36, find the indicated line. check by plotting the line and the curve.

The tangent line to $y = \sqrt{x^2 + 9}$ at $(4,5)$.

▸ $y' = \frac{x}{\sqrt{x^2 + 9}}$, $m = y'(4) = \frac{4}{\sqrt{4^2 + 9}} = \frac{4}{5}$. Tangent line: $y = \frac{4}{5}(x - 4) + 5 = \frac{4}{5}x + \frac{9}{5}$

The normal line to $y = \sqrt{16 + x^2}$ at the origin.

▸ $y' = \frac{x}{\sqrt{16 + x^2}}$. $y'(0) = 0$. The tangent line is horizontal; the normal line is $x = 0$.

The normal line to $9x^3 - y^3 = 1$ at $(1,2)$.

▸ $27x^2 - 3y^2y' = 0$; $y' = \frac{9x^2}{y^2}$. $m = -\frac{1}{y'(1)} = -\frac{2^2}{9\cdot 1^2} = -\frac{4}{9}$. Normal line: $y = -\frac{4}{9}(x - 1) + 2 = -\frac{4}{9}x + \frac{22}{9}$

36. The tangent line to the curve $16x^4 + y^4 = 32$ at the point $(1, 2)$.

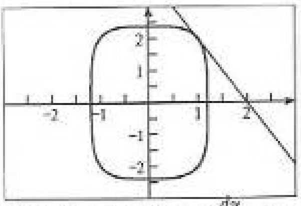
▷ Differentiating implicitly, we obtain

$64x^3 + 4y^3\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{64x^3}{4y^3} = -\frac{16x^3}{y^3}$. At $(1, 2)$: $\frac{dy}{dx} = -\frac{16}{8} = -2$

Equation of tangent line: $y - 2 = -2(x - 1)$; $2x + y - 4 = 0$

To plot the curve we solve for y. Then

$y^4 = 32 - 16x^4$; $y = \pm(32 - 16x^4)^{1/4}$



37. $xy = (1 - x - y)^2$; $y + x\frac{dy}{dx} = 2(1 - x - y)\left(-1 - \frac{dy}{dx}\right)$. The tangent line is parallel to the x axis when $\frac{dy}{dx} = 0$. Setting $\frac{dy}{dx} = 0$ gives $y = 2(x + y - 1)$; $y = 2 - 2x$. Substituting gives $x(2 - 2x) = (x - 1)^2$; $0 = 3x^2 - 4x + 1 = (3x - 1)(x - 1)$. At the points $(\frac{1}{3}, \frac{4}{3})$ and $(1, 0)$, the tangent line is parallel to the x axis. The graph is an ellipse in the first quadrant tangent to the y axis at $(0, 1)$ and the x axis at $(1, 0)$.

38. Find equation for the two lines through $(-1, 3)$ that are tangent to the curve $x^2 + 4y^2 - 4x - 8y + 3 = 0$.

▷ Let ℓ_1 be one of the required lines and let (x_1, y_1) be the point at which ℓ_1 is tangent to the curve. To find m_1, the slope of ℓ_1, we use Eq. (1) and differentiate implicitly with respect to x. Thus,

$2x + 8y\frac{dy}{dx} - 4 - 8\frac{dy}{dx} = 0$; $\quad \frac{dy}{dx} = \frac{-2x + 4}{8y - 8} = \frac{-x + 2}{4y - 4}$. Hence, $m_1 = \frac{-x_1 + 2}{4y_1 - 4}$ (2)

Because ℓ_1 contains $(-1, 3)$ and (x_1, y_1), by definition of slope we have $\quad m_1 = \frac{y_1 - 3}{x_1 + 1}$ (3)

Eliminating m_1 between Eqs. (2) and (3), we obtain $\quad {x_1}^2 + 4{y_1}^2 - x_1 - 16y_1 + 10 = 0$ (4)

The curve contains (x_1, y_1) so this pair must satisfy Eq. (1). That is, $\quad {x_1}^2 + 4y_1 - 4x_1 - 8y_1 + 3 = 0$ (5)

By subtracting terms of Eq. (5) from corresponding terms of Eq. (4), we get

$3x_1 - 8y_1 + 7 = 0$; $\quad x_1 = \frac{8y_1 - 7}{3}$ (6)

Substituting from Eq. (6) into Eq. (4), we have

$$\frac{64{y_1}^2 - 112y_1 + 49}{9} + 4{y_1}^2 - \frac{8y_1 - 7}{3} - 16y_1 + 10 = 0$$

$$100{y_1}^2 - 280y_1 + 160 = 0; \qquad 5{y_1}^2 - 14y_1 + 8 = 0; \qquad (y_1 - 2)(5y_1 - 4) = 0;\ y_1 = 2 \text{ and } y_1 = \tfrac{4}{5}$$

If $y_1 = 2$ then $x_1 = \frac{8(2) - 7}{3} = 3$, $m_1 = \frac{-3 + 2}{4(2) - 4} = -\frac{1}{4}$, and an equation of ℓ_1 is

$y - 2 = -\frac{1}{4}(x - 3)$; $\quad x + 4y - 11 = 0$

If $y_1 = \frac{4}{5}$ then $x_1 = \frac{8(\frac{4}{5}) - 7}{3} = -\frac{1}{5}$, $m_1 = \frac{\frac{1}{5} + 2}{4(\frac{4}{5}) - 4} = -\frac{11}{4}$, and an equation of ℓ_1 is

$y - \frac{4}{5} = -\frac{11}{4}(x + \frac{1}{5})$; $\quad 11x + 4y - 1 = 0$

In Exercises 39–42: (a) Find two functions defined by the equation; (b) sketch the graph of each and (c) the equation. (d) Find the derivative of each function and state its domain. (e) Find dy/dx by implicit differentiatio and check with (d). (f) Find an equation of each tangent line at x_1.

39. $y^2 = 4x - 8$; $x_1 = 3$

▷ (a) $y = \pm 2\sqrt{x - 2}$; $f_1(x) = 2\sqrt{x - 2}$, domain: $x \geq 2$; $f_2(x) = -2\sqrt{x - 2}$, domain $x \geq 2$

(d) ${f_1}'(x) = \frac{1}{\sqrt{x - 2}}$, domain: $x > 2$; ${f_2}'(x) = -\frac{1}{\sqrt{x - 2}}$, domain: $x > 2$

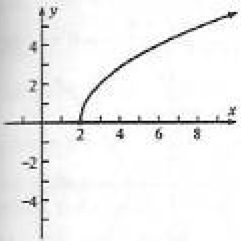
(e) $y^2 = 4x - 8$; $2y\frac{dy}{dx} = 4$; $\frac{dy}{dx} = \frac{2}{y}$

If $y = f_1(x) = 2\sqrt{x - 2}$, then $\frac{dy}{dx} = \frac{1}{y} = \frac{2}{2\sqrt{x - 2}} = \frac{1}{\sqrt{x - 2}} = {f_1}'(x)$

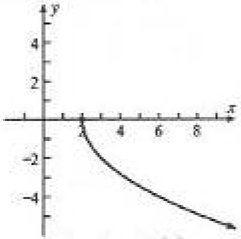
If $y = f_2(x) = -2\sqrt{x - 2}$, then $\frac{dy}{dx} = \frac{2}{y} = \frac{2}{-2\sqrt{x - 2}} = -\frac{1}{\sqrt{x - 2}} = {f_2}'(x)$

Therefore, the result in part (e) agrees with the results in part (d).

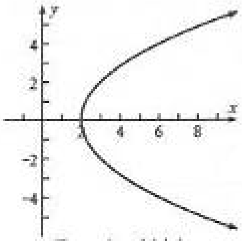
(f) $f_1(3) = 2$, ${f_1}'(3) = 1$. Tangent line to the graph of f_1: $y - 2 = 1(x - 3)$; $x - y - 1 = 0$
$f_2(3) = -2$, ${f_2}'(3) = -1$. Tangent line to the graph of f_2: $y + 2 = -1(x - 3)$; $x + y - 1 = 0$



Exercise 39(a)



Exercise 39(b)



Exercise 39(c)

$y^2 - x^2 = 16;\ x_1 = -3$

(a) Because $y = \pm\sqrt{x^2+16}$, the functions are given by $f_1(x) = \sqrt{x^2+16}$ and $f_2(x) = -\sqrt{x^2+16}$. The domain of each function is $(-\infty, +\infty)$.

(b) A sketch of the graph of f_1 is shown in fig. a and a sketch of the graph of f_2 is shown in fig. 3b.

(c) A sketch of the graph of the given equation is shown in fig. c.

(d)
$$f_1(x) = (x^2+16)^{1/2} \qquad f_2(x) = -(x^2+16)^{1/2}$$
$$f_1{}'(x) = \tfrac{1}{2}(x^2+16)^{-1/2}(2x) \qquad f_2{}'(x) = -\tfrac{1}{2}(x^2+16)^{-1/2}(2x)$$
$$f_1{}'(x) = \frac{x}{\sqrt{x^2+16}} \qquad f_2{}'(x) = \frac{-x}{\sqrt{x^2+16}}$$

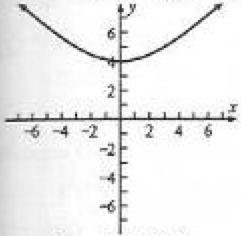
The domain of each derivative is $(-\infty, +\infty)$.

(e) Differentiating the given equation implicitly with respect to x, we have
$$2y\frac{dy}{dx} - 2x = 0; \qquad \frac{dy}{dx} = \frac{x}{y}$$
For $y = f_1(x) = \sqrt{x^2+16}$ and $y = f_2(x) = -\sqrt{x^2+16}$, from part (d),
$$\frac{dy}{dx} = f_1{}'(x) = \frac{x}{\sqrt{x^2+16}} = \frac{x}{y} \text{ and } \frac{dy}{dx} = f_2{}'(x) = \frac{-x}{\sqrt{x^2+16}} = \frac{x}{y}$$
which agree with the result found by implicit differentiation.

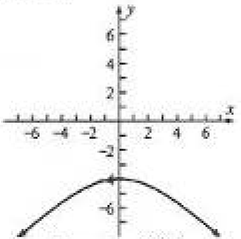
(f) For $y = f_1(x)$, if $x_1 = -3$, then $y_1 = f_1(x_1) = \sqrt{(-3)^2+16} = 5$; $m(x_1) = f_1{}'(x_1) = \frac{-3}{5}$

Thus an equation of the tangent line to the graph of f_1 at $(-3, 5)$ is
$$y - 5 = -\tfrac{3}{5}(x+3); \quad 3x + 5y - 16 = 0$$
For $y = f_2(x)$, if $x_1 = -3$, then $y_1 = f_2(x_1) = -\sqrt{(-3)^2+16} = -5$; $m(x_1) = f_2{}'(x_1) = \frac{-3}{-5} = \frac{3}{5}$

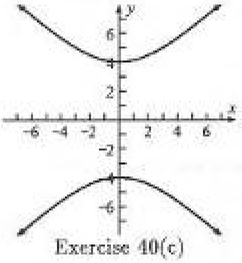
Thus an equation of the tangent line to the graph of f_2 at $(-3, -5)$ is
$$y + 5 = \tfrac{3}{5}(x+3); \qquad 3x - 5y + 16 = 0$$



Exercise 40(a)



Exercise 40(b)



Exercise 40(c)

$x^2 - y^2 = 9;\ x_1 = -5$

(a) $y^2 = x^2 - 9$; $y = \pm\sqrt{x^2-9}$, $f_1(x) = \sqrt{x^2-9}$, domain: $|x| \ge 3$; $f_2(x) = -\sqrt{x^2-9}$, domain: $|x| \ge 3$

(d) $f_1{}'(x) = x(x^2-9)^{-1/2}$, domain: $|x| > 3$; $f_2{}'(x) = -x(x^2-9)^{-1/2}$, domain: $|x| > 3$

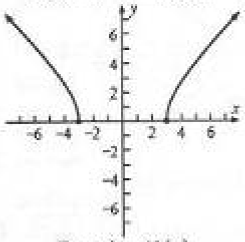
(e) $x^2 - y^2 = 9$; $2x - 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{x}{y}$

If $y = f_1(x) = \sqrt{x^2-9}$, then $\frac{dy}{dx} = \frac{x}{y} = \frac{x}{\sqrt{x^2-9}} = f_1{}'(x)$

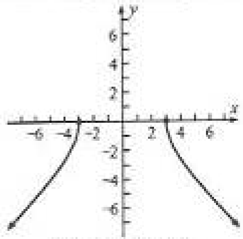
If $y = f_2(x) = \sqrt{x^2-9}$, then $\frac{dy}{dx} = \frac{x}{y} = \frac{x}{-\sqrt{x^2-9}} = f_2{}'(x)$

Therefore, the result in part (e) agrees with the results in part (d).

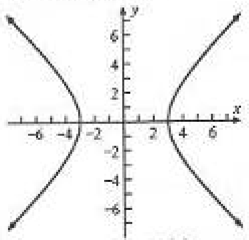
(f) $f_1(-5) = 4$, $f_1'(-5) = -\frac{5}{4}$. Tangent line to graph of f_1: $y - 4 = -\frac{5}{4}(x + 5)$; $5x + 4y + 9 = 0$
$f_2(-5) = -4$, $f_2'(-5) = \frac{5}{4}$. Tangent line to graph of f_2: $y + 4 = \frac{5}{4}(x + 5)$; $5x - 4y + 9 = 0$



Exercise 41(a)



Exercise 41(b)



Exercise 41(c)

42. $x^2 + y^2 = 25$, $x_1 = 4$

▷ (a) $y^2 = 25 - x^2$; $y = \pm\sqrt{25 - x^2}$, $f_1(x) = \sqrt{25 - x^2}$, domain $|x| \le 5$; $f_2(x) = -\sqrt{25 - x^2}$, domain $|x| \le 5$
(d) $f_1'(x) = -x(25 - x^2)^{-1/2}$, domain: $|x| < 5$; $f_2'(x) = x(25 - x^2)^{-1/2}$, domain: $|x| < 5$

(e) $x^2 + y^2 = 25$; $2x + 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{y}$

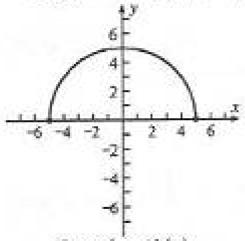
If $y = f_1(x) = \sqrt{25 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y} = \frac{-x}{\sqrt{25 - x^2}} = f_1'(x)$

If $y = f_2(x) = -\sqrt{25 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y} = \frac{x}{\sqrt{25 - x^2}} = f_2'(x)$

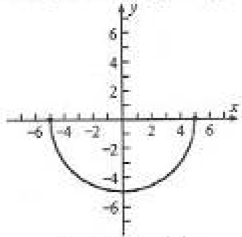
Therefore, the result in part (e) agrees with the results in part (d).

(f) $f_1(4) = 3$, $f_1'(4) = -\frac{3}{4}$. Tangent line to graph of f_1: $y = -\frac{3}{4}(x - 4) + 3 = -\frac{3}{4}x + 6$

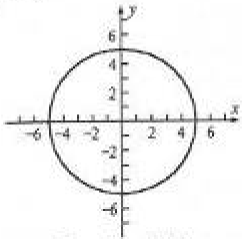
$f_2(4) = -3$, $f_2'(4) = \frac{3}{4}$. Tangent line to graph of f_2: $y = \frac{3}{4}(x - 4) - 3 = \frac{3}{4}x - 6$



Exercise 42(a)



Exercise 42(b)



Exercise 42(c)

43. $x^2 + y^2 = 1$; $2x + 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{y} = -xy^{-1}$

$$\frac{d^2y}{dx^2} = -y^{-1} + xy^{-2}\frac{dy}{dx} = -\frac{1}{y} + \frac{x}{y^2}\left(-\frac{x}{y}\right) = -\frac{1}{y^3}(y^2 + x^2) = -\frac{1}{y^3}, \text{ since } x^2 + y^2 = 1.$$

44. Given $x^{1/2} + y^{1/2} = 2$, show that $\frac{d^2y}{dx^2} = \frac{1}{x^{3/2}}$.

▷ We first solve for y. Thus,

$$y^{1/2} = 2 - x^{1/2}; \qquad y = (2 - x^{1/2})^2 = 4 - 4x^{1/2} + x$$
$$\frac{dy}{dx} = -2x^{-1/2} + 1; \qquad \frac{d^2y}{dx^2} = x^{-3/2} = \frac{1}{x^{3/2}}$$

45. $x^3 + y^3 = 1$; $3x^2 + 3y^2\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x^2}{y^2} = -x^2y^{-2}$

$$\frac{d^2y}{dx^2} = -2xy^{-2} + 2x^2y^{-3}\frac{dy}{dx} = -\frac{2x}{y^2} + \frac{2x^2}{y^3}\left(-\frac{x^2}{y^2}\right) = -\frac{2x}{y^5}(y^3 + x^3) = -\frac{2x}{y^5}, \text{ since } x^3 + y^3 = 1.$$

46. $x^2 + 25y^2 = 100$; $2x + 50y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{25y} = -\frac{1}{25}xy^{-1}$

$$\frac{d^2y}{dx^2} = -\frac{1}{25}\left(y^{-1} - xy^{-2}\frac{dy}{dx}\right) = -\frac{1}{25}\left[\frac{1}{y} + \frac{x}{y^2}\left(-\frac{x}{25y}\right)\right] = -\frac{1}{25 \cdot 25y^3}(25y^2 + x^2) = -\frac{4}{25y^3}, \text{ since } x^2 + 25y^2 = 100.$$

$s = \sqrt{4t^2+3}$, $t \geq 0$; $v = \frac{ds}{dt} = \frac{1}{2}(4t^2+3)^{-1/2}(8t) = \frac{4t}{\sqrt{4t^2+3}}$

(a) $v = 0$ when $\frac{4t}{\sqrt{4t^2+3}} = 0$; $t = 0$

(b) $v = 1$ when $\frac{4t}{\sqrt{4t^2+3}} = 1$; $4t = \sqrt{4t^2+3}$; $16t^2 = 4t^2+3$; $12t^2 = 3$; $t^2 = \frac{1}{4}$; $t = \frac{1}{2}$

(c) $v = 2$ when $\frac{4t}{\sqrt{4t^2+3}} = 2$; $2t = \sqrt{4t^2+3}$; $4t^2 = 4t^2+3$. There is no such value of t.

An object is moving along a line according to the equation of motion $s = \sqrt{5+t^2}$. Find the value of t for which the measure of the velocity is (a) 0; (b) 1.

» The velocity is given by $v = s'(t) = \frac{1}{2}(5+t^2)^{-1/2}(2t) = \frac{t}{\sqrt{5+t^2}}$

(a) We let $v = 0$. Thus, $\frac{t}{\sqrt{5+t^2}} = 0$; $t = 0$

(b) We let $v = 1$. Thus, $\frac{t}{\sqrt{5+t^2}} = 1$; $t = \sqrt{5+t^2}$; $t^2 = 5+t^2$; $0 = 5$

We conclude that the instantaneous velocity is never 1.

$C(x)$ dollars is the total cost of producing x liters of a liquid: $C(x) = 6 + 4\sqrt{x}$.

» (a) C′ is the marginal cost function: $C'(x) = \frac{2}{\sqrt{x}}$; $C'(16) = \frac{2}{\sqrt{16}} = \frac{1}{2}$.

Thus the marginal cost when 16 liters are produced is 50 cents per liter.

(b) We wish to find x for which $C'(x) = 0.40$: $\frac{2}{\sqrt{x}} = 0.40$; $\sqrt{x} = 5$; $x = 25$

Therefore 25 liters are produced when the marginal cost is \$0.40 per liter.

$C(x)$ dollars is the total cost of producing x units of a commodity: $C(x) = 40 + 3x + 9\sqrt{2x}$

» (a) C′ is the marginal cost function: $C'(x) = 3 + 9/\sqrt{2x}$; $C'(50) = 3.9$

Thus the marginal cost when 50 units are produced is \$3.90 per unit.

(b) We wish to find x for which $C'(x) = 4.50$: $\frac{9}{\sqrt{2x}} = 4.50$; $\sqrt{2x} = 6$; $x = 18$

Therefore 18 units are produced when the marginal cost is \$4.50 per liter.

$R = px = (30\sqrt{300-2x}) = 30x(300-2x)^{1/2}$, $x \in [0, 150]$. $R'(x) = 30(300-2x)^{1/2} + 30x[\frac{1}{2}(300-2x)^{-1/2}(-2)]$
$= 30(300-2x)^{-1/2}[300-2x-x] = \frac{30(300-3x)}{\sqrt{300-2x}} = \frac{90(100-x)}{\sqrt{300-2x}}$
$R'(x) = 0$ when $x = 100$; therefore 100 apartments must be rented.

The daily production of a particular factory is $f(x)$ units when the capital investment is x thousands of dollars, and $f(x) = 200\sqrt{2x+1}$. If the current capitalization is \$760,000, use the derivative to estimate the change in the daily production if the capital investment is increased by \$1000.

» $\frac{\Delta f}{\Delta x} \approx f'(x)$ and $\Delta x = 1$, then $\Delta f \approx f'(x)$.

$$f(x) = 200(2x+1)^{1/2}$$

$$f'(x) = 200(\tfrac{1}{2})(2)(2x+1)^{-1/2} = \frac{200}{\sqrt{2x+1}}$$

Because the current capitalization is \$760,000, then $x = 760$. Thus,

$$f'(760) = \frac{200}{\sqrt{2(760)+1}} = \frac{200}{39} \approx 5.13$$

The daily production will increase by approximately 5.1 units if the capital investment is increased by \$1000.

t minutes after the airplane was directly over the statue, the horizontal distance traveled is $\frac{9}{2}$t km and s km is the line of sight distance between the plane and the statue. Then from the Pythagorean theorem
$s^2 = 4 + (\frac{9}{2}t)^2$; $s = (4 + \frac{81}{4}t^2)^{1/2} = \frac{1}{2}(16+81t^2)^{1/2}$; $\frac{ds}{dt} = \frac{1}{4}(16+81t^2)^{-1/2}(162t) = \frac{81t}{2\sqrt{16+81t^2}}$

We wish to find $\frac{ds}{dt}$ when $t = \frac{1}{3}$. $\left.\frac{ds}{dt}\right|_{t=1/3} = \frac{81 \cdot \frac{1}{3}}{2\sqrt{16+81(\frac{1}{3})^2}} = \frac{27}{2\sqrt{25}} = 2.7$

Therefore, the line of sight distance between the plane and the statue is changing at the rate of 2.7km/min 20 sec after the plane was directly over the statue.

54. At t hours after 8 AM the first ship is 24t miles north of P, the second ship is $32(t-2)$ miles east of P and the distance between them is s miles. Then $s^2 = (24t)^2 + [32(t-2)]^2$. Differentiating implicitly,
$2s\frac{ds}{dt} = 2(24t)24 + 2[32(t-2)]32$, $\frac{ds}{dt} = \frac{24^2t + 32^2(t-2)}{s}$
(a) At 9 AM, $t = 1$, $s^2 = 24^2 + (-32)^2 = 1600$, $s = 40$, $\frac{ds}{dt} = \frac{24^2(1) + 32^2(-1)}{40} = -11.2$.

Thus at 9AM the distance between the ships is decreasing at 11.2 knots.

(b) At 11 AM, $t = 3$, $s^2 = (24 \cdot 3)^2 + (32 \cdot 1)^2 = 6208$, $s = \sqrt{6208}$, $\frac{ds}{dt} = \frac{24^2 \cdot 3 + 32^2 \cdot 1}{\sqrt{6208}} = \frac{2752}{\sqrt{6208}} \approx 34.9$

Thus at 11AM the distance between the ships is increasing at 34.9 knots.

55. $D_x(|x|) = D_x(x^2)^{1/2} = \frac{1}{2}(x^2)^{-1/2} \cdot 2x = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|} = \frac{|x|}{x} = \operatorname{sgn}(x)$, $x \neq 0$.

56. Find $D_x{}^2(|x|)$ when it exists.

▷ We use the definition. $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$, $D_x|x| = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$, $D_x{}^2|x| = 0$ if $x \neq 0$.

57. $f'(x) = D_x|x^2 - 4| = [(x^2-4)^2]^{1/2} = \frac{1}{2}[(x^2-4)^2]^{-1/2}[2(x^2-4)(2x)] = \frac{2x(x^2-4)}{[(x^2-4)^2]^{1/2}} = \frac{2x(x^2-4)}{|x^2-4|}$

Alternatively, $D_x|x^2-4| = \operatorname{sgn}(x^2-4)D_x(x^2-4) = 2x\operatorname{sgn}(x^2-4)$, $x \neq \pm 2$.

58. Find the derivative of the function $g(x) = x|x|$.

▷ Because $|x|$ is not differentiable at 0, we cannot use the product rule.
If $x > 0$ then $|x| = x$ and $g(x) = x(x) = x^2$ so that $g'(x) = 2x = 2|x|$.
If $x < 0$ then $|x| = -x$ and $g(x) = -x(x) = -x^2$ so that $g'(x) = -2x = 2|x|$.
If $x = 0$ then $|x| = 0$ and $g(x) = 0$. Furthermore,

$$\begin{aligned} g'_-(0) &= \lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0} \\ &= \lim_{x\to 0^-}\frac{-x^2-0}{x} \\ &= \lim_{x\to 0^-} -x \\ &= 0 \end{aligned} \qquad \begin{aligned} g'_+(0) &= \lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0} \\ &= \lim_{x\to 0^+}\frac{x^2-0}{x} \\ &= \lim_{x\to 0^+} x \\ &= 0 \end{aligned}$$

Therefore $g'(0) = 0 = 2|0|$. In each case we find $g'(x) = 2|x|$.

59. $f(x) = |x|^3 = x^3\operatorname{sgn} x$, $f'(x) = 3x^2\operatorname{sgn} x = 3x|x|$, $f''(x) = 6x\operatorname{sgn} x = 6|x|$

60. Given $g(x) = |f(x)|$. Prove that if $f'(x)$ and $g'(x)$ exist, then $|g'(x)| = |f'(x)|$.

▷ $g'(x) = D[f(x)^2]^{1/2} = \frac{1}{2}[f(x)^2]^{-1/2}D[f(x)^2] = |f(x)|^{-1}f(x)f'(x)$ and
$|g'(x)| = |f(x)|^{-1}|f(x)||f'(x)| = |f'(x)|$, provided $f(x) \neq 0$. Suppose $f(a) = 0$. Then
$g_+{}'(a) = \lim_{x\to a^+}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^+}\frac{|f(x)|}{x-a} \geq 0$ and $g_-{}'(x) = \lim_{x\to a^-}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^-}\frac{|f(x)|}{x-a} \leq 0$
Because $g'(a)$ exists, then $g'(a) = 0$ and $\lim_{x\to a}\frac{f(x)}{x-a} = 0$. Therefore
$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{f(x)}{x-a} = 0$ and so $|g'(a)| = |f'(a)|$.

61. $s(t) = h|\cos t|/(1+t)^2 = h(\cos^2 t)^{1/2}(1+t)^{-2}$; $s'(t) = h[D_t(\cos^2 t)^{1/2} \cdot (1+t)^{-2} + (\cos^2 t)^{1/2} \cdot D_t(1+t)^{-2}]$
$= h[-\sin t\cos t(\cos^2 t)^{-1/2}(1+t)^{-2} - 2(\cos^2 t)^{1/2}(1+t)^{-3}]$
$s'(1.4) = -0.1957h$ ft/sec, $s'(1.6) = 0.14454h$ ft/sec, $s'(1.8) = 0.1035h$ ft/sec, $s'(2.2) = 0.0430h$ ft/sec

62. Prove that the sum of the x and y intercepts of any tangent line to the parabola $x^{1/2} + y^{1/2} = k^{1/2}$ is k.

▷ Differentiating implicitly gives $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$; $y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}$.

The tangent at (x_0, y_0) is $y - y_0 = -\frac{y_0{}^{1/2}}{x_0{}^{1/2}}(x - x_0)$; $\frac{x}{x_0{}^{1/2}} + \frac{y}{y_0{}^{1/2}} = \frac{x_0}{x_0{}^{1/2}} + \frac{y_0}{y_0{}^{1/2}} = x_0{}^{1/2} + y_0{}^{1/2} = k^{1/2}$

The sum of the x-intercept and y-intercept is $x_0{}^{1/2}k^{1/2} + y_0{}^{1/2}k^{1/2} = (x_0{}^{1/2} + y_0{}^{1/2})k^{1/2} = k^{1/2}k^{1/2} = k$.

63. $x^{2/3} + y^{2/3} = 1$. Differentiating implicitly, $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$; $\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$; $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{2/3}}$. If $x = -\frac{1}{8}$ then $(-\frac{1}{8})^{2/3} + y^{2/3} = 1$; $y^{2/3} = 1 - \frac{1}{4} = \frac{3}{4}$. Therefore

(i) $y^{1/3} = \frac{1}{2}\sqrt{3}$ and $y = \frac{3}{8}\sqrt{3}$. $\frac{dy}{dx} = -\frac{\frac{1}{2}\sqrt{3}}{(-\frac{1}{8})^{1/3}} = -\frac{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = \sqrt{3}$.

An equation of the tangent line is $y - \frac{3}{8}\sqrt{3} = \sqrt{3}(x + \frac{1}{8})$; $\sqrt{3}x - y + \frac{1}{2}\sqrt{3} = 0$ or

(ii) $y^{1/3} = -\frac{1}{2}\sqrt{3}$ and $y = -\frac{3}{8}\sqrt{3}$. $\frac{dy}{dx} = -\frac{-\frac{1}{2}\sqrt{3}}{(-\frac{1}{8})^{1/3}} = -\frac{-\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -\sqrt{3}$.

An equation of the tangent line is $y + \frac{3}{8}\sqrt{3} = -\sqrt{3}(x + \frac{1}{8})$; $\sqrt{3}x + y + \frac{1}{2}\sqrt{3} = 0$.

64. Suppose that $g(x) = \sqrt{9 - x^2}$ and $h(x) = f(g(x))$, where f is differentiable at 3. Prove that $h'(0) = 0$.

▸ We have $h(x) = f((9 - x^2)^{1/2})$. By the chain rule,

$$h'(x) = f'((9-x^2)^{1/2})D_x(9 - x^2)^{1/2} = f'((9 - x^2)^{1/2}) \cdot \tfrac{1}{2}(9 - x^2)^{-1/2}(-2x) = f'((9 - x^2)^{1/2}) \cdot \frac{-x}{\sqrt{9 - x^2}}$$

Because $f'(3)$ exists, then $h'(0) = f'(3) \cdot 0 = 0$

65. Show that if $xy = 1$, then $\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = 4$.

▸ Because $xy = 1$, then

$$y = x^{-1} \qquad\qquad x = y^{-1}$$

$$\frac{dy}{dx} = -x^{-2} \qquad\qquad \frac{dx}{dy} = -y^{-2}$$

$$\frac{d^2y}{dx^2} = 2x^{-3} \qquad\qquad \frac{d^2x}{dy^2} = 2y^{-3}$$

Multiplying the last two equations, we have

$$\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = (2x^{-3})(2y^{-3}) = \frac{4}{(xy)^3} \tag{1}$$

Because $xy = 1$, Eq. (1) is equivalent to $\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = 4$

66. $f(x) = x^r$. Let $r = \frac{p}{q}$, q > 0 with p and q integers, and set $y = f(x)$. Then $y = x^{p/q}$ becomes $y^q = x^p$. Differentiating implicitly gives $qy^{q-1}\frac{dy}{dx} = px^{p-1}$; $\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^p x^{p/q}}{y^q x} = r \cdot \frac{x^p x^{p/q}}{x^p x} = rx^{p/q-1} = rx^{r-1}$

Therefore $f'(x) = rx^{r-1}$.

67. The proof given in Exercise 66 is not complete because it was assumed that f is differentiable.

68. Compute $(f \circ g)'(0)$ if $f(x) = x^6 + 7x^3$ and $g(x) = x^{1/3}$. Explain why the chain rule cannot be applied to perform this computation.

▸ $(f \circ g)(x) = f(g(x)) = (x^{1/3})^6 + 7(x^{1/3})^3 = x^2 + 7x$ and so $(f \circ g)'(x) = 2x + 7$; $(f \circ g)'(0) = 2(0) + 7 = 7$
However, $g'(x) = \frac{1}{3}x^{-2/3}$ and $g'(0)$ is not defined. Hence the chain does not apply at 0.

2.10 RELATED RATES

If x is a variable that is a function of time, which is represented by t, then the rate of change of x with respect to time is given by dx/dt. The following steps should be taken to solve problems involving the rate of change with respect to time for two or more related variables.

1. Select a letter to represent each variable.
2. Identify the constant rates of change that are given.
3. Identify the rate of change that must be found.
4. Find an equation that expresses the relation between the variables.
5. Differentiate with respect to t on both sides of the equation.
6. Replace the given rates of change with their constant values.
7. Replace the variables by their values at the particular time of interest.
8. Solve the resulting equation for the unknown rate of change.

Often it is helpful to draw a figure to find the equation in step 4. Be careful to distinguish the *variables*, which represent length, area, volume, etc., from the *rates of change* of these variables. Although a variable that represents length may appear as a dimension in the figure, the rate of change of this variable does not appear in the figure. To find the equation you may use any formulas from geometry for length, area, and volume. You may use the fact that corresponding sides of similar triangles are proportional.

Following are some of the formulas from geometry.

1. The Pythagorean theorem for a right triangle: $a^2 + b^2 = c^2$
2. Circumference of a circle: $C = 2\pi r$
3. Area formulas for plane figures:
   - a. Rectangle: $A = \ell w$
   - b. Triangle: $A = \frac{1}{2}bh$
   - c. Equilateral triangle of side s: $A = \frac{1}{4}\sqrt{3}\, s^2$
   - d. Parallelogram: $A = bh$
   - e. Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$
   - f. Circle: $A = \pi r^2$
4. Surface area formulas for solids:
   - a. Right circular cylinder:
     - (i) Lateral area: $S = 2\pi rh$
     - (ii) Total area: $S = 2\pi rh + 2\pi r^2$
   - b. Right circular cone:
     - (i) Lateral area: $S = \pi r\sqrt{r^2 + h^2}$
     - (ii) Total area: $S = \pi r\sqrt{r^2+h^2} + \pi r^2$
   - c. Sphere: $S = 4\pi r^2$
5. Volume formulas for solids:
   - a. Rectangular parallelepiped: $V = \ell wh$
   - b. Circular cylinder: $V = \pi r^2 h$
   - c. Circular cone: $V = \frac{1}{3}\pi r^2 h$
   - d. Sphere: $V = \frac{4}{3}\pi r^3$
   - e. Prism: $V = Bh$, where B is the area of the base.
   - f. Pyramid: $V = \frac{1}{3}Bh$, where B is the area of the base.

Exercises 2.10

In Exercises 1–8, x and y are functions of a third variable t.

1. $2x + 3y = 8$; $2\frac{dx}{dt} + 3\frac{dy}{dt} = 0$; $\frac{dx}{dt} = -\frac{3}{2}\frac{dy}{dt}$. Because $\frac{dy}{dt} = 2$, then $\frac{dx}{dt} = -\frac{3}{2}\cdot 2 = -3$.

2. $\frac{x}{y} = 10$; $x = 10y$; $\frac{dx}{dt} = 10\frac{dy}{dt}$. Because $\frac{dx}{dt} = -5$, then $-5 = 10\frac{dy}{dt}$, $\frac{dy}{dt} = -\frac{1}{2}$

3. $xy = 20$; $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$; $\frac{dx}{dt} = -\frac{x}{y}\frac{dy}{dt}$. When $x = 2$, $y = 10$. Because $\frac{dy}{dt} = 10$, then $\left.\frac{dx}{dt}\right|_{x=2} = -\frac{2}{10}\cdot 10 = -2$

4. If $2 \sin x + 4 \cos y = 3$ and $\frac{dy}{dt} = 3$, find $\frac{dx}{dt}$ at $(\frac{1}{6}\pi, \frac{1}{3}\pi)$.

▹ We differentiate with respect to t on both sides of the given equation. Because x and y are functions of t, we must apply the chain rule. Thus

$$2 \cos x \cdot \frac{dx}{dt} - 4 \sin y \cdot \frac{dy}{dt} = 0$$

We let $\frac{dy}{dt} = 3$, $x = \frac{1}{6}\pi$ and $y = \frac{1}{3}\pi$ and solve for $\frac{dx}{dt}$.

$$2(\cos \tfrac{1}{6}\pi)\frac{dx}{dt} - 4(\sin \tfrac{1}{3}\pi)(3) = 0; \qquad 2(\tfrac{1}{2}\sqrt{3})\frac{dx}{dt} - 4(\tfrac{1}{2}\sqrt{3})(3) = 0; \quad \sqrt{3}\frac{dx}{dt} - 6\sqrt{3} = 0; \qquad \frac{dx}{dt} = 6$$

5. $\sin^2 x + \cos^2 y = \frac{5}{4}$; $2 \sin x \cos x\frac{dx}{dt} + 2 \cos y(-\sin y)\frac{dy}{dt} = 0$; $\sin 2x\frac{dx}{dt} - \sin 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{\sin 2x}{\sin 2y}\frac{dx}{dt}$. Let $\frac{dx}{dt} = -1$, $x = \frac{3}{4}\pi$, $y = -\pi$. Then $\left.\frac{dy}{dt}\right|_{y=3\pi/4} = \frac{\sin \frac{4}{3}\pi}{\sin \frac{3}{2}\pi}(-1) = -\frac{-\frac{1}{2}\sqrt{3}}{-1} = -\frac{1}{2}\sqrt{3}$

6. $x^2 + y^2 = 25$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $y = 4$. ▹ Because the value of x is not given, we solve for x.

$x = \pm\sqrt{25 - y^2}$; $\frac{dx}{dt} = \pm\frac{y}{\sqrt{25 - y^2}}\cdot\frac{dy}{dt}$; $5 = \pm\frac{4}{3}\cdot\frac{dy}{dt}$; $\frac{dy}{dt} = \pm\frac{15}{4}$

7. $\sqrt{x} + \sqrt{y} = 5$; $x^{1/2} + y^{1/2} = 5$; $\frac{1}{2}x^{-1/2}\frac{dx}{dt} + \frac{1}{2}y^{-1/2}\frac{dy}{dt} = 0$; - $\frac{1}{x^{1/2}} = -\frac{1}{y^{1/2}}\frac{dy}{dt}$; $\frac{dx}{dt} = -\frac{x^{1/2}}{y^{1/2}}\frac{dy}{dt}$

Because $\frac{dy}{dt} = 3$ and when $x = 1$, $y = 16$, then $\left.\frac{dx}{dt}\right|_{x=1} = -\frac{1^{1/2}}{16^{1/2}} \cdot 3 = -\frac{3}{4}$.

8. If $y(\tan x + 1) = 4$ and $\frac{dy}{dt} = -4$, find $\frac{dx}{dt}$ when $x = \pi$.

▸ Because the value of y is not given, we solve for y.

$$y = 4(\tan x + 1)^{-1}$$

Differentiating with respect to t on both sides, we obtain

$$\frac{dy}{dt} = -4(\tan x + 1)^{-2}\sec^2 x \cdot \frac{dx}{dt}$$

We let $\frac{dy}{dt} = -4$ and $x = \pi$ and solve for $\frac{dx}{dt}$.

$$-4 = -4(\tan \pi + 1)^{-2}\sec^2 \pi \cdot \frac{dx}{dt}$$

$$-4 = -4\frac{dx}{dt}; \ \frac{dx}{dt} = 1$$

9. At t sec, let x ft be the horizontal distance from the child to the kite, $x > 0$, and let S ft be the length of the string. Then from the Pythagorean Theorem, $S^2 = 40^2 + x^2$; $2S\frac{dS}{dt} = 2x$; $\frac{dS}{dt} = \frac{x}{S}$.
When $S = 50$ we have $2500 = 1600 + x^2$; $x^2 = 900$; $x = 30$. Because $\frac{dx}{dt} = 3$, we have $\left.\frac{dS}{dt}\right|_{S=50} = \frac{30}{50} \cdot 3 = \frac{9}{5}$.
Therefore, when the length of the string released is 50 ft, the string is being paid out at the rate of $\frac{9}{5}$ ft/sec.

10. At t min, let r m be the radius and V ft^3 be the volume of the spherical balloon.
$V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^3\frac{dr}{dt}$. Because $\frac{dV}{dt} = 5$, $\frac{dr}{dt} = \frac{5}{4\pi r^2}$. Therefore $\left.\frac{dr}{dt}\right|_{r\,=\,6} = \frac{5}{4\pi \cdot 36} = \frac{5}{108\pi}$.
Hence when the balloon is 12m in diameter, the diameter is increasing at the rate of $\frac{5}{72\pi}$ m/min.

11. At t min, let r ft be the radius and V ft^3 be the volume of the spherical snowball.
$V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^3\frac{dr}{dt}$. Because $\frac{dV}{dt} = 8$, $\frac{dr}{dt} = \frac{8}{4\pi r^2} = \frac{2}{\pi r^2}$. Therefore $\left.\frac{dr}{dt}\right|_{r\,=\,2} = \frac{2}{\pi \cdot 4} = \frac{1}{2\pi}$.
Hence when the snowball is 4ft in diameter, the radius is increasing at the rate of $\frac{1}{2\pi}$ ft/min.

12. A spherical snowball with diameter 6 ft starts to melt at the rate of $\frac{1}{4}$ ft^3/min. Find the rate at which the radius is changing when the radius is 2 ft.

▸ At t min, let r ft be the radius and V ft^3 be the volume of the spherical snowball.

Because the snowball is melting at the rate of $\frac{1}{4}$ ft^3/min, we have $\quad \frac{dV}{dt} = -\frac{1}{4}$

We must find $\frac{dr}{dt}$ when $r = 2$. The volume of a sphere is given by the formula $\quad V = \frac{4}{3}\pi r^3$

Differentiating with respect to t on both sides, we have $\quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Substituting for $\frac{dV}{dt}$ and r, we obtain $\quad -\frac{1}{4} = 4\pi(2^2)\frac{dr}{dt}$; $\quad \frac{dr}{dt} = \frac{-1}{64\pi} \approx -0.0050$

We conclude that the radius is decreasing at the rate of 0.0050 ft/min when the radius is 2 ft.

13. At t min, let V cubic meters be the volume of the conical pile, let r meters be the radius of the base of the pile, and let h meters be the height of the pile.

Because $r = \frac{1}{2}h$, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{2}h)^2 = \frac{1}{12}\pi h$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$; $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$. Because $\frac{dV}{dt} = 10$, then $\left.\frac{dh}{dt}\right|_{h=8} = \frac{4 \cdot 10}{\pi \cdot 8^2} = \frac{5}{8\pi}$. Thus, when the pile is 8 m high, the height is increasing at the rate of $\frac{5}{8\pi}$ m/min.

In Exercises 14 and 15, a light hangs 15ft above a horizontal path and a man 6ft tall is walking away at 5 ft/sec.

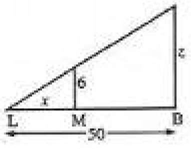
14. How fast is the man's shadow lengthening?

▸ At t sec, let x ft be the man's distance from the light and s ft the length of his shadow. Then $dx/dt = 5$. From similar triangles, $\frac{\text{base}}{\text{height}} = \frac{s}{6} = \frac{x+s}{15}$; $5s = 2x + 2s$; $3s = 2x$; $s = \frac{2}{3}x$; $\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3} \cdot 5 = \frac{10}{3}$. The man's shadow is lengthening at the rate of $\frac{10}{3}$ ft/sec.

15. How fast is the tip of his shadow moving?

▹ At t sec, let x ft be the distance of the man from the light and let y ft be the horizontal distance of the tip of the man's shadow from the bottom of the post holding the light. From similar triangles, $\frac{\text{base}}{\text{height}} = \frac{y-x}{6} = \frac{y}{15}$; $5y - 5x = 2y$; $3y = 5x$; $\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$. Because $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$.
Therefore, the tip of the man's shadow is moving at the rate of $\frac{25}{3}$ ft/sec.

16. A man 6 ft tall is walking toward a building at the rate of 5 ft/sec. If there is a light on the ground 50 ft from the building, how fast is his shadow on the building growing shorter when he is 30 ft from the building?



▹ At t sec, let x ft be the distance from the man to the light and z ft the length of his shadow on the building. The figure shows the man at point M, between point L (the light), and point B (the base of the building). Because the man is walking at the rate of 5 ft/sec, we are given that $dx/dt = 5$. Because dz/dt is the rate of change of the length of the shadow, we want to find dz/dt when the man is 30 ft from the building, that is, when $x = 50 - 30 = 20$. By similar triangles we have

$$\frac{z}{50} = \frac{6}{x}; \qquad z = 300x^{-1}$$

Differentiating on both sides with respect to t, we obtain $\frac{dz}{dt} = -300x^{-2}\frac{dx}{dt}$

Replacing x by 20 and $\frac{dx}{dt}$ by 5, we have $\frac{dz}{dt} = \frac{-300}{(20)^2}(5) = -\frac{15}{4}$

Therefore, the shadow is growing shorter at the rate of $\frac{15}{4}$ ft/sec when the man is 30 ft from the building.

17. At t days, r cm is the radius and V cm^3 is the volume of the spherical tumor. When $r = 0.5$, $\frac{dr}{dt} = 0.001$. $V = \frac{4}{3}\pi r^3$; $\left.\frac{dV}{dt}\right|_{r=0.5} = \left.4\pi r^2\frac{dr}{dt}\right|_{r=0.5} = 4\pi(0.5)^2(0.001) = 0.001\pi$
When the radius of the tumor is 0.5 cm, its volume is increasing at the rate of $0.001\pi \approx 0.003$ cm^3 per day.

18. At t days, r μm is the radius and V μm^3 is the volume of the spherical cell. When $r = 1.5$, $\frac{dr}{dt} = 0.01$. $V = \frac{4}{3}\pi r^3$; $\left.\frac{dV}{dt}\right|_{r=1.5} = \left.4\pi r^2\frac{dr}{dt}\right|_{r=1.5} = 4\pi(1.5)^2(0.01) = 0.09\pi$
When the radius of the cell is 1.5 μm, its volume is increasing at the rate of $0.09\pi \approx 0.028$ μm^3 per day.

19. At t days, r cm is the radius and S cm^2 is the surface area of the spherical tumor.
When $r = 0.5$, $\frac{dr}{dt} = 0.001$. $S = 4\pi r^2$; $\left.\frac{dS}{dt}\right|_{r=0.5} = \left.8\pi r\frac{dr}{dt}\right|_{r=0.5} = 8\pi(0.5)(0.001) = 0.004\pi$.
When the radius of the tumor is .5 cm, its surface area is increasing at the rate of $.004\pi \approx .012$ cm^2 per day.

20. A bacterial cell is spherical in shape. If the radius of the cell is increasing at the rate of 0.01 micrometers per day when it is 1.5 μm, what is the rate of increase of the surface area of the cell at that time?

▹ t days after the cell began to grow, let r μm be its radius and S μm^2 be its surface area.
Because the cell is growing at the rate of 0.01 μm/day, we are given that

$$\frac{dr}{dt} = 0.01$$

We must find $\frac{dS}{dt}$ when $r = 1.5$. The area of a sphere is given by the formula $S = 4\pi r^2$

Differentiating with respect to t on both sides, we have $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

Substituting for r and $\frac{dr}{dt}$, we obtain $\frac{dS}{dt} = 8\pi(1.5)0.01 = 0.12\pi \approx 0.377$

Thus the surface area is increasing at the rate of 0.377 μm^2/day when the radius is 1.5 μm.

21. At t min, let V cubic meters be the volume of water in the tank, let r meters be the radius of the surface of the water, and let h meters be the depth of the water. From similar triangles, $\frac{h}{r} = \frac{24}{12}$; $r = \frac{1}{2}h$.
The volume of water in the tank can be expressed in terms of the volume of a cone.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{2}h)^2 h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$; $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$

Because $\frac{dV}{dt} = -6$, we have $\left.\frac{dh}{dt}\right|_{h=10} = \frac{4(-6)}{\pi \cdot 10^2} = -\frac{6}{25\pi}$.
Hence, when the water is 10m deep, the water level is lowering at the rate of $\frac{6}{25\pi}$ m/min.

22. At t min, let x ft be the depth of the water, b ft its width and V ft^3 its volume. We are given $dV/dt = 2$. From similar triangles $b = x$ and so $V = \frac{1}{2} \cdot 12bh = 6x^2$. $\frac{dV}{dt} = 12x\frac{dx}{dt}$. When $x = 1$, $2 = 12(1)\frac{dx}{dt}$; $\frac{dx}{dt} = \frac{1}{6}$. When the water is 1 ft deep it is rising at the rate of $\frac{1}{6}$ ft/min.

23. At t min, P pounds per square foot is the pressure and V cubic feet is the volume of gas. C is a constant. $PV = C$; $P\frac{dV}{dt} + V\frac{dP}{dt} = 0$; $\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$. Because $\frac{dV}{dt} = 3$ and $P = 3000$ when $V = 5$, we have $\left.\frac{dP}{dt}\right|_{V=5} = -\frac{3000}{5} \cdot 3 = -1800$. Thus, when $V = 5$, the pressure is decreasing at the rate of 1800 lb/ft^2 per min.

24. The adiabatic law (no gain or loss of heat) for the expansion of air is $PV^{1.4} = C$, where P is the number of pounds per square unit of pressure, V is the number of cubic units of volume, and C is a constant. At a specific instant, the pressure is 40 lb/in. and is increasing at the rate of 8 lb/in.^{2} each second. If $C = \frac{5}{16}$, what is the rate of change of volume at this instant?

▸ Let t seconds be the time since the pressure began increasing. Then P and V are functions of t. When $P = 40$ we are given that $\frac{dP}{dt} = 8$ and we must find $\frac{dV}{dt}$. We are given that

$$PV^{1.4} = \frac{5}{16}; \quad V^{7/5} = \frac{5}{16}P^{-1}; \quad V = \left(\frac{5}{16}\right)^{5/7} P^{-5/7}$$

Differentiating with respect to t, we have

$$\frac{dV}{dt} = -\frac{5}{7}\left(\frac{5}{16}\right)^{5/7} P^{-12/7}\frac{dP}{dt}$$

When $P = 40 = 5 \cdot 2^3$ and $\frac{dP}{dt} = 8$ we have

$$\frac{dV}{dt} = -\frac{5}{7}\left(\frac{5}{2^4}\right)^{5/7}(5 \cdot 2^3)^{-12/7}(8) = -\frac{5}{7}\left(\frac{1}{5 \cdot 2^8}\right)(8) = -\frac{1}{224}$$

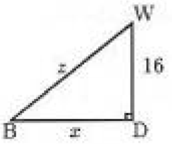
Thus, the volume is decreasing at the rate of $\frac{1}{224}$ cubic units per second at this instant.

25. At t sec, let r cm be the radius and let A cm^2 be the area of the disturbed region. $A = \pi r^2$; $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Because $\frac{dr}{dt} = 16$, $\left.\frac{dA}{dt}\right|_{r=4} = 2\pi(4)16 = 128\pi$. Thus, when the radius is 4 cm, the area of the disturbed region is increasing at the rate of 128π cm^2/sec.

26. At t min, let r m be the radius of the oil, h m its depth, V m^3 its volume. We are given $\frac{dV}{dt} = 3\pi$. From similar triangles, $\frac{r}{h} = \frac{2.5}{10}$; $r = \frac{1}{4}h$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{4}h)^2 h = \frac{1}{48}\pi h^3$. $\frac{dV}{dt} = \frac{1}{16}\pi h^2\frac{dh}{dt}$. $3\pi = \frac{1}{16}\pi(8)^2\frac{dh}{dt}$; $\frac{dh}{dt} = \frac{3}{4}$. Thus, when the depth of the oil is 8m, it is increasing at the rate of $\frac{3}{4}$m/min.

27. At t sec after the truck leaves the intersection, let x ft be the distance traveled by the truck, let y feet be the distance traveled by the automobile, and let s ft be the distance between the automobile and the truck. Then $s = \sqrt{x^2 + (120 - y)^2}$; $D_t s = \frac{xD_t x + (120 - y)(-D_t y)}{\sqrt{x^2 + (120 - y)^2}}$. Because $D_t x = 40$, $D_t y = 30$, and when $t = 2$, $x = 80$ and $y = 60$, we have $\left.D_t s\right|_{t=2} = \frac{40 \cdot 80 + 60(-30)}{\sqrt{80^2 + 60^2}} = \frac{1400}{100} = 14$. Thus, 2 sec after the truck leaves the intersection the automobile and the truck are separating at the rate of 14 ft/sec.

28. A rope is attached to a boat at water level, and a woman on the dock is pulling on the rope at the rate of 50 ft/min. If her hands are 16 ft above the water level, how fast is the boat approaching the dock when the amount of rope out is 20 ft?



▸ Let x feet be the distance between the boat and the dock and z feet the amount of rope out t min after the woman began to pull in the boat. In the figure the boat is at point B, the dock enters the water at point D, and the woman's hands are at point W. Because the woman is pulling in the rope at the rate of 50 ft/min, then z is decreasing, and $\frac{dz}{dt} = -50$. Since $\frac{dx}{dt}$ is the rate of change of the distance between the boat and the dock, we want to find $\frac{dx}{dt}$ when $z = 20$. By the Pythagorean theorem we have

$$z^2 = x^2 + 16^2 \tag{1}$$

Differentiating on both sides with respect to t, we obtain

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} \tag{2}$$

Substituting $z = 20$ in Eq. (1), we get

$$20^2 = x^2 + 16^2; \qquad 144 = x^2; \qquad x = 12$$

Substituting for z, $\frac{dz}{dt}$, and x in Eq. (2) we find

$$2(20)(-50) = 2(12)\frac{dx}{dt}; \qquad \frac{dx}{dt} = -\frac{250}{3}$$

The boat is approaching the dock at the rate of $\frac{250}{3}$ ft/min when 20 ft of rope is out.

29. C dollars is the cost of producing x units in t weeks. $\mathrm{C} = 0.08x^3 - x^2 + 10x + 48$;
$\mathrm{D}_t\mathrm{C} = (.24x^2 - 2x + 10)D_tx$. Because $\mathrm{D}_tx = 2$ and at the present time $x = 50$, we have

$$\mathrm{D}_t\mathrm{C}\Big|_{x=50} = (.24(50)^2 - 2(50) + 10)(2) = 1020$$

Thus the cost is increasing at the rate of \$1020 per week.

30. x thousand boxes are demanded if p cents is the price of a box, where $px + 50p = 16{,}000$; $x = 16{,}000p^{-1} - 50$.
$\frac{dx}{dt} = \frac{dx}{dp}\cdot\frac{dp}{dt} = -16{,}000p^{-2}\frac{dp}{dt} = -\frac{16{,}000}{160^2}\cdot 0.4 = -0.25$. Demand is decreasing at the rate of 250 boxes a week.

31. x units are supplied per month when p dollars is the price per unit. $x = 1000\sqrt{3p^2 + 20p}$;

$\frac{dx}{dp} = \frac{500(6p + 20)}{\sqrt{3p^2 + 20p}}\frac{dp}{dt}$. At the present, $p = 20$ and $\frac{dp}{dt} = \frac{1}{2}$. Thus, $\frac{dx}{dt}\Big|_{p=20} = \frac{500(120 + 20)\frac{1}{2}}{\sqrt{3\cdot 20^2 + 20\cdot 20}} = \frac{35000}{40} = 875$.

Hence, the supply is increasing at the rate of 875 units per month.

32. Suppose that y workers are needed to produce x units of a certain commodity, and $x = 4y^2$. If production of the commodity this year is 250,000 units and the production is increasing at the rate of 18,000 units per year, what is the current rate at which the labor force should be increased?

▹ Let t years be the time, and both x and y are functions of t. When $x = 250{,}000$, we are are given that $\frac{dx}{dt} = 18{,}000$ and we must find $\frac{dy}{dt}$. Solving the given equation for y, we have

$$x = 4y^2; \qquad y = \tfrac{1}{2}x^{1/2}$$

Differentiating with respect to t, we get $\qquad \frac{dy}{dt} = \frac{1}{4}x^{-1/2}\frac{dx}{dt}$

Substituting for x and $\frac{dx}{dt}$, we obtain $\qquad \frac{dy}{dt} = \frac{1}{4}(250{,}000^{-1/2})(18{,}000) = 9$

Thus, the labor force should be increased at the rate of 9 workers per year at present.

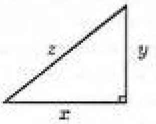
33. $100x$ shirts are demanded per week when p dollars is the price of a shirt.

$2\mathrm{p}x + 65\mathrm{p} - 4950 = 0$; $x = \frac{2475}{\mathrm{p}} - 32.5$; $\frac{dx}{dt} = -\frac{2475}{p^2}\frac{d\mathrm{p}}{dt}$

This week, $\mathrm{p} = 30$ and $\frac{d\mathrm{p}}{dt} = \frac{1}{5}$. Therefore, $\frac{dx}{dt}\Big|_{p=30} = -\frac{2475}{30^2}\frac{1}{5} = -\frac{11}{20}$.
Because $\frac{dx}{dt} = -\frac{11}{20}$, then $100\frac{dx}{dt} = -55$. Hence the demand is decreasing at the rate of 55 shirts per week.

34. The hypotenuse is 40 cm; α is such that $d\alpha/dt = \frac{1}{36}\pi$ rad/sec. The measures of sides are $40 \sin\alpha$ and $40\cos\alpha$.
A cm^2 is the area. $\mathrm{A} = \frac{1}{2}(40\sin\alpha)(40\cos\alpha) = 400\sin 2\alpha$. When $\alpha = \frac{1}{6}\pi$,
$\frac{d\mathrm{A}}{dt} = 800\cos 2\alpha\cdot\frac{d\alpha}{dt} = 800\cos\frac{1}{3}\pi\cdot\frac{1}{36}\pi = 800\cdot\frac{1}{2}\cdot\frac{1}{36}\pi = \frac{100}{9}\pi$. The area is increasing at about 34.9 cm^2/sec.

35. At t hr, let the distance from the intersection be x km for the first truck and y km for the second truck. See the figure. $z = \sqrt{x^2 + y^2}$; $\mathrm{D}_tz = \frac{x\mathrm{D}_tx + y\mathrm{D}_ty}{\sqrt{x^2 + y^2}}$.
Since $\mathrm{D}_tx = -\mathrm{k}$ and $\mathrm{D}_ty = -\mathrm{k}$, when $x = \mathrm{m}$ and $y = \mathrm{m}$, we have



$\mathrm{D}_tz = \frac{m(-\mathrm{k}) + m(-\mathrm{k})}{\sqrt{m^2 + m^2}} = \frac{-2mk}{\sqrt{2}m} = -\mathrm{k}\sqrt{2}$. Thus, when the two trucks are each m km from the intersection, they are approaching each other at the rate of $\mathrm{k}\sqrt{2}$ km/hr.

Exercises 36 and 37, a horizontal trough is 16 meters long, and its ends are isosceles trapezoids with an altitude 4 m, a lower base of 4 m, and and upper base of 6 m.

Water is being poured into the trough at the rate of 10 m^3/min. How fast is the water level rising when the water is 2 m deep?

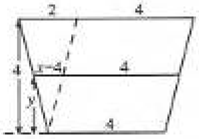
If the water level is decreasing at the rate of 25 cm/min when the water is 3 m deep, at what rate is water being drawn from the trough?

The figure illustrates one end of the trough. Let y m be the depth of the water, x m the width of the surface of the water, and V m^3 its volume, t minutes after water began pouring into the trough

Because water is being poured into the trough at the rate of 10 m^3/min, we are given that $dV/dt = 10$. Since dy/dt is the rate at which the depth of the water is changing, we want to find dy/dt when $y = 2$. The water that is in the trough is in the shape of a prism with altitude 16 m and base a trapezoid. Because the volume of a prism is the area of the base times its altitude, and the area of a trapezoid is given by the formula $\text{A} = \frac{1}{2}(b_1 + b_2) \cdot h$, we have

$$\text{V} = \tfrac{1}{2}(4 + x) \cdot y \cdot 16; \ \text{V} = 8y(x + 4) \qquad (1)$$

We express x as a function of y. By similar triangles in the figure,



$$\frac{x-4}{y} = \frac{2}{4} = \frac{1}{2}; \qquad x = \frac{y}{2} + 4$$

Substituting the value for x into Eq. (1), we obtain

$$\text{V} = 8y\left(\frac{y}{2} + 8\right); \qquad \text{V} = 4y^2 + 64y; \qquad \frac{d\text{V}}{dt} = \frac{d\text{V}}{dy} \cdot \frac{dy}{dt} = (8y + 64)\frac{dy}{dt}$$

In Exercise 36, we substitute $\frac{d\text{V}}{dt} = 10$ and $y = 2$. Then $\qquad 10 = 80\frac{dy}{dt}; \quad \frac{dy}{dt} = \frac{1}{8}$

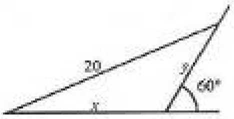
Thus, the water level is rising at the rate of $\frac{1}{8}$ m/min when the water is 2 m deep.

In Exercise 37, we substitute $\frac{dy}{dt} = -\frac{1}{4}$ and $y = 3$. Then $\qquad \frac{d\text{V}}{dt} = (8 \cdot 3 + 64)\left(-\frac{1}{4}\right) = -22.$

Thus, water is being drawn from the trough at the rate of 22m^3/min.

Let the base of the ladder be x m from the the wall, and the top y m from the ground after t sec. We are given $\frac{dx}{dt} = -1.5$. Because the ladder is 7 m long, then $y = \sqrt{49 - x^2}$; $\frac{dy}{dt} = \frac{-x}{\sqrt{49 - x^2}}$. When $x = 2$, $\frac{dy}{dt} = \frac{-2}{\sqrt{49 - 2^2}}(-1.5) = \frac{3}{\sqrt{45}} = \frac{\sqrt{5}}{5}$. The ladder is sliding up at about 0.447 m/sec.

At t sec since the bottom of the ladder started to be moved toward the embankment, let the bottom of the embankment be x ft from the bottom of the ladder and y ft from the top of the ladder. See the figure. From the law of cosines we have



$$x^2 + y^2 - 2xy \cos 120° = 400; \ x^2 + y^2 - 2xy\left(-\tfrac{1}{2}\right) = 400; \ y^2 + (x)y + (x^2 - 400) = 0$$

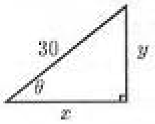
$$y = \frac{-x + \sqrt{x^2 - 4(x^2 - 400)}}{2} = \frac{\sqrt{1600 - 3x^2}}{2} - \frac{x}{2}; \ \frac{dy}{dt} = \left(\frac{-3x}{2\sqrt{1600 - 3x^2}} - \frac{1}{2}\right)\frac{dx}{dt}$$

Since $\frac{dx}{dt} = -1$, when $x = 4$ we get

$$\left.\frac{dy}{dt}\right|_{x=4} = \left(\frac{-12}{2\sqrt{1600 - 3 \cdot 4^2}} - \frac{1}{2}\right) - \ (-1) = \frac{12}{8\sqrt{97}} - \frac{1}{2} = \frac{3\sqrt{97} + 97}{194} \approx 0.65$$

Therefore, the top of the ladder is moving at the rate of $\frac{3\sqrt{97} + 97}{194}$ ft/sec ≈ 0.65 ft/sec at the given instant.

If a ladder of length 30 ft that is leaning against a wall has its upper end sliding down the wall at the rate of $\frac{1}{2}$ft/sec, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 ft above the ground?

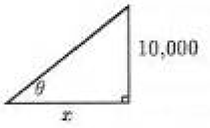


At t sec, let the distance from the bottom of the wall to the bottom of the ladder be x ft and to the top of the ladder be y ft. Let θ radians be the measure of the acute angle made by the ladder with the ground at t sec.

See the figure. $\sin\theta = \frac{y}{30}$; $\cos\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$; $\frac{d\theta}{dt} = \frac{1}{30\cos\theta}\frac{dy}{dt}$

When $y = 18$, $x = \sqrt{30^2 - 18^2} = \sqrt{576} = 24$; so $\cos\theta = \frac{24}{30}$; $30\cos\theta = 24$. Since $\frac{dy}{dt} = -\frac{1}{2}$, we have $\left.\frac{d\theta}{dt}\right|_{y=18} = \frac{1}{24}(-\frac{1}{2}) = -\frac{1}{48}$. Hence, the measure of the acute angle made by the ladder with the ground is decreasing at the rate of $\frac{1}{48}$ rad/sec at the given instant.

41. At t sec after takeoff, let x ft be the horizontal distance from the airplane to the point 10,000 ft directly above the observer, and let θ be the radian measure of the angle of elevation of the airplane. See the figure. $\theta = \pi/3$, $\frac{d\theta}{dt} = \frac{1}{60}$, and



$x = 10{,}000 \cot\theta$, $\frac{dx}{dt} = -10{,}000 \csc^2\theta \frac{d\theta}{dt}$, $\left.\frac{dx}{dt}\right|_{\theta=\pi/3} = -10{,}000 \cdot \frac{4}{3} \cdot \frac{1}{60} = -\frac{2000}{9}$

The minus sign implies that x is decreasing. The speed of the plane is $\frac{2000}{9}$ ft/sec.

42. At t min, the beam makes an angle θ with a perpendicular to the shore and is x mi from its foot. We are given $\frac{d\theta}{dt} = \frac{32 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} = \frac{64\pi \text{ rad}}{\text{min}}$. $x = 4\tan\theta$. When $\theta = 45°$, $\frac{dx}{dt} = 4\sec^2\theta \frac{d\theta}{dt} = 4(2)64\pi = 512\pi$. The beam is moving at about 1608.5 mi/min.

43. At t sec after lift-off the rocket is x yd high and the radar dish makes an angle θ with the ground. After 10 sec the average velocity is 50 yd/sec so its height is 500 yd. $x = 1000\tan\theta$, $\frac{dx}{dt} = 1000\sec^2\theta \frac{d\theta}{dt}$;

$\frac{d\theta}{dt} = \frac{dx/dt}{1000(\tan^2\theta + 1)} = \frac{100}{1000[(500/1000)^2 + 1]} = \frac{2}{25}$. The dish revolves at $\frac{2}{25}$ rad/sec.

44. Water is poured at the rate of 8 ft^3/min into a tank in the form of a cone. The cone is 20 ft deep and 10 ft in diameter at the top. If there is a leak in the bottom and the water is rising at the rate of 1 in./min, when the water is 16 ft deep, how fast is the water leaking?

▹ After t min, V cubic feet is the volume of the water, h feet is its depth, and r feet is the radius of its surface. Because the water level is rising at the rate of 1 in./min when the water is 16 ft deep, and 1 in. is $\frac{1}{12}$ ft, we are given that

$$\frac{dh}{dt} = \frac{1}{12} \text{ when } h = 16$$

We find dV/dt at this moment. Applying the formula for the volume of a cone, we have

$$V = \tfrac{1}{3}r^2h \tag{1}$$

Because the altitude of the tank is 20 ft and the radius of the tank is 5 ft, by similar triangles in the figure, we have $r/h = 5/20$ or, equivalently, $r = \frac{1}{4}h$. Substituting the value of r into Eq. (1), we obtain

$$V = \tfrac{1}{3}\pi(\tfrac{1}{4}h)^2h = \tfrac{1}{48}\pi h^3$$

Differentiating with respect to t, we have

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2\frac{dh}{dt}$$

We substitute $h = 16$ and $dh/dt = \frac{1}{12}$. Thus $\frac{dV}{dt} = \frac{1}{16}\pi(16^2)\frac{1}{12} = \frac{4}{3}\pi$

Because water is being poured into the tank at the rate of 8 ft^3/min and the rate of change of volume of the water in the tank is $\frac{4}{3}\pi$ ft^3/min, we conclude that water is leaking out of the tank at the rate of $8 - \frac{4}{3}\pi$, or approximately 3.81 ft^3/min.

45. The volume of a balloon is decreasing at a rate proportional to its surface area. Show that the radius of the balloon shrinks at a constant rate.

▹ r units is the radius of the balloon at time t, V cubic units is its volume and S square units is its surface area. We are given that

$$\frac{dV}{dt} = kS \tag{1}$$

where k is a negative constant. We must show that $\frac{dr}{dt}$ is a negative constant. Now $V = \frac{4}{3}\pi r^3$

Differentiating on both sides with respect to t, we have $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Substituting from $S = 4\pi r^2$ into the above, we get

$$\frac{dV}{dt} = S\frac{dr}{dt} \tag{2}$$

Substituting from Eq. (1) into Eq.(2), we obtain $kS = S\frac{dr}{dt}$; $\frac{dr}{dt} = k$

Because dr/dt is a negative constant k, we conclude that the radius shrinks at a constant rate.

Miscellaneous Exercises for Chapter 2

In Exercises 1–14, find the derivative of the function.

1. $f(x) = 5x^3 - 7x^2 + 2x - 3$ ▷ $f'(x) = 15x^2 - 14x + 2$

2. $g(x) = 5(x^4 + 3x^7)$ ▷ $g'(x) = 5(4x^3 + 21x^6)$

3. $g(x) = \frac{x^2}{4} + \frac{4}{x^2} = \frac{1}{4}x^2 + 4x^{-2}$ ▷ $g'(x) = \frac{1}{2}x - 8x^{-3} = \frac{x}{2} - \frac{8}{x^3}$

4. $f(x) = \frac{4}{x^2} - \frac{3}{x^4}$ ▷ We apply the power rule. Thus,
$f'(x) = \mathrm{D}_x(4x^{-2} - 3x^{-4}) = -8x^{-3} + 12x^{-5} = -\frac{8}{x^3} + \frac{12}{x^5}$

5. $\mathrm{F}(x) = 2x^{1/2} - \frac{1}{2}x^{-1/2}$ ▷ $\mathrm{F}'(x) = x^{-1/2} + \frac{1}{4}x^{-3/2}$

6. $\mathrm{G}(x) = \frac{x^2 - 4x + 4}{x - 1}$ ▷ $\mathrm{G}'(x) \overset{\text{quot}}{=} \frac{(2x-4)(x-1) - (x^2 - 4x + 4)1}{(x-1)^2} = \frac{2x^2 - 6x + 4 - x^2 + 4x - 4}{2} = \frac{x^2 - 2x}{(x-1)^2}$

7. $\mathrm{G}(t) = (3t^2 - 4)(4t^3 + t - 1)$ ▷ $\mathrm{G}'(t) = (3t^2 - 4)(12t^2 + 1) + (4t^3 + t - 1)(6t)$
$= 36t^4 - 45t^2 - 4 + 24t^4 + 6t^2 - 6t = 60t^4 - 39t^2 - 6t - 4$

8. $f(x) = (x^4 - 2x)(4x^2 + 2x + 5)$ ▷ We multiply and use the power rule.
$f(x) = 4x^6 + 2x^5 + 5x^4 - 8x^3 - 4x^2 - 10x$ $f'(x) = 24x^5 + 10x^4 + 20x^3 - 24x^2 - 8x - 10$

9. $g(x) = \frac{x^2 + 1}{x^3 - 1}$ ▷ $g'(x) = \frac{3x^2(x^3 - 1) - 3x^2(x^3 + 1)}{(x^3 - 1)^2} = \frac{-6x^2}{(x^3 - 1)^2}$

10. $h(y) = \frac{y^2}{y^3 + 8}$ ▷ $h'(y) \overset{\text{quot}}{=} \frac{2y(y^3 + 8) - y^2(3y^2)}{(y^3 + 8)^2} = \frac{2y^4 + 16y - 3y^4}{(y^3 + 8)^2} = \frac{16y - y^4}{(y^3 + 8)^2}$

11. $f(s) = (2s^3 - 2s + t)^4$ ▷ $f'(s) = 4(2s^3 - 3s + 7)(6s^2 - 3)$

12. $\mathrm{F}(x) = (4x^4 - 4x^2 + 1)^{-1/3}$ ▷ We factor and use the power chain rule.
$\mathrm{F}(x) = [(2x^2 - 1)^2]^{-1/3} = (2x^2 - 1)^{-2/3}$
$\mathrm{F}'(x) = -\frac{2}{3}(2x^2 - 1)^{-5/3}(4x) = -\frac{8}{3}x(2x^2 - 1)^{-5/3} = \frac{-8x}{3(2x^2 - 1)^{5/3}}$

13. $\mathrm{F}(x) = (x^2 - 1)^{3/2}(x^2 - 4)^{1/2}$ ▷ $\mathrm{F}'(x) = \frac{3}{2}(x^2 - 1)^{1/2}(2x)(x^2 - 4)^{1/2} + \frac{1}{2}(x^2 - 1)^{3/2}(x^2 - 4)^{-1/2}(2x)$
$= x(x^2 - 1)^{1/2}(x^2 - 4)^{-1/2}[3(x^2 - 4) + (x^2 - 1)] = x(x^2 - 1)^{1/2}(x^2 - 4)^{-1/2}(4x^2 - 13)$

14. $g(x) = (x^4 - x)^{-3}(5 - x^2)^{-1}$ ▷ $g'(x) = (x^4 - x)^{-3}\mathrm{D}_x(5 - x^2)^{-1} + (5 - x^2)^{-1}\mathrm{D}_x(x^4 - x)^{-3}$
$= (x^4 - x)^{-3}(-1)(5 - x^2)^{-2}(-2x) + (5 - x^2)^{-1}(-3)(x^4 - x)^{-4}(4x^3 - 1)$
$= (x^4 - x)^{-4}(5 - x^2)^{-2}[2x(x^4 - x) - 3(5 - x^2)(4x^3 - 1)]$
$= (x^4 - x)^{-4}(5 - x^2)^{-2}(14x^5 - 60x^3 - 5x^2 + 15) = \frac{14x^5 - 60x^3 - 5x^2 + 15}{x^4(x^3 - 1)^4(x^2 - 5)^2}$

In Exercises 15–20, compute the derivative.

15. $\mathrm{D}_x[(x + 1)\sin x - x\cos x] = \sin x + (x + 1)\cos x - \cos x + x\sin x = (x + 1)\sin x + x\cos x$

16. $\mathrm{D}_t(\sin^2 3t)$ ▷ We use the power rule first.
$= 2\sin 3t \cdot \mathrm{D}_t(\sin 3t) = 2\sin 3t \cdot 3\cos 3t = 6\sin 3t\cos 3t$

17. $\frac{d}{dt}(\sqrt{\tan 4t})$ ▷ $= \frac{d}{dt}(\tan 4t)^{1/2} = \frac{1}{2}(\tan 4t)^{-1/2}\sec^2 4t \cdot 4 = \frac{2\sec^2 4t}{\sqrt{\tan 4t}}$

18. $\frac{d}{dx}\left(x\cos\frac{1}{x}\right)$ ▷ $\cos\frac{1}{x} + x\left(-\sin\frac{1}{x} \cdot -\frac{1}{x^2}\right) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$

19. $\mathrm{D}_w[\sin(\cos 3w) - \sin w\cos 3w]$ ▷ $\cos(\cos 3w)(-3\sin 3w) - [\cos w\cos 3w + \sin w(-3\sin 3w)]$
$= -3\sin 3w\cos(\cos 3w) - \cos w\cos 3w + 3\sin w\sin 3w$

20. $\mathrm{D}_x[\tan 2x\sec x + \tan(2\sec x)$ ▷ We use the product rule in the first term; the chain rule in the second.
$= \mathrm{D}_x(\tan 2x) \cdot \sec x + \tan 2x \cdot \mathrm{D}_x(\sec x) + \sec^2(2\sec x) \cdot \mathrm{D}_x(2\sec x)$
$= 2\sec^2 2x\sec x + \tan 2x\tan x\sec x + \sec^2(2\sec x) \cdot 2\sec x\tan x$

In Exercises 21–24, compute the derivative and check by plotting your answer and NDER in the same window.

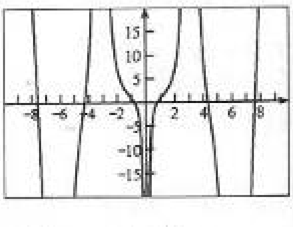
21. $f(x) = \left(\frac{2x}{x^2+1}\right)^2$ ▷ $f'(x) = 2\left(\frac{2x}{x^2+1}\right)\cdot\frac{2(x^2+1)-2x(2x)}{(x^2+1)^2} = \frac{8x(1-x^2)}{(x^2+1)^3}$

22. $g(x) = \sqrt{\frac{x}{4-x^2}} = \left(\frac{x}{4-x^2}\right)^{1/2}$; $g'(x) = \frac{1}{2}\left(\frac{x}{4-x^2}\right)^{-1/2}\frac{1(4-x^2)-x(-2x)}{(4-x^2)^2} = \frac{1}{2}\left(\frac{4-x^2}{x}\right)^{1/2}\frac{4+x^2}{(4-x^2)^2} = \frac{4+x^2}{2\sqrt{x(4-x^2)^3}}$

23. $g(x) = \frac{\tan x}{1+x}$ ▷ $g'(x) = \frac{\sec^2 x(1+x) - \tan x}{(1+x)^2}$

24. $f(x) = \frac{1+x^2}{\sin x}$

▷ $f'(x) = \frac{2x\sin x - (1+x^2)\cos x}{\sin^2 x}$



In Exercises 25–28, find $\frac{dy}{dx}$.

25. $4x^2 + 4y^2 - y^3 = 0$

▷ $8x + 8y\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$; $\frac{dy}{dx}(8y - 3y^2) = -8x$; $\frac{dy}{dx} = \frac{-8x}{8y-3y^2}$

26. $xy^2 + 2y^3 = x - 2y$

▷ $x\left(2y\frac{dy}{dx}\right) + y^2(1) + 6y^2\frac{dy}{dx} = 1 - 2\frac{dy}{dx}$; $(2xy + 6y^2 + 2)\frac{dy}{dx} = 1 - y^2$; $\frac{dy}{dx} = \frac{1-y^2}{2xy+6y^2+2}$

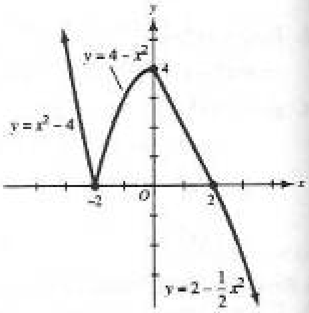
27. $\tan x + \tan y = xy$ ▷ $\sec^2 x + \sec^2 y\frac{dy}{dx} = y + x\frac{dy}{dx}$; $\frac{dy}{dx}(\sec^2 y - x) = y - \sec^2 x$; $\frac{dy}{dx} = \frac{y - \sec^2 x}{\sec^2 y - x}$

28. $\sin(x+y) + \sin(x-y) = 1$ ▷ We differentiate implicitly with respect to x on both sides.

$\cos(x+y)\left(1 + \frac{dy}{dx}\right) + \cos(x-y)\left(1 - \frac{dy}{dx}\right) = 0$; $[\cos(x+y) - \cos(x-y)]\frac{dy}{dx} = -\cos(x+y) - \cos(x-y)$;

$\frac{dy}{dx} = \frac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)} = \frac{\cos x\cos y}{\sin x\sin y}$

In Exercises 29 and 30, assume that each part of the accompanying graph of the continuous function f that appears to be a line segment is a line segment. (a) Define f piecewise; find (b) $f'_-(-2)$, (c) $f'_+(-2)$ (d) $f'_-(0)$, (e) $f'_+(0)$, (f) $f'_-(2)$, $f'_+(2)$. (h) At what numbers is f not differentiable?



29. (a) $f(x) = \begin{cases} x^2 - 4 & \text{if } x \le -2 \\ 4 - x^2 & \text{if } -2 < x \le 0 \\ 4 - 2x & \text{if } 0 < x \le 2 \\ 2 - \frac{1}{2}x^2 & \text{if } x > 2 \end{cases}$

(b) $f'_-(-2) = \lim_{x\to-2^-}\frac{(x^2-4)-0}{x+2} = \lim_{x\to-2^-}(x-2) = -4$

(c) $f'_+(-2) = \lim_{x\to-2^-}\frac{(4-x^2)-0}{x+2} = \lim_{x\to-2^+}(2-x) = 4$

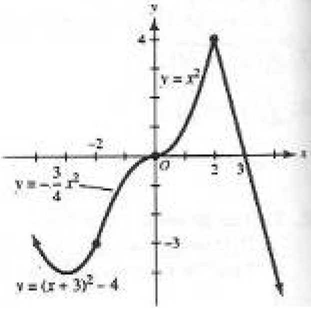
(d) $f'_-(0) = \lim_{x\to0^-}\frac{(4-x^2)-4}{x-0} = \lim_{x\to0^-}(-x) = 0$

(e) $f'_+(0) = \lim_{x\to0^+}\frac{(4-2x)-4}{x-0} = \lim_{x\to0^+}(-2) = -2$

(f) $f'_-(2) = \lim_{x\to2^-}\frac{(4-2x)-0}{x-2} = \lim_{x\to2^-}(-2) = -2$

(g) $f'_+(2) = \lim_{x\to2^+}\frac{(2-\frac{1}{2}x^2)-0}{x-2} = \lim_{x\to2^+}-\frac{1}{2}(x+2) = -2$

(h) f is not differentiable at -2 and 0.



30. (a) $\begin{cases} x^2 + 6x + 5 & \text{if } x \le -2 \\ -\frac{3}{4}x^2 & \text{if } -2 < x \le 0 \\ x^2 & \text{if } 0 < x \le 2 \\ 12 - 4x & \text{if } x > 2 \end{cases}$

(b) $f'_-(-2) = \lim_{x\to-2^-}\frac{(x^2+6x+5)+3}{x+2} = \lim_{x\to-2^-}(x+4) = 2$

(c) $f'_+(-2) = \lim_{x\to-2^-}\dfrac{-\frac{3}{4}x^2+3}{x+2} = \lim_{x\to-2^+} -\frac{3}{4}(x-2) = 3$

(d) $f'_-(0) = \lim_{x\to0^-}\dfrac{-\frac{3}{4}x^2-0}{x-0} = \lim_{x\to0^-}(-\frac{3}{4}x) = 0$

(e) $f'_+(0) = \lim_{x\to0^+}\dfrac{x^2-0}{x-0} = \lim_{x\to0^+} x = 0$

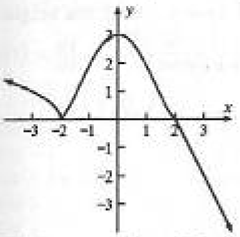
(f) $f'_-(2) = \lim_{x\to2^-}\dfrac{x^2-4}{x-2} = \lim_{x\to2^-}(x+2) = 4$

(g) $f'_+(2) = \lim_{x\to2^+}\dfrac{(12-4x)-4}{x-2} = \lim_{x\to2^+} -4 = -4$

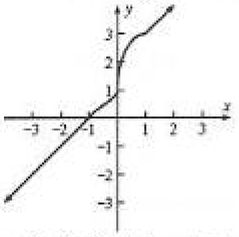
(h) f is not differentiable at -2 and 2.

In Exercises 31 and 32, sketch the graph of a continuous function f defined on R and having the given properties.

31. f is differentiable except at -2, 2; $f(x) > 0$ if $x < -2$; $f(-2) = 0$; $0 < f(x) \le 3$ if $-2 < x < 2$; $f(0) = 3$; $f(2) = 0$; $f(x) < 0$ if $x > 2$; $f'_+(-2) = 1$; $f'(0) = 0$; $f'_-(2) = -1$; $f'_+(2) = -2$; $\lim_{x\to-2^-}\dfrac{f(x)-f(-2)}{x+2} = -\infty$



32. f is differentiable except at -1, 0, and 1; the range of f is $(-\infty, +\infty)$; $f(-1) = 0$; $f(0) = 1$; $f(1) = 3$; $f'_-(-1) = 1$; $f'_+(-1) = 2$; $f'_-(1) = 0$; $f'_+(1) = 1$; $\lim_{x\to0}\dfrac{f(x)-f(0)}{x} = +\infty$



33. Find an equation of the tangent line to $y = x^3 - 3x - 1$ at $(2,1)$. Check by plotting the curve and the tangent.

▸ $f'(x) = 3x^2 - 3$. $m = f'(2) = 3(2)^2 - 3 = 9$. Equation of tangent line: $y = 9(x-2)+1$; $y = 9x - 17$

34. Find an equation of the normal line to $y = \dfrac{8x}{x^2+3}$ at $(3,2)$. Check by plotting the curve and the normal.

▸ $f'(x) = 8\dfrac{1(x^2+3)-x(2x)}{(x^2+3)^2} = 8\dfrac{3-x^2}{(x^2+3)^2}$. $m = -\dfrac{1}{f'(3)} = -\dfrac{12^2}{8(-6)} = 3$. Normal line: $y = 3(x-3)+2$; $y = 3x - 7$

35. Find equations of the tangent lines to $y = 2x^3 + 4x^2 - x$ having slope $\frac{1}{2}$. Check by plotting the lines and curve.

▸ $y' = 6x^2 + 8x - 1 = \frac{1}{2}$ when $0 = 12x^2 + 16x - 3 = (6x-1)(2x+3)$; so $x = \frac{1}{6}, x = -\frac{3}{2}$.

At $x = \frac{1}{6}$, $y = 2(\frac{1}{6})^3 + 4(\frac{1}{6})^2 - \frac{1}{6} = -\frac{5}{108}$ Equation of tangent line: $y = -\frac{5}{108} + \frac{1}{2}(x - \frac{1}{6})\frac{1}{2}x - \frac{7}{54}$

At $x = -\frac{3}{2}$, $y = 2(-\frac{3}{2})^3 + 4(-\frac{3}{2})^2 - (-\frac{3}{2}) = \frac{15}{4}$ Equation of tangent line: $y = \frac{15}{4} + \frac{1}{2}(x + \frac{3}{2}) = \frac{1}{2}x + \frac{9}{2}$

36. Find an equation of the normal line to the curve $x - y = \sqrt{x+y}$ at the point $(3,1)$.

▸ Differentiating implicitly with respect to x on both sides of the given equation, we get

$$1 - \mathrm{D}_x y = \tfrac{1}{2}(x+y)^{-1/2}(1 + \mathrm{D}_x y)$$

Replacing x by 3 and y by 1 and solving for $\mathrm{D}_x y$, we have

$$1 - \mathrm{D}_x y = \tfrac{1}{2}(4)^{-1/2}(1 + \mathrm{D}_x y) = \tfrac{1}{4}(1 + \mathrm{D}_x y)$$

$$4 - 4\mathrm{D}_x y = 1 + \mathrm{D}_x y; \qquad \mathrm{D}_x y = \tfrac{3}{5}$$

Hence, the slope of the tangent to the curve at the point $(3,1)$ is $\frac{3}{5}$ and so the slope of the normal line there is $-\frac{5}{3}$. Thus an equation for the normal line is

$$y - 1 = -\tfrac{5}{3}(x-3); \quad 5x + 3y - 18 = 0$$

37. Find equations of the tangent and normal lines to the curve $2x^3 + 2y^3 - 9xy = 0$ at the point $(2,1)$.

▸ $6x^2 + 6y^2\dfrac{dy}{dx} = 9y - 9x\dfrac{dy}{dx} = 0$; $(6y^2 - 9x)\dfrac{dy}{dx} = 9y - 6x^2$; $\dfrac{dy}{dx} = \dfrac{3y - 2x^2}{2y^2 - 3x}$; $\left.\dfrac{dy}{dx}\right|_{(2,1)} = \dfrac{3\cdot1 - 2\cdot2^2}{2\cdot1^2 - 3\cdot2} = \dfrac{5}{4}$

At $(2,1)$ the tangent line has slope $\frac{5}{4}$. Its equation is $y - 1 = \frac{5}{4}(x-2)$; $5x + 4y - 6 = 0$.
At $(2,1)$ the normal line has slope $-\frac{4}{5}$.Its equation is $y - 1 = -\frac{4}{5}(x-2)$; $4x + 5y - 13 = 0$.

38. Find equations of the tangent and normal to $y = 8\sin^3 2x$ at $(\frac{1}{12}\pi, 1)$. Check by plotting the lines and curve.

▷ $y' = 48\sin^2 2x\cos 2x$. $m_{tangent} = f'(\frac{1}{12}\pi) = 48\sin^2\frac{1}{6}\pi\cos\frac{1}{6}\pi = 6\sqrt{3}$. Tangent line: $y = 6\sqrt{3}(x - \frac{1}{12}\pi) + 1$
$m_{normal} = -\dfrac{1}{6\sqrt{3}} = -\frac{1}{18}\sqrt{3}$. Equation of normal line: $y = -\frac{1}{18}\sqrt{3}(x - \frac{1}{12}\pi) + 1$

39. Prove that the line tangent to $y = -x^4 + 2x^2 + x$ at $(1, 2)$ is tangent to the curve at another point and find it.

▷ $y'(x) = -4x^3 + 4x + 1$; $y'(1) = 1$. At $(1, 2)$ the tangent line has slope 1. Its equation is $y - 2 = 1(x - 1)$; $y = x + 1$. To determine the intersections of the line and the original curve we solve the first and last equations simultaneously.
$-x^4 + 2x^2 + x = x + 1$; $x^4 - 2x^2 + 1 = 0$; $(x^2 - 1)^2 = 0$; $(x - 1)^2(x + 1)^2 = 0$; $x = 1, 1, -1, -1$
When $x = -1$, $y = x + 1 = 0$. There are two ways to proceed.
Method 1: $y'(-1) = 1$. Hence the tangent line at $(-1, 0)$ has equation $y - 0 = 1(x + 1)$ or equivalently, $y = x + 1$, which is the tangent line at $(1, 2)$.
Method 2: Since the roots $1, -1$ are repeated, the line is tangent at $(1, 0)$ and $(-1, 0)$.

40. Prove that the tangent lines to the curves $4y^3 - x^2y - x + 5y = 0$ and $x^4 - 4y^3 + 5x + y = 0$ at the origin are perpendicular.

▷ We show that the product of the slopes of the tangent lines at $(0, 0)$ is -1. The first curve is

$$4y^3 - x^2y - x + 5y = 0$$

Differentiating implicitly with respect to x, using the product rule on the second term, we have

$$12y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy - 1 + 5\frac{dy}{dx} = 0$$

We let $x = 0$ and $y = 0$ and solve for $\frac{dy}{dx}$.

$$-1 + 5\frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = \frac{1}{5}$$

Thus, $m_1 = \frac{1}{5}$ is the slope of the tangent line to the first curve at the origin. The second curve is

$$x^4 - 4y^3 + 5x + y = 0$$

Differentiating implicitly with respect to x, we have

$$4x^3 - 12y^2\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0$$

When $x = 0$ and $y = 0$, we have

$$5 + \frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = -5$$

Thus, $m_2 = -5$. Because $m_1m_2 = \frac{1}{5}(-5) = -1$ then the tangent lines at the origin are perpendicular.

41. $y = \sqrt{3 - 2x} = (3 - 2x)^{1/2}$; $\frac{dy}{dx} = \frac{1}{2}(3 - 2x)^{-1/2}(-2) = -(3 - 2x)^{-1/2}$
$\dfrac{d^2y}{dx^2} = -(3 - 2x)^{-3/2}(-2) = -(3 - 2x)^{-3/2}$; $\dfrac{d^3y}{dx^3} = -(3 - 2x)^{-5/2}(-2) = -3(3 - 2x)^{-5/2}$

42. If $\frac{dy}{dx} = y^k$, find $\frac{d^3y}{dx^3}$. ▷ $\dfrac{d^2y}{dx^2} = ky^{k-1}\dfrac{dy}{dx} = ky^{k-1}y^k = ky^{2k-1}$
$\dfrac{d^3y}{dx^3} = k(2k - 1)y^{2k-2}\dfrac{dy}{dx} = k(2k - 1)y^{2k-2}y^k = k(2k - 1)y^{3k-2}$

43. $f(x) = \frac{1}{12}x^4 + \frac{2}{3}x^3 + \frac{3}{2}x^2 + 8x + 2$; $f'(x) = \frac{1}{3}x^3 + 2x^2 + 3x + 8$; $f''(x) = x^2 + 4x + 3$
We wish to find when $f''(x) > 0$, that is when $x^2 + 4x + 3 > 0$; $(x + 3)(x + 1) > 0$.
Case 1: $x + 3 > 0$ and $x + 1 > 0$; $x > -3$ and $x > -1$. Hence $x > -1$.
Case 2: $x + 3 < 0$ and $x + 1 < 0$; $x < -3$ and $x < -1$. Hence $x < -3$.
Therefore $f''(x) > 0$ when either $x < -3$ or $x > -1$.

44. Find the rate of change of y with respect to x at the point $(3, 2)$ if $7y^2 - xy^3 = 4$.

▷ Differentiating implicitly with respect to x, using the product rule on the second term, we have

$$14y\frac{dy}{dx} - y^3 - 3xy^3\frac{dy}{dx} = 0$$

We let $x = 3$ and $y = 2$ and solve for $\frac{dy}{dx}$: $\quad -8 - 44\frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = -\frac{2}{11}$

The rate of change of y with respect to x is $-\frac{2}{11}$.

Exercise 45 and 46, a particle is moving along a horizontal line according to the given equation where s meters the directed distance of the particle from a point O at t seconds. The positive direction is to the right. Determine the intervals of time when the particle is moving to the right and when it is moving to the left. Also determine when it reverses its direction.

$s = 2t^3 + 3t^2 - 12t - 5$

▸ $v(t) = \frac{ds}{dt} = 6t^2 + 6t - 12 = 6(t^2 + t - 2) = 6(t+2)(t-1)$. $v(t) = 0$ when $t = -2$ and $t = 1$.

	$t+2$	$t-1$	Conclusion
$t < -2$	$-$	$-$	v is positive and the particle is moving to the right
$t = -2$	0	$-$	v is zero; particle is changing direction from right to left
$-2 < t < 1$	$+$	$-$	v is negative and the particle is moving to the left
$t = 1$	$+$	0	v is zero and the particle is changing direction from left to right
$1 < t$	$+$	$+$	v is positive and the particle is moving to the right

$s = \frac{t-1}{t^2 - 2t + 5}$

▸ $v(t) = \frac{ds}{dt} = \frac{1(t^2 - 2t + 5) - (t-1)(2t-2)}{(t^2 - 2t + 5)^2} = \frac{-(t^2 - 2t - 3)}{(t^2 - 2t + 5)^2} = \frac{-(t+1)(t-3)}{(t^2 - 2t + 5)^2}$. $v(t) = 0$ when $t = -1$ and $t = 3$.

	$-(t+1)$	$t-3$	Conclusion
$t < -1$	$+$	$-$	v is negative and the particle is moving to the left
$t = -1$	$+$	0	v is zero and the particle is changing direction from left to right
$-1 < t < 3$	$-$	$-$	v is positive and the particle is moving to the right
$t = 3$	$-$	0	v is zero and the particle is changing direction from left to right
$3 < t$	$-$	$+$	v is negative and the particle is moving to the left

Exercises 47 and 48, make a table giving s, v, a. Include the intervals of time when the particle is moving to the left and right, and when the velocity or speed is increasing or decreasing. Plot position as a function of t.

$s = 4 - 9t + 6t^2 - t^3 = -(t-1)^2(t-4)$; $v = \frac{ds}{dt} = -9 + 12t - 3t^2 = -3(t^2 - 4t + 3) = -3(t-1)(t-3)$;

$a = \frac{dv}{dt} = 12 - 6t = 6(2-t)$

	s	v	a	Conclusion
$0 \le t < 1$	$+$	$-$	$+$	Particle is right of the origin and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 1$	0	0	$+$	Particle is at the origin and it is changing its direction of motion from left to right. The velocity is increasing. The speed is increasing.
$1 < t < 2$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right. The velocity is increasing. The speed is increasing.
$t = 2$	$+$	$+$	0	Particle is right of the origin, and it is moving to the right. The velocity is not changing; so the speed is not changing.
$2 < t < 3$	$+$	$+$	$-$	Particle is right of the origin, and it is moving to the right. The velocity is decreasing. The speed is decreasing.
$t = 3$	$+$	0	$-$	Particle is right of the origin, and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$3 < t < 4$	$+$	$-$	$-$	Particle is right of the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.
$t = 4$	0	$-$	$-$	Particle is at the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.
$4 < t$	$-$	$-$	$-$	Particle is left of the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.

48. $s = t^3 - 3t^2 - 9t + 13 = (t - t_0)(t - t_1)(t - t_2)$, where $t_0 \approx -2.5446$, $t_1 \approx 1.1671$, $t_2 \approx 4.3776$

$v = \frac{ds}{dt} = 3t^2 - 6t - 9 = 3(t^2 - 2t - 3) = 3(t + 1)(t - 3)$; $a = \frac{ds}{dt} = 6t - 6 = 6(t - 1)$

	s	v	a	Conclusion
$0 \leq t < 1$	+	−	−	Particle is right of the origin and it is moving to the left. The velocity is decreasing. The speed is increasing.
$t = 1$	+	−	0	Particle is right of the origin and it is moving to the left. The velocity is not changing; so the speed is not changing.
$1 < t < t_1$	+	−	+	Particle is right of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = t_1$	0	−	+	Particle is at the origin. and it is moving to the left The velocity is increasing. The speed is decreasing.
$t_1 < t < 3$	−	−	+	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 3$	−	0	+	Particle is left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is increasing.
$3 < t < t_2$	−	+	+	Particle is left of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t = t_2$	0	+	+	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_2 < t$	+	+	+	Particle is right of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.

In Exercises 49 and 50, a particle is moving along a line where s feet is the directed distance at t seconds. Find th time when the instantaneous acceleration is zero and the position and velocity at this time.

49. $s = 9t^2 + 2\sqrt{2t} + 1 = 9t^2 + 2\sqrt{2}t^{1/2} + 1$; $v = ds/dt = 18t + \sqrt{2}t^{-1/2}$; $a = dv/dt = 18 - \frac{1}{2}\sqrt{2}t^{-3/2}$

$a = 0$ when $t^{3/2} = 2^{-3/2}3^{-2}$, $t = 2^{-1}3^{-4/3}$. Then $s = \frac{1}{4}3^{4/3} + 1$, $v = 3^{5/3}$

50. $s = \frac{4}{9}t^{3/2} + 2t^{1/2}$, $v = ds/dt = \frac{2}{3}t^{1/2} + t^{-1/2}$, $a = dv/dt = \frac{1}{3}t^{-1/2} - \frac{1}{2}t^{-3/2}$

$a = 0$ when $t = \frac{3}{2}$. Then $s = \frac{4}{3}\sqrt{6}$, $v = \frac{2}{3}\sqrt{6}$

51. A bag is dropped from 200 ft. (a) Write an equation of motion. (b) Find the velocity at 1 sec and 3 sec. (c How long does it take to hit the ground? (d) What is its speed when it hits the ground?

▹ (a) $s = 200 - 16t^2$ (b) $v = ds/dt = -32t$. $v(1) = -32$ ft/sec. $v(3) = -32 \cdot 3 = -96$ ft/sec

(c) $s = 0$ when $t^2 = \frac{200}{16} = \frac{50}{4}$, $t = \frac{5}{2}\sqrt{2} \approx 3.54$ sec (d) $v(\frac{5}{2}\sqrt{2}) = -80\sqrt{2} \approx -113$. Its speed is 113 ft/sec.

52. A bag is thrown downward from an altitude of 200 ft with a velocity of 20 ft/sec. (a) Use Equation (10) i Exercises 2.5 to write an equation of motion and simulate the motion on your calculator. (b) Find the velocit at 1 sec and 3 sec. (c) How long does it take to hit the ground? (d) What is its speed when it hits the ground

▹ (a) Equation (10) is $s = -16t^2 + v_0t + s_0$. Because $v_0 = -20$ and $s_0 = 200$, the equation of motion is $s = -16t^2 - 20t + 200$
(b) $v = D_ts = D_t(-16t^2 - 20t + 200) = -32t - 20$. $v(1) = -32 - 20 = -52$. $v(3) = -32 \cdot 3 - 20 = -116$
The velocity at 1 sec is −52 ft/sec; the velocity at 3 sec is −116 ft/sec (if it falls into a deep hole).
(c) When $s = 0$; $-16t_1{}^2 - 20t_1 + 200 = 0$; $t_1 = \frac{5}{8}(\sqrt{33} - 1)$. The bag hits the ground after about 2.97 sec.
(d) $v(t_1) = -32 \cdot \frac{5}{8}(\sqrt{33} - 1) - 20 = -20\sqrt{33}$. Its speed is about 114.9 ft/sec when it hits the ground.

53. A ball is thrown upward from a height of 112 ft with a velocity of 96 ft/sec. (a) Write an equation of moti and simulate the motion. (b) Estimate and (c) calculate how high the ball will go and when it gets there. (Estimate and (e) calculate how long it takes the ball to reach the ground. (f) Find the velocity at 2 sec and sec. (g) Find the speed at 2 sec and 4 sec. (h) Find the velocity when it hits the ground.

▹ (a) $s = -16t^2 + 96t + 112$ (c) $v = -32t + 96 = 0$ when $t = 3$. $s(3) = -16 \cdot 3^2 + 96 \cdot 3 + 112 = 256$ ft
(d) $s = 0$ when $0 = -16(t^2 - 6t - 7) = -16(t - 7)(t + 1)$, $t = 7$ sec. (f) $v(2) = -32 \cdot 2 + 96 =$
$v(4) = -32 \cdot 4 + 96 = -32$ (g) The speed is 32 ft/sec at both times. (h) $v(7) = -32 \cdot 7 + 96 = -128$

In Exercises 54–56, a particle is moving along a line where at t sec, s cm is the directed distance form the origin, v cm/sec is the velocity and a cm/sec^2 is the acceleration. (a) Find v and a. (b) Show that the motion is simple harmonic. (c) Simulate the motion on your graphics calculator.

54. $s = 5 - 2\cos^2 t = 5 - (1 + \cos 2t) = 4 - \cos 2t$ (a) $v = ds/dt = 2\sin 2t$, $a = dv/dt = 4\cos 2t$. (b) The motion is simple harmonic because a is proportional to the distance from a fixed point ($s = 4$) and oppositely directed.

$s = \cos 2t + 2\sin 2t$ (a) $v = ds/dt = -2\sin 2t + 4\cos 2t$, $a = dv/dt = -4\cos 2t - 8\sin 2t$
(b) The motion is simple harmonic because $a = -4s$.

$s = \sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)$

► (a) $v = ds/dt = D_t[\sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)] = 4[\cos(4t + \frac{1}{3}\pi) + \cos(4t + \frac{1}{6}\pi)]$
$a = dv/dt = D_t 4[\cos(4t + \frac{1}{3}\pi) + \cos(4t + \frac{1}{6}\pi)] = -16[\sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)]$
(b) Because -16 is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because -16 is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic.

57. Profit on an item is \$200 if not more than 800 are produced each week and decreases \$0.20 per item for each item over 800. (a) Express the profit as a function f of the number x of items sold. (b) Prove that f is continuous. (c) Determine if f is differentiable at 800.

► (a) If $0 \le x \le 800$, then $f(x) = 200x$. Otherwise, $f(x) = [200 - .2(x - 800)]x = (360 - .2x)x = 360x - .2x^2$, $800 < x \le 1800$ (b) $\lim_{x \to 800^-} f(x) = \lim_{x \to 800^-} 200x = 200 \cdot 800 = f(200)$ and $\lim_{x \to 800^+} f(x) = \lim_{x \to 800^+} (360 - .2x)x = 200 \cdot 800$ so f is continuous at 800. (c) $\lim_{x \to 800^-} f'(x) = \lim_{x \to 800^-} 200 = 200$ and $\lim_{x \to 800^+} f'(x) = \lim_{x \to 800^+} (360 - .4x) = 40$. Hence f is not differentiable at 800.

If $R = kT^4$, find (a) the average rate of change of R as T increases from 200 to 300; (b) the instantaneous rate of change of R with respect to T when T is 200.

► (a) $\dfrac{\Delta R}{\Delta T} = \dfrac{k300^4 - k200^4}{300 - 200} = 65{,}000{,}000k$ (b) $R'(T) = 4kT^3$; $R'(200) = 4k \cdot 200^3 = 32{,}000{,}000k$

A square units is the area of an isosceles right triangle for which the length of each leg is x units. Thus $A = \frac{1}{2}x^2$; $A'(x) = x$. (a) The average rate of change of A with respect to x as x changes from 8 to 8.01 is

$$\frac{\frac{1}{2} \cdot 8.01^2 - \frac{1}{2} \cdot 8^2}{8.01 - 8} = \frac{0.08005}{0.01} = 8.005$$

(b) The instantaneous rate of change of A with respect to x when $x = 8$ is $A'(8) = 8$.

If $y = x^{2/3}$, find the relative rate of change of y with respect to x when (a) $x = 8$, and (b) $x = c$, where c is a constant.

► The relative rate of change of y with respect to x is $\dfrac{dy/dx}{y}$. Now, $\dfrac{dy}{dx} = \frac{2}{3}x^{-1/3}$

Thus, $\dfrac{dy/dx}{y} = \dfrac{\frac{2}{3}x^{-1/3}}{x^{2/3}} = \dfrac{2}{3x}$

(a) If $x = 8$, we have $\dfrac{dy/dx}{y} = \dfrac{2}{3(8)} = \dfrac{1}{12}$

The relative rate of change of y with respect to x is $\frac{1}{12}$ when $x = 8$.

(b) If $x = c$, we have $\dfrac{dy/dx}{y} = \dfrac{2}{3c}$

The relative rate of change of y with respect to x is $2/3c$ when $x = c$.

$100y$ calculators are supplied when m dollars is the price per calculator. $y = m^2 + \sqrt{m}$
(a) The average rate of change of y with respect to m when m increases from 16 to 17 is

$$\frac{(17^2 + \sqrt{17}) - (16^2 - \sqrt{16})}{17 - 16} \approx \frac{33.123}{1} = 33.123$$

Therefore the average rate of change of $100y$ with respect to m when m increases from 16 to 17 is 3,312.3. Thus, the average rate of change of the supply when the price increases from \$16 to \$17 is 3312.3 calculators per \$1 increase in price.

(b) $y'(\mathrm{m}) = \mathrm{m} + \frac{1}{2}m^{-1/2}$. The instantaneous rate of change of y with respect to m when m is 16 is $y'(16) = 2(16) + \frac{1}{2}(16)^{-1/2} = 32.125$. Therefore, the instantaneous rate of change of $100y$ when $\mathrm{m} = 16$ is 3212.5. Hence, the instantaneous rate of change of the supply with respect to the price when the price is \$16 is 3212.5 calculators per \$1 increase in price.

62. The remainder theorem of algebra states that if $\mathrm{P}(x)$ is a polynomial in x, and r is any real number, then there is a polynomial $\mathrm{Q}(x)$ such that $\mathrm{P}(x) = \mathrm{Q}(x)(x-r) + \mathrm{P}(r)$. What is $\lim\limits_{x \to r} \mathrm{Q}(x)$?

▹ Solving the given equation for $\mathrm{Q}(x)$, we obtain $\mathrm{Q}(x) = \dfrac{\mathrm{P}(x) - \mathrm{P}(r)}{x - r}$. Because P is a polynomial function, then P is differentiable at every real number. Thus we may apply the Definition 2.1.3′ to conclude that

$$\lim_{x \to r} \mathrm{Q}(x) = \lim_{x \to r} \frac{\mathrm{P}(x) - \mathrm{P}(r)}{x - r} = \mathrm{P}'(r)$$

63. $f(x) = \dfrac{3}{x+2}$; $f'(x) = \lim\limits_{x \to -5} \dfrac{f(x) - f(x)}{x+5} = \lim\limits_{x \to -5} \dfrac{1}{x+5}\left(\dfrac{3}{x+2} - (-1)\right) = \lim\limits_{x \to -5} \dfrac{x+5}{(x+2)(x+5)} = \lim\limits_{x \to -5} \dfrac{1}{x+2} = -\dfrac{1}{3}$

64. Use the definition of a derivative to find $f'(x)$ if $f(x) = 3x^2 - 5x + 1$.

▹ $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 - 5(x+\Delta x) + 1] - (3x^2 - 5x + 1)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{3(2x\Delta x + \Delta x^2) - 5\Delta x}{\Delta x} = \lim_{\Delta x \to 0} [3(2x + \Delta x) - 5] = 6x - 5$$

65. $f(x) = \sqrt{4x-3}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{4(x+\Delta x) - 3} - \sqrt{4x-3}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(\sqrt{4x + 4\Delta x - 3} - \sqrt{4x-3})(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x-3})}{\Delta x(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x-3})}$$
$$= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 3 - 4x + 3}{\Delta x(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x-3})} = \lim_{\Delta x \to 0} \frac{4}{\sqrt{4x + 4\Delta x - 3} + \sqrt{4x-3}} = \frac{2}{\sqrt{4x-3}}$$

66. Use the definition of a derivative to find $f'(5)$ if $f(x) = \sqrt{3x+1}$.

▹ $$f'(5) = \lim_{x \to 5} \frac{\sqrt{3x+1} - 4}{x - 5} = \lim_{x \to 5} \frac{(\sqrt{3x+1} - 4)(\sqrt{3x+1} + 4)}{(x-5)(\sqrt{3x+1} + 4)} = \lim_{x \to 5} \frac{(3x+1) - 16}{(x-5)(\sqrt{3x+1} + 4)} = \lim_{x \to 5} \frac{3}{\sqrt{3x+1} + 4} = \frac{3}{8}$$

67. $f(x) = \sqrt{2 + \cos x} = (2 + \cos x)^{1/2}$; $f'(x) = \frac{1}{2}(2 + \cos x)^{-1/2}(-\sin x)$

$f''(x) = \frac{1}{2}[-\frac{1}{2}(2 + \cos x)^{-3/2}(-\sin x)^2 + (2 + \cos x)^{-1/2}(-\cos x)]$

$f''(\pi) = \frac{1}{2}[-\frac{1}{2}(2-1)^{-3/2}(0) + (2-1)^{-1/2}(1)] = \frac{1}{2}$

68. Find $f''(x)$ if $f(x) = 3\sin^2 x - 4\cos^2 x$.

▹ Because we will be taking two derivatives, it pays to simplify first.

$$f(x) = 3 \cdot \tfrac{1}{2}(1 - \cos 2x) - 4 \cdot \tfrac{1}{2}(1 + \cos 2x) = -\tfrac{1}{2} - \tfrac{7}{2}\cos 2x$$
$$f'(x) = 7 \sin 2x$$
$$f''(x) = 14 \cos 2x$$

69. $f(x) = (|x+1| - |x|)^2$; $f'(x) = 2(|x+1| - |x|)\mathrm{D}_x(\sqrt{(x+1)^2} - \sqrt{x^2})$
$= 2(|x+1| - |x|)\left(\dfrac{x+1}{\sqrt{(x+1)^2}} - \dfrac{x}{\sqrt{x^2}}\right) = 2(|x+1| - |x|)\left(\dfrac{x+1}{|x+1|} - \dfrac{x}{|x|}\right)$. Alternatively,

$$f(x) = \begin{cases} (-x-1+x)^2 = 1 & \text{if } x < -1 \\ (x+1+x)^2 = (2x+1)^2 & \text{if } -1 \le x \le 0; \\ (x+1-x)^2 = 1 & \text{if } x > 0 \end{cases} \quad f'(x) = \begin{cases} 0 & \text{if } x < -1 \\ 8x+4 & \text{if } -1 < x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

70. Find $f'(-3)$ if $f(x) = (|x| - x)\sqrt[3]{9x}$.

▹ Because x is negative, then $|x| = -x$. Therefore

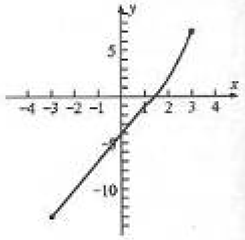
$$f(x) = -2x(9x)^{1/3} = -2 \cdot 3^{2/3}x^{4/3}; \quad f'(x) = -8 \cdot 3^{-1/3}x^{1/3}; \quad f'(-3) = -8 \cdot 3^{-1/3}(-3)^{1/3} = 8$$

In Exercises 71 and 72, s cm is the distance of a weight from its central position at t seconds, and the positive direction is upward. (a) Find the velocity and acceleration. (b) Show that the motion is simple harmonic. (c) Find

55. $F(x) = \begin{cases} 3x-4 & \text{if } -3 \le x < 1 \\ x^2-2 & \text{if } 1 \le x \le 3 \end{cases}$, $I = [-3,3]$; $F'(x) = \begin{cases} 3 & \text{if } -3 < x < 1 \\ 2x & \text{if } 1 < x < 3 \end{cases}$

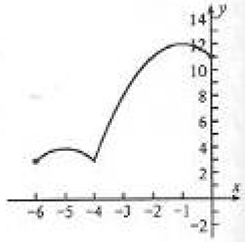
$\lim\limits_{x\to1^-} F(x) = \lim\limits_{x\to1^-} (3x-4) = -1$; $\lim\limits_{x\to1^+} F(x) = \lim\limits_{x\to1^+} (x^2-2) = -1$.
Thus F is continuous at 1. Because F is continuous on I, F has an absolute maximum value and an absolute minimum value on I.
$F'(x)$ is never 0 and $F'(x)$ exists except perhaps at 1; $F'_-(1) = 3$ and $F'_+(1) = 2$; therefore $F'(1)$ does not exist; thus 1 is a critical number of F.
$F(-3) = -13$, $F(1) = -1$, $F(3) = 7$

▪ The absolute minimum value is -13; the absolute maximum value is 7.



56. $G(x) = \begin{cases} 4-(x+5)^2 & \text{if } -6 \le x \le -4 \\ 12-(x+1)^2 & \text{if } -4 < x \le 0 \end{cases}$; $I = [-6,0]$

▹ $G'(x) = \begin{cases} -2(x+5) & \text{if } -6 < x < -4 \\ -2(x+1) & \text{if } -4 < x < 0 \end{cases}$



Because $G'(-5) = 0$ and $G'(-1) = 0$, then -5 and -1 are critical numbers of G. Furthermore,

$$G'_-(-4) = \lim_{x\to-4^-} G'(x) = \lim_{x\to-4^-} -2(x+5) = -2$$

and

$$G'_+(-4) = \lim_{x\to-4^+} G'(x) = \lim_{x\to-4^+} -2(x+1) = 6$$

Therefore, $G'(-4)$ is not defined, and so -4 is a critical number of G. We evaluate G at each endpoint of $[-6,0]$ and at each critical number of G.

$$G(-6) = 3 \quad G(-5) = 4 \quad G(-4) = 3 \quad G(-1) = 12 \quad G(0) = 11$$

The absolute maximum value of G on I is 12 which occurs at -1, and the absolute minimum value of G on I is 3 which occurs at both -6 and -4. The graph is shown at the right.

57. If $\lim\limits_{x\to c} f(x) = L < 0$ then there is an an open interval I containing c such that $f(x) < 0$ for every $x \ne c$ in I.

▹ Choose $\epsilon = -\frac{1}{2}L$ in the definition of limit. The there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$ then
$|f(x) - L| < -\frac{1}{2}L \Leftrightarrow \frac{1}{2}L < f(x) - L < -\frac{1}{2}L \Rightarrow f(x) < \frac{1}{2}L$
Hence $f(x) < 0$ if $x \in I = (c-\delta, c+\delta)$ and $x \ne c$.

58. If the function f has a relative maximum value at c then $-f$ has a relative minimum value at c.
By the first part of the proof, $-f'(c) = 0$; hence $f'(c) = 0$.

59. If f is differentiable everywhere and $f'(c) = 0$, can we conclude that f has a local extremum at c?

▹ No. If $f(x) = x^3$ then $f'(x) = 3x^2$ and $f'(0) = 0$ but f does not have a local extremum at 0.

60. If the function f has a local extremum at a number c, can we we conclude that $f'(c) = 0$?

▹ No. $f'(c)$ may not exist. If $f(x) = |x|$, then 0 is a local minimum of f but $f'(0)$ does not exist.

3.2 APPLICATIONS INVOLVING AN ABSOLUTE EXTREMUM ON A CLOSED INTERVAL

The procedure for solving the Exercises in this section is as follows.

1. Identify the variable for which you want to find an absolute extreme value. This is the dependent variable.
2. Express the dependent variable as a function of some other variable.
3. Verify that the function found in step 2 has a closed interval I for its domain and that the function is continuous on that closed interval.
4. Follow the steps given in Section 3.1 for finding the absolute extrema of a function on a closed interval.

Often it is helpful to first express the dependent variable as a function of two other variables. In this case, you must find another equation involving those two variables and then use the equation to eliminate one of the variables. Sometimes a figure is helpful, and sometimes the formulas from geometry given in Section 2.10 are helpful.

Exercises 3.2

1. Find a number in $I = [\frac{1}{3}, 2]$ such that the sum of the number and its reciprocal is (a) minimum; (b) maximum.

■ Let x be the number. We seek the extrema of $f(x) = x + \frac{1}{x}$; $f'(x) = 1 - \frac{1}{x^2}$. f is continuous and $f'(x)$ exists on any interval that does not contain 0. $f'(x) = 0$ when $1 = \frac{1}{x^2}$, $x^2 = 1$, $x = \pm 1$, but only $x = 1$ is in I. $f(\frac{1}{3}) = \frac{1}{3} + 3 = \frac{10}{3}$, $f(1) = 1 + 1 = 2$, $f(2) = 2 + \frac{1}{2} = \frac{5}{2}$. (a) $x = 1$ minimizes the sum (b) $x = \frac{1}{3}$ maximizes

2. Find the number in $I = [-1, 1]$ such that the number minus its square is (a) maximum; (b) minimum.

■ Let x be the number. We seek the extrema of $f(x) = x - x^2$; $f'(x) = 1 - 2x$. f is continuous and $f'(x)$ exists on any interval. $f'(x) = 0$ when $2x = 1$; $x = \frac{1}{2} \in I$. $f(-1) = -2$, $f(\frac{1}{2}) = \frac{1}{4}$, $f(1) = 0$ (a) $x = \frac{1}{2}$ maximizes the difference (b) $x = -1$ minimizes the difference.

Use calculus to solve the indicated exercise of Exercises 1.3.

3. Ex. 13. A field of length x m is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - 2x) = 120 - x$ m and its area is $a(x) = (120 - x)x = 120x - x^2$, $0 \le x \le 120$. $a'(x) = 120 - 2x = 0$ when $x = 60$. $a(0) = 0$, $a(60) = 3600$, $a(120) = 0$. The area is greatest when $x = 60$ and the field is 60 m × 60 m.

4. Ex. 14. A garden of length x ft is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - 2x) = 50 - x$ ft and its area is $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$. $a'(x) = 50 - 2x = 0$ when $x = 25$. $a(0) = 0$, $a(25) = 625$, $a(50) = 0$. The area is greatest when $x = 25$ and the garden is 25 ft × 25 ft.

5. Ex. 15. A field of length x m parallel to a river is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - x)$ m and its area is $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$. $a'(x) = 120 - x = 0$ when $x = 120$. $a(0) = 0$, $a(120) = 7200$, $a(240) = 0$. The area is greatest when $x = 120$ and the field is 120 m × 60 m.

6. Ex. 16. A garden of length x ft parallel to a house is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - x)$ ft and its area is $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$. $a'(x) = 50 - x = 0$ when $x = 50$. $a(0) = 0$. $a(50) = 1250$, $a(100) = 0$. The area is greatest when $x = 50$ and the garden is 50 ft × 25 ft.

7. Ex. 17. When x in. squares are cut from the corners of 8 in. by 15 in. sheet and the sides are turned up, the volume is $\mathrm{V}(x)$ in^3 where $\mathrm{V}(x) = \ell wh = (8 - 2x)(15 - 2x)x = 4x^3 - 46x^2 + 120x$, $0 \le x \le 4$.
$\mathrm{V}'(x) = 12x^2 - 92x + 120 = 4(3x^2 - 23x + 30) = 4(3x - 5)(x - 6)$. $\mathrm{V}'(x) = 0$ when $x = \frac{5}{3} \in I$ and $x = 6 \notin I$.
$\mathrm{V}(0) = 0$, $\mathrm{V}(\frac{5}{3}) = \frac{4900}{27}$, $\mathrm{V}(4) = 0$. The maximum volume is $\frac{4900}{27} \approx 90.74$ in^3 when $\frac{5}{3}$ in. squares are cut off.

8. Ex. 18. When x cm squares are cut from the corners of a 12 cm square and the sides are turned up, the volume is $\mathrm{V}(x)$ cm^3 where $\mathrm{V}(x) = \ell wh = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x$, $0 \le x \le 6$.
$\mathrm{V}'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6)$. $\mathrm{V}'(x) = 0$ when $x = 2 \in I$ and $x = 6 \notin I$.
$\mathrm{V}(0) = 0$, $\mathrm{V}(2) = 128$, $\mathrm{V}(6) = 0$. The maximum volume is 128 cm^3 when 2 cm squares are cut off each corner.

9. Ex. 19. When x in. squares are cut from the corners of 12 in. by 15 in. sheet and the sides are turned up, the volume is $\mathrm{V}(x)$ in^3 where $\mathrm{V}(x) = \ell wh = (12 - 2x)(15 - 2x)x = 4x^3 - 54x^2 + 180x$, $0 \le x \le 6$. $\mathrm{V}'(x) = 12x^2 - 108x + 180 = 12(x^2 - 9x + 15)$. $\mathrm{V}'(x) = 0$ when $x = \frac{1}{2}(9 + \sqrt{21}) \approx 6.79 \notin I$ and $x = \frac{1}{2}(9 - \sqrt{21}) \approx 2.21 \in I$. $\mathrm{V}(0) = 0$, $\mathrm{V}(2.21) \approx 177$, $\mathrm{V}(6) = 0$. The maximum volume is 177 in^3 when 2.21 in. squares are cut off.

10. Ex. 20. When x in. squares are cut from the corners of 40 cm by 50 cm sheet and the sides are turned up, the volume is $\mathrm{V}(x)$ cm^3 where $\mathrm{V}(x) = \ell wh = (40 - 2x)(50 - 2x)x = 4x^3 - 180x^2 + 2000x$, $0 \le x \le 20$. $\mathrm{V}'(x) = 12x^2 - 360x + 2000 = 4(3x^2 - 90x + 500)$. $\mathrm{V}'(x) = 0$ when $x = \frac{5}{3}(9 + \sqrt{21}) \approx 22.64 \notin I$ and $x = \frac{5}{3}(9 + \sqrt{21}) \approx 7.36 \in I$. $\mathrm{V}(0) = 0$, $\mathrm{V}(7.36) \approx 6564.2$, $\mathrm{V}(20) = 0$. The maximum volume is 6564.2 cm^3; cut 7.36 cm squares.

11. Ex. 25. A box of length x in. with square cross section has 100 in. as the sum of its length and girth has volume $\mathrm{V}(x)$ in^3. The width of the box is $\frac{1}{4}(100 - x)$ in. $\mathrm{V}(x) = x[\frac{1}{4}(100 - x)]^2 = \frac{1}{16}(x^3 - 200x^2 + 10000x)$, $20 \le x \le 100$ (length ≥ width). $\mathrm{V}'(x) = \frac{1}{16}(3x^2 - 400x + 10000) = \frac{1}{16}(3x - 100)(x - 100)$. $\mathrm{V}'(x) = 0$ when $x = \frac{100}{3} \in I$ and $x = 100 \in I$. $\mathrm{V}(20) = 8000$, $\mathrm{V}(\frac{100}{3}) = \frac{1}{32}(\frac{200}{3})^2 \approx 9259$, $\mathrm{V}(100) = 0$.

■ The largest box is about 33 by 17 by 17 in.

12. Ex. 26. The growth rate f bacteria/min of a colony is jointly proportional to the number x of bacteria and the number $1,000,000 - x$ of capacity. $f(x) = kx(1,000,000 - x) = k(1,000,000x - x^2)$, $0 \le x \le 1,000,000$. We do not need to determine k. $f'(x) = k(1,000,000 - 2x)$. $f'(x) = 0$ when $x = 500,000 \in \mathrm{I}$. $f(0) = 0$, $f(500,000) = 500,000^2k$, $f(1,000,000) = 0$. The growth rate is greatest when there are 500,000 bacteria.

13. Ex. 27. The growth rate f infected/day of an epidemic is jointly proportional to the number x of infected and the number $5,000 - x$ of capacity. $f(x) = kx(5,000 - x) = k(5000x - x^2)$, $0 \le x \le 5,000$. We do not need to determine k. $f'(x) = k(5000 - 2x)$. $f'(x) = 0$ when $x = 2500 \in \mathrm{I}$. $f(0) = 0$, $f(2500) = 2500^2k$, $f(5000) = 0$.

■ The growth rate is greatest when 2,500 are infected.

14. Ex. 28. The base of a pyramidal tent is $2x$ m square and a triangular side has height $2.5 - x$ m. Because the slant height of the tent is $2.5 - x$, the height h satisfies $h^2 + x^2 = (2.5 - x)^2$, $h = \sqrt{6.25 - 5x}$. The volume is $\mathrm{V}(x)$ m^3, where $\mathrm{V}(x) = \frac{1}{3}(2x)^2\sqrt{6.25 - 5x} = \frac{4}{3}x^2(6.25 - 5x)^{1/2}$, $0 \le x \le 1.25$. We maximize $f(x) = \mathrm{V}(x)^2 = \frac{16}{9}x^4(6.25 - 5x) = \frac{80}{9}x^4(1.25 - x)$. $f'(x) = \frac{80}{9}[4x^3(1.25 - x) + x^4(-1)] = \frac{80}{9}x^3(5 - 5x)$. $f'(x) = 0$ when $x = 0 \in \mathrm{I}$, $x = 1 \in \mathrm{I}$. $f(0) = 0$, $f(1) = \frac{20}{9}$, $f(1.25) = 0$. The volume is maximum when the base is 2m square.

15. In Exercise 2.2.37, a trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.05 per student for each student over 150. If $f(x)$ dollars is the gross income if x students make the trip, then $f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 22.5x - .05x^2 & \text{if } 150 < x \le 250 \end{cases}$. $f'(x) = \begin{cases} 15 & \text{if } 0 < x < 150 \\ 22.5 - .1x & \text{if } 150 < x < 250 \end{cases}$.

$f'(150)$ might not exist and $f'(225) = 0$. $f(0) = 0$, $f(150) = 2250$, $f(225) = 2531.25$, $f(250) = 2500$

■ The gross income is greatest when there are 225 students.

16. In Exercise 2.2.38, a trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.07 per student for each student over 150. How many students yield the largest income?

▷ We showed that if $f(x)$ dollars is the gross income if x students make the trip, then

$$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 25.5x - .07x^2 & \text{if } 150 < x \le 250 \end{cases}. \quad f'(x) = \begin{cases} 15 & \text{if } 0 < x < 150 \\ 25.5 - .14x & \text{if } 150 < x < 250 \end{cases}$$

$f'(150)$ might not exist and $f'(182.1) = 0$. Because the number of students is an integer, we test 182 and 183.

$$f(0) = 0,\ f(150) = 2250,\ f(182) = 2322.32,\ f(183) = 2322.27\ f(250) = 2000$$

■ The gross income is greatest when there are 182 students.

17. In Exercise 2.2.39, each tree produces 600 oranges, at up to 20 trees per acre. For each additional tree per acre, yield per tree decreases by 15 oranges. If $f(x)$ oranges are produced when there are number x of trees per acre, then $f(x) = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ 900x - 15x^2 & \text{if } 20 < x \le 60 \end{cases}$. $f'(x) = \begin{cases} 600 & \text{if } 0 < x < 20 \\ 900 - 30x & \text{if } 20 < x < 60 \end{cases}$

$f'(20)$ might not exist and $f'(30) = 0$. $f(0) = 0$, $f(20) = 12,000$, $f(30) = 13,500$, $f(60) = 0$

■ The number of oranges is greatest when 30 trees per acre are planted.

18. In Exercise 2.2.40, a club's annual dues is \$100 per member, less \$0.50 for each member of 600 and plus \$0.50 for each member less than 600. If $f(x)$ dollars is the clubs revenue when there are x members, then $f(x) = 400x - \frac{1}{2}x^2$, $0 \le x \le 800$. $f'(x) = 400 - x = 0$ when $x = 400$. $f(0) = 0$, $f(400) = 80000$, $f(800) = 0$

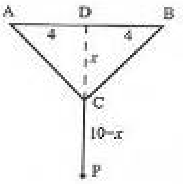
■ The revenue is greatest when there are 400 members.

19. Find two nonnegative numbers whose sum is 12 such that (a) their product is an absolute maximum; (b) the sum of their squares is an absolute minimum.

▷ If x is the smaller number then $12 - x$ is the larger, $0 \le x \le 6$. (a) $f(x) = x(12 - x) = 12x - x^2$, $f'(x) = 12 - 2x = 0$ when $x = 6 \in \mathrm{I}$. $f(0) = 0$, $f(6) = 36$. The numbers are 6 and 6. (b) $g(x) = x^2 + (12 - x)^2$, $g'(x) = 2x - 2(12 - x) = 4x - 24 = 0$ when $x = 6$. $g(0) = 144$, $g(6) = 72$. The numbers are 6 and 6.

Suppose a weight is to be held 10 ft below a horizontal line AB by a wire in the shape of a Y. If the points A and B are 8 ft apart, what is the shortest total length of wire that can be used?

» See the figure at the right. The weight is at point P and the line PD is perpendicular to AB. We are given that $|\overline{AB}| = 8$ and $|\overline{PD}| = 10$. Because triangle ABC is isosceles, then $|\overline{AD}| = |\overline{DB}| = 4$. Let



$x =$ the number of feet in $|\overline{CD}|$
$z =$ the number of feet in the total length of the wire

We want to find the absolute minimum value of z. We first express z as a function of x. We note that

$$z = |\overline{AC}| + |\overline{BC}| + |\overline{PC}| \tag{1}$$

Because triangles ADC and BDC are right triangles, we have $|\overline{AC}| = \sqrt{x^2+16}$ and $|\overline{BC}| = \sqrt{x^2+16}$. And because $|\overline{PC}| = 10 - x$, from Eq. (1) we have

$$z(x) = 2\sqrt{x^2+16} + 10 - x$$

Because $|\overline{PC}| \geq 0$, $z(x)$ is defined and continuous on the closed interval $[0, 10]$.

$$z'(x) = \frac{2x}{\sqrt{x^2+16}} - 1$$

We note that $z'(x)$ is defined for all x. If $z'(x) = 0$, then

$$\frac{2x}{\sqrt{x^2+16}} = 1; \qquad 4x^2 = x^2 + 16; \qquad x^2 = \tfrac{16}{3}; \qquad x = \tfrac{4}{3}\sqrt{3}$$

We evaluate $z(x)$ at the critical number $\frac{4}{3}\sqrt{3}$ and at the endpoints 0 and 10.

$$z(0) = 2(4) + 10 = 18, \ \ z(\tfrac{4}{3}\sqrt{3}) = 2\sqrt{\tfrac{64}{3}} + 10 - \tfrac{4}{3}\sqrt{3} = 10 + 4\sqrt{3} \approx 16.9, \ \ z(10) = 2\sqrt{116} \approx 21.5$$

■ The shortest total length of wire that can be used is $(10 + 4\sqrt{3})$ ft.

A woman on an island at A, 4 km from the nearest point B on a beach, wishes to go in the least time to C, 6 km down the beach from B. She rows at 5 km/hr to point P between B and C, then walks at 8 km/hr to C.

» Let x km be the distance from B to P. Then $\sqrt{x^2+16}$ km is the distance from P to A and $(6 - x)$ km is the distance from P to C. The time of the trip equals the time of the part by water plus the time of the part by land. If the trip takes $f(x)$ hours, then

$$f(x) = \tfrac{1}{5}\sqrt{x^2+16} + \tfrac{1}{8}(6 - x), \ 0 \leq x \leq 6; \ f'(x) = \frac{x}{5\sqrt{x^2+16}} - \frac{1}{8}$$

f is continuous on $[0, 6]$. Therefore f has an absolute minimum value on $[0, 6]$.

Set $f'(x) = 0$: $\dfrac{x}{5\sqrt{x^2+16}} - \dfrac{1}{8} = 0$; $8x = 5\sqrt{x^2+16}$; $64x^2 = 25x^2 + 400$; $39x^2 = 400$; $x^2 = \dfrac{400}{39}$; $x = \pm\dfrac{20}{\sqrt{39}}$

Because $f'(x)$ exists everywhere, $20/\sqrt{39} \approx 3.20$ is the only critical number of f in $[0, 6]$.

$$f(0) = \tfrac{31}{20} = 1.55, \ f(20/\sqrt{39}) = \tfrac{1}{10}\sqrt{39} + \tfrac{3}{4} \approx 1.37, \ f(6) = \tfrac{2}{5}\sqrt{13} \approx 1.44$$

The absolute minimum value of f is 1.37 when $x = 20/\sqrt{39}$. For the fastest route, P is 3.2 km from B.

A woman on an island at A, 4 km from the nearest point B on a beach, wishes to go in the least time to C, 3 km down the beach from B. She rows at 5 km/hr to point P between B and C, then walks at 8 km/hr to C.

» Let x km be the distance from B to P. Then $\sqrt{x^2+16}$ km is the distance from P to A and $(3 - x)$ km is the distance from P to C. The time of the trip equals the time of the part by water plus the time of the part by land. If the trip takes $f(x)$ hours, then

$$f(x) = \tfrac{1}{5}\sqrt{x^2+16} + \tfrac{1}{8}(3 - x), \ 0 \leq x \leq 3; \ f'(x) = \frac{x}{5\sqrt{x^2+16}} - \frac{1}{8}$$

f is continuous on $[0, 3]$. Therefore f has an absolute minimum value on $[0, 3]$.

Set $f'(x) = 0$: $\dfrac{x}{5\sqrt{x^2+16}} - \dfrac{1}{8} = 0$; $8x = 5\sqrt{x^2+16}$; $64x^2 = 25x^2 + 400$; $39x^2 = 400$; $x^2 = \dfrac{400}{39}$; $x = \pm\dfrac{20}{\sqrt{39}}$

Because $f'(x)$ exists everywhere and $20/\sqrt{39} \approx 3.20$, there is no critical number of f in $[0, 3]$.

$$f(0) = \tfrac{47}{40} = 1.175, \ f(3) = 1$$

■ The absolute minimum value of f is 1 when $x = 3$. Hence, for the fastest route the point P should be C.

23. Find the dimensions of the right circular cylinder of greatest lateral surface area that can be inscribed in sphere with a radius of 6 in.

▹ When r in. is the radius of the cylinder, let h in. be its height and $A(r)$ in^2 its lateral surface area. See th figure for Ex. 24. By the Pythagorean Theorem $r^2 + (\frac{1}{2}h)^2 = 36$; $h = 2\sqrt{36 - r^2}$.

Hence $A(r) = 2\pi rh = 4\pi r\sqrt{36 - r^2} = 4\pi\sqrt{36r^2 - r^4}$, $0 \le r \le 6$

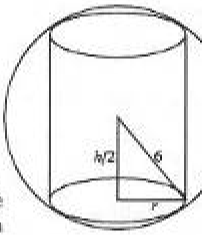
$$A'(r) = \frac{4\pi(72r - 4r^2)}{2\sqrt{36r^2 - r^4}} = \frac{8\pi(18 - r^2)}{\sqrt{36 - r^2}}$$

$A'(-3\sqrt{2}) = 0$ and $A'(3\sqrt{2}) = 0$, and $A'(x)$ exists on $(0,6)$, so $3\sqrt{2}$ is the critical number. $A(0) = 0$ $A(3\sqrt{2}) = 72\pi$, $A(6) = 0$. Hence A has an absolute maximum value when $r = 3\sqrt{2}$ and

$h = 2\sqrt{36 - r^2} = 2\sqrt{36 - 18} = 2\sqrt{18} = 6\sqrt{2}$.

▪ The right-circular cylinder of greatest lateral surface area has a radius of $3\sqrt{2}$ in. and a height of $6\sqrt{2}$ in.

24. Find the dimensions of the right circular cylinder of greatest volume that can be inscribed in a sphere with radius of 6 in.



▹ See the figure at the right. Let

r inches be the radius of the cylinder
h inches be the altitude of the cylinder
V cubic inches be the volume of the cylinder

We want to determine r and h so that V has an absolute maximum value.

The volume of a circular cylinder is given by

$$V = \pi r^2 h \qquad (1)$$

To express V in terms of one variable, we find another equation involving r and h. Because the center of the sphere bisects the altitude of the cylinder and the altitude of the cylinder is perpendicular to the base, we have from the Pythagorean theorem

$$(\tfrac{1}{2}h)^2 + r^2 = 6^2; \qquad r^2 = 36 - \tfrac{1}{4}h^2 \qquad (2)$$

Substituting from Eq. (2) into Eq. (1) we have

$$V(h) = \pi h(36 - \tfrac{1}{4}h^2) = \pi(36h - \tfrac{1}{4}h^3)$$

Because h cannot exceed the diameter of the sphere, V is defined on the closed interval $[0, 12]$. We have

$$V'(h) = \pi(36 - \tfrac{3}{4}h^2)$$

We note that $V'(h)$ is defined for all h. If $V'(h) = 0$, then

$$36 - \tfrac{3}{4}h^2 = 0; \qquad h^2 = 48; \qquad h = 4\sqrt{3}$$

We evaluate V at the critical number $4\sqrt{3}$ and at the endpoints 0 and 12.

$$V(0) = 0 \qquad V(4\sqrt{3}) = 4\sqrt{3}\pi \cdot 24 = 96\pi\sqrt{3} \qquad V(12) = 0$$

Therefore the greatest possible volume is $96\pi\sqrt{3}$ in^3. Furthermore, from Eq. (2), if $h^2 = 48$, then $r^2 = 24$ an so $r = 2\sqrt{6}$. Hence for the cylinder of greatest volume the altitude is $4\sqrt{3}$ in., and the radius is $2\sqrt{6}$ in.

25. The square of the distance from the point (x,y) on the circle $x^2 + y^2 = 9$ to $(4,5)$ is
$(x-4)^2 + (y-5)^2 = x^2 - 8x + 16 + y^2 - 10y + 25 = (x^2 + y^2 + 41) - 8x - 10y = 50 - 8x - 10y$
If $y \ge 0$, $y = \sqrt{9 - x^2}$. If $S_1(x)$ is the square of the distance from $(4,5)$ when $y \ge 0$, then
$S_1(x) = 50 - 8x - 10\sqrt{9 - x^2}$, $-3 \le x \le 3$; $S_1'(x) = -8 + \dfrac{10x}{\sqrt{9 - x^2}}$
If $y \le 0$, $y = -\sqrt{9 - x^2}$. If $S_2(x)$ is the square of the distance from $(4,5)$ when $y \le 0$, then
$S_2(x) = 50 - 8x + 10\sqrt{9} - x^2$, $-3 \le x \le 3$; $S_2'(x) = -8 - \dfrac{10x}{\sqrt{9 - x^2}}$
Both S_1 and S_2 are continuous on $[-3, 3]$. Therefore the extreme-value theorem applies.
Set $S_1'(x) = 0$: $\dfrac{10x}{\sqrt{9 - x^2}} = 8 \Rightarrow x > 0$; $5x = 4\sqrt{9 - x^2}$; $25x^2 = 16(9 - x^2)$; $41x^2 = 144$, $x = \dfrac{12}{\sqrt{41}}$.
Set $S_2'(x) = 0$: $\dfrac{10x}{\sqrt{9 - x^2}} = -8 \Rightarrow x < 0$; $5x = -4\sqrt{9 - x^2}$; $25x^2 = 16(9 - x^2)$; $41x^2 = 144$, $x = -\dfrac{12}{\sqrt{41}}$.

When $x = \pm 3$, $\sqrt{9-x^2} = 0$. Thus $S_1(-3) = S_2(-3) = 50 - 8(-3) = 74$, $S_1(3) = S_2(3) = 50 - 8(3) = 26$.

When $x = \pm\frac{12}{\sqrt{41}}$, $\sqrt{9-x^2} = \sqrt{9 - \frac{144}{41}} = \sqrt{\frac{225}{41}} = \frac{15}{\sqrt{41}}$. Thus

$S_1\left(\frac{12}{\sqrt{41}}\right) = 50 - 8\cdot\frac{12}{\sqrt{41}} - 10\cdot\frac{15}{\sqrt{41}} = 50 - \frac{246}{\sqrt{41}} = 50 - 6\sqrt{41}$ and

$S_2\left(-\frac{12}{\sqrt{41}}\right) = 50 - 8\left(-\frac{12}{\sqrt{41}}\right) + 10\cdot\frac{15}{\sqrt{41}} = 50 + \frac{246}{\sqrt{41}} = 50 + 6\sqrt{41}$.

Hence the absolute minimum distance is $\sqrt{50 - 6\sqrt{41}} = \sqrt{41 - 2\cdot 3\sqrt{41} + 9} = \sqrt{41} - 3$

and the absolute maximum distance is $\sqrt{50 + 6\sqrt{41}} = \sqrt{41} + 3$.

Alternatively, let $f(\theta)$ be the distance $|AP|$ when the measure of angle POA is θ. Then $|OA|^2 = 4^2 + 5^2 = 41$ and $|OP|^2 = 9$. By the law of cosines,

$|AP|^2 = |OP|^2 + |OA|^2 - 2|OP||OA|\cos\theta = 9 + 41 - 2\cdot 3\sqrt{41}\cos\theta = 50 - 6\sqrt{41}\cos\theta$

Hence $f(\theta) = \sqrt{50 - 6\sqrt{41}\cos\theta}$, $0 \le \theta \le \pi$. Because $-1 \le \cos\theta \le 1$,

(a) $f(\theta) \ge \sqrt{50 - 6\sqrt{41}(1)} = \sqrt{41} - 3 = a$ on $[0,\pi]$ and $f(0) = a$. a is the absolute minimum value.

(b) $f(\theta) \le \sqrt{50 - 6\sqrt{41}(-1)} = \sqrt{41} + 3 = b$ on $[0,\pi]$ and $f(\pi) = b$. b is the absolute minimum value.

[illegible] Find the area of the largest rectangle with two vertices on the x-axis and two on the parabola $y = 9 - x^2$.

► Because the parabola is symmetric with respect to the y-axis, if one vertex is at $(x,0)$, $0 \le x \le 3$, another is at $(-x,0)$. The two on the parabola are $(\pm x, 9-x^2)$. We wish to maximize the area $A(x) = 2x(9-x^2)$ $= 18x - 2x^3$; $A'(x) = 18 - 6x^2$. $A'(x) = 0$ when $6x^2 = 18$, $x^2 = 3$, $x = \sqrt{3} \in I$ and $x = -\sqrt{3} \notin I$.

$$A(0) = 0,\ A(\sqrt{3}) = 2\sqrt{3}(9-3) = 12\sqrt{3},\ A(3) = 0$$

■ The largest rectangle has area $12\sqrt{3} \approx 20.78$.

[illegible] $f(x) = \frac{1}{2}x^2(k-x) = \frac{1}{2}kx^2 - \frac{1}{2}x^3$, $0 \le x \le k$; $f'(x) = kx - \frac{3}{2}x^2 = x(k - \frac{3}{2}x)$

f is continuous on $[0,k]$ so f has an absolute maximum value on $[0,k]$.

$f'(0) = 0$, $f'(\frac{2}{3}k) = 0$ and $f'(x)$ exists on $[0,k]$. Hence $\frac{2}{3}k$ is the only critical number.

$$f(0) = 0,\ f(\tfrac{2}{3}k) = \tfrac{2}{27}k^3,\ f(k) = 0$$

Thus f has an absolute maximum value at $\frac{2}{3}k$.

■ The greatest decrease in blood pressure occurs when $\frac{2}{3}k$ mg of the drug is taken.

[illegible] If $V(r)$ cm/sec is the velocity of air during a cough when r cm is the radius of the trachea, then $V(r) = kr^2(R-r)$ where R is the normal radius, k is a positive constant, and r is in $[\frac{1}{2}R, R]$. Determine the radius of the trachea which maximizes the velocity.

► $V(r) = k(Rr^2 - r^3)$. $V'(r) = k(2Rr - 3r^2) = kr(2R - 3r)$

V is continuous on $[\frac{1}{2}R, R]$ so f has an absolute maximum value on $[\frac{1}{2}R, R]$.

$V'(0) = 0$, $f'(\frac{2}{3}R) = 0$ and $V'(x)$ exists on $[\frac{1}{2}R, R]$. Hence $\frac{2}{3}R$ is the only critical number.

$$V(\tfrac{1}{2}R) = k(R\cdot\tfrac{1}{4}R^2 - \tfrac{1}{8}R^3) = \tfrac{1}{8}kR^3,\ V(\tfrac{2}{3}R) = k(R\cdot\tfrac{4}{9}R^2 - \tfrac{8}{27}R^3) = \tfrac{4}{27}kR^3,\ V(R) = 0$$

Because $\frac{4}{27} > \frac{1}{8}$, V has an absolute maximum value at $\frac{2}{3}R$.

■ The greatest velocity occurs when the radius is $\frac{2}{3}$ the normal radius.

[illegible] Let $S(x)$ be the strength of the beam when its breadth is x cm. Because a diagonal of the beam is a diameter of the log, its depth is $\sqrt{144^2 - x^2}$ cm, and for some positive constant k,

$S(x) = kx(144^2 - x^2) = k(144^2x - x^3)$, $0 \le x \le 144$; $S'(x) = k(144^2 - 3x^2) = 3k(3\cdot 48^2 - x^2)$

S is continuous on $[0,144]$. Therefore S has an absolute maximum value on $[0,144]$.

$S'(x) = 0$ when $x = \pm 48\sqrt{3}$ and $S'(x)$ exists on $[0,144]$, so $48\sqrt{3}$ is the critical number.

$$S(0) = 0,\ S(48\sqrt{3}) = 663{,}552\sqrt{3}\ k,\ S(144) = 0$$

Thus S has an absolute maximum value when $x = 48\sqrt{3}$ and $\sqrt{144^2 - x^2} = 48\sqrt{6}$.

■ The dimensions of the strongest beam are $48\sqrt{3} \approx 83.14$ cm by $48\sqrt{6} \approx 117.58$ cm.

[illegible] Let $S(x)$ be the stiffness of the beam when its breadth is x cm. Because a diagonal of the beam is a diameter of the log, its depth is $\sqrt{4a^2 - x^2}$ cm, and for some positive constant k, $S(x) = kx(4a^2 - x^2)^{3/2}$, $0 \le x \le 2a$;

$S'(x) = k[(4a^2 - x^2)^{3/2} + \frac{3}{2}x(4a^2 - x^2)^{1/2}(-2x)] = k\sqrt{4a^2 - x^2}[(4a^2 - x^2) - x^2] = 4k\sqrt{4a^2 - x^2}(a^2 - x^2)$

S is continuous on $[0, 2a]$. Therefore S has an absolute maximum value on $[0, 2a]$.
$S'(x) = 0$ when $x = \pm a$ and $S'(x)$ exists on $[0, 2a]$, so a is the critical number.

$$S(0) = 0,\ S(a) = ka(3a^2)^{3/2} = 3\sqrt{3}ka^4,\ S(2a) = 0$$

Thus S has an absolute maximum value when $x = 2a$ and $\sqrt{4a^2 - x^2} = \sqrt{3}a$.

- The dimensions of the stiffest beam are $2a$ cm by $\sqrt{3}a$.

31. Let x ft be the length of the piece of wire bent into the shape of a circle of radius $x/2\pi$. Then $(10 - x)$ ft is the length of the piece of wire bent into the shape of a square of side length $\frac{1}{4}(10 - x)$ft. Let $A(x)$ ft^2 be the combined area of the two figures.

$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{10 - x}{4}\right)^2,\ 0 \le x \le 10;\ A'(x) = \frac{1}{2\pi}x - \frac{1}{8}(10 - x)$

A is continuous on $[0, 10]$ so A has an absolute maximum and an absolute minimum value.

Set $A'(x) = 0$: $\frac{1}{2\pi}x - \frac{5}{4} + \frac{1}{8}x = 0;\ 4x + \pi x = 10\pi;\ x = \frac{10\pi}{\pi + 4}$

$A'\left(\frac{10\pi}{\pi + 4}\right) = 0$ and $A'(x)$ exists on $[0, 10]$ so the only critical number of A is $\frac{10\pi}{\pi + 4}$.

$A(0) = \frac{25}{4} = 6.25,\ A\left(\frac{10\pi}{\pi + 4}\right) = \frac{25}{\pi + 4} \approx 3.5,\ A(10) = \frac{25}{\pi} \approx 8.0$

Thus A has an absolute minimum value when $x = \frac{10\pi}{\pi + 4}$ and an absolute maximum when $x = 10$.

(a) The combined area is smallest when $x = \frac{10\pi}{\pi + 4}$; that is, when the radius of the circle is $\frac{5}{\pi + 4}$ ft and the length of a side of the square is $\frac{10}{\pi + 4}$ ft.

(b) The combined area is largest when $x = 10$; that is, when the radius of the circle is $\frac{5}{\pi}$ ft and there is no square.

32. A piece of wire 10 ft long is cut into two pieces. One piece is bent into the shape of an equilateral triangle and the other piece is bent into the shape of a square. How should the wire be cut so that (a) the combined area of the two figures is as small as possible; (b) the combined area of the two figures is as large as possible?

▹ If each side of the equilateral triangle is x ft long, then $(10 - 3x)$ ft is left for the square. Hence each side of the square has length $\frac{1}{4}(10 - 3x)$ ft. The number of square feet in the area of the equilateral triangle is $\frac{1}{4}\sqrt{3}\,x^2$ and the number of square feet in the area of the square is $[\frac{1}{4}(10 - 3x)]^2$. Hence, if A ft^2 is the combined area of the triangle and the square, then

$$A(x) = \tfrac{1}{4}\sqrt{3}\,x^2 + \tfrac{1}{16}(10 - 3x)^2$$

Because $x \ge 0$ and $10 - 3x \ge 0$, we find the absolute maximum and minimum values of A on the closed interval $[0, \frac{10}{3}]$. Differentiating, we have

$$A'(x) = \tfrac{1}{2}\sqrt{3}\,x + \tfrac{1}{8}(10 - 3x)(-3)$$

We note that $A'(x)$ is defined for all x. If $A'(x) = 0$, we have

$$\begin{aligned} 0 &= \tfrac{1}{2}\sqrt{3}\,x - \tfrac{1}{8}(10 - 3x)(-3) \\ 0 &= 4\sqrt{3}\,x + (10 - 3x)(-3) \\ 0 &= 4\sqrt{3}\,x - 30 + 9x \\ 30 &= (4\sqrt{3} + 9)x \\ x &= \frac{30}{9 + 4\sqrt{3}} \cdot \frac{9 - 4\sqrt{3}}{9 - 4\sqrt{3}} = \frac{30(9 - 4\sqrt{3})}{81 - 48} = \tfrac{10}{11}(9 - 4\sqrt{3}) \approx 1.88 \end{aligned}$$

We evaluate A at the critical number 1.88 and at the endpoints 0 and $\frac{10}{3}$.

$A(0) = \frac{25}{4} = 6.25 \qquad A(1.88) \approx 2.72 \qquad A(\frac{10}{3}) = \frac{25}{9}\sqrt{3} \approx 4.8$

The absolute minimum and maximum of A are 2.72 and 6.25 respectively. Thus the combined area of the two figures is as small as possible if the side of the triangle is exactly $\frac{10}{11}(9 - 4\sqrt{3})$ ft long. Hence, the wire should be cut so that the piece that is bent into an equilateral triangle is $\frac{30}{11}(9 - 4\sqrt{3})$ ft ≈ 5.65 ft long. And to make the combined area as large as possible, all of the wire should be used for the square.

$R(\theta) = \dfrac{v_0{}^2 \sin 2\theta}{g}$, $0 \le \theta \le \frac{1}{2}\pi$, $0 \le 2\theta \le \pi$. The absolute maximum value of the sine function on $[0, \pi]$ is 1 when the independent variable is $\frac{1}{2}\pi$. Thus the absolute maximum value of $R(\theta)$ is $\dfrac{v_0{}^2}{g}$ when $\theta = \frac{1}{4}\pi$.

If a body of weight W pounds is dragged along a horizontal floor at a constant velocity by means of a force of magnitude F pounds and directed at an angle of θ radians with the plane of the floor, then F is given by the equation $F = \dfrac{kW}{k \sin \theta + \cos \theta}$ where k is a constant called the coefficient of friction and $0 < k < 1$. If $0 \le \theta \le \frac{1}{2}\pi$, find $\cos \theta$ when F is least.

» Because $0 \le \theta \le \frac{1}{2}\pi$, both $\sin \theta$ and $\cos \theta$ are positive. Furthermore, k and W are positive. Therefore F is least when $g(\theta) = k \sin \theta + \cos \theta$ is greatest.

$$g'(\theta) = k \cos \theta - \sin \theta$$

Note that $g'(\theta)$ is defined for all θ. If $g'(\theta) = 0$, then

$$k \cos \theta = \sin \theta; \quad k^2 \cos^2 \theta = \sin^2 \theta; \quad k^2 \cos^2 \theta = 1 - \cos^2 \theta; \qquad (k^2 + 1)\cos^2 \theta = 1$$

$$\cos^2 \theta = \frac{1}{k^2 + 1}; \qquad \cos \theta = \frac{1}{\sqrt{k^2 + 1}}$$

The only critical number of F occurs when $\cos \theta = (k^2 + 1)^{-1/2}$. We evaluate $g(\theta)$ at the endpoints of the interval $[0, \frac{1}{2}\pi]$ and at the critical number.

$$g(0) = k \sin 0 + \cos 0 = 1$$

$$g(\tfrac{1}{2}\pi) = k \sin \tfrac{1}{2}\pi + \cos \tfrac{1}{2}\pi = k$$

Because $\sin \theta = k \cos \theta$ at the critical number, then

$$g(\theta) = k \sin \theta + \cos \theta = k(k \cos \theta) + \cos \theta = (k^2 + 1)\cos \theta = \frac{k^2 + 1}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1}$$

Because $\sqrt{k^2 + 1}$ is greater than both k and 1, we conclude that the maximum value of g and the minimum value of F occur when $\cos \theta = (k^2 + 1)^{-1/2}$.

C dollars is the cost when x machines are used to produce A and y machines are used to produce B, where $C = 3x^2 + 42y$. If 15 machines are working, how many should be used to produce each product for the cost to be least?

» Because $y = 15 - x$, $C(x) = 3x^2 + 42(15 - x)$, $0 \le x \le 15$. $C'(x) = 6x - 42 = 0$ when $x = 7 \in I$. $C(0) = 630$, $C(7) = 483$, $C(15) = 675$. The cost is least when 7 machines produce A and 8 produce B.

(a) In Example 1, $C(x) = 12{,}500\sqrt{9 + x^2} + 10{,}000(k - x)$, where $k \ge 0$ and $C'(x) = 0$ when $x = \pm 4$. For what values of k will the absolute minimum value of C occur at a number in the open interval $(0, k)$?
(b) More generally, Let $f(x) = u\sqrt{a^2 + x^2} + v(b - x)$, $x \in [0, b]$, $u > v > 0$. If the absolute minimum value of f occur at a number in the open interval $(0, b)$, show that $av < b\sqrt{u^2 - v^2}$.

» (a) If the absolute minimum value of C occurs in the interval $(0, k)$ then the critical number 4 must be in that interval, that is $k > 4$.
(b) To locate the critical number, we find the derivative, set it to zero and solve for x.

$f'(x) = \dfrac{ax}{\sqrt{a^2 + x^2}} - v = 0$ when $\dfrac{ux}{\sqrt{a^2 + x^2}} = v$; $ux = v\sqrt{a^2 + x^2}$; $u^2x^2 = v^2(a^2 + x^2)$; $u^2x^2 = a^2v^2 + v^2x^2$;
$u^2x^2 - v^2x^2 = a^2v^2$; $(u^2 - v^2)x^2 = a^2v^2$; $x^2 = \dfrac{a^2v^2}{u^2 - v^2}$; $x = \pm\dfrac{av}{\sqrt{u^2 - v^2}}$

If the absolute minimum value of f occurs in the interval $I = (0, b)$ then the critical number must be in I, that is, the positive root must be less than b.

$\dfrac{av}{\sqrt{u^2 - v^2}} < b$; $av < \sqrt{u^2 - v^2}$

3.3 ROLLE'S THEOREM AND THE MEAN-VALUE THEOREM

3.3.1 Rolle's Theorem Let f be a function such that

(i) it is continuous on the closed interval $[a, b]$
(ii) it is differentiable on the open interval (a, b)
(iii) $f(a) = 0$ and $f(b) = 0$

Then there is a number c in the open interval (a, b) such that $f'(c) = 0$.

3.3.2 Mean-Value Theorem Let f be a function such that

(i) it is continuous on the closed interval $[a,b]$
(ii) it is differentiable on the open interval (a,b)

Then there is a number c in the open interval (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ or equivalently, } f(b)-f(a) = (b-a)f'(c)$$

The conclusion of Rolle's theorem states that on the graph of f, between the points where $x = a$ and $x = b$, there is a point at which the line tangent to the curve is horizontal. The conclusion of the mean-value theorem states that on the graph of f, between the points where $x = a$ and $x = b$, there is a point at which the line tangent to the curve is parallel to the line joining these points. Note that Rolle's theorem is actually a special case of the mean-value theorem. The mean-value theorem is among the most important theorems of the calculus because it is used in the proof of many theorems. In particular:

3.3.3 Theorem If f is a function such that $f'(x) = 0$ for all values of x in an interval I, then f is constant on I.

Also, taking the limit as b approaches a, and so c approaches a, we have

3.3.4 Theorem If f is continuous at a and $\lim\limits_{x\to a} f'(x)$ exists, then $f'(a)$ exists and $\lim\limits_{x\to a} f'(x) = f'(a)$.

Theorem 3.3.4 also applies to one-sided limits as stated in Section Summary 2.2.

Exercises 3.3

In Exercises 1–4, verify that conditions (i), (ii), and (iii) of the hypothesis of Rolle's theorem are satisfied by the function on the indicated interval. Then find a suitable value for c that satisfies the conclusion of Rolle's theorem. Support your choice of c by plotting the graph of f and the horizontal tangent line at $(c, f(c))$.

1. $f(x) = x^2 - 4x + 3$, $[1,3]$; $f'(x) = 2x - 4$
(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[1,3]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(1,3)$.
(iii) $f(1) = 1 - 4 + 3 = 0$, and $f(3) = 9 - 12 + 3 = 0$.
By Rolle's theorem there exists a number c in $(1,3)$ such that $f'(c) = 0$.
$f'(c) = 2c - 4$. Set $f'(c) = 0$: $2c - 4 = 0$; $c = 2$. Because $2 \in (1,3)$, 2 qualifies as c.

2. $f(x) = x^3 - 2x^2 - x + 2$, $[1,2]$; $f'(x) = 3x^2 - 4x - 1$
(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[1,2]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(1,2)$.
(iii) $f(1) = 1 - 2 - 1 + 2 = 0$, and $f(2) = 8 - 8 - 2 + 2 = 0$
By Rolle's theorem there exists a number c in $(1,3)$ such that $f'(c) = 0$.
Set $f'(c) = 0$: $3c^2 - 4c - 1 = 0$; $c = c_1 = \frac{1}{3}(4 - \sqrt{7}) \approx 0.45$ and $c = c_2 = \frac{1}{3}(4 + \sqrt{7}) \approx 2.21$.
Because $c_1 \in (1,2)$, c_1 qualifies as c.

3. $f(x) = \sin 2x$, $[0, \frac{1}{2}\pi]$; $f'(x) = 2\cos 2x$
(i) Because the sine function is continuous everywhere, then f is continuous everywhere. Therefore f is continuous on $[0, \frac{1}{2}\pi]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(0, \frac{1}{2}\pi)$.
(iii) $f(0) = \sin 0 = 0$, and $f(\frac{1}{2}\pi) = \sin \pi = 0$
By Rolle's theorem there exists a number c in $(0, \frac{1}{2}\pi)$ such that $f'(c) = 0$.
$f'(c) = 2\cos 2c$. Set $f'(c) = 0$: $2\cos 2c = 0$; $2c = \frac{1}{2}\pi$; $c = \frac{1}{4}\pi$.
Because $\frac{1}{4}\pi \in (0, \frac{1}{2}\pi)$, $\frac{1}{4}\pi$ qualifies as c.

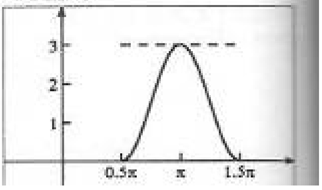
4. $f(x) = 3\cos^2 x$; $[\frac{1}{2}\pi, \frac{3}{2}\pi]$

▹ Because the cosine function is continuous for all x, then f is continuous on the closed interval $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ and condition (i) of the hypothesis of Rolle's theorem is satisfied. Differentiating f we have

$$f'(x) = -6\cos x \sin x$$

Because both the sine and cosine function are defined everywhere, then $f'(x)$ is defined in the open interval $(\frac{1}{2}\pi, \frac{3}{2}\pi)$ and condition (ii) of the hypothesis is satisfied. Furthermore,

$$f(\tfrac{1}{2}\pi) = 3\cos^2 \tfrac{1}{2}\pi = 3(0^2) \qquad f(\tfrac{3}{2}\pi) = 3\cos^2 \tfrac{3}{2}\pi = 3(0^2)$$



and so condition (iii) is satisfied. To find a suitable value for c,
we set $f'(c) = 0$ and get

$$0 = -6 \cos c \sin c$$

$$\cos c = 0 \text{ or } \sin c = 0$$

Because $\sin \pi = 0$ and π is in the open interval $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, we take $c = \pi$.

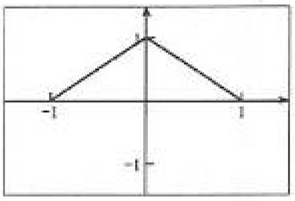
In Exercises 5–10, (a) Plot the graph of the function on the indicated interval; (b) test the three conditions (i), (ii), and (iii) of the hypothesis of Rolle's theorem, and determine which conditions are satisfied and which, if any, are not satisfied; and (c) if the three conditions in part (b) are satisfied, determine a point at which there is a horizontal tangent line.

5. $f(x) = x^{4/3} - 3x^{1/3}$, $[0, 3]$; $f'(x) = \frac{4}{3} x^{1/3} - x^{-2/3}$
   (b) (i) f is continuous everywhere. Thus f is continuous on $[0, 3]$.
   (ii) f is differentiable at every number but 0, so f is differentiable on $(0, 3)$.
   (iii) $f(0) = 0$ and $f(3) = 0$.
   (c) By Rolle's theorem there is a number c in $(0, 3)$ such that $f'(c) = 0$; that is $\frac{4}{3}c^{1/3} - c^{-2/3} = 0$; $4c - 3 = 0$; $c = \frac{3}{4}$
   Thus there is a horizontal tangent line at the point $(\frac{3}{4}, -\frac{9}{8}\sqrt[3]{6})$.

6. $f(x) = x^{3/4} - 2x^{1/4}$, $[0, 4]$; $f'(x) = \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4}$
   (b) (i) f is continuous everywhere. Thus f is continuous on $[0, 4]$.
   (ii) f is differentiable at every number but 0, so f is differentiable on $(0, 3)$.
   (iii) $f(0) = 0$ and $f(4) = 2\sqrt{2} - 2\sqrt{2} = 0$.
   (c) By Rolle's theorem there is a number c in $(0, 4)$ such that $f'(c) = 0$; that is
   $\frac{3}{4}c^{-1/4} - \frac{1}{2}c^{-3/4} = 0$; $3c^{1/2} - 2 = 0$; $c^{1/2} = \frac{2}{3}$; $c = \frac{4}{9}$
   Thus there is a horizontal tangent line at the point $(\frac{4}{9}, -\frac{4}{9}\sqrt{6})$

7. $f(x) = \dfrac{x^2 - x - 12}{x - 3}$, $[-3, 4]$
   (b) Because $f(3)$ does not exist, f is not continuous on $[-3, 4]$.
   Thus condition (i) of Rolle's theorem is not satisfied.

8. $f(x) = 1 - |x|$; $[-1, 1]$
   (a) A plot of the graph of f on $[-1, 1]$ is shown at the right.
   (b) Because f is continuous in $[-1, 1]$, condition (i) of the hypothesis of Rolle's theorem is satisfied. Because the absolute-value function is not differentiable at 0, f is not differentiable on $(-1, 1)$, and thus condition (ii) is not satisfied. Because $f(1) = 0$ and $f(-1) = 0$, condition (iii) is satisfied.
   (c) Because one of the conditions of the hypothesis of Rolle's theorem is not satisfied, we cannot conclude that there is a point at which there is a horizontal tangent. Indeed, as the figure illustrates, there is no such point.



9. $f(x) = \begin{cases} x^2 - 4 & \text{if } x < 1 \\ 5x - 8 & \text{if } 1 \le x \end{cases}$, $[-2, \frac{4}{5}]$; $f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ 5 & \text{if } 1 < x \end{cases}$
   (b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 - 4) = -3$; $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (5x - 8) = -3$
   Because $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$, then $\lim\limits_{x \to 1} f(x) = -3 = f(1)$.
   Thus f is continuous at 1 and so f is continuous on $[-2, \frac{4}{5}]$.

$$f'_-(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 4 - (-3)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$$

$$f'_+(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{5x - 8 - (-3)}{x - 1} = \lim_{x \to 1^+} \frac{5x - 5}{x - 1} = \lim_{x \to 1^+} \frac{5(x-1)}{x - 1} = \lim_{x \to 1^+} 5 = 5$$

Alternative computation for $f'_-(1)$ and $f'_+(1)$:

$f'_-(1) = \lim\limits_{x \to 1^-} f'(x) = \lim\limits_{x \to 1^-} 2x = 2$; $f'_+(1) = \lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^+} 5 = 5$

Because $f'_-(1) \neq f'_+(1)$, then $f'(1)$ does not exist. Therefore f is not differentiable on$(-2, \frac{8}{5})$. Thus condition (ii) of Rolle's theorem is not satisfied. However, $f'(0) = 0$.

10. $f(x) = \begin{cases} 3x+6 & \text{if } x < 1 \\ x-4 & \text{if } x \geq 1 \end{cases}; [-2, 4]$

(b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (3x+6) = 9$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x-4) = -3$

Because f is not continuous at 1, then f is not continuous on $[-2, 4]$.

Thus condition (i) of Rolle's theorem is not satisfied.

In Exercises 11–20, verify that the hypothesis of the mean-value theorem is satisfied for the function on the indicated interval. Then find a suitable value for c that satisfies the conclusion of the mean-value theorem. Check by plotting the graph of f on $[a, b]$, the tangent line at $(c, f(c))$, and the secant line through $(a, f(a))$ and $(b, f(b))$ and noting that the lines are parallel.

▹ On your graphics calculator, use TanLn(expression,value) and Line(xbeg,ybeg,xend,yend), respectively.

11. $f(x) = x^2 + 2x - 1$, $[0, 1]$; $f'(x) = 2x + 2$

(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[0, 1]$.

(ii) f is differentiable everywhere. Hence f is differentiable on $(0, 1)$.

By the mean-value theorem, there exists a number c in $(0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0};\ 2c + 2 = \frac{2 - (-1)}{1};\ 2c + 2 = 3;\ c = \tfrac{1}{2}$$

Because $\frac{1}{2} \in (0, 1)$, $\frac{1}{2}$ qualifies as c.

12. $f(x) = x^3 + x^2 - x$, $[-2, 1]$

▹ $f'(x) = 3x^2 + 2x - 1$

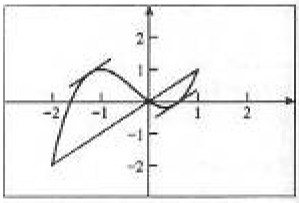
(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[-2, 1]$.

(ii) f is differentiable everywhere. Hence f is differentiable on $(-2, 1)$.

By the mean-value theorem, there exists a number c in $(0, 1)$ such that

$f'(c) = \dfrac{f(1) - f(-2)}{1 - (-2)};\ 3c^2 + 2c - 1 = \dfrac{1 - (-2)}{3} = 1;\ 3c^2 + 2c - 2 = 0;$

$c = c_+ = \frac{1}{3}(-1 + \sqrt{7}) \approx 0.55$ and $c = c_- = \frac{1}{3}(-1 - \sqrt{7}) \approx -1.22$



Because c_+ and $c_- \in (-2, 1)$, both c_+ and c_- qualify as c.

13. $f(x) = x^{2/3}$, $[0, 1]$; $f'(x) = \frac{2}{3}x^{-1/3}$

(i) f is continuous everywhere, and so continuous on $[0, 1]$.

(ii) f is differentiable everywhere except at 0. Thus f is differentiable on $(0, 1)$.

By the mean-value theorem, there exists a number c in $(0, 1)$ such that

$f'(c) = \dfrac{f(1) - f(0)}{1 - 0};\ \dfrac{2}{3c^{1/3}} = \dfrac{1 - 0}{1};\ \frac{2}{3} = c^{1/3};\ c = \frac{8}{27}$. Because $\frac{8}{27} \in (0, 1)$, $\frac{8}{27}$ qualifies as c.

14. $f(x) = \dfrac{x^2 + 4x}{x - 7}$, $[2, 6]$. $f'(x) = \dfrac{(2x + 4)(x - 7) - (x^2 + 4x)}{(x - 7)^2} = \dfrac{x^2 - 14x - 28}{(x - 7)^2}$

(i) f is continuous at every number but 7, and so continuous on $[2, 6]$.

(ii) f is differentiable everywhere except at 7. Thus f is differentiable on $(2, 6)$.

By the mean-value theorem, there exists a number c in $(2, 6)$ such that

$f'(c) = \dfrac{f(6) - f(2)}{6 - 2};\ \dfrac{c^2 - 14c - 28}{(c - 7)^2} = \dfrac{-60 + 12/5}{4};\ 1 - \dfrac{77}{(c - 7)^2} = -\dfrac{72}{5};\ -\dfrac{77}{(c - 7)^2} = -\dfrac{77}{5};\ (c - 7)^2 = 5$

$c = 7 + \sqrt{5}$ and $c = 7 - \sqrt{5}$. Because $7 - \sqrt{5} \in (2, 6)$, $7 - \sqrt{5}$ qualifies as c.

15. $f(x) = \sqrt{1 + \cos x}$, $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$; $f'(x) = \dfrac{-\sin x}{2\sqrt{1 + \cos x}}$

(i) $1 + \cos x \geq 0$ on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Thus f is continuous on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

(ii) $f'(x)$ exists for all x in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Thus f is differentiable on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

By the mean-value theorem, there exists a number c in $(-\frac{1}{2}\pi, \frac{1}{2}x)$ such that

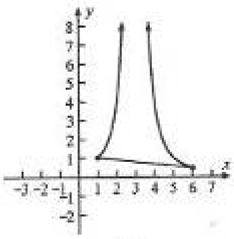
$$f'(c) = \frac{f(\frac{1}{2}\pi) - f(-\frac{1}{2}\pi)}{\frac{1}{2}\pi - (-\frac{1}{2}\pi)};\ \frac{-\sin c}{2\sqrt{1 + \cos c}} = \frac{1 - 1}{\pi};\ \frac{-\sin c}{2\sqrt{1 + \cos c}} = 0;\ \sin c = 0;\ c = 0$$

Because $0 \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, 0 qualifies as c.

In Exercises 21–24, for the given function, there is no number c in the open interval (a, b) that satisfies th conclusion of the mean-value theorem. Determine which part of the hypothesis of the mean-value theorem fails t hold. Sketch the graph of f and the line through the points $(a, f(a))$ and $(b, f(b))$.

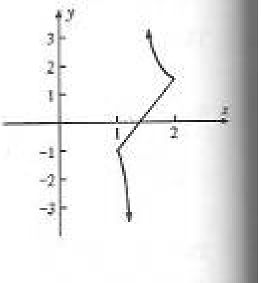
21. $f(x) = \dfrac{4}{(x-3)^2}$, $a = 1$, $b = 6$

▷ f is not defined at 3.
Thus f is discontinuous on $[1, 6]$.



22. $f(x) = \dfrac{2x-1}{3x-4}$; $a = 1$, $b = 2$.

▷ f is not defined at $\frac{4}{3}$.
Thus f is discontinuous on $[1, 2]$.

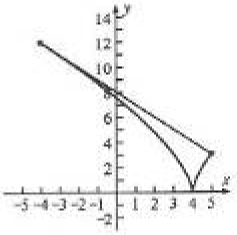


23. $f(x) = 3(x-4)^{2/3}$, $a = -4$, $b = 5$
$f'(x) = 2(x-4)^{-1/3}$

$$f'_-(4) = \lim_{x \to 4^-} \frac{3(x-4)^{2/3} - 0}{x-4} = \lim_{x \to 4^-} \frac{3}{(x-4)^{1/3}} = -\infty$$

$$f'_+(4) = \lim_{x \to 4^+} \frac{3(x-4)^{2/3}}{x-4} = \lim_{x \to 4^+} \frac{3}{(x-4)^{1/3}} = +\infty$$

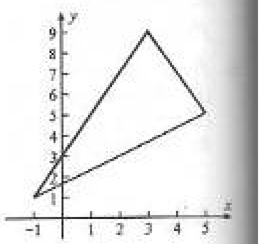
Therefore, f is not differentiable at 4;
so f is not differentiable on $(-4, 5)$.



24. $$f(x) = \begin{cases} 2x+3 & \text{if } x < 3 \\ 15-2x & \text{if } 3 \leq x \end{cases}; \; a = -1, \; b = 5$$

▷ $$f'(x) = \begin{cases} 2 & \text{if } x < 3 \\ -2 & \text{if } x > 3 \end{cases}$$

By Theorem 3.3.4, $f'_-(3) = 2$ and $f'_+(3) = -2$. Thus $f'(3)$ does not exist and so f is not differentiable on the open interval $(-1, 5)$. Hence, condition (ii) of the hypothesis of the mean-value theorem is not satisfied. The graph of f on $[-1, 5]$ is shown at the right. Note that there is no point on the graph of f at which the tangent line to the graph is parallel to L, the line through the endpoints of the graph.



25. $f(x) = x^4 - 2x^3 + 2x^2 - x$; $f'(x) = 4x^3 - 6x^2 + 4x - 1$
Because f is a polynomial function, f is continuous and differentiable everywhere; $f(0) = 0$ and $f(1) =$ Thus Rolle's theorem holds on $[0, 1]$. Hence, there is some number $c \in (0, 1)$ such that $f'(c) = 0$. Thus, there at least one real root of the equation $4x^3 - 6x^2 + 4x - 1 = 0$ in the open interval $(0, 1)$.

26. $f(x) = x^3 + 2x + k$; $f'(x) = 3x^2 + 2$
Because f is a polynomial function, f is continuous and differentiable everywhere. Suppose $f(a) = 0$ an $f(b) = 0$. Thus Rolle's theorem holds on $[a, b]$. Hence, there is some number $c \in (a, b)$ such that $f'(c) = 0$, tha is, $3c^2 + 2 = 0$. Because this is impossible, f cannot have two real roots.

27. $f(x) = 4x^5 + 3x^3 + 3x - 2$; $f'(x) = 20x^4 + 9x^2 + 3$
$f(0) = -2$ and $f(1) = 8$. Because f is continuous on $[0, 1]$ and 0 is between -2 and 8, $f(x_1) = 0$ for som number x_1 between 0 and 1. Hence x_1 is a root of the equation. Now suppose that the equation has anothe root; call this root r. Then $f(r) = 0$. If $r < x_1$, consider the interval $[r, x_1]$ and if $r > x_1$, consider the interva $[x_1, r]$. In either case $f(r) = f(x_1) = 0$ and Rolle's theorem holds. Therefore there is a number c between r an x_1 such that $f'(c) = 0$. But $f'(x)$ is always greater than or equal to 3. Hence our assumption leads to contradiction. Therefore, the equation cannot have another root.

Use the mean-value theorem to prove that if $x > 0$, then $\cos x > 1 - \frac{1}{2}x^2$.

» Let $f(x) = \cos x - \left(1 - \frac{1}{2}x\right)$. Then $f'(x) = x - \sin x$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[\cos x - (1 - \tfrac{1}{2}x^2)] - 0 = x(c - \sin c)$$

Because $c > 0$, then $\sin c < c$ and so $x(c - \sin c) > 0$. Therefore

$$\cos x - (1 - \tfrac{1}{2}x^2) > 0$$

$$\cos x > 1 - \tfrac{1}{2}x^2$$

Because $\cos x - (1 - \frac{1}{2}x)$ is an even function, the above inequality is also true if $x < 0$.

Use the mean-value theorem to prove that if $x > 0$, then $\sin x > x - \frac{1}{6}x^3$.

» Let $f(x) = \sin x - (x - \frac{1}{6}x^3)$. Then $f'(x) = \cos x - (1 - \frac{1}{2}x^2)$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[\sin x - (x - \tfrac{1}{6}x^3)] - 0 = x[\cos c - (1 - \tfrac{1}{2}c^2)]$$

Because $c > 0$, by Exercise 28 , $\cos c - (1 - \frac{1}{2}c^2) > 0$. Because $x > 0$,

$$\sin x - (x - \tfrac{1}{6}x^3) > 0$$

$$\sin x > x - \tfrac{1}{6}x^3$$

Because $\sin x - (x - \frac{1}{6}x^3)$ is an odd function, the above inequality is reversed if $x < 0$.

Use the mean-value theorem to prove Bernoulli's inequality: if $x > 0$ and $r > 1$, where r is rational, then $(1 + x)^r > 1 + rx$.

» Let $f(x) = (1 + x)^r - (1 + rx)$. Then $f'(x) = r(1 + x)^{r-1} - r = r[(1 + x)^{r-1} - 1]$. Because f is continuous and differentiable if $x > -1$, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[(1 + x)^r - (1 + rx)] - 0 = xr[(1 + c)^{r-1} - 1]$$

Because $c > 0$, then $1 + c > 1$; because $r > 1$ then $r - 1 > 0$. Therefore $(1 + c)^{r-1} > 1$. Hence

$$(1 + x)^r - (1 + rx) > 0$$

$$(1 + x)^r > 1 + rx$$

If $-1 < x < 0$, then $-1 < c < 0$, $0 < 1 + c < 1$ and $(1 + c)^{r-1} < 1$ so that $xr[(1 + c)^{r-1} - 1] > 0$ and the above inequality is still true. The inequality remains true if $r < 0$. The inequality is reversed if $0 < r < 1$.

Use the mean-value theorem to prove that if $a < b$, the arithmetic mean $\frac{1}{2}(a + b)$ is in the open interval (a, b).

» Let $f(x) = x^2$. Then $f'(x) = 2x$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[a, b]$. Hence there is a number $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

$$b^2 - a^2 = (b - a) \cdot 2c$$

$$\tfrac{1}{2}(b + a) = c$$

That is, $\frac{1}{2}(a + b)$ is in the open interval (a, b).

Use the mean-value theorem to prove that if $0 < a < b$, the geometric mean $\sqrt{ab}$ of a and b is in the open interval (a, b).

» Let $f(x) = 1/x$. Then $f'(x) = -1/x^2$. Because f is continuous and differentiable for $x \neq 0$, the mean-value theorem holds on $[a, b]$. Hence there is a number $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

$$\frac{1}{b} - \frac{1}{a} = (b - a) \cdot \frac{-1}{c^2}$$

$$\frac{a - b}{ab} = \frac{a - b}{c^2}$$

$$ab = c^2$$

$$\sqrt{ab} = c$$

That is, $\sqrt{ab}$ is in the open interval (a, b).

33. The mean-value theorem says that at some point of an interval the instantaneous velocity is equal to the average velocity. (See Exercise 39.) Because the average velocity is $35/\frac{1}{2} = 70$ mph, at some point the velocity was 70 mph.

34. $f(x) = \sin^2 x + \cos^2 x$; $f'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) \equiv 0$ in any interval I. By Theorem 3.3.3, f is constant in I. Because $f(0) = 1$, then $f(x) = 1$ in I.

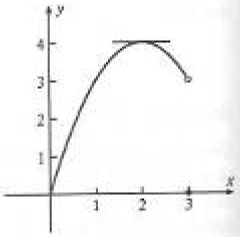
35. f is continuous on $[a,b]$ and $f'(x) = 1$ for all x in (a,b). Let x be in the interval $(a,b]$. Then the hypothesis of the mean-value theorem is satisfied by f on $[a,x]$. Hence, there is a number c in (a,x) such that $f(x) - f(a) = (x-a)f'(c)$. But $f'(c) = 1$ because $a < c < x$. Therefore, if x is in $(a,b]$, $f(x) - f(a) = x - a$; $f(x) = x - a + f(a)$. The last equation is also valid if $x = a$. Hence for all x in $[a,b]$, $f(x) = x - a + f(a)$.

36. The converse of Rolle's theorem is not true. Make up an example of a function and an interval for which the conclusion of Rolle's theorem is true and for which
(a) condition (i) is not satisfied but conditions (ii) and (iii) are satisfied;
(b) condition (ii) is not satisfied but conditions (i) and (iii) are satisfied;
(c) condition (iii) is not satisfied but conditions (i) and (ii) are satisfied.
Sketch the graph showing the horizontal tangent line for each case.

(a) We want to define f so that f is discontinuous on the closed interval $[a,b]$; f is differentiable on the open interval (a,b); $f(a) = f(b) = 0$; and so that there is some number c in the open interval (a,b) for which $f'(c) = 0$. If f is differentiable in the open interval (a,b), then f is continuous in the open interval (a,b). Thus f must be either discontinuous from the right at a or discontinuous from the left at b. Let f be defined by

$$f(x) = \begin{cases} 4x - x^2 & \text{if } 0 \le x < 3 \\ 0 & \text{if } x = 3 \end{cases}$$

The graph of f is shown in at the right. We note that



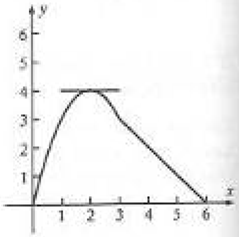
$$f'(x) = 4 - 2x \quad \text{if } 0 < x < 3$$

Thus, $f'(2) = 0$ and so the number $c = 2$ satisfies the conclusion of Rolle's theorem for the interval $[0,3]$. Furthermore, f is discontinuous from the left at $x = 3$; f is differentiable in the open interval $(0,3)$; $f(0) = 0$ and $f(3) = 0$.

(b) We want to define f so that f is continuous on the closed interval $[a,b]$; f is not differentiable on the open interval (a,b); $f(a) = f(b) = 0$; and there is a number c in (a,b) such that $f'(c) = 0$. f will not be differentiable at a point where its graph has a corner. Let

$$f(x) = \begin{cases} 4x - x^2 & \text{if } 0 \le x \le 3 \\ 6 - x & \text{if } 3 < x \le 6 \end{cases}$$

The graph of f is shown at the right. There is a horizontal tangent line at the point where $x = 2$; f is continuous on the closed interval $[0,6]$; $f(0) = 0$ and $f(6) = 0$. However, because



$$f'(x) = \begin{cases} 4 - 2x & \text{if } 0 < x < 3 \\ -1 & \text{if } 3 < x < 6 \end{cases}$$

then by Theorem 3.3.4

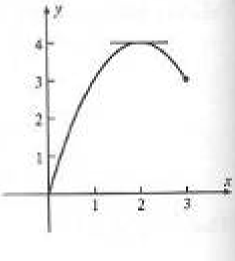
$$f'_-(3) = 4 - 2(3) = -2 \text{ and } f'_+(3) = -1$$

Thus, $f'(3)$ does not exist and f is not differentiable at $x = 3$.

(c) We want to define f so that f is continuous on the closed interval $[a,b]$; f is differentiable in the open interval (a,b); either $f(a) \ne 0$ or $f(b) \ne 0$; and so there is a number c in the open interval (a,b) such that $f'(c) = 0$. Let

$$f(x) = 4x - x^2;\ 0 \le x \le 3$$

The graph of f is shown at the right. Note that $f'(x) = 4 - 2x$ and thus there is a horizontal tangent line at the point where $x = 2$. Moreover, f is continuous on the closed interval $[0,3]$; f is differentiable in the open interval $(0,3)$; $f(0) = 0$, but $f(3) \ne 0$.



Use Rolle's theorem to prove that if every polynomial function of the second degree has at most two real roots, then every polynomial of the third degree has at most three real roots.

» Assume $f(x)$ is a third-degree polynomial with four real roots: x_1, x_2, x_3, x_4 and $x_1 < x_2 < x_3 < x_4$. Because the degree of f is 3, the degree of f' is 2. Because f is a polynomial, f is continuous and differentiable everywhere. Because $f(x_1) = f(x_2) = 0$, by Rolle's theorem there is a number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Because $f(x_2) = f(x_3) = 0$, there is a number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. Because $f(x_3) = f(x_4) = 0$, there is a number c_3 in (x_3, x_4) such that $f'(c_3) = 0$. We have numbers c_1, c_2, and c_3, $c_1 < c_2 < c_3$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$. Hence f' is a polynomial of degree 2 which has three real roots. This contradicts the hypothesis that a second-degree polynomial has at most two real roots. Hence our assumption is false. Therefore a third-degree polynomial has at most three roots.

Use mathematical induction to prove that a polynomial of the nth degree has at most n real real roots.

» A linear polynomial has 1 real root and a quadratic polynomial has at most 2 real roots. Suppose the result is true for a polynomial of degree k. Assume $f(x)$ is a polynomial of degree $k+1$ with $k+2$ real roots: x_1, $\ldots x_{k+2}$ and $x_1 < \ldots < x_{k+2}$. Because the degree of f is $k+1$, the degree of f' is k. Because f is a polynomial, f is continuous and differentiable everywhere. Let j be an integer from 1 to $k+1$. Because $f(x_j) = f(x_{j+1}) = 0$, by Rolle's theorem there is a number c_j in (x_j, x_{j+1}) such that $f'(c_j) = 0$. We have numbers $c_1, \ldots, c_{k+1}$, $c_1 < \ldots < c_{k+1}$ such that $f'(c_1) = \ldots = f'(c_{k+1}) = 0$. Hence f' is a polynomial of degree k which has $k+1$ real roots. This contradicts the hypothesis that a polynomial of degree k has at most k real roots. Hence our assumption is false. Therefore a polynomial degree $k+1$ has at most $k+1$ roots. Because we have shown the result is true when $n = 1$ and if it is true when $n = k$, then it is true for $n = k+1$, it follows from the principle of mathematical induction the the result is true for every positive integer n.

Suppose $s = f(t)$ is an equation of the motion of a particle moving in a straight line. Show that the conclusion of the mean-value theorem assures that during any time interval where f satisfies the hypothesis of the mean-value theorem there will be some instant when the instantaneous velocity will equal the average velocity during that time interval.

» Suppose f satisfies the hypothesis of the mean-value theorem on the time interval $[t_1, t_2]$. Therefore there is some c with $t_1 < c < t_2$ such that

$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

Now, $f'(c)$ is the instantaneous velocity at the instant when $t = c$ and $\Delta s / \Delta t$ is the average velocity of the particle during the time interval $[t_1, t_2]$. Therefore, there is some instant during the time interval when the instantaneous velocity equals the average velocity during that time interval.

3.4 INCREASING AND DECREASING FUNCTIONS AND THE FIRST-DERIVATIVE TEST

3.4.1 Definition A function f defined on an interval is said to be *increasing* on that interval if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, where x_1 and x_2 are any numbers in the interval.

3.4.2 Definition A function f defined on an interval is said to be *decreasing* on that interval if and only if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, where x_1 and x_2 are any numbers in the interval.

If a function is either increasing on an interval or decreasing on an interval then it is said to be *monotonic* on the interval.

3.4.3 Theorem Let the function f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b):

(i) if $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$
(ii) if $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$

Note that the hypothesis of Theorem 3.4.3 may be satisfied even when $f'(a) = 0$ and $f'(b) = 0$ or when f is not differentiable at a or b. By letting points where $f'(x)$ is zero or undefined be endpoints of subintervals, we have:

(i) if $f'(x) > 0$ at all but finitely many points of an interval I, then f is increasing on I
(ii) if $f'(x) < 0$ at all but finitely many points of an interval I, then f is decreasing on I

3.4.4 First-Derivative Test for Relative Extrema Let the function f be continuous at all points of the open interval (a, b) containing the numbe c, and suppose that f' exists at all points of (a, b) except possibly at c:

(i) if $f'(x) > 0$ for all values of x in some open interval having c as its right endpoint, an if $f'(x) < 0$ for all values of x in some open interval having c as its left endpoint, the f has a relative maximum value at c

(ii) if $f'(x) < 0$ for all values of x in some open interval having c as its right endpoint, an if $f'(x) > 0$ for all values of x in some open interval having c as its left endpoint, the f has a relative minimum value at c

The following steps make use of the first-derivative test to sketch the graph of a function f.

1. Find each number at which $f'(x)$ is either zero or undefined, often by factoring $f'($ Arrange the numbers in increasing order.
2. Use these numbers to partition the number line into open intervals.
3. For each interval (number) determine whether $f'(x)$ is positive or negative (zero o undefined). $f'(x)$ will be positive on an interval if the number of negative factors even, and negative if the number of negative factors is odd.
4. For each interval state whether f is increasing or decreasing.
5. For each number, state whether f has a maximum, a minimum or neither, an evaluate $f(x)$. Determine if the graph of f has a horizontal or vertical tangent line, corner, or a vertical asymptote.

In step 5 we may need to use Definition 3.1.1(ii) which states that $x = c$ is a vertical tange to the graph of f at $P = (c, f(c))$ if both $f'_-(c)$ and $f'_+(c)$ are infinite. If, in some ope interval containing c, the curve lies on one side of the normal line at P, then the graph is sa to have a *cusp* at P. Figures 3.4.24 and 3.4.28 illustrate vertical tangents; in the latter the is a cusp.

Intermediate-Value Theorem for Derivatives. If $f'(x)$ exists at each point of a closed in the closed interval $[a,$ and k is a number strictly between $f'(a)$ and $f'(b)$ then there exists a number c in (a, b) su that $f(c) = k$. We do *not* assume that f' is continuous. See Exercise 56.

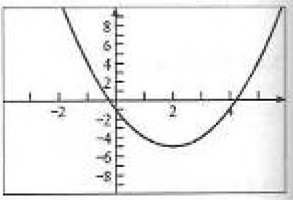
Exercises 3.4

In Exercises 1–18, (a) plot the graph, and determine (b) the relative extrema of f, (c) the values of x at which th relative extrema occur, (d) the intervals on which f is increasing, and (e) the intervals on which f is decreasin Confirm by calculus.

1. $f(x) = x^2 - 4x - 1$; $f'(x) = 2x - 4$

Set $f'(x) = 0$ and obtain the critical number 2.

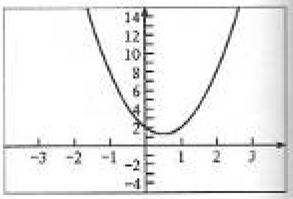
	$f(x)$	$f'(x)$	Conclusion
$x < 2$		$-$	f is decreasing on $(-\infty, 2]$
$x = 2$	-5	0	f has a relative minimum value
$2 < x$		+	f is increasing on $[2, +\infty)$



2. $f(x) = 3x^2 - 3x + 2$; $f'(x) = 6x - 3$

Set $f'(x) = 0$ and obtain the critical number $\frac{1}{2}$.

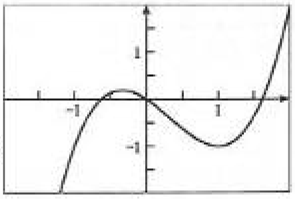
	$f(x)$	$f'(x)$	Conclusion
$x < \frac{1}{2}$		$-$	f is decreasing on $(-\infty, \frac{1}{2}]$
$x = \frac{1}{2}$	$\frac{5}{4}$	0	f has a relative minimum value
$\frac{1}{2} < x$		+	f is increasing on $[\frac{1}{2}, +\infty)$



3. $f(x) = x^3 - x^2 - x$; $f'(x) = 3x^2 - 2x - 1$

Set $f'(x) = 0$: $(3x+1)(x-1) = 0$; $x = -\frac{1}{3}$, $x = 1$

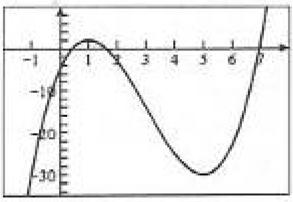
	$f(x)$	$f'(x)$	Conclusion
$x < -\frac{1}{3}$		$+$	f is increasing on $(-\infty, -\frac{1}{3}]$
$x = -\frac{1}{3}$	$\frac{5}{27}$	0	f has a relative maximum value
$-\frac{1}{3} < x < 1$		$-$	f is decreasing on $[-\frac{1}{3}, 1]$
$x = 1$	-1	0	f has a relative minimum value
$1 < x$		$+$	f is increasing on $[1, +\infty)$



4. $f(x) = x^3 - 9x^2 + 15x - 5$

A plot of f is shown at the right. We find the derivative of f and factor it.

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$$



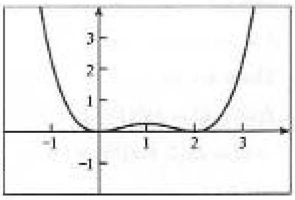
Because $f'(x) = 0$ when $x = 1$ or $x = 5$, the critical numbers of f are 1 and 5. We consider the three intervals $x < 1$, $1 < x < 5$, and $x > 5$ that are determined by the critical numbers; these are shown in the first row of the table. The factor $x - 1$ is 0 for $x = 1$, negative for smaller values of x, and positive for larger values of x; this is shown in the second row of the table. The third row shows the signs of the factor $x - 5$. In the fourth row, the sign of $f'(x)$ is the product of the signs in rows 2 and 3. The conclusion for the intervals is found using Theorem 3.4.3 and the conclusion for the critical numbers is found using Theorem 3.4.4. $f(x)$ is evaluated at the critical numbers and at one point of each infinite branch.

	$x < 1$	$x = 1$	$1 < x < 5$	$x = 5$	$x > 5$
$3(x-1)$	$-$	0	$+$	$+$	$+$
$x - 5$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
conclusion	f is increasing on $(-\infty, 1]$	f has a relative maximum	f is decreasing on $[1, 5]$	f has a relative minimum	f is increasing on $[5, -\infty)$
$f(x)$	$f(0) = 5$	2		-30	$f(7) = 2$

5. $f(x) = \frac{1}{4}x^4 - x^3 + x^2$; $f'(x) = x^3 - 3x^2 + 2x$

Set $f'(x) = 0$: $x(x-1)(x-2) = 0$; $x = 0$, $x = 1$, $x = 2$

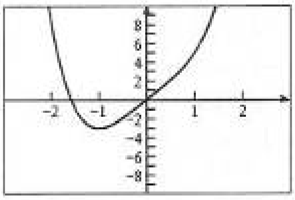
	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$-$	f is decreasing on $(-\infty, 0]$
$x = 0$	0	0	f has a relative minimum value
$0 < x < 1$		$+$	f is increasing on $[0, 1]$
$x = 1$	$\frac{1}{4}$	0	f has a relative maximum value
$1 < x < 2$		$-$	f is decreasing on $[1, 2]$
$x = 2$	0	0	f has a relative minimum value
$2 < x$		$+$	f is increasing on $[2, +\infty)$



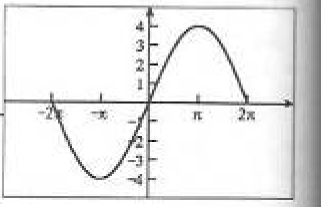
6. $f(x) = x^4 + 4x$; $f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x+1)(x^2 - x + 1)$

Set $f'(x) = 0$: $x = -1$

	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$-$	f is decreasing on $(-\infty, -1]$
$x = -1$	-3	0	f has a relative minimum value
$-1 < x$		$+$	f is increasing on $[-1, +\infty)$

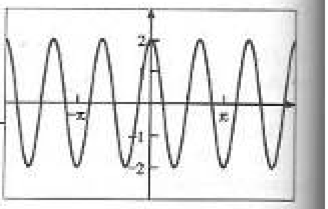


7. $f(x) = 4\sin\frac{1}{2}x,\ x \in [-2\pi, 2\pi];\ f'(x) = 2\cos\frac{1}{2}x$
Because the sine function has period 2π, f has period 4π.
Set $f'(x) = 0$: $2\cos\frac{1}{2}x = 0$; $\frac{1}{2}x = -\frac{1}{2}\pi$ or $\frac{1}{2}x = \frac{1}{2}\pi$; $x = -\pi,\ x = \pi$.



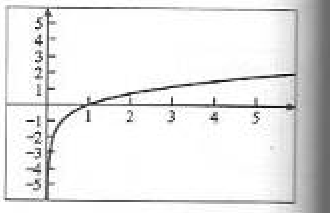
	$f(x)$	$f'(x)$	f is/has
$-2\pi \le x < -\pi$		$-$	decreasing on $[-2\pi, -\pi]$
$x = -\pi$	-4	0	a relative minimum value
$-\pi < x < \pi$		$+$	increasing on $[-\pi, \pi]$
$x = \pi$	4	0	a relative maximum value
$\pi < x \le 2\pi$		$-$	decreasing on $[\pi, 2\pi]$

8. $f(x) = 2\cos 3x,\ x \in [-2\pi, 2\pi];\ f'(x) = -6\sin 3x$
Set $f'(x) = 0$: $-6\sin 3x = 0$; $3x = 0$ or $3x = \pi$; $x = 0,\ x = \frac{1}{3}\pi$.
Because the cosine function has period 2π, f has period $\frac{2}{3}\pi$.
Thus we add $\frac{2}{3}k\pi$ where k is any integer.



	$f(x)$	$f'(x)$	f is/has
$\frac{2}{3}k\pi < x < \frac{1}{3}\pi + \frac{2}{3}k\pi$		$-$	decreasing on $[\frac{2}{3}k\pi, \frac{1}{3}\pi + \frac{2}{3}k\pi]$
$x = \frac{1}{3}\pi + \frac{2}{3}k\pi$	-2	0	a relative minimum value
$\frac{1}{3}\pi + \frac{2}{3}k\pi < x < \frac{2}{3}\pi + \frac{2}{3}k\pi$		$+$	increasing on $[\frac{1}{3}\pi + \frac{2}{3}k\pi, \frac{2}{3}\pi + \frac{2}{3}k\pi]$
$x = \frac{2}{3}\pi + \frac{2}{3}k\pi$	2	0	a relative maximum value
$\frac{2}{3}\pi + \frac{2}{3}k\pi < x < \pi + \frac{2}{3}k\pi$		$-$	decreasing on $[\frac{2}{3}\pi + \frac{2}{3}k\pi, \pi + \frac{2}{3}k\pi]$

9. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2};\ f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x+1)$
Set $f'(x) = 0$: $x = -1$. The domain of f is $(0, +\infty)$ and $f'(x)$ exists for all x in domain of f. -1 is not in the domain so there are no critical numbers and no relative extrema.



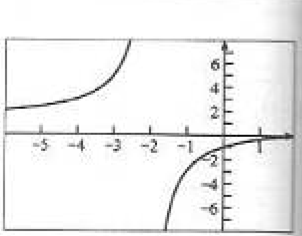
	$f(x)$	$f'(x)$	Conclusion
$0 < x$		$+$	f is increasing on $(0, +\infty)$

10. $f(x) = \dfrac{x-2}{x+2} = 1 - \dfrac{4}{x+2};\ f'(x) = \dfrac{4}{(x+2)^2}$
f and f' are not defined at 2.

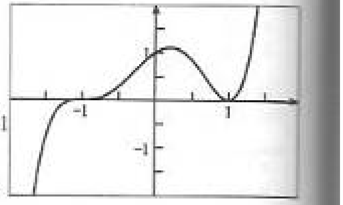
Because $f'(x) > 0$ at every number in its domain,

f is increasing on $(-\infty, -2)$ and $(2, +\infty)$.

There are no relative extrema.



11. $f(x) = (1-x)^2(1+x)^3$. $f'(x) = -2(1-x)(1+x)^3 + 3(1+x)^2(1-x)^2$
$= (1-x)(1+x)^2[-2(1+x) + 3(1-x)] = (1-x)(1+x)^2(-5x+1)$
Set $f'(x) = 0$: $x = 1,\ x = -1,\ x = \frac{1}{5}$



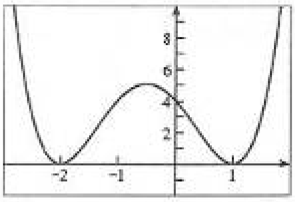
	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$+$	f is increasing on $(-\infty, \frac{1}{5}]$ no relative extremum at $x = -1$
$x = -1$	0	0	
$-1 < x < \frac{1}{5}$		$+$	
$x = \frac{1}{5}$	$\frac{3456}{3125}$	0	f has a relative maximum value
$\frac{1}{5} < x < 1$		$-$	f is decreasing on $[\frac{1}{5}, 1]$
$x = 1$	0	0	f has a relative minimum value
$1 < x$		$+$	f is increasing $[1, +\infty)$

$f(x) = (x+2)^2(x-1)^2$

A plot of the graph is shown at the right.

$$\begin{aligned} f'(x) &= 2(x+2)(x-1)^2 + 2(x+2)^2(x-1) \\ &= 2(x+2)(x-1) + [(x-1)+(x+2)] \\ &= 2(x+2)(x-1)(2x+1) \end{aligned}$$

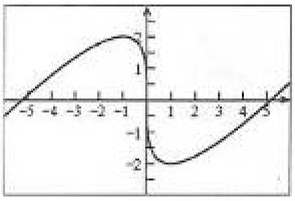
The critical numbers are -2, $-\frac{1}{2}$ and 1.
The table is filled in one row at a time.



	$x < -2$	$x = -2$	$-2 < x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$x = 1$	$x > 1$
$2(x+2)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$2x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
conclusion	f is decreasing on $(-\infty, -2]$	f has a relative minimum	f is increasing on $[-2, -\frac{1}{2}]$	f has a relative maximum	f is decreasing on $[-\frac{1}{2}, 1]$	f has a relative minimum	f is increasing on $[1, +\infty)$
$f(x)$	$f(-3) = 16$	0	$f(-1) = 4$	$\frac{81}{16}$=5.0625	$f(0) = 4$	0	$f(2) = 16$

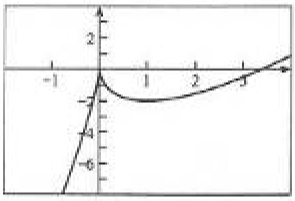
$f(x) = x - 3x^{1/3}$; $f'(x) = 1 - x^{-2/3}$. Set $f'(x) = 0$: $x^{-2/3} = 1$; $x^{2/3} = 1$; $x^2 = 1$; $x = \pm 1$
$f'(x)$ does not exist when $x = 0$. The critical numbers of f are -1, 0, and 1.

	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$+$	f is increasing on $(-\infty, -1]$
$x = -1$	2	0	f has a relative maximum value
$-1 < x < 0$		$-$	
$x = 0$	0	does not exist	f is decreasing on $[-1, 1]$ no relative extremum at 0
$0 < x < 1$		$-$	
$x = 1$	-2	0	f has a relative minimum value
$1 < x$		$+$	f is increasing



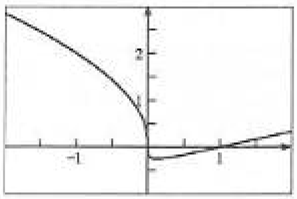
$f(x) = 4x - 6x^{2/3}$; $f'(x) = 4 - 4x^{-1/3} = 4(1 - x^{-1/3})$. $f'(x) = 0$ when $x = 1$ and $f'(x)$ does not exist when $x = 0$. The critical numbers of f are 0 and 1. There is a cusp at the origin.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	d.n.e.	$-$	0	$+$
f is/has a	increasing	relative maximum	decreasing	relative minimum	increasing
$f(x)$		0		-2	



$f(x) = x^{2/3} - x^{1/3}$; $f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-2/3} = \frac{1}{3}x^{-2/3}(2x^{1/3} - 1)$

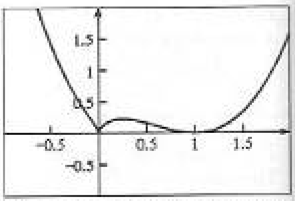
Set $f'(x) = 0$: $2x^{1/3} = 1$; $x^{1/3} = \frac{1}{2}$; $x = \frac{1}{8}$
$f'(0)$ does not exist and 0 is in the domain of f.
The critical numbers are 0 and $\frac{1}{8}$.



	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$-$	
$x = 0$	0	does not exist	f is decreasing on $(-\infty, \frac{1}{8}]$ no relative extremum at 0
$0 < x < \frac{1}{8}$		$-$	
$x = \frac{1}{8}$	$-\frac{1}{4}$	0	f has a relative minimum value
$\frac{1}{8} < x$		$+$	f is increasing on $[\frac{1}{8}, +\infty)$

16. $f(x) = x^{2/3}(x-1)^2$

▷ A plot is shown at the right. f is continuous for all x.



$$\begin{aligned} f'(x) &= x^{2/3}(2)(x-1) + (x-1)^2(\tfrac{2}{3}x^{-1/3}) \\ &= \tfrac{2}{3}x^{-1/3}(x-1)[3x + (x-1)] \\ &= \frac{2(x-1)(4x-1)}{3x^{1/3}} \end{aligned}$$

The critical numbers of f are 0, $\frac{1}{4}$, and 1. Because $f'_-(0) = -\infty$ and $f'_+(0) = +\infty$, the graph of f has vertical tangent line at $x = 0$. Because the x axis is the normal line at $x = 0$ and the curve lies on one side the normal line, the graph has a cusp at the origin. The table is filled in by rows.

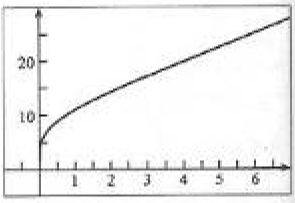
	$x < 0$	$x = 0$	$0 < x < \frac{1}{4}$	$x = \frac{1}{4}$	$\frac{1}{4} < x < 1$	$x = 1$	$x > 1$
$2(x-1)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$4x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$1/3x^{1/3}$	$-$	doesn't exist	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	doesn't exist	$+$	0	$-$	0	$+$
f is/has a	decreasing on $(-\infty, 0]$	relative minimum	increasing on $[0, \frac{1}{4}]$	relative maximum	decreasing on $[\frac{1}{4}, 1]$	relative minimum	increasing on $[1, +\infty)$
$f(x)$	$f(-.25) \approx .62$	0		$9/2^{16/3} \approx .22$		0	$f(1.5) \approx .32$

17. $f(x) = x^{5/4} + 10x^{1/4}$

$f'(x) = \frac{5}{4}x^{1/4} + \frac{10}{4}x^{-3/4}$

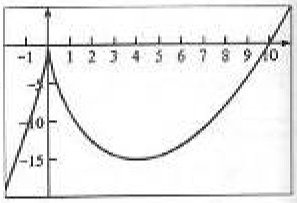
The domain of f is $[0, \infty)$. $f'(x) > 0$ if $x > 0$.

f is increasing on $[0, \infty)$.



18. $f(x) = x^{5/3} - 10x^{2/3}$; $f'(x) = \frac{5}{3}x^{2/3} - \frac{20}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-4)$

$f'(4) = 0$ and $f'(0)$ does not exist. The graph has a cusp at the origin.



	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$x > 4$
$\frac{5}{3}x^{-1/3}$	$-$	doesn't exist	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	doesn't exist	$-$	0	$+$
f is/has a	increasing	relative maximum	decreasing	relative minimum	increasing
$f(x)$		0		$-6 \cdot 4^{2/3} \approx -15.1$	

In Exercises 19–32, compute (a) the relative extrema of f, (b) the values of x at which the relative extrema occur (c) the intervals on which f is increasing, and (d) the intervals on which f is decreasing. Check by plotting.

19. $f(x) = 2x^3 - 9x^2 + 2$; $f'(x) = 6x^2 - 18x$

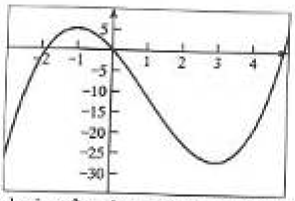
Set $f'(x) = 0$: $6x(x-3) = 0$; $x = 0$, $x = 3$

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$+$	f is increasing on $(-\infty, 0]$
$x = 0$	2	0	f has a relative maximum value
$0 < x < 3$		$-$	f is decreasing on $[0, 3]$
$x = 3$	-25	0	f has a relative minimum value
$3 < x$		$+$	f is increasing on $[3, +\infty)$

$f(x) = x^3 - 3x^2 - 9x$

A plot of f is shown at the right. We find the derivative of f and factor it.

$$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$$



Because $f'(x) = 0$ when $x = -1$ or $x = 3$, the critical numbers of f are -1 and 3. We consider the three intervals $x < -1$, $-1 < x < 3$, and $x > 3$ that are determined by the critical numbers; these are shown in the first row of the table. The sign of each factor is show in the next two rows. In the fourth row, the sign of $f'(x)$ is the product of the signs in rows 2 and 3. The conclusion for the intervals is found using Theorem 3.4.3 and the conclusion for the critical numbers is found using Theorem 3.4.4. $f(x)$ is evaluated at the critical numbers and at one point of each infinite branch.

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
$3(x+1)$	−	0	+	+	+
$x - 3$	−	−	−	0	+
$f'(x)$	+	0	−	0	+
f is/has a	increasing on $(-\infty, 1]$	relative maximum	decreasing on $[1, 5]$	relative minimum	increasing on $[5, -\infty)$
$f(x)$	$f(0) = 5$	2		-30	$f(7) = 2$

$f(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + 1$; $f'(x) = x^4 - 5x^2 + 4$
Set $f'(x) = 0$: $(x^2-1)(x^2-4) = 0$; $(x+1)(x-1)(x+2)(x-2) = 0$;
$x = -1$, $x = 1$, $x = -2$, $x = 2$

	$f(x)$	$f'(x)$	Conclusion
$x < -2$		+	f is increasing on $(-\infty, -2]$
$x = -2$	$-\frac{1}{15}$	0	f has a relative maximum value
$-2 < x < -1$		−	f is decreasing on $[-2, -1]$
$x = -1$	$-\frac{23}{15}$	0	f has a relative minimum value
$-1 < x < 1$		+	f is increasing on $[-1, 1]$
$x = 1$	$\frac{53}{15}$	0	f has a relative maximum value
$1 < x < 2$		−	f is decreasing on $[1, 2]$
$x = 2$	$\frac{31}{15}$	0	f has a relative minimum value
$2 < x$		+	f is increasing on $[2, +\infty)$

$f(x) = x^5 - 5x^3 - 20x - 2$

$$f'(x) = 5x^4 - 15x^2 - 20 = 5(x+2)(x-2)(x^2+1)$$

Because $x^2 + 1 > 0$, the critical numbers are -2 and 2. The table is filled in one row at a time.

	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
$5(x+2)$	−	0	+	+	+
$x - 2$	−	−	−	0	+
$x^2 + 1$	+	+	+	+	+
$f'(x)$	+	0	−	0	+
f is/has a	increasing on $(-\infty, -2]$	relative maximum	decreasing on $[-2, 2]$	relative minimum	increasing on $[2, +\infty)$
$f(x)$	$f(-2.7) \approx -7$	46		-50	$f(2.7) \approx -11$

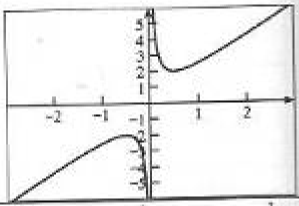
23. $f(x) = x + \frac{1}{x^2} = x + x^{-2}$; $f'(x) = 1 - 2x^{-3}$. Set $f'(x) = 0$: $\frac{x^3 - 2}{x^3} = 0$; $x = \sqrt[3]{2}$
The domain of f is $\{x \mid x \neq 0\}$, and $f'(x)$ exists for all x in its domain.

The only critical number is $\sqrt[3]{2} \approx 1.26$.

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$+$	f is increasing on $(-\infty, 0)$
$x = 0$	not defined	not defined	vertical asymptote
$0 < x < \sqrt{2}$		$-$	f is decreasing on $(0, \sqrt{2}]$
$x = \sqrt{2}$	$\frac{3}{2}\sqrt{2} \approx 1.89$	0	f has a relative minimum value
$\sqrt{2} < x$		$+$	f is increasing on $[\sqrt{2}, +\infty)$

24. $f(x) = 2x + \frac{1}{2x}$

▸ A plot is shown at the right. Note that $f(0)$ is not defined.



$$f'(x) = 2 - \frac{1}{2x^2} = \frac{4x^2 - 1}{2x^2} = \frac{(2x+1)(2x-1)}{2x^2}$$

The critical numbers are $-\frac{1}{2}$ and $\frac{1}{2}$. The table is filled in one row at a time.

	$x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$-\frac{1}{2} < x < 0$	$x = 0$	$0 < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$2x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$2x - 1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$1/2x^2$	$+$	$+$	$+$	doesn't exist	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	doesn't exist	$-$	0	$+$
f is/has a	increasing on $(-\infty, -\frac{1}{2}]$	relative maximum	decreasing on $[-\frac{1}{2}, 0)$	vertical asymptote	decreasing on $(0, \frac{1}{2}]$	relative minimum	increasing on $[\frac{1}{2}, +\infty)$
$f(x)$	$f(-2)=-4.25$	-2	$f(-.5)=-4.25$		$f(.5)=4.25$	2	$f(2)=4.25$

25. $f(x) = 2x\sqrt{3 - x}$; $f'(x) = 2\sqrt{3 - x} + 2x\left(-\frac{1}{2\sqrt{3 - x}}\right) = \frac{2(3 - x) - x}{\sqrt{3 - x}} = \frac{6 - 3x}{\sqrt{3 - x}}$
Set $f'(x) = 0$: $6 - 3x = 0$; $x = 2$

	$f(x)$	$f'(x)$	Conclusion
$x < 2$		$+$	f is increasing on $(-\infty, 2]$
$x = 2$	4	0	f has a relative maximum value
$2 < x < 3$		$-$	f is decreasing on $[2, 3]$

26. $f(x) = x\sqrt{5 - x^2}$. Because $5 - x^2 \geq 0$ if $|x| \leq \sqrt{5}$, the domain of f is $[-\sqrt{5}, \sqrt{5}]$. $f(x) = x(5 - x^2)^{1/2}$
$$f'(x) = (5 - x^2)^{1/2}(1) + x(\tfrac{1}{2})(5 - x^2)^{-1/2}(-2x) = (5 - x^2)^{-1/2}[(5 - x^2) - x^2] = \frac{5 - 2x^2}{\sqrt{5 - x^2}}$$
Let $c = \sqrt{\frac{5}{2}}$. The critical numbers are $\pm c$. The table is filled in one row at a time. Values at the endpoints i
rows 3 and 4 are one-sided limits.

	$x=-\sqrt{5}$	$-\sqrt{5} < x < -c$	$x=-c$	$-c < x < c$	$x=c$	$c < x < \sqrt{5}$	$x=\sqrt{5}$
$5 - 2x^2$	$-$	$-$	0	$+$	0	$-$	$-$
$1/\sqrt{5 - x^2}$	$+\infty$	$+$	$+$	$+$	$+$	$+$	$+\infty$
$f'(x)$	$-\infty$	$-$	0	$+$	0	$-$	$-\infty$
f is/has a	vertical tangent	decreasing on $[-\sqrt{5}, -c]$	relative minimum	increasing on $[-c, c]$	relative maximum	decreasing on $[c, \sqrt{5}]$	vertical tangent
$f(x)$	0		-2.5		2.5		0

$f(x) = 2 - 3(x-4)^{2/3}$; $f'(x) = -2(x-4)^{-1/3}$. $f'(x) \neq 0$ for any value of x. $f'(4)$ does not exist and 4 is in the domain of f. Therefore 4 is a critical number.

	$f(x)$	$f'(x)$	Conclusion
$x < 4$		$+$	f is increasing on $(-\infty, 4]$
$x = 4$	2	doesn't exist	f has a relative maximum value
$4 < x$		$-$	f is decreasing on $[4, +\infty)$

$f(x) = 2 - (x-1)^{1/3}$

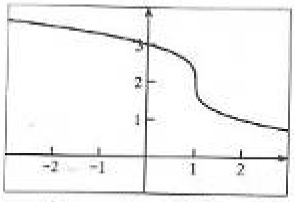
f is continuous at all real numbers.

$$f'(x) = \frac{-1}{3(x-1)^{2/3}}$$

Because $f'(x) < 0$ if $x \neq 1$, we conclude that f is decreasing on $(-\infty, +\infty)$ and does not have a relative extremum. Because

$$\lim_{x \to 1} f'(x) = -\infty$$

the tangent line to the graph of f is vertical at the point $(1, 2)$. Other points on the graph are $(0, 3)$ and $(2, 1)$. A plot of the graph is shown at the right.



$f(x) = \frac{1}{2} \sec 4x$, $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$; $f'(x) = \frac{1}{2} \sec 4x \tan 4x(4) = 2 \sec 4x \tan 4x$

Set $f'(x) = 0$: $\tan 4x = 0$; $4x = 0$ or $4x = \pi$; $x = 0$, $x = \frac{1}{4}\pi$.

Because the secant function has period 2π, f has period $\frac{1}{2}\pi$. Thus we add $\frac{1}{2}k\pi$, where k is -1 or 0.

	$f(x)$	$f'(x)$	Conclusion
$x = \frac{1}{2}k\pi$	$\frac{1}{2}$	0	f has a relative minimum value
$\frac{1}{2}k\pi < x < \frac{1}{8}\pi + \frac{1}{2}k\pi$		$+$	f is increasing
$x = \frac{1}{8}\pi + \frac{1}{2}k\pi$	not defined	not defined	
$\frac{1}{8}\pi + \frac{1}{2}k\pi < x < \frac{1}{4}\pi + \frac{1}{2}k\pi$		$+$	f is increasing
$x = \frac{1}{4}\pi + \frac{1}{2}k\pi$	$-\frac{1}{2}$	0	f has a relative maximum value
$\frac{1}{4}\pi + \frac{1}{2}k\pi < x < \frac{3}{8}\pi + \frac{1}{2}k\pi$		$-$	f is decreasing
$x = \frac{3}{8}\pi + \frac{1}{2}k\pi$	not defined	not defined	
$\frac{3}{8}\pi + \frac{1}{2}k\pi < x < \frac{1}{2}\pi + \frac{1}{2}k\pi$		$-$	f is decreasing

$f(x) = 3 \csc 2x$, $[-\pi, \pi]$; $f'(x) = -6 \csc 2x \cot 2x = -6 \cos 2x / \sin^2 2x$

Set $f'(x) = 0$: $\cos 2x = 0$; $2x = \frac{1}{2}\pi$ or $2x = \frac{3}{2}\pi$; $x = \frac{1}{4}\pi$, $x = \frac{3}{4}\pi$.

Because the cosecant function has period 2π, f has period π. Thus we add $k\pi$, where k is any integer.

	$f(x)$	$f'(x)$	Conclusion
$x = k\pi$	not defined	not defined	f has a vertical asymptote
$k\pi < x < \frac{1}{4}\pi + k\pi$		$-$	f is decreasing
$x = \frac{1}{4}\pi + k\pi$	3	0	f has a relative minimum value
$\frac{1}{4}\pi + k\pi < x < \frac{1}{2}\pi + k\pi$		$+$	f is increasing
$x = \frac{1}{2}\pi + k\pi$	not defined	not defined	f has a vertical asymptote
$\frac{1}{2}\pi + k\pi < x < \frac{3}{4}\pi + k\pi$		$+$	f is increasing
$x = \frac{3}{4}\pi + k\pi$	-3	0	f has a relative maximum value
$\frac{3}{4}\pi + k\pi < x < \pi + k\pi$		$-$	f is decreasing

31. $f(x) = x^{1/3}(x+4)^{-2/3}$. $f(-4)$ is not defined.

$f'(x) = \frac{1}{3}x^{-2/3}(x+4)^{-2/3} - \frac{2}{3}x^{1/3}(x+4)^{-5/3} = \frac{1}{3}x^{-2/3}(x+4)^{-5/3}[(x+4) - 2x] = \frac{1}{3}x^{-2/3}(x+4)^{-5/3}(4-x)$

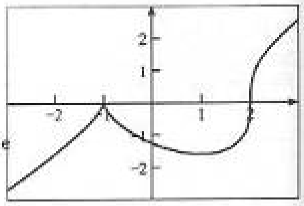
$f'(4) = 0$ and $f'(-4)$ and $f'(0)$ do not exist.

	$x < -4$	$x = -4$	$-4 < x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$x > 4$
$\frac{1}{3}x^{-2/3}$	+	+	+	doesn't exist	+	+	+
$(x+4)^{-5/3}$	−	doesn't exist	+	+	+	+	+
$4 - x$	+	+	+	+	+	0	−
$f'(x)$	−	doesn't exist	+	doesn't exist	+	0	−
f is/has a	decreasing	vertical asymptote	increasing	vertical tangent	increasing	relative maximum	decreasing

32. $f(x) = (x+1)^{2/3}(x-2)^{1/3}$. $f'(x) = \frac{2}{3}(x+1)^{-1/3}(x-2)^{1/3} + \frac{1}{3}(x-2)^{-2/3}(x+1)^{2/3}$

$= \frac{1}{3}(x+1)^{-1/3}(x-2)^{-2/3}[2(x-2) + (x+1)] = (x+1)^{-1/3}(x-2)^{-2/3}(x-1)$

Set $f'(x) = 0$: $x = 1$. $f'(-1)$ and $f'(2)$ do not exist and -1 and 2 are in the domain of f. The critical numbers of f are -1, 1, and 2.



	$f(x)$	$f'(x)$	Conclusion
$x < -1$		+	f is increasing on $(-\infty, -1]$
$x = -1$	0	does not exist	f has a relative maximum value graph has a cusp
$-1 < x < 1$		−	f is decreasing on $[-1, 1]$
$x = 1$	$-\sqrt[3]{4}$	0	f has a relative minimum value
$1 < x < 2$		+	
$x = 2$	0	does not exist	f is increasing on $[1, +\infty)$ no relative extremum at 2
$2 < x$		+	

In Exercises 33–38, compute (a) the relative extrema of f, (b) the values of x at which the relative extrema occur, (c) the intervals on which f is increasing, and (d) the intervals on which f is decreasing. (e) Use (a)-(d) to sketch

33. $f(x) = \{2x + 9$

11. $x = \tan^{-1}t$, $y = \frac{1}{2}\ln(t^2+1)$; from $t = 0$ to $t = 1$. $x'(t) = \dfrac{1}{1+t^2}$, $y'(t) = \dfrac{t}{1+t^2}$. $\mathrm{L} = \int_0^1 \sqrt{x'^2 + y'^2}\,dt$

$$= \int_0^1 \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \left(\frac{t}{1+t^2}\right)^2}\,dt = \int_0^1 \sqrt{\frac{1+t^2}{(1+t^2)^2}}\,dt = \int_0^1 \frac{dt}{\sqrt{1+t^2}} = \ln\left|t + \sqrt{1+t^2}\right|\Big]_0^1 = \ln(1+\sqrt{2})$$

12. $x = 2(\cos t + t\sin t)$, $y = 2(\sin t - t\cos t)$ from $t = 0$ to $t = \frac{1}{3}\pi$.

▷ $$\mathrm{L} = \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{\pi/3} \sqrt{[2(-\sin t + t\cos t + \sin t)]^2 + [2(\cos t - \cos t + t\sin t)]^2}\,dt$$
$$= \int_0^{\pi/3} 2\sqrt{t^2(\cos^2 t + \sin^2 t)}\,dt = \int_0^{\pi/3} 2|t|\,dt = \int_0^{\pi/3} 2t\,dt = t^2\Big]_0^{\pi/3} = \tfrac{1}{9}\pi^2$$

13. $x = 4\sin 2t$, $y = 4\cos 2t$; from $t = 0$ to $t = \pi$. $\mathrm{L} = \int_0^{\pi} \sqrt{x'^2 + y'^2}\,dt = \int_0^{\pi} \sqrt{(8\cos 2t)^2 + (-8\sin 2t)^2}\,dt$

$$= \int_0^{\pi} \sqrt{64\cos^2 2t + 64\sin^2 2t}\,dt = \int_0^{\pi} 8\sqrt{\cos^2 2t + \sin^2 2t}\,dt = 8\int_0^{\pi} dt = 8t\Big]_0^{\pi} = 8\pi$$

14. $x = e^{-t}\cos t$, $y = e^{-t}\sin t$; from $t = 0$ to $t = \pi$. By a product rule, $\mathrm{L} = \int_0^{\pi} \sqrt{x'^2 + y'^2}\,dt$

$$= \int_0^{\pi} \sqrt{[\sqrt{2}e^{-t}\cos(t + \tfrac{3}{4}\pi)]^2 + [\sqrt{2}e^{-t}\sin(t + \tfrac{3}{4}\pi)]^2}\,dt = \int_0^{\pi} \sqrt{2}e^{-t}\,dt = -\sqrt{2}e^{-t}\Big]_0^{\pi} = \sqrt{2}(1 - e^{-\pi})$$

In Exercises 15–22, use NINT to find to four significant digits the length L of arc of the parametric curve.

15. $x = t + 2$, $y = 4t^2 + t$; from $t = 0$ to $t = 3$. $\mathrm{L} = \int_0^3 \sqrt{x'^2 + y'^2}\,dt = \int_0^3 \sqrt{1^2 + (8t+1)^2}\,dt = 39.194 \approx 39.19$

16. $x = 2t^2 + 3t$, $y = 2t - 1$; from $t = 1$ to $t = 2$

▷ $$\mathrm{L} = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_1^2 \sqrt{(4t+3)^2 + 2^2}\,dt$$

Using NINT, we find $\mathrm{L} = 9.223$ to four significant digits.

17. $x = 3\cos t$, $y = 2\sin t$; from $t = 0$ to $t = \frac{1}{2}\pi$. $\mathrm{L} = \int_0^{\pi/2} \sqrt{(-3\sin t)^2 + (2\cos t)^2}\,dt = 3.9664 \approx 3.966$

18. $x = 2\sec t$, $y = 3\tan t$; from $t = 0$ to $t = \frac{1}{4}\pi$. $\mathrm{L} = \int_0^{\pi/4} \sqrt{(2\sec t\tan t)^2 + (3\sec^2 t)^2}\,dt = 3.1381 \approx 3.138$

19. $x = 8\tan t$, $y = 6\sec t$; from $t = \frac{3}{4}\pi$ to $t = \pi$. $\mathrm{L} = \int_{3\pi/4}^{\pi} \sqrt{(8\sec^2 t)^2 + (6\sec t\tan t)^2}\,dt = 8.4622 \approx 8.462$

20. $x = e^t$, $y = \ln t$; from $t = 1$ to $t = 5$

▷ $$\mathrm{L} = \int_1^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_1^5 \sqrt{(e^t)^2 + \left(\frac{1}{t}\right)^2}\,dt$$

Using NINT, we find $\mathrm{L} = 145.8$ to four significant digits.

21. $x = 4 - t^2$, $y = t^2 + 4t$; from $t = -4$ to $t = 4$. $\mathrm{L} = \int_{-4}^{4} \sqrt{(-2t)^2 + (2t+4)^2}\,dt = 55.314 \approx 55.31$

22. $x = 3t$, $y = 4t^3$; from $t = -1$ to $t = 1$. $\mathrm{L} = \int_{-1}^{1} \sqrt{9 + (12t^2)^2}\,dt = 10.960 \approx 10.96$

23. Parametric equations of the hypocycloid of four cusps are $x = a\cos^3 t$ and $y = a\sin^3 t$.

$$\mathrm{L} = \int_0^{\pi/2} \sqrt{x'^2 + y'^2}\,dt = 4\int_0^{\pi/2} \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\,dt$$
$$= 4\int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^2 t}\sqrt{\cos^2 t + \sin^2 t}\,dt = 12a\int_0^{\pi/2} \sin t\cos t\,dt = 6a\sin^2 t\Big]_0^{\pi/2} = 6a$$

24. One arch of the cycloid: $x = a(t - \sin t)$, $y = a(1 - \cos t)$

▷ We use Theorem 9.2.3. Thus,

$\dfrac{dx}{dt} = a(1 - \cos t)$ and $\dfrac{dy}{dt} = a\sin t$

Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2(1 - 2\cos t + \cos^2 t) + a^2\sin^2 t} = a\sqrt{2 - 2\cos t} \qquad (1)$$

Because

$$\sin^2 \tfrac{1}{2}t = \tfrac{1}{2}(1 - \cos t)$$

then

$$4\sin^2\tfrac{1}{2}t = 2 - 2\cos t \tag{2}$$

Substituting from (2) into (1), we obtain

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a\sqrt{4\sin^2\tfrac{1}{2}t} = 2a\left|\sin\tfrac{1}{2}t\right| \tag{3}$$

Because one arch of the cycloid is defined for $0 \le t \le 2\pi$, we have by (3)

$$\text{L} = \int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2a\int_0^{2\pi}\left|\sin\tfrac{1}{2}t\right|dt \tag{4}$$

Because $0 \le t \le 2\pi$, then $0 \le \frac{1}{2}t \le \pi$. Thus, $\sin\frac{1}{2}t \ge 0$. Hence, from (4)

$$\text{L} = 2a\int_0^{2\pi}\sin\tfrac{1}{2}t\,dt = -4a\cos\tfrac{1}{2}t\Big]_0^{2\pi} = 8a$$

25. $x = t - a\tanh\frac{t}{a}$, $y = a\operatorname{sech}\frac{t}{a}$; from $t = \text{a}$ to $t = 2\text{a}$. Let $u = \frac{t}{a}$. Then

$x = au - a\tanh u$ and $u = a\operatorname{sech} u$ from $u = 1$ to $u = 2$.

$$\text{L} = \int_1^2\sqrt{(\text{D}_u x)^2 + (\text{D}_u y)^2}\,du = \int_1^2\sqrt{(\text{a} - a\operatorname{sech}^2 u)^2 + (-a\operatorname{sech} u\tanh u)^2}\,du$$
$$= a\int_1^2\sqrt{(1-\operatorname{sech}^2 u)^2 + \operatorname{sech}^2 u\tanh^2 u}\,du = a\int_1^2\sqrt{\tanh^4 u + (1-\tanh^2 u)\tanh^2 u}\,du = a\int_1^2\sqrt{\tanh^2 u}\,du$$
$$= a\int_1^2 \tanh u\,du = a\ln|\cosh u|\Big]_1^2 = a(\ln\cosh 2 - \ln\cosh 1)$$

26. Using meters, the path of the thumbtack is a cycloid having the parametric equations $x = 0.4(t - \sin t)$ and $y = 0.4(1 - \cos t)$. If the bicycle goes a distance of 50π m and the circumference of the tire is 0.8π m, then the number of revolutions made by the thumbtack is $\frac{50\pi}{0.8\pi} = 62.5$. If L m is the distance traveled by the thumbtack during 62.5 revolutions, then

$$\text{L} = 62.5\int_0^{2\pi}\sqrt{x'^2 + y'^2}\,dt = 62.5\int_0^{2\pi}\sqrt{[0.4(1-\cos t)]^2 + (0.4\sin t)^2}\,dt$$
$$= 62.5(0.4)\int_0^{2\pi}\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt = 25\int_0^{2\pi}\sqrt{2 - 2\cos t}\,dt = 25(2)\int_0^{2\pi}\sqrt{\tfrac{1}{2}(1-\cos t)}\,dt$$
$$= 50\int_0^{2\pi}\sin\tfrac{1t}{2}\,dt = 50\Big[-2\cos\tfrac{1}{2}t\Big]_0^{2\pi} = -100(-1-1) = 200$$

27. (a) $x = a\sin t$ and $y = b\cos t$. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sin^2 t + \cos^2 t$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse.

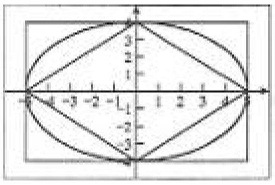
(b) $$\text{C} = 4\int_0^{\pi/2}\sqrt{(\text{D}_t x)^2 + (\text{D}_t y)^2}\,dt = 4\int_0^{\pi/2}\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt = 4\int_0^{\pi/2}\sqrt{a^2(1-\sin^2 t) + b^2\sin^2 t}\,dt$$
$$= 4\int_0^{\pi/2}\sqrt{a^2 - (a^2-b^2)\sin^2 t}\,dt = 4\int_0^{\pi/2}a\sqrt{1 - k^2\sin^2 t}\,dt$$ where $k^2 = (a^2-b^2)/a^2 < 1$, an elliptic integral

28. (a) Use the formula of Exercise 27(b) and NINT to determine the circumference C of the ellipse defined by $x = 5\sin t$ and $y = 4\cos t$. (b) Plot the ellipse. support your answer in part (a) by finding the perimeters of the inscribed rhombus and circumscribed rectangle and showing that C is between these two perimeters.

▹ (a) From Exercise 27, we have $\text{C} = 4\int_0^{\pi/2}a\sqrt{1 - k^2\sin^2 t}\,dt$ where $k^2 = (a^2-b^2)/a^2$. Because $a = 5$ and $b = 4$, then $k^2 = (5^2 - 4^2)/5^2 = \frac{9}{25}$ and

$$\text{C} = 4\int_0^{\pi/2}5\sqrt{1 - \tfrac{9}{25}\sin^2 t}\,dt = 28.36$$

using NINT.



(b) From the figure we see the part of the inscribed rhombus in the first quadrant has length $\sqrt{4^2 + 5^2} = \sqrt{41}$ and so its perimeter is $4\sqrt{41} \approx 25.6$ while the part of the circumscribed rectangle in the first quadrant has length $4 + 5 = 9$ and so its perimeter is $4 \times 9 = 36$, and s is between these values.

9.3 POLAR COORDINATES AND POLAR GRAPHS

Pole, Polar axis The pole is the origin, the polar axis is the positive x axis

Polar Coordinates $P(r,\theta)$ is located r units from the pole on the ray with direction angle θ.

There is not a unique pair of polar coordinates for a point. If (r,θ) is a polar coordinate representation of point P, then we may add any even multiple of π to θ and obtain another polar coordinate representation of P. That is, for any integer n, $(r,\theta)=(r,\theta+2n\pi)$.

If we add an odd multiple of π to θ and also replace r by $-r$, we obtain another polar coordinate representation of P. That is, for any integer n, $(r,\theta)=(-r,\theta+(2n-1)\pi)$.

If $(x,y)^c$ is the Cartesian coordinate representation of P and $(r,\theta)^p$ is a polar coordinate representation of P, then
$x=r\cos\theta$, $y=r\sin\theta$ and $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$

You should be able to identify the graph and make a quick sketch by plotting only a few points for any polar equation of a type listed below. The letters a and b are any constants, but n is a positive integer in each of the following equations.

Special Graphs

TYPE OF EQUATION	NAME OF CURVE	See Exercise
1. $r=a$	Circle, center at pole, radius r	16(b)
2. $\theta=a$	Line, of angle a with polar axis	16(a)
3. $r=a\theta$	Spiral of Archimedes	40
4. $r=a/\cos(\theta-b)$	Line through $P(a,b)^p$ and $\perp$ OP	
5. $r=2a\cos\theta+2b\sin\theta$	Circle through O, center at $(a,b)^c$	
6. $r^2=a^2\cos 2\theta$ or $r=a^2\sin 2\theta$	Lemniscate	44
7. $r=a\cos n\theta$ or $r=a\sin n\theta$	Rose, $2n$ leaves if n is even	35
8. $r=a\cos n\theta$ or $r=a\sin n\theta$	Rose, n leaves if n is odd	36
9. $r=a+b\cos\theta$ or $a+b\sin\theta$	Limaçon, points to furthest point	
$\lvert a\rvert\geq 2\lvert b\rvert$	Convex	29
$2\lvert b\rvert>\lvert a\rvert>\lvert b\rvert$	With a dent	27
$\lvert a\rvert=\lvert b\rvert$	Cardioid	24
$0<\lvert a\rvert<\lvert b\rvert$	With a loop	28

Symmetry Tests Let n be any integer. A polar graph is symmetric with respect to

(i) the polar axis if an equivalent equation is obtained when (r,θ) is replaced by either $(r,-\theta+2n\pi)$ or $(-r,\pi-\theta+2n\pi)$.

(ii) $\frac{1}{2}\pi$ axis if an equivalent equation is obtained when (r,θ) is replaced by either $(r,\pi-\theta+2n\pi)$ or $(-r,-\theta+2n\pi)$.

(iii) the pole if an equivalent equation is obtained when (r,θ) is replaced by either $(r,\theta+\pi+2n\pi)$ or $(-r,\theta+2n\pi)$.

Intersection Often we may find all the points of intersection of two graphs by solving the equations simultaneously. However, we must treat the pole as a special case. We test each equation to see whether there is a replacement for θ (not necessarily the same replacement) that gives r the value 0. If so, then each graph contains the pole and hence the intersection contains the pole.

Because two or more distinct polar equations may represent the same graph, we must sometimes take additional steps to find all points of intersection. If the curve C is the graph of the polar equation $r=f(\theta)$, then C is also the graph of any equation of the form $r=(-1)^n f(\theta+n\pi)$, where n is any integer. Thus, to find all the points of intersection of a curve C_1 and a curve C_2, do the following:

1. Find all the distinct equations of curve C_2.
2. Solve the equation of curve C_1 simultaneously with each equation of C_2.
3. Test to see whether the pole is a point of intersection.

Tangents The slope of the tangent line to $r=f(\theta)$ at the point (r,θ) is $m=\dfrac{\sin\theta(dr/d\theta)+r\cos\theta}{\cos\theta(dr/d\theta)-r\sin\theta}$.

The angle α between the tangent line and the radius satisfies $\tan\alpha=r/(dr/d\theta)$.

If α is constant, then $r=r_0e^{\theta/r}$, a logarithmic spiral. See Exercises 37 and 38.

To find an equation of a tangent line to a curve at the pole, let $r=0$ in the equation and solve for θ. If $\theta=\theta_1$ is a solution, then the line $\theta=\theta_1$ is a tangent line to the curve at the pole

Parametric Plot The polar graph of $r = f(\theta)$ yields the cartesian graph $x = f(t)\cos t$, $y = f(t)\sin t$.

Exercises 9.3

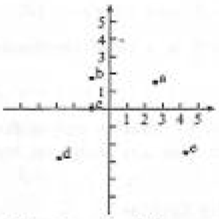
In Exercises 1–4, locate the point having the given set of polar coordinates.

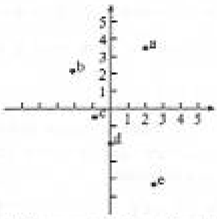
1. (a) $(3, \frac{1}{6}\pi)$ (b) $(2, \frac{2}{3}\pi)$ (c) $(1, \pi)$ (d) $(4, \frac{5}{4}\pi)$ (e) $(5, \frac{11}{6}\pi)$

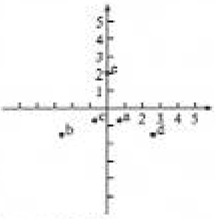
2. (a) $(4, \frac{1}{3}\pi)$ (b) $(3, \frac{3}{4}\pi)$ (c) $(1, \frac{7}{6}\pi)$ (d) $(2, \frac{3}{2}\pi)$ (e) $(5, \frac{5}{3}\pi)$

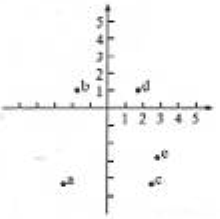
3. (a) $(1, -\frac{1}{4}\pi)$ (b) $(3, -\frac{5}{6}\pi)$ (c) $(-1, -\frac{1}{4}\pi)$ (d) $(-3, \frac{5}{6}\pi)$ (e) $(-2, -\frac{1}{2}\pi)$

4. (a) $(5, -\frac{2}{3}\pi)$ (b) $(2, -\frac{7}{6}\pi)$ (c) $(-5, \frac{2}{3}\pi)$ (d) $(-2, \frac{7}{6}\pi)$ (e) $(-4, -\frac{5}{4}\pi)$









In Exercises 5 and 6, convert from polar to rectangular coordinates.

5. (a) $(3, \pi) = (3\cos\pi, 3\sin\pi)^c = (-3, 0)^c$

(b) $(\sqrt{2}, -\frac{3}{4}\pi) = (\sqrt{2}\cos(-\frac{3}{4}\pi), \sqrt{2}\sin(-\frac{3}{4}\pi))^c = (-\sqrt{2}\cdot 1/\sqrt{2}, -\sqrt{2}\cdot 1/\sqrt{2})^c = (-1, 1)^c$

(c) $(-4, \frac{2}{3}\pi) = (-4\cos\frac{2}{3}\pi, -4\sin\frac{2}{3}\pi)^c = (-4(-\frac{1}{2}), -3(\frac{1}{2}\sqrt{3}))^c = (2, -2\sqrt{3})^c$

(d) $(-1, -\frac{7}{6}\pi) = (1, -\frac{1}{6}\pi) = (\cos(-\frac{1}{6}\pi), \sin(-\frac{1}{6}\pi))^c = (\frac{1}{2}\sqrt{3}, -\frac{1}{2})^c$

6. (a) $(-2, -\frac{1}{2}\pi) = (2, \frac{1}{2}\pi) = (2\cos\frac{1}{2}\pi, 2\sin\frac{1}{2}\pi)^c = (0, 2)^c$

(b) $(-1, \frac{1}{4}\pi) = (-\cos\frac{1}{4}\pi, -\sin\frac{1}{4}\pi)^c = (-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})^c$

(c) $(2, -\frac{7}{6}\pi) = (2, \frac{5}{6}\pi) = (2\cos\frac{5}{6}\pi, 2\sin\frac{5}{6}\pi)^c = (2\cdot -\frac{1}{2}\sqrt{3}, 2\cdot\frac{1}{2})^c = (-\sqrt{3}, 1)^c$

(d) $(2, \frac{7}{4}\pi) = (2\cos\frac{7}{4}\pi, 2\sin\frac{7}{4}\pi)^c = (2\cdot\frac{1}{2}\sqrt{2}, 2\cdot -\frac{1}{2}\sqrt{2})^c = (\sqrt{2}, -\sqrt{2})^c$

In Exercises 7 and 8, convert from rectangular to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

7. (a) $(1, -1)^c$. $r = \sqrt{2}$. Q4, $\theta = \tan^{-1}(-1/1) + 2\pi = \frac{7}{4}\pi$. $(\sqrt{2}, \frac{7}{4}\pi)$

(b) $(-\sqrt{3}, 1)^c$. $r = 2$. Q2, $\theta = \tan^{-1}(1/-\sqrt{3}) + \pi = \frac{5}{6}\pi$. $(2, \frac{5}{6}\pi)$

(c) $(2, 2)^c$. $r = 2\sqrt{2}$. Q1, $\theta = \tan^{-1}(2/2) = \frac{1}{4}\pi$. $(2\sqrt{2}, \frac{1}{4}\pi)$

(d) $(-5, 0)^c$. quadrantal. $(5, \pi)$

8. (a) $(3, -3)$; (b) $(-1, \sqrt{3})$; (c) $(0, -2)$; (d) $(-2, -2\sqrt{3})$

▹ (a) $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$ and $\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1$

Because $\tan\frac{1}{4}\pi = 1$, the reference angle is $\frac{1}{4}\pi$. Because $x > 0$ and $y < 0$, the point is in the fourth quadrant. Thus, $\theta = 2\pi - \frac{1}{4} = \frac{7}{4}\pi$. The polar coordinates of the point are $(3\sqrt{2}, \frac{7}{4}\pi)$

(b) $r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$ $\tan\theta = \frac{y}{x} = -\sqrt{3}$

Because $x < 0$ and $y > 0$, the point is in the second quadrant. Because $\tan\frac{1}{3}\pi = \sqrt{3}$, then $\theta = \pi - \frac{1}{3}\pi = \frac{2}{3}\pi$. Thus, the polar coordinates are $(2, \frac{2}{3}\pi)$.

(c) $r = \sqrt{x^2 + y^2} = \sqrt{4} = 2$. Because $x = 0$, then $\tan\theta$ is not defined. However, because the point is on the negative y axis, then $\theta = \frac{3}{2}\pi$. Thus, the polar coordinates are $(2, \frac{3}{2}\pi)$.

(d) $r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$ $\tan\theta = \frac{y}{x} = \sqrt{3}$

The point is in the third quadrant, and $\tan\frac{1}{3}\pi = \sqrt{3}$, so $\theta = \pi + \frac{1}{3}\pi = \frac{4}{3}\pi$. The polar coordinates are $(4, \frac{4}{3}\pi)$.

In Exercises 9–12, find a Cartesian equation of the graph having the given polar equation.

9. $r^2 = 2\sin 2\theta = 4\sin\theta\cos\theta$ ▹ $r^4 = 4(r\sin\theta)(r\cos\theta)$. $(x^2 + y^2)^2 = 4xy$

(b) $r^2 = \cos\theta$ ▹ (pole for $\theta = \frac{1}{2}\pi$) $r^3 = r\cos\theta$, $(r^2)^3 = (r\cos\theta)^2$. $(x^2 + y^2)^3 = x^2$

10. (a) $r^2\cos 2\theta = 10$ ▷ $r^2(\cos^2\theta - \sin^2\theta) = 10$, $x^2 - y^2 = 10$

(b) $r^2 = 4\cos 2\theta$ ▷ (pole for $\theta = \frac{1}{4}\pi$) $r^4 = 4r^2(\cos^2\theta - \sin^2\theta)$, $(x^2+y^2)^2 = 4x^2 - 4y^2$

11. (a) $r\cos\theta = -1$ ▷ $x = -1$

(b) $r = 6/(2 - 3\sin\theta)$ ▷ $2r - 3r\sin\theta = 6$. $2r = 3r\sin\theta + 6$, $4r^2 = (3r\sin\theta + 6)^2$, $4(x^2+y^2) = (3y+6)^2$, $4x^2 + 4y^2 = 9y^2 + 36y + 36$, $4x^2 - 5y^2 - 36y - 36 = 0$

12. (a) $r = 2\sin 3\theta$; (b) $r = \dfrac{4}{3 - 2\cos\theta}$

▷ (a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Thus, the given equation may be written

$r = 6\sin\theta - 8\sin^3\theta$

Because the graph of this equation contains the pole, we may multiply both sides by r^3 without affecting the graph. Thus,

$$r^4 = 6r^3\sin\theta - 8r^3\sin^3\theta$$

$$(r^2)^2 = 6r^2(r\sin\theta) - 8(r\sin\theta)^3$$

Because $r^2 = x^2 + y^2$ and $r\sin\theta = y$, this becomes

$$(x^2+y^2)^2 = 6(x^2+y^2)y - 8y^3$$

$$x^4 + y^4 + 2x^2y^2 - 6x^2y + 2y^3 = 0$$

(b) Eliminating the fraction, we have

$$3r - 2r\cos\theta = 4$$

$$3r = 2(r\cos\theta + 2)$$

$$9r^2 = 4(r\cos\theta + 2)^2$$

Because $r^2 = x^2 + y^2$ and $r\cos\theta = x$, we have

$$9(x^2+y^2) = 4(x+2)^2$$

$$5x^2 + 9y^2 - 16x - 16 = 0$$

In Exercises 13–20, sketch the graph of the equation.

13. (a) $\theta = \frac{1}{3}\pi$ ▷ line through the pole with direction 60°

(b) $r = \frac{1}{3}\pi$ ▷ circle centered at the pole of radius $\frac{1}{3}\pi \approx 1.05$

14. (a) $\theta = \frac{3}{4}\pi$ ▷ line through the pole with direction 135°

(b) $r = \frac{3}{4}\pi$ ▷ circle centered at the pole of radius $\frac{3}{4}\pi \approx 2.36$

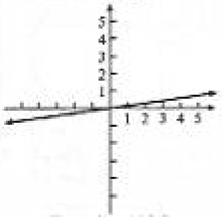
15. (a) $\theta = 2$ ▷ line through the pole with direction $2 \cdot 180°/\pi \approx 114.6°$

(b) $r = 2$ ▷ circle centered at the pole of radius 2

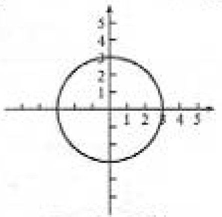
16. (a) $\theta = -3$ (b) $r = -3$

▷ (a) This equation is of Type 2. The graph is a line through the pole that makes an angle of -3 radians with the polar axis. A sketch of the graph is shown below.

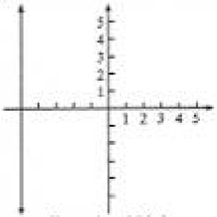
(b) This equation is of Type 1. Because we may replace r by $-r$ and θ by $\theta + \pi$, then the equation $r = -3$ has the same graph as the equation $r = 3$. Thus, the graph is a circle with center at the pole and radius 3.



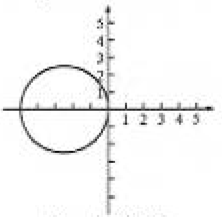
Exercise 16(a)



Exercise 16(b)



Exercise 20(a)



Exercise 20(b)

17. (a) $r\cos\theta = 4$ ▷ line $x = 4$

(b) $r = 4\cos\theta$ ▷ circle centered at $(2, 0)$ of radius 2

18. (a) $r\sin\theta = 2$ ▷ line $y = 2$

(b) $r = 2\sin\theta$ ▷ circle centered at $(1, \frac{1}{2}\pi)$ of radius 1

19. (a) $r\sin\theta = -4$ ▷ line $y = -4$

(b) $r = -4\sin\theta$ ▷ circle centered at $\left(2, \frac{3}{2}\pi\right)$ of radius 2

20. (a) $r\cos\theta = -5$ (b) $r = -5\cos\theta$

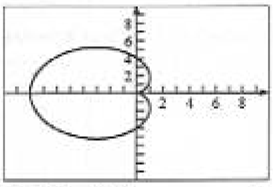
▹ (a) Because $r\cos\theta = x$, then a Cartesian equation is $x = -5$. The graph is the line shown above.
(b) This is a Type 5 equation with $a = -\frac{5}{2}$ and $b = 0$. Thus the graph is the circle passing through the origin and centered at $(-\frac{5}{2}, 0)^c$ shown above.

In Exercises 21–30, determine the type of limaçon, its symmetry S, and the direction it points. Plot the limaçon.

21. $r = 4(1 - \cos\theta)$

▹ $\frac{a}{b} = \frac{4}{4} = 1$, cardioid

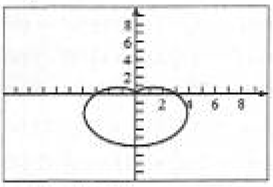
S: polar axis, points left



22. $r = 3(1 - \sin\theta)$

▹ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid

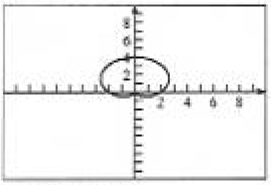
S: $\frac{1}{2}\pi$ axis, points downward



23. $r = 2(1 + \sin\theta)$

▹ $\frac{a}{b} = \frac{2}{2} = 1$, cardioid

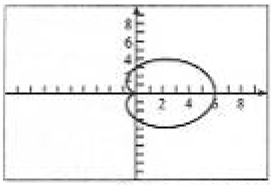
S: $\frac{1}{2}\pi$ axis, points upward



24. $r = 3(1 + \cos\theta)$

▹ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid

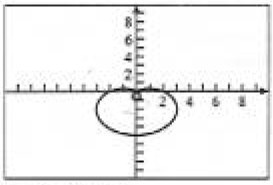
S: polar axis, points right



25. $r = 2 - 3\sin\theta$

▹ $\frac{a}{b} = \frac{2}{3} \in (0, 1)$, a loop

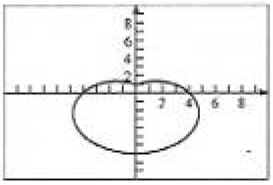
S: $\frac{1}{2}\pi$ axis, points downward



26. $r = 4 - 3\sin\theta$

▹ $\frac{a}{b} = \frac{4}{3} \in (1, 2)$, dented

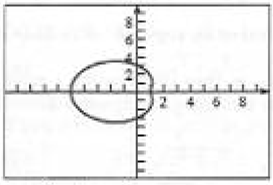
S: $\frac{1}{2}\pi$ axis, points downward



27. $r = 3 - 2\cos\theta$

▹ $\frac{a}{b} = \frac{3}{2} \in (1, 2)$, dented

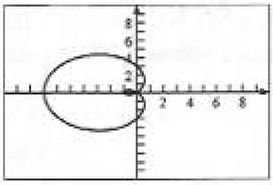
S: polar axis, points left



28. $r = 3 - 4\cos\theta$

▹ $\frac{a}{b} = \frac{3}{4} \in (0, 1)$, with a loop

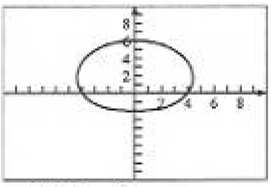
S: polar axis, points left



29. $r = 4 + 2\sin\theta$

▹ $\frac{a}{b} = \frac{4}{2} \geq 2$, convex

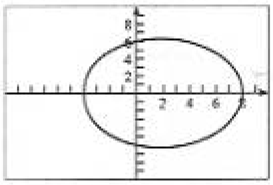
S: $\frac{1}{2}\pi$ axis, points upward



30. $r = 6 + 2\cos\theta$

▹ $\frac{a}{b} = \frac{6}{2} \geq 2$, convex

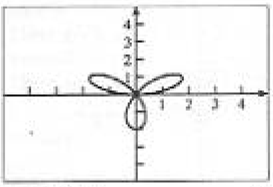
S: polar axis, points right



In Exercises 31–50, describe and plot the graph of the equation.

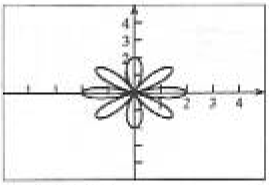
31. $r = 2\sin 3\theta$

▹ 3-leafed rose



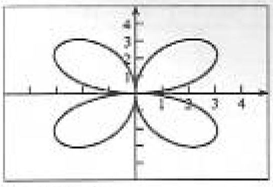
33. $r = 2\cos 4\theta$

▹ 8-leafed rose



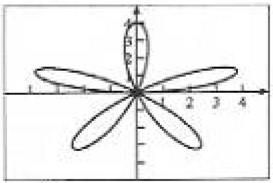
35. $r = 4\sin 2\theta$

▹ 4-leafed rose



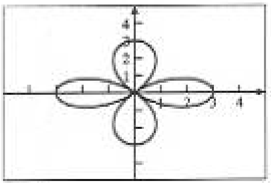
32. $r = 4\sin 5\theta$

▹ 5-leafed rose



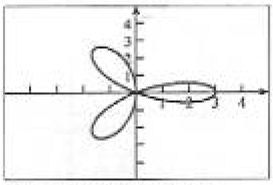
34. $r = 3\cos 2\theta$

▹ 4-leafed rose



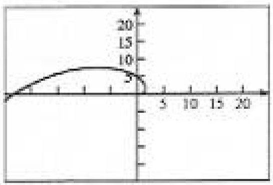
36. $r = 3\cos 3\theta$

▹ 3-leafed rose



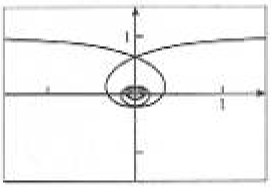
37. $r = e^{\theta}$

▹ logarithmic spiral



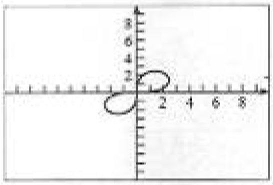
39. $r = \frac{1}{\theta}$

▹ reciprocal spiral



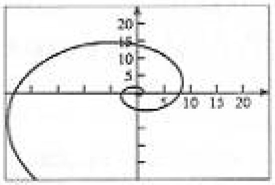
41. $r^2 = 9\sin 2\theta$ (lemniscate)

▹ $X_1 = 3\cos t\sqrt{\sin 2t}$,
$Y_1 = 3\sin t\sqrt{\sin 2t}$, $0 \le t \le 6.3$



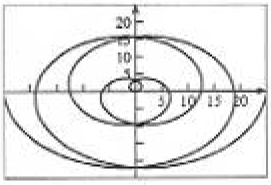
38. $r = e^{\theta/3}$

▹ logarithmic spiral



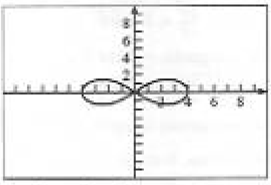
40. $r = 2\theta$

▹ spiral of Archimedes



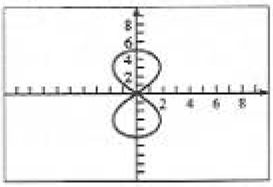
42. $r^2 = 16\cos 2\theta$ (lemniscate)

▹ $X_1 = 4\cos t\sqrt{\cos 2t}$,
$Y_1 = 4\sin t\sqrt{\cos 2t}$, $0 \le t \le 6.3$



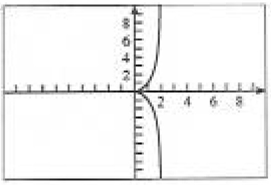
43. $r^2 = -25\cos 2\theta$ (lemniscate)

▹ $X_1 = 5\cos t\sqrt{-\cos 2t}$,
$Y_1 = 5\sin t\sqrt{-\cos 2t}$, $0 \le t \le 6.3$



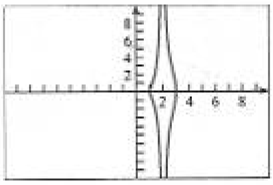
45. $r = 2\sin\theta\tan\theta$

▹ cissoid

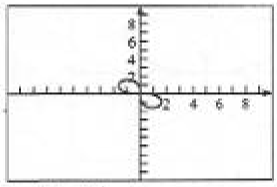


47. $r = 2\sec\theta - 1$

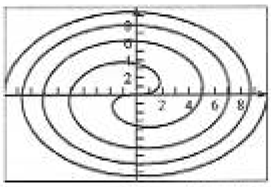
▹ conchoid of Nicomedes
coefficient (2) > constant (1), no loop



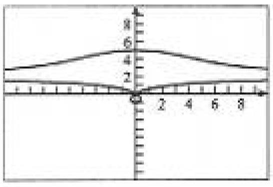
44. $r^2 = -4 \sin 2\theta$ (lemniscate)
▹ $X_1 = 2 \cos t \sqrt{-\sin 2t}$,
$Y_1 = 2 \sin t \sqrt{-\sin 2t}$,
$0 \le t \le 6.3$



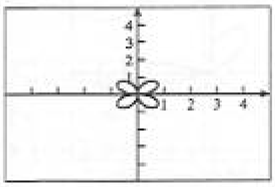
46. $r^2 = 8\theta$ (Fermat spiral)
▹ $X_1 = \cos t \sqrt{8t}$, $Y_1 = \sin t \sqrt{8t}$
$X_2 = -\cos t \sqrt{8t}$, $Y_1 = -\sin t \sqrt{8t}$
$0 \le t \le 6.3$



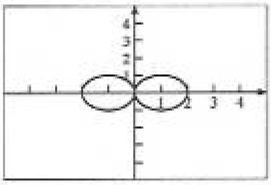
48. $r = 2 \csc \theta + 3$
conchoid of Nicomedes
coefficient (2) < constant (3), loop



49. $r = |\sin 2\theta|$
▹ 4-leafed rose, same as $r = \sin 2\theta$

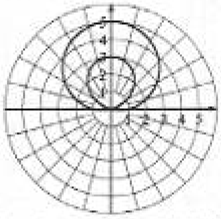


50. $r = 2|\cos \theta|$
▹ two circles

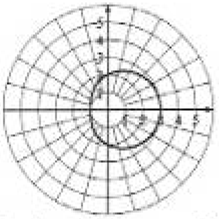


In Exercises 51 and 52, construct a polar coordinate system and sketch on it the graph of the function.

51. $r = 1 + 4 \sin \theta$
▹ limaçon with a loop



52. $r = 2 + \cos \theta$
▹ convex limaçon



In Exercises 53 through 60, determine the horizontal HT and vertical tangents VT of the graph, and sketch them.

53. $r = 4 + 3 \sin \theta$; $\dfrac{dr}{d\theta} = 3 \cos \theta$. $m = \dfrac{\sin \theta(3 \cos \theta) + (4 + 3 \sin \theta)\cos \theta}{\cos \theta(3 \cos \theta) - (4 + 3 \sin \theta)\sin \theta} = \dfrac{\cos \theta(4 + 6 \sin \theta)}{3 - 4 \sin \theta - 6 \sin^2\theta}$

Horizontal tangents. $-2 \cos \theta(3 \sin \theta + 2) = 0$. $\cos \theta = 0$; $\theta = \frac{1}{2}\pi$, $\theta = \frac{3}{2}\pi$ and $4 + 6 \sin \theta = 0$; $\sin \theta = -\frac{2}{3}$;

$\theta = \pi - \sin^{-1}(-\frac{2}{3}) \approx 3.87$, $\theta = 2\pi + \sin^{-1}(-\frac{2}{3}) \approx 5.55$

There are horizontal tangent lines at the points $(7, \frac{1}{2}\pi)$, $(1, \frac{3}{2}\pi)$, $(2, 3.87)$, $(2, 5.55)$.

Vertical tangents. $6 \sin^2\theta + 4 \sin \theta - 3 = 0$; $\sin \theta = \frac{1}{12}(-4 \pm \sqrt{16 + 72}) = \frac{1}{12}(-4 \pm 2\sqrt{22}) = \frac{1}{6}(-2 \pm \sqrt{22})$

$\sin \theta \approx 0.448$; $\theta = \sin^{-1} 0.448 = 0.46$, $\theta = \pi - \sin^{-1} 0.448 = 2.68$. $\sin \theta \approx -1.115$; no solution.

There are vertical tangent lines at the points $(5.35, 0.46)$ and $(5.35, 2.68)$.

54. $r = 2 + \cos \theta$. $\dfrac{dr}{d\theta} = -\sin \theta$. $m = \dfrac{\sin \theta(-\sin \theta) + (2 + \cos \theta)\cos \theta}{\cos \theta(-\sin \theta) - (2 + \cos \theta)\sin \theta} = \dfrac{\cos^2\theta - 1 + 2 \cos \theta + \cos^2\theta}{-2 \sin \theta(1 + \cos \theta)}$

HT: $2 \cos^2\theta + 2 \cos \theta - 1 = 0$, $\cos \theta = \frac{1}{4}(-2 \pm \sqrt{12}) = \frac{1}{2}(-1 \pm \sqrt{3})$. $\cos \theta = \frac{1}{2}(-1 - \sqrt{3})$, no solution.

$\theta = \cos^{-1}[\frac{1}{2}(\sqrt{3} - 1)] = 1.196$ or $2\pi - 1.196 = 5.087$, $r = 2 + \frac{1}{2}(\sqrt{3} - 1) = 2.366$

Vertical tangents: $\sin \theta = 0$, $r = 3$ and $\theta = \pi$, $r = 1$. $\cos \theta = -1$, $\theta = \pi$, duplicate

55. $r = 4 - 2\cos\theta;\ \dfrac{dr}{d\theta} = 2\sin\theta.\ m = \dfrac{\sin\theta(2\sin\theta) + (4 - 2\cos\theta)\cos\theta}{\cos\theta(2\sin\theta) - (4 - 2\cos\theta)\sin\theta} = \dfrac{2\sin^2\theta + 4\cos\theta - 2\cos^2\theta}{2\sin\theta\cos\theta - 4\sin\theta + 2\sin\theta\cos\theta}$
$= \dfrac{2 - 4\cos^2\theta + 4\cos\theta}{4\sin\theta(\cos\theta - 1)}$. HT: $2\cos^2\theta - 2\cos\theta - 1 = 0$; $\cos\theta = \frac{1}{4}(2 \pm \sqrt{4 + 8}) = \frac{1}{4}(2 \pm 2\sqrt{3}) = \frac{1}{2}(1 \pm \sqrt{3})$.
$\cos\theta \approx 1.366$, no solution. $\cos\theta \approx = -0.366$; $\theta = \cos^{-1}(-0.366) = 1.95$, $\theta = 2\pi - \cos^{-1}(-0.366) = 4.34$
There are horizontal tangent lines at the points $(4.73, 1.95)$ and $(4.73, 4.34)$.
Vertical tangents. $-2\sin\theta(\cos\theta - 1) = 0$. $\sin\theta = 0$; $\theta = 0$, $\theta = \pi$ and $\cos\theta = 1$; $\theta = 0$.
There are vertical tangent lines at the points $(2, 0)$ and $(6, \pi)$.

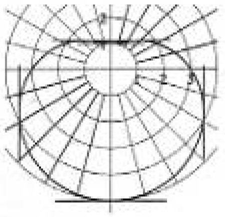
56. $r = 3 - 2\sin\theta$

▷ We have

$\dfrac{dr}{d\theta} = -2\cos\theta$

Therefore, if m is the slope of the tangent line to the curve, then

$$m = \frac{\sin\theta(dr/d\theta) + r\cos\theta}{\cos\theta(dr/d\theta) - r\sin\theta} = \frac{\sin\theta(-2\cos\theta) + (3 - 2\sin\theta)\cos\theta}{\cos\theta(-2\cos\theta) - (3 - 2\sin\theta)\sin\theta}$$
$$= \frac{-2\sin\theta\cos\theta + 3\cos\theta - 2\sin\theta\cos\theta}{-2\cos^2\theta - 3\sin\theta + 2\sin^2\theta}$$
$$= \frac{-4\sin\theta\cos\theta + 3\cos\theta}{-2(1 - \sin^2\theta) + 2\sin^2\theta - 3\sin\theta} = \frac{\cos\theta(3 - 4\sin\theta)}{4\sin^2\theta - 3\sin\theta - 2} \tag{1}$$



Because $m = 0$ at a point where the tangent line is horizontal, we set the numerator to 0 and solve for θ. Thus
$\cos\theta = 0$, $\theta = \frac{1}{2}\pi$ or $\theta = \frac{3}{2}\pi$; $\sin\theta = \frac{3}{4}$, $\theta = \sin^{-1}\frac{3}{4} = 0.848$ or $\theta = \pi - \sin^{-1}\frac{3}{4} = 2.294$
There is a horizontal tangent line at the points with polar coordinates $(1, \frac{1}{2}\pi)$, $(5, \frac{3}{2}\pi)$, $(\frac{3}{2}, 0.848)$, and $(\frac{3}{2}, 2.294)$.
Because m is not defined at a point where the tangent line is vertical, we set the denominator of (1) to 0 and solve for θ. Thus,
$4\sin^2\theta - 3\sin\theta - 2 = 0$
$\sin\theta = \frac{1}{8}(3 + \sqrt{41}) = 1.18$, no solution or $\sin\theta = \frac{1}{8}(3 - \sqrt{41}) \approx -0.4254$,
$r = \frac{1}{4}(9 + \sqrt{41}) = 3.85$, $\theta = \pi + \sin^{-1}(0.4254) = 3.58$, $\theta = 2\pi - \sin^{-1}(0.4254) = 5.84$
Therefore, the curve has a vertical tangent at the points with polar coordinates $(3.85, 3.58)$ and $(3.85, 5.84)$.
Because $a/b = 1.5$ is close to 2, the limçon is nearly convex and the 3 tangents near the dent are hard to see.

57. $r = \cos 2\theta = \cos^2\theta - \sin^2\theta;\ \dfrac{dr}{d\theta} = -2\sin 2\theta = -4\sin\theta\cos\theta$.
$m = \dfrac{\sin\theta(-4\sin\theta\cos\theta) + (\cos^2\theta - \sin^2\theta)\cos\theta}{\cos\theta(-4\sin\theta\cos\theta) - (\cos^2\theta - \sin^2\theta)\sin\theta} = \dfrac{\cos\theta(-4\sin^2\theta + \cos^2\theta - \sin^2\theta)}{\sin\theta(-4\cos^2\theta - \cos^2\theta + \sin^2\theta)} = \dfrac{\cos\theta(1 - 6\sin^2\theta)}{\sin\theta(1 - 6\cos^2\theta)}$
Horizontal tangents: $\cos\theta = 0$; $\theta = \frac{1}{2}\pi$, $\theta = \frac{3}{2}\pi$ and $1 - 6\sin^2\theta = 0$
$\sin\theta = \pm\frac{1}{6}\sqrt{6}$; $\theta = \sin^{-1}(\frac{1}{6}\sqrt{6}) \approx 0.42$, $\theta = \pi - \sin^{-1}(-\frac{1}{6}\sqrt{6}) \approx 3.56$, $\theta = 2\pi + \sin^{-1}(-\frac{1}{6}\sqrt{6}) \approx 5.86$
Horizontal tangents are at $(-1, \frac{1}{2}\pi)$, $(-1, \frac{3}{2}\pi)$, $(\frac{2}{3}, 0.42)$, $(\frac{2}{3}, 2.72)$, $(\frac{2}{3}, 3.56)$, $(\frac{2}{3}, 5.86)$.
Vertical tangents: $\sin\theta = 0$; $\theta = 0$, $\theta = \pi$ and $1 - 6\cos^2\theta = 0$; $\cos\theta = \pm\frac{1}{6}\sqrt{6}$
$\theta = \cos^{-1}(\frac{1}{6}\sqrt{6}) \approx 1.15$, $\theta = \cos^{-1}(-\frac{1}{6}\sqrt{6}) \approx 1.99$, $\theta = 2\pi - \cos^{-1}(-\frac{1}{6}\sqrt{6})$, $\theta = 2\pi - \cos^{-1}(\frac{1}{6}\sqrt{6}) \approx 5.13$
Vertical tangents are at $(1, 0)$, $(1, \pi)$, $(-\frac{2}{3}, 1.15)$, $(-\frac{2}{3}, 1.99)$, $(-\frac{2}{3}, 4.29)$, $(-\frac{2}{3}, 5.13)$.

58. $r = 2\sin 3\theta.\ \dfrac{dr}{d\theta} = 6\cos 3\theta$.
$m = \dfrac{\sin\theta(6\cos 3\theta) + (2\sin 3\theta)\cos\theta}{\cos\theta(6\cos 3\theta) - (2\sin 3\theta)\sin\theta} = \dfrac{6\sin\theta(4\cos^3\theta - 3\cos\theta) + 2(3\sin\theta - 4\sin^3\theta)\cos\theta}{6\cos\theta(4\cos^3\theta - 3\cos\theta) - 2(3\sin\theta - 4\sin^3\theta)\sin\theta}$
HT: $0 = 2\sin\theta\cos\theta[(12\cos^2\theta - 9) + (3 - 4\sin^2\theta)] = 2\sin\theta\cos\theta(6 - 16\sin^2\theta)$. $\sin\theta = 0$, $\theta = 0$, $r = 0$.
$\cos\theta = 0$, $\theta = \frac{1}{2}\pi$, $r = -3$. $\sin^2\theta = \frac{6}{16}$, $\sin\theta = \pm\frac{1}{4}\sqrt{6}$, $\theta = \sin^{-1}\frac{1}{4}\sqrt{6} = 0.659$, $r = 1.837$.
$\theta = \pi - \sin^{-1}\frac{1}{4}\sqrt{6} = 2.483$, $r = 1.837$
VT: $0 = 2[3\cos^2\theta(4\cos^2\theta - 3) - (3\sin^2\theta - 4\sin^4\theta)] = 2[(3 - 3\sin^2\theta)(1 - 4\sin^2\theta) - (3\sin^2\theta - 4\sin^4\theta)]$
$= 2(16\sin^4\theta - 18\sin^2\theta + 3)$. $\sin^2\theta = \frac{1}{16}(9 \pm \sqrt{33})$. $\sin\theta = 0.9600$, $\theta = 1.286$, $r = -1.33$.
$\theta = \pi - 1.286 = 1.855$, $r = -1.33$. $\sin\theta = 0.4511$, $\theta = 0.467$, $r = 1.96$. $\theta = \pi - 0.467 = 2.674$, $r = 1.96$

59. $r^2 = 4 \sin 2\theta = 8 \sin \theta \cos \theta$, $\theta \in [0, \frac{1}{2}\pi]$; $r \frac{dr}{d\theta} = 8 \cos^2\theta - 8 \sin^2\theta$

$$m = \frac{\sin \theta(dr/d\theta) + r \cos \theta}{\cos \theta(dr/d\theta) - r \sin \theta} = \frac{\sin \theta(r\, dr/d\theta) + r^2\cos \theta}{\cos \theta(r\, dr/d\theta) - r^2\sin \theta} = \frac{\sin \theta(4 \cos^2\theta - 4 \sin^2\theta + 8 \cos^2\theta)}{\cos \theta(4 \cos^2\theta - 4 \sin^2\theta + 8 \sin^2\theta)} = \frac{\sin \theta(4 \cos^2\theta - 1)}{\cos \theta(1 - 4 \sin^2\theta)}$$

Horizontal tangents. $\sin \theta = 0$; $\theta = 0$ and $4 \cos^2\theta - 1 = 0$; $\cos \theta = \pm \frac{1}{2}$; $\theta = \frac{1}{3}\pi$.

There are horizontal tangent lines at the points $(0, 0)$, $(\sqrt[4]{12}, \frac{1}{3}\pi)$, $(-\sqrt[4]{12}, \frac{1}{3}\pi)$.

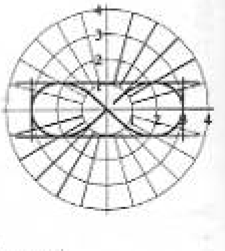
Vertical tangents. $\cos \theta = 0$; $\theta = \frac{1}{2}\pi$ and $1 - 4 \sin^2\theta = 0$; $\sin \theta = \pm \frac{1}{2}$; $\theta = \frac{1}{6}\pi$.

There are vertical tangent lines at the points $(0, \frac{1}{2}\pi)$, $(\sqrt{2}, \frac{1}{6}\pi)$, $(-\sqrt{2}, \frac{1}{6}\pi)$.

60. $r^2 = 9 \cos 2\theta$

▸ The graph is a lemniscate. We have

$$2r \frac{dr}{d\theta} = -18 \sin 2\theta; \quad r \frac{dr}{d\theta} = -9 \sin 2\theta$$



If m is the slope of the tangent line to the curve, then

$$\begin{aligned} m &= \frac{\sin \theta(dr/d\theta) + r \cos \theta}{\cos \theta(dr/d\theta) - r \sin \theta} = \frac{\sin \theta(r\, dr/d\theta) + r^2\cos \theta}{\cos \theta(r\, dr/d\theta) - r^2\sin \theta} \\ &= \frac{\sin \theta(-9 \sin 2\theta) + (9 \cos 2\theta)\cos \theta}{\cos \theta(-9 \sin 2\theta) - (9 \cos 2\theta)\sin \theta} = \frac{\cos \theta \cos 2\theta - \sin \theta \sin 2\theta}{-(\sin \theta \cos 2\theta + \cos \theta \sin 2\theta)} \\ &= -\frac{\cos 3\theta}{\sin 3\theta} \end{aligned}$$

Setting the numerator equal to zero, and bearing in mind that we need $\cos 2\theta \geq 0$, we get
$\cos 3\theta = 0$, $3\theta = \frac{1}{2}\pi$, $\theta = \frac{1}{6}\pi$, $r = \pm\frac{1}{2}\sqrt{2}$; $3\theta = \frac{3}{2}\pi$, $\theta = \frac{1}{2}\pi$, no r; $3\theta = \frac{5}{2}\pi$, $\theta = \frac{5}{6}\pi$, $r = \pm\frac{1}{2}\sqrt{2}$

Thus, there is a horizontal tangent line at the points $(\frac{3}{2}\sqrt{2}, \frac{1}{6}\pi)$, $(-\frac{3}{2}\sqrt{2}, \frac{1}{6}\pi)$, $(\frac{3}{2}\sqrt{2}, \frac{5}{6}\pi)$ and $(-\frac{3}{2}\sqrt{2}, \frac{5}{6}\pi)$.
Setting the denominator equal to zero, we get
$\sin 3\theta = 0$, $3\theta = 0$, $\theta = 0$, $r = \pm 3$; $3\theta = \pi$, $\theta = \frac{1}{3}\pi$, no r.
Therefore the curve has a vertical tangent at the points with polar coordinates $(3, 0)$ and $(-3, 0)$.

In Exercises 61–64, find the Cartesian coordinates of the intersection graphically to two significant digits. Then find the polar coordinates algebraically and compare.

61. The circle $r = 3$ and the cardioid $r = 2(1 + \cos \theta)$. Solving the equations simultaneously, we have $2 + 2 \cos \theta = 3$; $\cos \theta = \frac{1}{2}$. $\theta = \frac{1}{3}\pi$, $-\frac{1}{3}\pi$

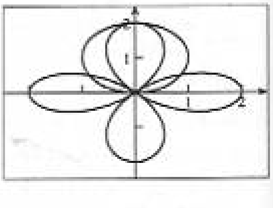
We have the points of intersection $(3, \frac{1}{3}\pi)^p = (\frac{3}{2}, \frac{3}{2}\sqrt{3})^c \approx (1.5, 2.6)$ and $(3, -\frac{1}{3}\pi)^p = (\frac{3}{2}, -\frac{3}{2}\sqrt{3})^c \approx (1.5, -2.6)$.

62. The circles $r = 2 \cos \theta$ and $r = 2 \sin \theta$. Dividing corresponding members of the two equations, we obtain $\frac{2 \sin \theta}{2 \cos \theta} = \frac{r}{r}$; $\tan \theta = 1$. Hence, $\theta = \frac{1}{4}\pi$ and $\frac{5}{4}\pi$. Substituting these values of θ into one of the two given equations we obtain the corresponding values of r. We then have the following points of intersection:
$(\sqrt{2}, \frac{1}{4}\pi)^p = (2, 2)^c$ and $(-\sqrt{2}, \frac{5}{4}\pi)^p = (2, 2)^c$, the same point. The pole lies on the graph of $r = 2 \cos \theta$ because $r = 0$ when $\theta = \frac{1}{2}\pi$. The pole also lies on the graph of $r = 2 \sin \theta$ because $r = 0$ when $\theta = 0$. Thus, the two curves intersect at the pole.

63. The 3-leafed rose $r = 2 \sin 3\theta$ and the circle $r = 4 \sin \theta$. $2(3 \sin \theta - 4 \sin^3\theta) = 4 \sin \theta$,
$0 = 8 \sin^3\theta - 2 \sin \theta = 8 \sin \theta(\sin^2\theta - \frac{1}{4})$. $\sin \theta = 0$, $\theta = 0$, $r = 0$. $\sin \theta = \frac{1}{2}$, $\theta = \frac{1}{6}\pi$, $r = 2$. $\sin \theta = \pi - \frac{1}{6}\pi = \frac{5}{6}\pi$, $r = 2$. $(0, 0)^p = (0, 0)^c$, $(2, \frac{1}{6}\pi)^p = (\sqrt{3}, 1) \approx (1.7, 1)$, $(2, \frac{5}{6}\pi) = (-\sqrt{3}, 1) \approx (-1.7, 1)$

64. $r = 2 \cos 2\theta$ and $r = 2 \sin \theta$

▸ The graph of $r = 2 \cos 2\theta$ is a four-leafed rose, and the graph of $r = 2 \sin \theta$ is a circle as shown. First we solve the given equations simultaneously. Thus,



$$\begin{aligned} 2 \cos 2\theta &= 2 \sin \theta \\ 2(1 - 2 \sin^2\theta) &= 2 \sin \theta \\ 2 \sin^2\theta + \sin \theta - 1 &= 0 \\ (\sin \theta + 1)(2 \sin \theta - 1) &= 0 \end{aligned}$$

$\sin \theta = -1$ or $\sin \theta = \frac{1}{2}$
$\theta = \frac{3}{2}\pi$, $r = -2$ or $\theta = \frac{1}{6}\pi$, $r = 1$ or $\theta = \frac{5}{6}\pi$, $r = 1$

Hence $(-2, \frac{3}{2}\pi)^p = (0, 2)^c$, $(1, \frac{1}{6}\pi)^p = (\frac{1}{2}\sqrt{3}, \frac{1}{2})^c \approx (0.87, 0.5)^c$ and $(1, \frac{5}{6}\pi)^p = (-\frac{1}{2}\sqrt{3}, \frac{1}{2})^c \approx (-0.87, 0.5)$ are points of intersection. Because r can be 0 in each of the given equations, the pole is also point of intersection.

9.4 LENGTH OF ARC AND AREA OF A REGION FOR POLAR GRAPHS

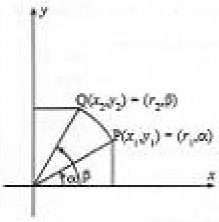
Arc Length The formula for the number of units in the length of arc of a curve defined by the polar equation $r = f(\theta)$ from the point where $\theta = \alpha$ ro the point where $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}$$

9.4.1 Theorem Area Let R be the region bounded by the lines $\theta = \alpha$ and $\theta = \beta$ and the curve whose polar equation is $r = r(\theta)$, where r is continuous and nonnegative on the closed interval $[\alpha, \beta]$. If A square units is the area of region R, then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[r(w_i)]^2 \Delta_i \theta = \tfrac{1}{2}\int_{\alpha}^{\beta} [r(\theta)]^2 \, d\theta$$

Proof Because area has already been defined, the limit of this sum is not the definition; rather, it is a consequence of the integral formula. Refer to the figure. Let $m_1 = y_1/x_1$ and $m_2 = y_2/x_2$ and let $y = y(x)$ and $x = x(y)$ be the cartesian equations of the curve. (If necessary, we divide $[\alpha, \beta]$ into subintervals on which the curve is the cartesian graph of a function.) By Definition 4.5.4 (vertical elements of area), we have



$$\begin{aligned} A &= \int_0^{x_2} m_2 x \, dx + \int_{x_2}^{x_1} y \, dx - \int_0^{x_1} m_2 x \, dx \\ &= \tfrac{1}{2}x_2 y_2 + \int_{x_2}^{x_1} y \, dx - \tfrac{1}{2}x_1 y_1 \end{aligned} \tag{1}$$

By the theorem of Section 4.8 (horizontal elements of area), we have

$$\begin{aligned} A &= \int_0^{y_1} \frac{1}{m_1} dy + \int_{y_1}^{y_2} x \, dy - \int_0^{y_2} \frac{1}{m_2} dy \\ &= \tfrac{1}{2}x_1 y_1 + \int_{y_1}^{y_2} x \, dy - \tfrac{1}{2}x_2 y_2 \end{aligned} \tag{2}$$

Adding Eqs. (1) and (2) and dividing by 2, we get

$$\begin{aligned} A &= \tfrac{1}{2}\left[\int_{x_2}^{x_1} y \, dx + \int_{y_1}^{y_2} x \, dy\right] \\ &= \tfrac{1}{2}\left[\int_{y_1}^{y_2} x \, dy - \int_{x_1}^{x_2} y \, dx\right] \end{aligned} \tag{3}$$

Now $x = r \cos \theta$, $y = r \sin \theta$ and so

$$dx = (r' \cos \theta - r \sin \theta) d\theta \quad \text{and} \quad dy = (r' \sin \theta + r \cos \theta) d\theta$$

Furthermore, when $x = x_1$, then $y = y_1$ and $\theta = \alpha$; when $x = x_2$, then $y = y_2$ and $\theta = \beta$. Making these replacements in (3), we obtain

$$\begin{aligned} A &= \tfrac{1}{2}\int_{\alpha}^{\beta} [(r \cos \theta)(r' \sin \theta + r \cos \theta) - (r \sin \theta)(r' \cos \theta - r \sin \theta)] \, d\theta \\ &= \tfrac{1}{2}\int_{\alpha}^{\beta} r^2(\sin^2 \theta + \cos^2 \theta) \, d\theta \\ &= \tfrac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta \end{aligned}$$

Area Between Curves If $f(\theta) \geq g(\theta)$ for $\alpha \leq \theta \leq \beta$, then the area of the region bounded by the curves $r = f(\theta)$ and $r = g(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$\begin{aligned} A &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}\{[f(w_i)]^2 - [g(w_i)]^2)\Delta_i \theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} \{[f(\theta)]^2 - [g(\theta)]^2\} d\theta \end{aligned}$$

Exercises 9.4

In Exercises 1–4, use the length of arc formula to find the circumference of the circle having the polar equation.

1. If L units is the circumference of the circle $r = 5\cos\theta$, then

$$\text{L} = \int_0^{\pi} \sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi} \sqrt{(-5\sin\theta)^2 + (5\cos\theta)^2}\, d\theta = \int_0^{\pi} 5\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = 5\int_0^{\pi} d\theta = 5\theta\Big]_0^{\pi} = 5\pi$$

2. $r = 4\sin\theta$. $\text{L} = \int_0^{\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi}\sqrt{(4\cos\theta)^2 + (4\sin\theta)^2 d\theta} = \int_0^{\pi} 4\sqrt{\cos^2\theta + \sin^2\theta d\theta} = 4\int_0^{\pi} d\theta = 4\pi$

3. If L units is the circumference of the circle $r = a$, then

$$\text{L} = \int_0^{2\pi} \sqrt{r'^2 + r^2}\, d\theta = \int_0^{2\pi} \sqrt{0^2 + a^2}\, d\theta = \int_0^{2\pi} a\, d\theta = a\theta\Big]_0^{2\pi} = 2\pi a$$

4. $r = a\sin\theta$

▹ We use the formula

$$\text{L} = \int_{\alpha}^{\beta}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}$$

Because $r = 0$ when $\theta = 0$; $r = 0$ when $\theta = \pi$; and $r > 0$ for $0 < \theta < \pi$, we have $\alpha = 0$ and $\beta = \pi$. That is, the entire circle is traced out when θ takes on all values in the closed interval $[0, \pi]$. Thus,

$$\text{L} = \int_0^{\pi}\sqrt{(a\cos\theta)^2 + (a\sin\theta)^2 d\theta} = a\int_0^{\pi} d\theta = \pi a$$

We note that the diameter of the circle is a units. Thus, the circumference is π times the diameter.

In Exercises 5–12, find the exact length of arc of the given polar graph.

5. The entire cardioid $r = 4 + 4\cos\theta$.

$$\text{L} = 2\int_0^{\pi}\sqrt{r'^2 + r^2} d\theta = 2\int_0^{\pi}\sqrt{(-4\sin\theta)^2 + (4 + 4\cos\theta)^2 d\theta} = 8\int_0^{\pi}\sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}\, d\theta$$

$$= 8\int_0^{\pi}\sqrt{2 + 2\cos\theta d\theta} = 16\int_0^{\pi}\sqrt{\tfrac{1}{2}(1 + \cos\theta) d\theta} = 16\int_0^{\pi}\cos\tfrac{1}{2}\theta\, d\theta = 32\sin\tfrac{1}{2}\theta\Big]_0^{\pi} = 32$$

6. The entire cardioid $r = 1 - \sin\theta$. $\text{L} = 2\int_{-\pi/2}^{\pi/2}\sqrt{r'^2 + r^2} d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{(-\cos\theta)^2 + (1 - \sin\theta)^2 d\theta}$

$$= 2\int_{-\pi/2}^{\pi/2}\sqrt{\cos^2\theta + 1 - 2\sin\theta + \sin^2\theta d\theta} = 2\int_{-\pi/2}^{\pi/2}\sqrt{2 - 2\sin\theta d\theta} = 2\int_{-\pi/2}^{\pi/2}\sqrt{2 + 2\cos(\theta + \tfrac{1}{2}\pi) d\theta}$$

$$= 4\int_{-\pi/2}^{\pi/2}\sqrt{\tfrac{1}{2}[1 + \cos(\theta + \tfrac{1}{2}\pi)] d\theta} = 4\int_{-\pi/2}^{\pi/2}\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) d\theta = 8\sin(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)\Big]_{-\pi/2}^{\pi/2} = 8$$

7. If L units is the length of the entire cardioid $r = 3\cos^2\frac{1}{2}\theta = \frac{3}{2}(1 + \cos\theta)$, then

$$\text{L} = 2\int_0^{\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi}\sqrt{(-3\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta)^2 + 9\cos^4\tfrac{1}{2}\theta}\, d\theta = 2\int_0^{\pi}\sqrt{9\cos^2\tfrac{1}{2}\theta}\sqrt{\sin^2\tfrac{1}{2}\theta + \cos^2\tfrac{1}{2}\theta}\, d\theta$$

$$= 6\int_0^{\pi}\cos\tfrac{1}{2}\theta\, d\theta = 12\sin\tfrac{1}{2}\theta\Big]_0^{\pi} = 12$$

8. $r = 3\theta$, from $\theta = 0$ to $\theta = 2\pi$

▹
$$\text{L} = \int_0^{2\pi}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta} = \int_0^{2\pi}\sqrt{3^2 + (3\theta)^2 d\theta} = 3\int_0^{2\pi}\sqrt{1 + \theta^2 d\theta}$$

$$= \tfrac{3}{2}\Big[\theta\sqrt{1 + \theta^2} + \ln(\theta + \sqrt{1 + \theta^2})\Big]_0^{2\pi} = \tfrac{3}{2}[2\pi\sqrt{1 + 4\pi^2} + \ln(2\pi + \sqrt{1 + 4\pi^2})]$$

with the help of Integral Formula 28.

9. If L units is the length of the arc of the curve $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 4$, then

$$\text{L} = \int_0^4\sqrt{r'^2 + r^2} d\theta = \int_0^4\sqrt{(2e2^{\theta})^2 + 4e^{4\theta} d\theta} = \int_0^4\sqrt{4e^{4\theta} + e^{4\theta}}\, d\theta = \sqrt{5}\int_0^4 e^{2\theta} d\theta = \tfrac{1}{2}\sqrt{5}\Big[e^{2\theta}\Big]_0^4 = \tfrac{1}{2}\sqrt{5}(e^8 - 1)$$

10. $r = 3\theta^2$. $\text{L} = \int_0^{\pi}\sqrt{r'^2 + r^2} d\theta = \int_0^{\pi}\sqrt{(6\theta)^2 + (3\theta^2)^2 d\theta} = \int_0^{\pi} 3\theta\sqrt{4 + \theta^2} d\theta = (4 + \theta^2)^{3/2}\Big]_0^{\pi} = (4 + \pi^2)^{3/2} - 8$

11. If L units is the length of the arc of the curve $r = 2\sin^3\frac{1}{3}\theta$ from $\theta = 0$ to $\theta = 6\pi$, then

$$\text{L} = \int_0^{6\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{6\pi}\sqrt{(2\sin^2\tfrac{1}{3}\theta\cos\tfrac{1}{3}\theta)^2 + 2^2\sin^6\tfrac{1}{3}\theta}\, d\theta = 2\int_0^{6\pi}\sqrt{\sin^4\tfrac{1}{3}\theta(\cos^2\tfrac{1}{3}\theta + \sin^2\tfrac{1}{3}\theta) d\theta}$$

$$= 2\int_0^{6\pi}\sin^2\tfrac{1}{3}\theta\, d\theta = \int_0^{6\pi}(1 - \cos\tfrac{2}{3}\theta) d\theta = \Big[\theta - \tfrac{3}{2}\sin\tfrac{2}{3}\theta\Big]_0^{6\pi} = 6\pi$$

12. $r = \sin^2 \frac{1}{2}\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$

▹ Because $r = \frac{1}{2}(1 - \cos\theta)$, then

$$\mathrm{L} = \int_0^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^{\pi/2} \sqrt{(\tfrac{1}{2}\sin\theta)^2 + [\tfrac{1}{2}(1 - \cos\theta)]^2}\,d\theta = \int_0^{\pi/2} \sqrt{\tfrac{1}{4}(\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta)}\,d\theta$$

$$= \int_0^{\pi/2} \sqrt{\tfrac{1}{2}(1 - \cos\theta)}\,d\theta = \int_0^{\pi/2} \sin\tfrac{1}{2}\theta\,d\theta = -2\cos\tfrac{1}{2}\theta\Big]_0^{\pi/2} = -2(\cos\tfrac{1}{4}\pi - \cos 0) = -2(\tfrac{1}{2}\sqrt{2} - 1) = 2 - \sqrt{2}$$

In Exercises 13–20, use NINT to approximate to four significant digits the length of arc of the given polar graph.

13. $r = 3 + \cos\theta$. $\mathrm{L} = 2\int_0^{\pi} \sqrt{r'^2 + r^2}\,d\theta = 2\int_0^{\pi/2} \sqrt{(-\sin\theta)^2 + (3 + \cos\theta)^2}\,d\theta = 2\int_0^{\pi} \sqrt{10 + 6\cos\theta}\,d\theta = 19.377$

14. $r = 3 - 2\sin\theta$.

$\mathrm{L} = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-2\cos\theta)^2 + (3 - 2\sin\theta)^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{13 - 12\sin\theta}\,d\theta = 21.01$

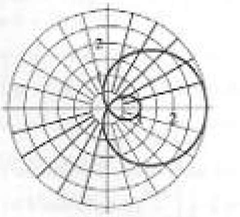
15. The loop of the limaçon $r = 2 - 3\sin\theta$. Let $a = \sin^{-1}\frac{2}{3}$.

$\mathrm{L} = 2\int_a^{\pi/2} \sqrt{(-3\cos\theta)^2 + (2 - 3\sin\theta)^2}\,d\theta = 2\int_a^{\pi/2} \sqrt{13 - 12\sin\theta}\,d\theta = 2.505$

16. The loop of the limaçon $r = 1 + 2\cos\theta$.

▹ See the figure. The midpoint of the loop is at $\theta = \pi$. The loop begins when

$$\begin{aligned} r &= 0 \\ 2\cos\theta &= -1 \\ \cos\theta &= -\tfrac{1}{2} \\ \theta &= \tfrac{2}{3}\pi \end{aligned}$$



By symmetry,

$$\mathrm{L} = 2\int_{2\pi/3}^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 2\int_{2\pi/3}^{\pi} \sqrt{(-2\sin\theta)^2 + (1 + 2\cos\theta)^2}\,d\theta$$

$$= 2\int_{2\pi/3}^{\pi} \sqrt{5 + 4\cos\theta}\,d\theta$$

Using NINT we find $\mathrm{L} = 2.6824 = 2.682$ to four significant digits.

17. One leaf of rose $r = 2\sin 3\theta$. $\mathrm{L} = 2\int_0^{\pi/6} \sqrt{(6\cos 3\theta)^2 + (2\sin 3\theta)^2}\,d\theta = 4\int_0^{\pi/6} \sqrt{9\cos^2 3\theta + \sin^2 3\theta}\,d\theta = 4.455$

18. 1 leaf of rose $r = 3\cos 4\theta$. $\mathrm{L} = 2\int_0^{\pi/8} \sqrt{(-12\sin 4\theta)^2 + (3\cos 4\theta)^2}\,d\theta = 6\int_0^{\pi/8} \sqrt{16\sin^2 4\theta + 9\cos^2 4\theta}\,d\theta = 7.203$

19. The lemniscate $r^2 = 25\cos 2\theta$. $2rr' = -50\sin 2\theta$. $r' = \dfrac{-25\sin 2\theta}{r}$. $r'^2 = \dfrac{25^2\sin^2 2\theta}{25\cos 2\theta}$

$r'^2 + r^2 = 25\left(\dfrac{\sin^2 2\theta}{\cos 2\theta} + \cos 2\theta\right) = \dfrac{25}{\cos 2\theta}$. $\mathrm{L} = 4\int_0^{\pi/4} \dfrac{5\,d\theta}{\sqrt{\cos 2\theta}} = 26.22$, an improper integral.

20. The lemniscate $r^2 = 4\sin 2\theta$.

▹ We have

$$\begin{aligned} r &= 2\sqrt{\sin 2\theta} \\ \frac{dr}{d\theta} &= \frac{2\cos 2\theta}{\sqrt{\sin 2\theta}} \\ \left(\frac{dr}{d\theta}\right)^2 + r^2 &= \frac{4\cos^2 2\theta}{\sin 2\theta} + 4\sin 2\theta = \frac{4}{\sin 2\theta} \end{aligned}$$

By symmetry

$$\mathrm{L} = 4\int_0^{\pi/4} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 8\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Some calculators can evaluate this improper integral to get $10.488 \approx 10.49$. The substitution $\theta = x^2$, $d\theta = 2x\,dx$ yields the proper integral

$$\mathrm{L} = 16\int_0^{\sqrt{\pi/2}} \frac{x\,dx}{\sqrt{\sin 2x^2}}$$

In Exercises 21–26, find the exact area of the region enclosed by the graph of the equation.

21. The graph of the equation $r = 3\cos\theta$ is a circle. We obtain the region enclosed by a semicircle if $\theta \in [0, \frac{1}{2}\pi]$.

$\mathrm{A} = 2\lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} \frac{1}{2}(3\cos w_i)^2 \Delta_i\theta = 9\int_0^{\pi/2} \cos^2\theta\,d\theta = 9\int_0^{\pi/2} \dfrac{1 + \cos 2\theta}{2}\,d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\Big]_0^{\pi/2} = \frac{9}{4}\pi$

22. $r = 2 - \sin\theta$, a convex limaçon. $\mathrm{A} = 2 \cdot \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2 - \sin\theta)^2\,d\theta = \int_{-\pi/2}^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)\,d\theta =$

$$\int_{-\pi/2}^{\pi/2}[4-4\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)]d\theta=\int_{-\pi/2}^{\pi/2}(\tfrac{9}{2}-4\sin\theta-\tfrac{1}{2}\cos 2\theta)d\theta=\tfrac{9}{2}\pi+0+0=\tfrac{9}{2}\pi$$

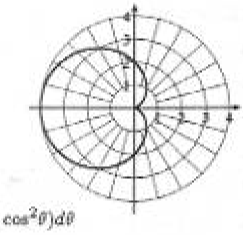
23. The graph of $r=4\cos 3\theta$ is a three-leafed rose. We get the region enclosed by one half of a leaf if $\theta\in[0,\frac{1}{6}\pi]$.

$$A=6\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(4\cos 3w_i)^2\Delta_i\theta=48\int_0^{\pi/6}\cos^2 3\theta\,d\theta=24\int_0^{\pi/6}(1+\cos 6\theta)d\theta=24\Big[\theta+\tfrac{1}{6}\sin 6\theta\Big]_0^{\pi/6}=4\pi$$

24. $r=4\sin^2\frac{1}{2}\theta$.

▸ First, we simplify the given equation.

$$r=4\sin^2\tfrac{1}{2}\theta=2(1-\cos\theta)$$



The graph is the cardioid shown in the figure. Because the region R is symmetric with respect to the polar axis, the area of the region R_1 above the horizontal axis is one-half the area of the entire region R. Because $r=0$ when $\theta=0$, we take the line $\theta=0$ as one of the boundaries or R_1. Thus R_1 is bounded by the lines $\theta=0$, $\theta=\pi$, and the curve $r=2(1-\cos\theta)$. Hence, the area of R is given by

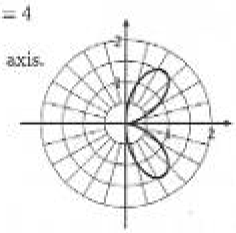
$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[2(1-\cos w_i)]^2d\theta=\int_0^{\pi}[2(1-\cos\theta(]^2d\theta=4\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$=4\int_0^{\pi}[1-2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)]d\theta=\int_0^{\pi}(6-8\cos\theta+2\cos 2\theta)d\theta=\Big[6\theta-8\sin\theta+\sin 2\theta\Big]_0^{\pi}=6\pi$$

25. The graph of $r^2=4\sin 2\theta$ is a lemniscate. We obtain the area of one-half of one of the two loops if $\theta\in[0,\frac{1}{4}\pi]$.

$$A=4\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(4\sin 2w_i)\Delta_i\theta=8\int_0^{\pi/4}\sin 2\theta\,d\theta=-4\cos 2\theta\Big]_0^{\pi/4}=-4(0-1)=4$$

26. $r=4\sin^2\theta\cos\theta$. The folium (see figure) is symmetrical with respect to the polar axis.



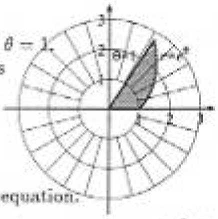
$$A=2\cdot\tfrac{1}{2}\int_0^{\pi/2}16\sin^4\theta\cos^2\theta\,d\theta=\int_0^{\pi/2}4(2\sin\theta\cos\theta)^2\sin^2\theta\,d\theta$$

$$=\int_0^{\pi/2}2\sin^2 2\theta(1-\cos 2\theta)d\theta=\int_0^{\pi/2}(1-\cos 4\theta-2\sin^2 2\theta\cos 2\theta)d\theta$$

$$=\Big[\theta-\tfrac{1}{4}\sin 4\theta-\tfrac{1}{3}\sin^3 2\theta\Big]_0^{\pi/2}=\tfrac{1}{2}\pi$$

27. $A=\lim\limits_{\|\Delta\|\to 0}\sum\limits_{i=1}^{n}\frac{1}{2}(w_i)^2\Delta_i\theta=\frac{1}{2}\int_0^{3\pi/2}\theta^2d\theta=\frac{1}{6}\theta^3\Big]_0^{3\pi/2}=\frac{1}{6}\Big(\frac{3}{2}\pi\Big)^3=\frac{9}{16}\pi^3$

28. Find the area of the region enclosed by the graph of $r=e^{\theta}$ and the lines $\theta=0$ and $\theta=1$.



▸ The curve is an equiangular spiral with endpoints $(1,0)$ and $(e,1)$. The figure shows the region. If A square units is the area of the region, then

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[e^{w_i}]^2\Delta_i\theta=\tfrac{1}{2}\int_0^1 e^{2\theta}d\theta=\tfrac{1}{4}e^{2\theta}\Big]_0^1=\tfrac{1}{4}(e^2-1)$$

■ The area of the region is $\frac{1}{4}(e^2-1)$ square units.

In Exercises 29–32, find the area of the region enclosed by one loop of the graph of the equation.

29. The graph of $r=3\cos 2\theta$ is a four-leafed rose. We get the region enclosed by one-half of a leaf if $\theta\in[0,\frac{1}{4}\pi]$.

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(3\cos 2w_i)^2\Delta_i\theta=9\int_0^{\pi/4}\cos^2 2\theta\,d\theta=9\int_0^{\pi/4}\frac{1+\cos 4\theta}{2}\,d\theta=\frac{9}{2}\theta+\frac{9}{8}\sin 4\theta\Big]_0^{\pi/4}=\frac{9}{8}\pi$$

30. $r=4(1-2\cos\theta)$ is a limaçon with a loop. $A=2\cdot\frac{1}{2}\int_0^{\pi/3}[4(1-2\cos\theta)]^2d\theta=16\int_0^{\pi/3}(1-4\cos\theta+4\cos^2\theta)$

$$=16\int_0^{\pi/3}(1-4\cos\theta+2+2\cos 2\theta)d\theta=16\Big[3\theta-4\sin\theta+\sin 2\theta\Big]_0^{\pi/3}=16(\pi-\tfrac{3}{2}\sqrt{3})16\pi-24\sqrt{3}$$

31. The graph of $r=1+3\sin\theta$ is a limaçon with a loop. The graph is symmetric with respect to the $\frac{1}{2}\pi$ axis. We get the region enclosed by half of the loop if $\theta\in[-\frac{1}{2}\pi,b]$, where $\sin b=-\frac{1}{3}$, $\cos b=\frac{2}{3}\sqrt{2}$, and $b=\sin^{-1}(-\frac{1}{3})$.

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(1+3\sin w_i)^2\Delta_i\theta=\int_{-\pi/2}^{b}(1+3\sin\theta)^2d\theta=\int_{-\pi/2}^{b}(1+6\sin\theta+9\sin^2\theta)d\theta$$

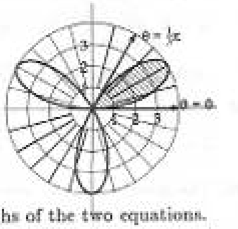
$$=\int_{-\pi/2}^{b}\Big(1+6\sin\theta+\frac{9}{2}-\frac{9}{2}\cos 2\theta\Big)d\theta=\frac{11}{2}\theta-6\cos\theta-\frac{9}{4}\sin 2\theta\Big]_{-\pi/2}^{b}$$

$$=\frac{11}{2}\sin^{-1}\Big(-\frac{1}{3}\Big)-6\Big(\frac{2}{3}\sqrt{2}\Big)-\frac{9}{2}(2)\Big(-\frac{1}{3}\Big)\Big(\frac{2}{3}\sqrt{2}\Big)+\frac{11}{4}\pi=\frac{11}{4}\pi-\frac{11}{2}\sin^{-1}\frac{1}{3}-3\sqrt{2}$$

32. $r = 4 \sin 3\theta$.

▸ The graph is a three-leafed rose. We take the first quadrant loop. Because $r = 0$ when $3\theta = \pi$, or equivalently, $\theta = \frac{1}{3}\pi$, we take the lines $\theta = 0$ and $\theta = \frac{1}{3}\pi$ as boundaries. See the figure. Thus, the area of the loop is given by

$$\begin{aligned} A &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[4 \sin 3w_i]^2 \Delta_i\theta \\ &= \frac{1}{2}\int_0^{\pi/3} [4 \sin 3\theta]^2 d\theta \\ &= 8\int_0^{\pi/3} \sin^2 3\theta \; d\theta \\ &= 8\int_0^{\pi/3} (1 - \cos 6\theta) d\theta = 8\Big[\theta - \frac{1}{6}\sin 6\theta\Big]_0^{\pi/3} = \tfrac{4}{3}\pi \end{aligned}$$



▪ The area of the loop is $\frac{4}{3}\pi$ square units.

In Exercises 33–36, find the area of the intersection of the regions enclosed by the graphs of the two equations.

33. The equations are the circle $r = 2$ and the limaçon with a dent $r = 3 - 2 \cos \theta$. Solving the two equations simultaneously, we get $3 - 2\cos\theta = 2$; $\cos \theta = \frac{1}{2}$; so $\theta = \frac{1}{3}\pi, \frac{5}{3}\pi$. Therefore the points of intersection are $(2, \frac{1}{3}\pi)$ and $(2, \frac{5}{3}\pi)$. We obtain one-half of the required area if $\theta \in [0, \frac{1}{3}\pi]$ for the limaçon and $\theta \in [\frac{1}{3}\pi, \pi]$ for the circle.

$$A = 2\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(3 - 2\cos w_i)^2 \Delta_i\theta + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(2)^2 \Delta_i\theta\Big] = \int_0^{\pi/3} (9 - 12\cos\theta + 4\cos^2\theta) d\theta + 4\int_{\pi/3}^{\pi} d\theta$$

$$= \Big[9\theta - 12 \sin\theta + 4\int \frac{1 + \cos 2\theta}{2}\, d\theta\Big]_0^{\pi/3} + 4\theta\Big]_0^{\pi/3} = 9\theta - 12 \sin\theta + 2\theta + \sin 2\theta\Big]_0^{\pi/3} + \Big(4\pi - \frac{4}{2}\pi\Big)$$

$$= \Big[\frac{11}{3}\pi - 12\Big(\frac{1}{2}\sqrt{3}\Big) + \frac{1}{2}\sqrt{3}\Big] + \frac{8}{3}\pi = \frac{19}{3}\pi - \frac{11}{2}\sqrt{3}$$

34. $r = 4 \sin \theta$, $r = 4 \cos \theta$ are congruent circles point upward and to the right. By symmetry,

$$A = 2 \cdot \tfrac{1}{2}\int_0^{\pi/4} (4 \sin\theta)^2 d\theta = 8\int_0^{\pi/4} (1 - \cos 2\theta) d\theta = 8\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4} = 8(\tfrac{1}{4}\pi - \tfrac{1}{2}) = 2\pi - 4$$

35. The given equations are the four-leafed roses $r = 3 \sin 2\theta$ and $r = 3 \cos 2\theta$. We obtain one-eighth of the area if $\theta \in [0, \frac{1}{8}\pi]$ for the first graph and $\theta \in [\frac{1}{8}\pi, \frac{1}{4}\pi]$ for the second.

$$A = 8\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(3 \sin 2w_i)^2 \Delta_i\theta + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(3 \cos 2w_i)^2 \Delta_i\theta\Big] = 4\int_0^{\pi/8} 9 \sin^2 2\theta \, d\theta + 4\int_{\pi/8}^{\pi/4} 9 \cos^2 2\theta \, d\theta$$

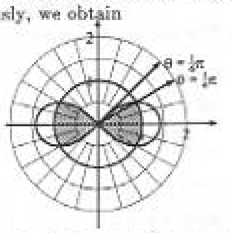
$$= 18\int_0^{\pi/8} (1 - \cos 4\theta) d\theta + 18\int_{\pi/8}^{\pi/4} (1 + \cos 4\theta) d\theta = 18\Big[\theta - \frac{1}{4}\sin 4\theta\Big]_0^{\pi/8} + 18\Big[\theta + \frac{1}{4}\sin 4\theta\Big]_{\pi/8}^{\pi/4}$$

$$= 18\Big(\frac{\pi}{8} - \frac{1}{4}\Big) + 18\Big(\frac{\pi}{4} + 0 - \frac{\pi}{8} - \frac{1}{4}\Big) = \frac{9}{4}\pi - \frac{9}{2} + \frac{9}{2}\pi - \frac{9}{4}\pi - \frac{9}{2} = \frac{9}{2}\pi - 9$$

36. $r^2 = 2 \cos 2\theta$ and $r = 1$.

▸ The region is the intersection of the regions bounded by the lemniscate $r^2 = 2 \cos 2\theta$ and the circle $r = 1$, as shown in the figure. Let R be the first quadrant portion of the required region. Because of symmetry, the area of the entire region is four times the area of R. Solving the equations simultaneously, we obtain

$$1 = 2 \cos 2\theta, \quad \cos 2\theta = \tfrac{1}{2}, \qquad 2\theta = \tfrac{1}{3}\pi, \qquad \theta = \tfrac{1}{6}\pi$$



Thus, the line $\theta = \frac{1}{6}\pi$ contains the first-quadrant intersection of the lemniscate and the circle. Let R_1 be the part of region R that is between the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. Because the region R_1 is bounded by the circle $r = 1$ and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$, we have

$$A_1 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}\Delta_i\theta = \frac{1}{2}\int_0^{\pi/6} d\theta = \frac{1}{2}\theta\Big]_0^{\pi/6} = \frac{1}{12}\pi \qquad (1)$$

Let R_2 be the part of the region R that is not in region R_1. Setting $r = 0$ in the equation $r^2 = 2 \cos 2\theta$, we get $\theta = \frac{1}{4}\pi$. Thus, R_2 is bounded by the lines $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{4}\pi$ and the lemniscate $r^2 = 2 \cos 2\theta$. Hence,

$$A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[2 \cos 2w_i]\Delta_i\theta = \frac{1}{2}\int_{\pi/6}^{\pi/4} (2 \cos 2\theta) d\theta = \tfrac{1}{2}\sin 2\theta\Big]_{\pi/6}^{\pi/4} = \tfrac{1}{2}(1 - \tfrac{1}{2}\sqrt{3}) \qquad (2)$$

Using (1) and (2), the area of the entire region, is given by

$$A = 4(A_1 + A_2) = 4(\tfrac{1}{12}\pi + \tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}) = \tfrac{1}{3}\pi + 2 - \sqrt{3}$$

▪ The area of the region is $\frac{1}{3}\pi + 2 - \sqrt{3}$ square units.

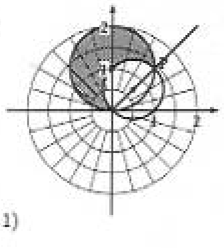
In Exercises 37–44, find the area inside the graph of the first equation and outside the graph of the second.
In Exercises 41–44, (a) find all points of intersection, then (b) find the required area.

37. The given equations are the circle $r = 3$ and the cardioid $r = 3(1 - \cos\theta)$. Each graph is symmetric with respect to the polar axis. We get one-half the required area if $\theta \in [0, \frac{1}{2}\pi]$.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}[9 - 9(1 - \cos w_i)^2]\Delta_i\theta = \int_0^{\pi/2} [9 - 9(1 - 2\cos\theta + \cos^2\theta)]d\theta$$
$$= 9\int_0^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta = 9\left[2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = 9(2 - \tfrac{1}{4}\pi + \tfrac{1}{4} - \tfrac{1}{4}) = 9(2 - \tfrac{1}{4}\pi)$$

38. The lemniscate $r^2 = 4\sin 2\theta$ and the circle $r = \sqrt{2}$ are symmetric with respect to $\theta = \frac{1}{4}\pi$ meet when $2 = 4\sin 2\theta$, $\sin 2\theta = \frac{1}{2}$, $2\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{12}\pi$. $A = 4 \cdot \frac{1}{2}\int_{\pi/12}^{\pi/4}(4\sin 2\theta - 2)d\theta = 2\Big[-2\cos 2\theta - 2\theta\Big]_{\pi/12}^{\pi/4} = 2\sqrt{3} - \frac{2}{3}\pi$

39. The given equations are the circle $r = 2\sin\theta$ with center at $(1, \frac{1}{2}\pi)$ and radius 1, and the circle $r = \sin\theta + \cos\theta$ with center at $(\frac{1}{2}\sqrt{2}, \frac{1}{4}\pi)$ and radius $\frac{1}{2}\sqrt{2}$. The two circles intersect at the pole and at $P = (\sqrt{2}, \frac{1}{4}\pi)$. The required region is that enclosed by the first circle except for two regions, one of which is one-half of the region enclosed by the second circle and the other is a segment of the first circle formed by the diameter of the second circle from the pole to P. See the figure.



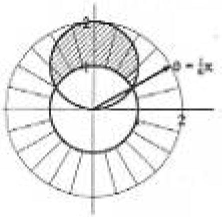
$$A = \pi - \left[\frac{1}{4}\pi + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(2\sin w_i)^2\Delta_i\theta\right] = \pi - \frac{1}{4}\pi - 2\int_0^{\pi/4}\sin^2\theta\, d\theta$$
$$= \frac{3}{4}\pi - \int_0^{\pi/4}(1 - \cos 2\theta)d\theta = \frac{3}{4}\pi - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{3}{4}\pi - \left(\frac{1}{4}\pi - \frac{1}{2}\sin\frac{\pi}{2}\right) = \frac{1}{2}(\pi + 1)$$

40. $r = 4\sin\theta$ and $r = 2$.

▸ The figure shows the region R. Let R_1 be the first-quadrant part of R. Solving the equations simultaneously, we have

$$4\sin\theta = 2, \quad \sin\theta = \tfrac{1}{2}, \qquad \theta = \tfrac{1}{6}\pi$$

Thus, the region R_1 is bounded by the lines $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{2}\pi$ and by the curves $r = 2$ and $r = 4\sin\theta$, where $2 \leq 4\sin\theta$ when $\frac{1}{6}\pi \leq \theta \leq \frac{1}{2}\pi$. Thus, the area of the entire region is given by



$$A = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[(4\sin w_i)^2 - 2^2]\Delta_i\theta$$
$$= \int_{\pi/6}^{\pi/2}[(4\sin\theta)^2 - 2^2]d\theta$$
$$= 4\int_{\pi/6}^{\pi/2}[2(1 - \cos 2\theta) - 1]d\theta = 4\int_{\pi/6}^{\pi/2}(1 - 2\cos 2\theta)d\theta$$
$$= 4\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/2} = 4[\tfrac{1}{2}\pi - (\tfrac{1}{6}\pi - \tfrac{1}{2}\sqrt{3})] = 4(\tfrac{1}{3}\pi + \tfrac{1}{2}\sqrt{3})$$

▪ The area of the region is $4(\frac{1}{3}\pi + \frac{1}{2}\sqrt{3})$ square units.

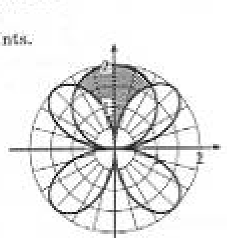
41. $r = 1 + 4\cos\theta$ (loop) and $r = 1$. (a) $1 + 4\cos\theta = 1$, $4\cos\theta = 0$, $\theta = \pm\frac{1}{2}\pi$ and $1 + 4\cos\theta = -1$, $4\cos\theta = -2$, $\cos\theta = -\frac{1}{2}$, $\theta = \pm\frac{2}{3}\pi$. (b) $A = 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi}[(1 + 4\cos\theta)^2 - 1]d\theta = 8\int_{2\pi/3}^{\pi}(\cos\theta + 2\cos^2\theta)d\theta$
$$= 8\int_{2\pi/3}^{\pi}(\cos\theta + 1 + \cos 2\theta)d\theta = 8\Big[\sin\theta + \theta + \tfrac{1}{2}\sin 2\theta\Big]_{2\pi/3}^{\pi} = \tfrac{8}{3}\pi - 2\sqrt{3}$$

42. $r = 1 - 3\sin\theta$ (loop) and $r = 1$. (a) $1 - 3\sin\theta = 1$, $-3\sin\theta = 0$, $\theta = 0$, π and $1 - 3\sin\theta = -1$, $3\sin\theta = 2$, $\sin\theta = \frac{2}{3}$, $\theta = a = \sin^{-1}\frac{2}{3}$, $\theta = \pi - \sin^{-1}\frac{2}{3}$. (b) $\cos a = \sqrt{1 - \frac{4}{9}} = \frac{1}{3}\sqrt{5}$, $\sin 2a = 2\sin a\cos a = \frac{4}{9}\sqrt{5}$
$$A = 2 \cdot \tfrac{1}{2}\int_a^{\pi/2}[(1 - 3\sin\theta)^2 - 1]d\theta = 3\int_a^{\pi/2}(3\sin^2\theta - 2\sin\theta)d\theta = 3\int_a^{\pi/2}(\tfrac{3}{2} - \tfrac{3}{2}\cos 2\theta - 2\sin\theta)d\theta$$
$$= 3\Big[\tfrac{3}{2}\theta - \tfrac{3}{4}\sin 2\theta + 2\cos\theta\Big]_a^{\pi/2} = 3(\tfrac{3}{4}\pi - \tfrac{3}{2}\sin^{-1}\tfrac{2}{3} + \tfrac{1}{3}\sqrt{5} - \tfrac{2}{3}\sqrt{5}) = 3(\tfrac{3}{4}\pi - \tfrac{3}{2}\sin^{-1}\tfrac{2}{3} - \tfrac{1}{3}\sqrt{5})$$

43. $r = 4\cos 2\theta$ (4 leaves) and $r = 2$. (a) $4\cos 2\theta = 2$; $\cos 2\theta = \frac{1}{2}$; $2\theta = \pm\frac{1}{3}\pi$, $\pm\frac{7}{3}\pi$; $\theta = \pm\frac{1}{6}\pi$, $\pm\frac{7}{6}\pi$ and $4\cos 2\theta = -2$; $\cos 2\theta = -\frac{1}{2}$; $2\theta = \pm\frac{2}{3}\pi$, $\pm\frac{4}{3}\pi$; $\theta = \pm\frac{1}{3}\pi$, $\pm\frac{2}{3}\pi$. $A = 8 \cdot \frac{1}{2}\int_0^{\pi/6}[(4\cos 2\theta)^2 - 2^2]d\theta$
$$= 16\int_0^{\pi/6}(1 + 2\cos 4\theta)d\theta = 16\Big[\theta + \tfrac{1}{2}\sin 4\theta\Big]_0^{\pi/6} = 16(\tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3}) = \tfrac{8}{3}\pi + 4\sqrt{3}$$

44. (a) Find the coordinates of all points of intersection of the circle $r = 2 \sin \theta$ and the rose $r = 2 \sin 2\theta$.
(b) Find the area of the region inside the circle and outside the rose.

▹ (a) From the figure we see that the curves intersect in the pole and two other points.
We use $\sin 2\theta = 2 \sin \theta \cos \theta$ and solve simultaneously.



$$4 \sin \theta \cos \theta = 2 \sin \theta$$
$$4 \sin \theta \cos \theta - 2 \sin \theta = 0$$
$$4 \sin \theta(\cos \theta - \tfrac{1}{2}) = 0$$

$\sin \theta = 0$, $\theta = 0, \pi$ give the pole
$\cos \theta = \frac{1}{2}$, $\theta = \frac{1}{3}\pi, \frac{2}{3}\pi$
Thus, the points of intersection are the pole, $(\sqrt{3}, \frac{1}{3}\pi)$ and $(\sqrt{3}, \frac{2}{3}\pi)$.
(b) By symmetry, we find

$$A = 2 \cdot \tfrac{1}{2} \int_{\pi/3}^{\pi/2} [(2 \sin \theta)^2 - (2 \sin 2\theta)^2] d\theta = 2 \int_{\pi/3}^{\pi/2} [(1 - \cos 2\theta) - (1 - \cos 4\theta)] d\theta$$
$$= 2\Big[\tfrac{1}{4} \sin 4\theta - \tfrac{1}{2} \sin 2\theta\Big]_{\pi/3}^{\pi/2} = \tfrac{1}{2}(\sin 2\pi - \sin \tfrac{4}{3}\pi) - (\sin \pi - \sin \tfrac{2}{3}\pi) = \tfrac{1}{2} \cdot \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3} = \tfrac{3}{4}\sqrt{3}$$

▪ The area of the region is $\frac{3}{4}\sqrt{3}$ square units.

45. We obtain one-fourth of the area of the face of the bow tie when $\theta \in [0, \frac{1}{4}\pi]$.

$$A = 4 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(4 \cos 2w_i)\Delta_i\theta = 8 \int_0^{\pi/4} \cos 2\theta \, d\theta = 4 \sin 2\theta \Big]_0^{\pi/4} = 4$$

46. $9\pi = 2 \cdot \frac{1}{2} \int_0^{\pi} [a(1 - \cos \theta)]^2 d\theta = a^2 \int_0^{\pi} [1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta = a^2\Big[\frac{3}{2}\theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta\Big]_0^{\pi} = \frac{3}{2}\pi a^2$
$a^2 = 6$, $a = \sqrt{6}$

47. $A = \frac{1}{2} \int_0^{2\pi} [a(\theta + 2\pi)^2 - (a\theta)^2] d\theta = \frac{1}{2}a^2 \int_0^{2\pi} (4\pi\theta + 4\pi^2) d\theta = \frac{1}{2}a^2\Big[2\pi\theta^2 + 4\pi^2\theta\Big]_0^{2\pi} = 8\pi^3 a^2$

48. Find the area of the region swept out by the radius vector of the spiral $r = a\theta$ during its third revolution that was not swept out during its second revolution.

▹ If the radius vector sweeps out a region of area A_3 square units during its third revolution when $\theta \in [4\pi, 6\pi]$ and A_2 square units during its second revolution when $\theta \in [2\pi, 4\pi]$, then

$$A = A_3 - A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(aw_i)^2 \Delta_i\theta - \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(aw_i)^2 \Delta_i\theta = \frac{1}{2} a^2 \int_{4\pi}^{6\pi} \theta^2 d\theta - \frac{1}{2} a^2 \int_{2\pi}^{4\pi} \theta^2 d\theta$$
$$= \frac{1}{6} a^2\theta^3 \Big]_{4\pi}^{6\pi} - \frac{1}{6} a^2\theta^3 \Big]_{2\pi}^{4\pi} = \frac{1}{6} a^2(216\pi^3 - 64\pi^3 - 64\pi^3 + 8\pi^3) = 16a^2\pi^3$$

49. We are given the cardioid $r = a(1 + \cos\theta)$ and the circle $r = 2a\cos\theta$ of radius a. The circle is contained in the cardioid. We get top half of the cardioid if $\theta \in [0, \pi]$ and the top half of the circle if $\theta \in [0, \frac{1}{2}\pi]$.

$$A = 2\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(a + a \cos w_i)^2 \Delta_i\theta - \tfrac{1}{2}\pi a^2\Big] = a^2 \int_0^{\pi} (1 + 2 \cos \theta + \cos^2\theta) d\theta - \pi a^2$$
$$= a^2 \int_0^{\pi} \Big(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta\Big) d\theta - \pi a^2 = a^2\Big[\frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta\Big]_0^{\pi} - \pi a^2 = \frac{3}{2}\pi a^2 - \pi a^2 = \frac{1}{2}\pi a^2$$

50. $[F'(\theta) \cos \theta - F(\theta) \sin \theta]^2 + [F'(\theta) \sin \theta + F(\theta) \cos \theta]^2$
$= [F'(\theta)^2\cos^2\theta - 2F'(\theta)F(\theta)\sin \theta \cos \theta + F(\theta)^2\sin^2\theta] + [F'(\theta)^2\sin^2\theta + 2F'(\theta)F(\theta)\sin \theta \cos \theta + F(\theta)^2\cos^2\theta]$
$= F'(\theta)^2(\cos^2\theta + \sin^2\theta) + F(\theta)^2(\sin^2\theta + \cos^2\theta) = F'(\theta)^2 + F(\theta)^2$

9.5 A UNIFIED TREATMENT OF CONIC SECTIONS AND POLAR EQUATIONS OF CONICS

9.5.1 Theorem A conic section can be defined as the set of all points P in a plane such that the ratio of the undirected distance of P from a fixed point to the undirected distance of P from a fixed line that does not contain the fixed point is a positive constant e. Furthermore, if $e = 1$, the conic is a parabola; if $0 < e < 1$, it is an ellipse; and if $e > 1$, it is a hyperbola.

The positive constant of Theorem 9.5.1 is called the *eccentricity* of the conic. (It has nothing to do with 2.718….) The fixed point is a focus of the conic, and the fixed line is called a *directrix* of the conic. A parabola has only one focus and one directrix. An ellipse and a hyperbola, the *central conics*, have two foci and two corresponding directrices. A circle has eccentricity 0.

9.5.2 Theorem The central conic having the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

where $a > 0$, has a focus at $(-ae, 0)$, whose corresponding directrix is $x = -a/e$, and a focus at $(ae, 0)$, whose corresponding directrix is $x = a/e$.

Note that if $0 < e < 1$, then $1 - e^2 > 0$ and the central conic of Theorem 9.5.2 is an ellipse. However, if $e > 1$, then $1 - e^2 < 0$, and the conic is a hyperbola. If x and y are interchanged in the equation of Theorem 9.5.2, then there will be a focus at $(0, -ae)$ with corresponding directrix $y = -a/e$, and a focus at $(0, ae)$ with corresponding directrix $y = a/e$.

Standard Form For a central conic in standard form, $e = \frac{c}{a}$ and the directrix is $\delta = \frac{a}{e} = \frac{a^2}{c}$ units from C.

Asymptotes Lines through center in two directions for which r is not defined.

For polar equations of the conics let e be the eccentricity of a conic and let d be the undirected distance from a focus to the corresponding directrix. If the focus is at the pole and the axis is the line $\theta = \theta_0$ then the vertex nearest the pole is in the direction θ, the equation of the conic is

$$r = \frac{ed}{1 + e\cos(\theta - \theta_0)}$$

and the equation of the directrix nearest the pole is $r\cos(\theta - \theta_0) = d$, obtained by setting the constant of the denominator to 0. We have the following four special cases.

θ	Equation of conic	Directrix nearest pole	Vertex nearest pole
0	$r = \frac{ed}{1 + e\cos\theta}$	$r\cos\theta = d$	To the right of the pole
$\frac{1}{2}\pi$	$r = \frac{ed}{1 + e\sin\theta}$	$r\sin\theta = d$	Above the pole
π	$r = \frac{ed}{1 - e\cos\theta}$	$r\cos\theta = -d$	To the left of the pole
$\frac{3}{2}\pi$	$r = \frac{ed}{1 - e\sin\theta}$	$r\sin\theta = -d$	Below the pole

Exercises 9.5

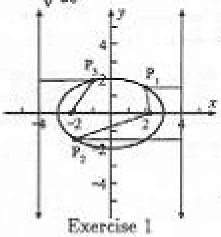
In Exercises 1–8, (a) find the eccentricity e, foci F, and directrices ℓ of the central conic. (b) Sketch the conic and show foci and directrices. Also choose a point P in each of three quadrants and draw the line segments to a focus and corresponding directrix. Observe that the ratio of their lengths is e.

▸ XA means the x axis is the principal axis; YA means the y axis is the principal axis.

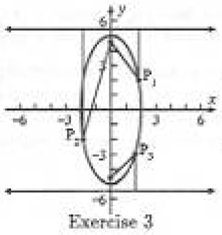
1. $4x^2 + 9y^2 = 36$ ▸ $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Ellipse, XA. $a = 3$, $b = 2$, $c = \sqrt{9-4} = \sqrt{5}$. $e = \frac{\sqrt{5}}{3}$, $\delta = \frac{9}{\sqrt{5}} = \frac{9}{5}\sqrt{5}$. Foci $(\pm\sqrt{5}, 0)$, directrices: $x = \pm\frac{9}{5}\sqrt{5}$

3. $25x^2 + 4y^2 = 100$ ▸ $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Ellipse, YA. $a = 5$, $b = 2$, $c = \sqrt{25-4} = \sqrt{21}$. $e = \frac{\sqrt{21}}{5}$, $\delta = \frac{25}{\sqrt{21}} = \frac{25}{21}\sqrt{21}$. Foci: $(0, \pm\sqrt{21})$, directrices: $y = \pm\frac{25}{21}\sqrt{21}$

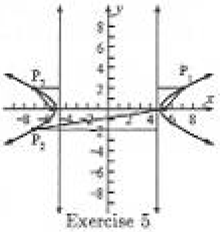
5. $4x^2 - 25y^2 = 100$ ▸ $\frac{x^2}{25} - \frac{y^2}{4} = 1$. Hyperbola, XA. $a = 5$, $b = 2$, $c = \sqrt{25+4} = \sqrt{29}$. $e = \frac{\sqrt{29}}{5}$, $\delta = \frac{25}{\sqrt{29}} = \frac{25}{29}\sqrt{29}$. Foci: $(\pm\sqrt{29}, 0)$, directrices: $x = \pm\frac{25}{29}\sqrt{29}$



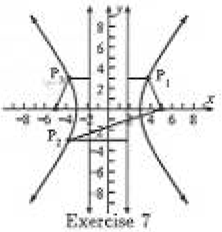
Exercise 1



Exercise 3



Exercise 5



Exercise 7

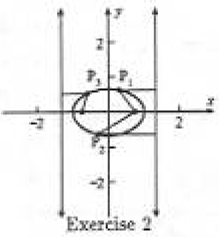
7. $16x^2 - 9y^2 = 144$ ▹ $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Hyperbola, XA. $a = 3$, $b = 4$, $c = \sqrt{9+16} = 5$. $e = \frac{5}{3}$. $\delta = \frac{9}{5}$. Foci: $(\pm 5, 0)$, directrices: $x = \pm\frac{9}{5}$

2. $4x^2 + 9y^2 = 4$ ▹ $\frac{x^2}{1} + \frac{y^2}{\frac{4}{9}} = 1$. Ellipse, XA. $a = 1$, $b = \frac{2}{3}$, $c = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{1}{3}\sqrt{5}$. $e = \frac{1}{3}\sqrt{5}$, $\delta = \frac{1}{\sqrt{5/3}} = \frac{3}{5}\sqrt{5}$. Foci: $(\pm\frac{1}{3}\sqrt{5}, 0)$, directrices: $x = \pm\frac{3}{5}\sqrt{5}$

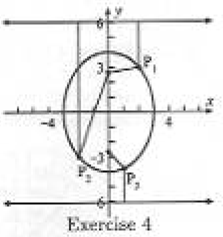
4. $16x^2 + 9y^2 = 144$ ▹ $\frac{x^2}{9} + \frac{y^2}{16} = 1$. Ellipse, YA. $a = 4$, $b = 2$, $c = \sqrt{16-9} = \sqrt{7}$. $e = \frac{1}{4}\sqrt{7}$, $\delta = \frac{16}{\sqrt{7}} = \frac{16}{7}\sqrt{7}$. Foci: $(0, \pm\sqrt{7})$, directrices: $y = \pm\frac{16}{7}\sqrt{7}$

6. $x^2 - 9y^2 = 9$ ▹ $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Hyperbola, XA. $a = 3$, $b = 1$, $c = \sqrt{9+1} = \sqrt{10}$. $e = \frac{1}{3}\sqrt{10}$, $\delta = \frac{9}{\sqrt{10}} = \frac{9}{10}\sqrt{10}$. Foci: $(\pm\sqrt{10}, 0)$, directrices: $x = \pm\frac{9}{10}\sqrt{10}$

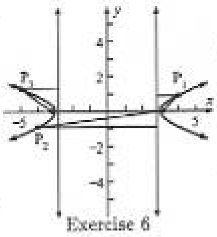
8. $4y^2 - x^2 = 16$ ▹ $\frac{y^2}{4} - \frac{x^2}{16} = 1$. Hyperbola, YA. $a = 2$, $b = 4$, $c = \sqrt{4+16} = 2\sqrt{5}$. $e = \frac{2\sqrt{5}}{2} = \sqrt{5}$, $\delta = \frac{4}{2\sqrt{5}} = \frac{2}{5}\sqrt{5}$. Foci: $(0, \pm 2\sqrt{5})$, directrices: $y = \pm\frac{2}{5}\sqrt{5}$.



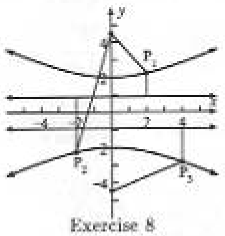
Exercise 2



Exercise 4



Exercise 6



Exercise 8

In Exercises 9 and 10, the conic has a focus F at the pole. Identify the conic.

9. (a) $r = \frac{3}{1 - \cos\theta}$ ▹ $e = 1$: parabola (b) $r = \frac{6}{4 + 5\sin\theta}$ ▹ $e = \frac{5}{4}$: hyperbola

(c) $r = \frac{5}{4 - \cos\theta}$ ▹ $e = \frac{1}{4}$: ellipse (d) $r = \frac{4}{1 + \sin\pi}$ ▹ circle of radius 4

10. (a) $r = \frac{1}{1 - \sin\theta}$ ▹ $e = 1$: parabola (b) $r = \frac{2}{3 + \sin\theta}$ ▹ $e = \frac{1}{3}$: ellipse

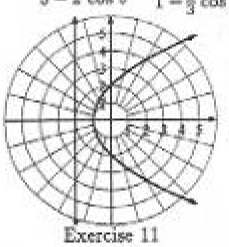
(c) $r = \frac{3}{2 + 4\cos\theta}$ ▹ $e = 2$: hyperbola (d) $5 = \frac{5}{1 - \cos\pi}$ ▹ circle of radius $\frac{5}{2}$

In Exercises 11–22, the conic has a focus F at the pole. Find (a) the eccentricity e, (b) the type of conic, (c) an equation of the directrix ℓ of F. (d) Sketch the conic and check by plotting.

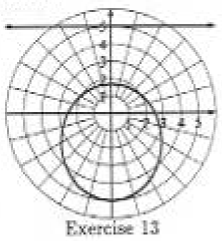
11. $r = \frac{2}{1 - \cos\theta} = \frac{ed}{1 - e\cos\theta}$ with $e = 1$ and $d = \frac{2}{1} = 2$. A parabola. ℓ is to the left of F: $r\cos\theta = -2$

13. $r = \frac{5}{2 + \sin\theta} = \frac{\frac{5}{2}}{1 + \frac{1}{2}\sin\theta} = \frac{ed}{1 + e\sin\theta}$ with $e = \frac{1}{2}$ and $d = \frac{5}{1} = 5$. An ellipse. ℓ is above F: $r\sin\theta = 5$

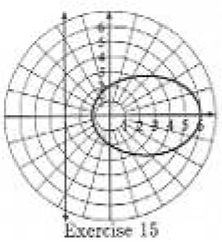
15. $r = \frac{6}{3 - 2\cos\theta} = \frac{2}{1 - \frac{2}{3}\cos\theta} = \frac{ed}{1 - e\cos\theta}$ with $e = \frac{2}{3}$ and $d = \frac{6}{2} = 3$. Ellipse. ℓ is left of F: $r\cos\theta = -3$



Exercise 11



Exercise 13

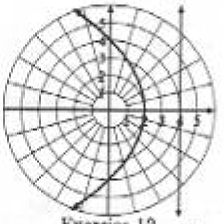


Exercise 15

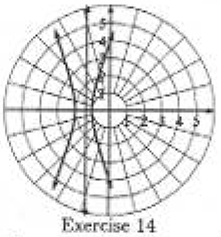
12. $r = \frac{4}{1+\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = 1$ and $d = \frac{4}{1} = 4$. A parabola. ℓ is to the right of F: $r\cos\theta = 4$

14. $r = \frac{4}{1-3\cos\theta} = \frac{ed}{1-e\cos\theta}$ with $e = 3$ and $d = \frac{4}{3}$. A hyperbola. ℓ is to the left of F: $r\cos\theta = -\frac{4}{3}$

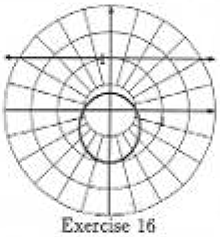
16. $r = \frac{1}{2+\sin\theta} = \frac{\frac{1}{2}}{1+\frac{1}{2}\sin\theta} = \frac{ed}{1+e\sin\theta}$ with $e = \frac{1}{2}$ and $d = 1$. An ellipse. ℓ is above F: $r\sin\theta = 1$



Exercise 12



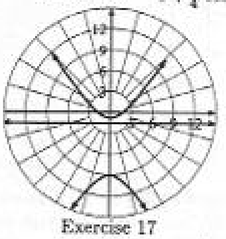
Exercise 14


Exercise 16

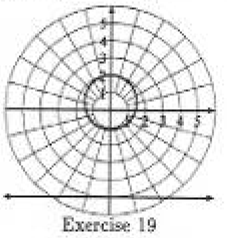
17. $r = \frac{9}{5-6\sin\theta} = \frac{\frac{9}{5}}{1-\frac{6}{5}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{6}{5}$ and $d = \frac{9}{6} = \frac{3}{2}$. Hyperbola. ℓ is below F: $r\sin\theta = -\frac{3}{2}$

19. $r = \frac{10}{7-2\sin\theta} = \frac{\frac{10}{7}}{1-\frac{2}{7}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{2}{7}$ and $d = \frac{10}{2} = 5$. Ellipse. ℓ is below F: $r\sin\theta = -5$

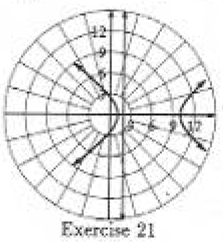
21. $r = \frac{10}{4+5\cos\theta} = \frac{\frac{5}{2}}{1+\frac{5}{4}\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = \frac{5}{4}$ and $d = \frac{10}{5} = 2$. Hyperbola. ℓ is right of F: $r\cos\theta = 2$



Exercise 17



Exercise 19

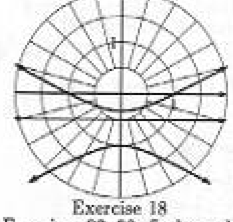


Exercise 21

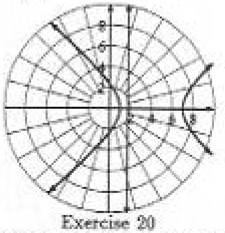
18. $r = \frac{1}{1-2\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = 2$ and $d = \frac{1}{2}$. Hyperbola. ℓ is below F: $r\sin\theta = -\frac{1}{2}$

20. $r = \frac{7}{3+4\cos\theta} = \frac{\frac{7}{3}}{1+\frac{4}{3}\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = \frac{4}{3}$ and $d = \frac{7}{4}$. Hyperbola. ℓ is right of F: $r\cos\theta = \frac{7}{4}$

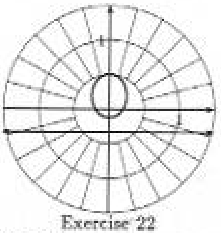
22. $r = \frac{1}{5-3\sin\theta} = \frac{\frac{1}{5}}{1-\frac{3}{5}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{3}{5}$ and $d = \frac{1}{3}$. Ellipse. ℓ is below F: $r\sin\theta = -\frac{1}{3}$



Exercise 18



Exercise 20



Exercise 22

In Exercises 23–30, find a polar equation of the conic having a focus at the pole.

23. Parabola; vertex at $(4, \frac{3}{2}\pi)$. ▹ $e = 1$, ℓ is horizontal $4 + 4 = 8$ units below F. $r = \frac{8}{1-\sin\theta}$

24. Ellipse; $e = \frac{1}{2}$; the corresponding vertex at $(4, \pi)$.

▹ ℓ is vertical, $4 + 4/\frac{1}{2} = 12$ units left of F. $r = \frac{6}{1-\frac{1}{2}\cos\theta} = \frac{12}{2-\cos\theta}$

25. Hyperbola; $e = \frac{4}{3}$; $r\cos\theta = 9$ is the corresponding directrix.

▷ ℓ is vertical 9 units right of pole. $r = \dfrac{\frac{4}{3}\cdot 9}{1 + \frac{4}{3}\cos\theta} = \dfrac{36}{3 + 4\cos\theta}$

26. Hyperbola; vertices at $\left(1, \frac{1}{2}\pi\right)$ and $\left(3, \frac{1}{2}\pi\right) = \left(-3, \frac{3}{2}\pi\right)$.

▷ ℓ is horizontal, above F. $\theta = \frac{1}{2}\pi$: $1 = \dfrac{ed}{1+e}$. A: $1 + e = ed$. $\theta = \frac{3}{2}\pi$: $-3 = \dfrac{ed}{1-e}$. B: $1 - e = -\frac{1}{3}ed$

A + 3B: $4 - 2e = 0$, $e = 2$, $ed = 3$. $r = \dfrac{3}{1 + 2\sin\theta}$

27. Ellipse; vertices at $(3, 0)$ and $(1, \pi)$.

ℓ is vertical, left of F. $r = \dfrac{ed}{1 - e\cos\theta}$. $\theta = 0$: $3 = \dfrac{ed}{1-e}$. A: $1 - e = \frac{1}{3}ed$. $\theta = \pi$: $1 = \dfrac{ed}{1+e}$. B: $1 + e = ed$

B − 3a: $-2 + 4e = e$, $e = \frac{1}{2}$, $ed = \frac{3}{2}$. $r = \dfrac{\frac{3}{2}}{1 - \frac{1}{2}\cos\theta} = \dfrac{3}{2 - \cos\theta}$

28. Parabola; vertex at $\left(6, \frac{1}{2}\pi\right)$. ▷ $e = 1$, ℓ is horizontal $6 + 6 = 12$ units above F. $r = \dfrac{12}{1 + \sin\theta}$

29. (a) $e = 3$, ℓ is left of the pole and $\left(2, \frac{4}{3}\pi\right)$ is on the hyperbola. (b) Write an equation of ℓ.

▷ (a) $r = \dfrac{3d}{1 - 3\cos\theta}$. $2 = \dfrac{3d}{1 - 3\cos\frac{4}{3}\pi} = \dfrac{3d}{1 + \frac{3}{2}}$, $3d = 5$, $d = \frac{5}{3}$. $r = \dfrac{5}{1 - 3\cos\theta}$ (b) $r\cos\theta = -\frac{5}{3}$

30. (a) $e = 3$, ℓ is $r\sin\theta = 3$. (b) Find equations of the lines through the pole parallel to the asymptotes.

▷ (a) $d = 3$. $r = \dfrac{3(3)}{1 + 3\sin\theta} = \dfrac{9}{1 + 3\sin\theta}$ (b) $\sin\theta = -\frac{1}{3}$, $\theta = -0.34$ and $\theta = 3.48$

31. A units is the area inside the ellipse $r = 6/(2 - \sin\theta)$ and above the parabola $r = 3/(1 + \sin\theta)$. The two curves intersect at the points $(3, 0)$ and $(3, \pi)$ and each is symmetric with respect to the $\frac{1}{2}\pi$ axis. See the figure for Exercise 32. Then

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}\left[\left(\frac{6}{2 - \sin w_i}\right)^2 - \left(\frac{3}{1 + \sin w_i}\right)^2\right]\Delta_i\theta = \int_0^{\pi/2}\left[\left(\frac{6}{2 - \sin\theta}\right)^2 - \left(\frac{3}{1 + \sin\theta}\right)^2\right]d\theta$$

Because cosines are easier to integrate, let $\theta = \frac{1}{2}\pi - x$, $\sin\theta = \cos x$. Therefore

$A = \displaystyle\int_0^{\pi/2}\left[\frac{36}{(2 - \cos x)^2} - \frac{9}{(1 + \cos x)^2}\right]dx$. Now let $z = \tan\frac{1}{2}\theta$ so $\cos\theta = \dfrac{1 - z^2}{1 + z^2}$ and $d\theta = \dfrac{2\,dz}{1 + z^2}$. Thus

$$A = \int_0^1\left[\frac{36\cdot\frac{2}{1+z^2}}{\left(2 - \frac{1 - z^2}{1 + z^2}\right)^2} - \frac{9\cdot\frac{2}{1+z^2}}{\left(1 + \frac{1 - z^2}{1 + z^2}\right)^2}\right]dz = 72\int_0^1\frac{1 + z^2}{(3z^2 + 1)^2}\,dz - \frac{9}{2}\int_0^1(1 + z^2)dz = I - J$$

In I, let $z = \frac{1}{3}\sqrt{3}\tan u$. Then $dz = \frac{1}{3}\sqrt{3}\sec^2 u\,du$, and $u = \frac{1}{3}\pi$ when $z = 1$. Hence

$$I = 72\int_0^{\pi/3}\frac{1 + \frac{1}{3}\tan^2 u}{\sec^4 u}\left(\frac{1}{3}\sqrt{3}\sec^2 u\,du\right) = 24\sqrt{3}\int_0^{\pi/3}\left(\cos^2 u + \frac{1}{3}\sin^2 u\right)du$$

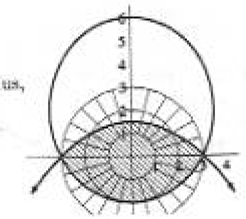
$$= 24\sqrt{3}\int_0^{\pi/3}\left[\frac{1 + \cos 2u}{2} + \frac{1 - \cos 2u}{6}\right]du = 4\sqrt{3}\int_0^{\pi/3}(4 + 2\cos 2u)du = 4\sqrt{3}\Big[4u + \sin 2u\Big]_0^{\pi/3} = \frac{16}{3}\sqrt{3}\pi + 6$$

$J = \frac{9}{2}\Big[z + \frac{1}{3}z^3\Big]_0^1 = \frac{9}{2}\cdot\frac{4}{3} = 6$. Therefore $A = \left(\frac{16}{3}\sqrt{3}\pi + 6\right) - 6 = \frac{16}{3}\sqrt{3}\pi$.

32. Find the area of the region inside the ellipse $r = 6/(2 - \sin\theta)$ and below the parabola $r = 3/(1 + \sin\theta)$.

▷ The two curves intersect at the points $(3, 0)$ and $(3, \pi)$ and each is symmetric with respect to the $\frac{1}{2}\pi$ axis. The figure shows the region R. Let R_1 be that part of the region R in the fourth quadrant. Then R_1 is bounded by the ellipse $r = 6/(2 - \sin\theta)$ and the lines $\theta = -\frac{1}{2}\pi$ and $\theta = 0$. Thus,

$$A_1 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}\left[\frac{6}{2 - \sin w_i}\right]^2\Delta_i\theta = \frac{1}{2}\int_{-\pi/2}^{0}\frac{36\,d\theta}{(2 - \sin\theta)^2}$$



Because cosines lead to simpler integrals, we let $\theta = x - \frac{1}{2}\pi$ so $\sin\theta = -\cos x$. Thus,

$$A_1 = 18\int_0^{\pi/2}\frac{dx}{(2 + \cos x)^2}$$

Now we let $z = \tan\frac{1}{2}\theta$, $\cos\theta = \dfrac{1 - z^2}{1 + z^2}$ and $d\theta = \dfrac{2\,dz}{1 + z^2}$. Thus

$$A_1 = 18\int_0^1\frac{\frac{2\,dz}{1 + z^2}}{\left(2 + \frac{1 - z^2}{1 + z^2}\right)^2} = 36\int_0^1\frac{1 + z^2}{(3 + z^2)^2}\,dz$$

We let $z = \sqrt{3}\tan t$ and $dz = \sqrt{3}\sec^2 t\,dt$; when $z = 0$, $t = 0$; when $z = 1$, $t = \pi/6$. Thus

$$\begin{aligned} A_1 &= 36\int_0^{\pi/6} \frac{1+3\tan^2 t}{9\sec^4 t}\sqrt{3}\sec^2 dt = 4\sqrt{3}\int_0^{\pi/6}(\cos^2 t + 3\sin^2 t)\,dt \\ &= 2\sqrt{3}\int_0^{\pi/6}[(1+\cos 2t) + 3(1-\cos 2t)]\,dt = 2\sqrt{3}\int_0^{\pi/6}(4 - 2\cos 2t)\,dt \\ &= 2\sqrt{3}\Big[4t - \sin 2t\Big]_0^{\pi/6} = 2\sqrt{3}[\tfrac{2}{3}\pi - \tfrac{1}{2}\sqrt{3}] = \tfrac{4}{3}\sqrt{3}\pi - 3 \qquad (1) \end{aligned}$$

Next, let R_2 be the part of the region R in first quadrant. Then R_2 is bounded by the parabola $r = 3/(1+\sin\theta)$ and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$. Thus,

$$A_2 = \frac{1}{2}\int_0^{\pi/2}\frac{9\,d\theta}{(1+\sin\theta)^2} \qquad (2)$$

We let $\theta = \frac{1}{2}\pi - t$. Then

$$(1+\sin\theta)^2 = [1+\sin(\tfrac{1}{2}\pi - t)]^2 = (1+\cos t)^2 = (2\cos^2\tfrac{1}{2}t)^2 = 4\cos^4\tfrac{1}{2}t$$

Furthermore, $d\theta = -dt$; $t = \frac{1}{2}\pi$ when $\theta = 0$; and $t = 0$ when $\theta = \frac{1}{2}\pi$. Thus, from (2) we have

$$A_2 = \frac{1}{2}\int_{\pi/2}^{0}\frac{-9\,dt}{4\cos^4\frac{1}{2}t} = \frac{9}{8}\int_0^{\pi/2}\sec^4\tfrac{1}{2}t\,dt = \frac{9}{4}\int_0^{\pi/2}(1+\tan^2\tfrac{1}{2}t)\sec^2\tfrac{1}{2}t(\tfrac{1}{2}dt) = \frac{9}{4}\Big[\tan\tfrac{1}{2}t + \tfrac{1}{3}\tan^3\tfrac{1}{2}t\Big]_0^{\pi/2} = 3 \qquad (3)$$

Because the region R is symmetric with respect to the line $\theta = \frac{1}{2}\pi$, then the area of R is given by $A = 2(A_1 + A_2)$. Substituting from (1) and (3), we get

$$A = 2[(-3 + \tfrac{4}{3}\sqrt{3}\pi) + 3] = \tfrac{8}{3}\sqrt{3}\pi$$

Thus, the area of region R is $\frac{8}{3}\sqrt{3}\pi$ square units.

33. Derive the third form of the polar equation. ▷ Rotate π: $r = \dfrac{ed}{1 - e\cos(\theta+\pi)} = \dfrac{ed}{1 + e\cos\theta}$

34. Derive the fourth form of the polar equation. ▷ Rotate $\frac{3}{2}\pi$: $r = \dfrac{ed}{1 - e\cos(\theta + \frac{3}{2}\pi)} = \dfrac{ed}{1 + e\sin\theta}$

35. Derive the second form of the polar equation. ▷ Rotate $\frac{1}{2}\pi$: $r = \dfrac{ed}{1 - e\cos(\theta - \frac{1}{2}\pi)} = \dfrac{ed}{1 - e\sin\theta}$

36. Show that the equation $r = k\csc^2\frac{1}{2}\theta$, where k is a constant, is a polar equation of a parabola.

▷ $r = \dfrac{k}{\sin^2\frac{1}{2}\theta} = \dfrac{k}{\frac{1}{2}(1-\cos\theta)} = \dfrac{2k}{1-\cos\theta}$, a parabola

37. The orbit of a comet is a parabola with the sun as its focus. When the comet is 80 million miles from the sun, the line from the sun makes an angle of $\frac{1}{3}\pi$ with axis of the orbit. (a) Find an equation of the orbit. (b) How close does it come to the sun?

▷ (a) Choose the focus at the pole and the polar axis along the axis of the parabola. Let the unit be 1 million miles. Because $e = 1$, then $r = \dfrac{d}{1-\cos\theta}$. Set $\theta = \frac{1}{3}\pi$: $80 = \dfrac{d}{1-\frac{1}{2}} = 2d$, $d = 40$. $r = \dfrac{40}{1-\cos\theta}$

(b) r is least when $\cos\theta = -1$. Then $r = \frac{40}{2} = 20$. The comet comes within 20 million miles of the sun.

38. Let $z = \tan\frac{1}{2}\theta$. $\bar{r} = \dfrac{1}{\pi}\displaystyle\int_{\theta=0}^{\pi}\frac{p\,d\theta}{1+e\cos\theta} = \frac{p}{\pi}\int_{z=0}^{\infty}\frac{2\,dz/(1+z^2)}{1+e(1-z^2)/(1+z^2)} = \frac{p}{\pi}\int_0^{\infty}\frac{2\,dz}{(1+e)+(1-e)z^2}$

$$= \frac{2p}{(1-e)\pi}\int_0^{\infty}\frac{dz}{z^2+(1+e)/(1-e)} = \frac{2p}{(1-e)\pi}\sqrt{\frac{1-e}{1+e}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}z\Big]_0^{\infty} = \frac{2p}{\pi\sqrt{1-e^2}}\cdot\frac{\pi}{2} = \frac{p}{\sqrt{1-e^2}}$$

39. The orbit of a satellite is an ellipse with the center of the earth as a focus and $e = \frac{1}{3}$. The closest it gets to the earth is 300 mi. Find the farthest it gets from the earth. Assume the earth's radius is 4000 mi.

▷ From the center, $300 + 4000 = 4300 = a - c = 3c - c = 2c$. $a + c = (a-c) + 2c = 4300 + 4300 = 8600$, $8600 - 4000 = 4600$. The greatest distance from the surface is 4600 miles.

40. Show that the point F mentioned in the proof of Theorem 9.5.1 is one of the foci when the conic is either an ellipse or a hyperbola.

▹ We are asked to show that if $h = \frac{e^2d}{1-e^2}$ and $a = \frac{ed}{1-e^2}$, then the origin is a focus of $\frac{(x-h)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$. The focus is c units from the center $(h, 0)$ where $c = ae = \frac{ed}{1-e^2}e = h$.

41. Find the angle α between the asymptotes of a hyperbola in terms of its eccentricity e.

▹ From the auxiliary rectangle we have $\tan \frac{1}{2}\alpha = \frac{b}{a} = \frac{\sqrt{c^2-a^2}}{a} = \frac{\sqrt{a^2e^2-a^2}}{a} = \sqrt{e^2-1}$.

42. Describe how the shape of a conic changes as the eccentricity e increases from 0 to 2.

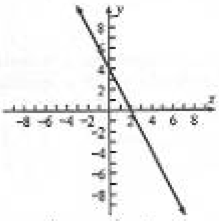
▹ Start with a circle of diameter 1" and keep the left edge fixed. Because $a = b/\sqrt{1-e^2}$, when $e = .01$, the major axis is 1.00005; when $e = 0.10$, it is 1.005"; when $e = 0.5$ it is 1.15", a noticeable elongation; when $e = .99$ it is 7.1". As e approaches 1, the right edge moves off the paper and when $e = 1$, there is no right edge: we have a parabola. When $e = 1.01$ we have a hyperbola the angle between whose asymptotes is $\alpha = 2\tan^{-1}\sqrt{0.0201} = 16.1°$. When $e = \sqrt{2}$, $\alpha = 2\tan^{-1}1 = 90°$ and we have a rectangular (equilateral) hyperbola. When $e = 2$, α has increased to $2\tan^{-1}\sqrt{3} = 120°$.

Miscellaneous Exercises for Chapter 9

In Exercises 1–4, (a) sketch the parametric equations; check by plotting. (b) Find a cartesian equation.

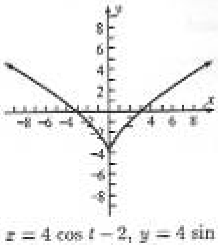
1. $x = 2 - t$, $y = 2t$

▹ $2x + y = 4 - 2t + 2t = 4$



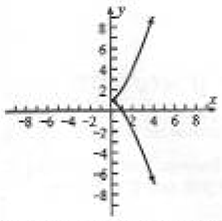
3. $x = 2t^3$, $y = 3t^2 - 4$

▹ $t = (\frac{1}{2}x)^{1/3}$. $y = 3(\frac{1}{2}x)^{2/3} - 4$



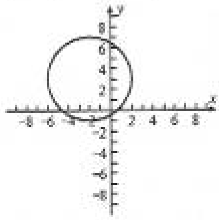
2. $x = t^2$, $y = t^3 + 1$

▹ $t = (y-1)^{1/3}$. $x = (y-1)^{2/3}$



4. $x = 4\cos t - 2$, $y = 4\sin t + 3$

▹ $(x+2)^2 + (y-3)^2 = 16\cos^2 t + 16\sin^2 t = 16$



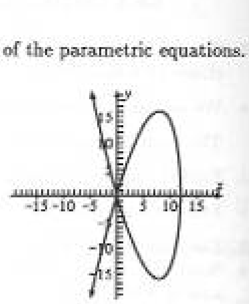
In Exercises 5 and 6, find dy/dx and d^2y/dx^2 without eliminating the parameter.

5. $x = 9t^2 - 1$ and $y = 3t + 1$. Therefore $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{18t} = \frac{1}{6t}$ and $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{6t^2}}{18t} = -\frac{1}{108t}$

6. $x = e^{2t}$ and $y = e^{-3t}$. Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3e^{-3t}}{2e^{2t}} = -\frac{3}{2}e^{-5t}$ and $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{15}{2}e^{-5t}}{2e^{2t}} = \frac{15}{4}e^{-7t}$

In Exercises 7 and 8, find the horizontal and vertical tangent lines. Sketch the graph of the parametric equations.

7. $x = 12 - t^2$ and $y = 12t - t^3$. $\frac{dx}{dt} = -2t$ and $\frac{dy}{dt} = 12 - 3t^2$.
Set $\frac{dy}{dt} = 0$: $12 - 3t^2 = 0$; $t = \pm 2$; $y = \pm 16$ are the horizontal tangent lines.
Set $\frac{dx}{dt} = 0$: $-2t = 0$; $t = 0$; $x = 12$ is the vertical tangent line.



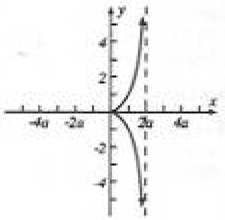
8. $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$, $a > 0$ (the cissoid of Diocles)

▸ Differentiating with respect to t, we obtain

$$\frac{dx}{dt} = \frac{4at}{(1+t^2)^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{2at^2(t^2+3)}{(1+t^2)^2}$$

Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2at^2(t^2+3)}{(1+t^2)^2}}{\frac{4at}{(1+t^2)^2}} = \tfrac{1}{2}t(t^2+3),\ t \neq 0$$



Although $\frac{dy}{dx}$ is not defined for $t = 0$, because $\lim_{t\to 0} \frac{dy}{dx} = 0$, there is a horizontal tangent at $t = 0$. Moreover, $0 \le \frac{2at^2}{1+t^2} < 2a$ for all t. Thus, x is in the interval $(0, 2a)$. Furthermore,

$$\lim_{t\to+\infty} x = \lim_{t\to+\infty} \frac{2at^2}{1+t^2} = 2a \quad \text{and} \quad \lim_{t\to+\infty} y = \lim_{t\to+\infty} \frac{2at^3}{1+t^2} = +\infty$$

Therefore, the line $x = 2a$ is a vertical asymptote. If t is replaced by $-t$, then x is unchanged and y is replaced by $-y$. Thus, for every point (x, y) on the curve we have the point $(x, -y)$, and hence the graph is symmetric with respect to the x axis. We use the above information to sketch the graph, as shown in the figure.

In Exercises 9 and 10, locate the point in polar coordinates. Rename twice, once with r, once with $-r$.

9. (a) $(2, \frac{3}{4}\pi)$. $(-2, \frac{7}{4}\pi)$, $(2, \frac{11}{4}\pi)$ (b) $(-3, \frac{7}{6}\pi)$. $(3, \frac{13}{6}\pi)$, $(-3, \frac{19}{6}\pi)$

10. (a) $(4, -\frac{1}{3}\pi)$. $(-4, \frac{2}{3}\pi)$, $(4, \frac{5}{3}\pi)$ (b) $(-1, \frac{1}{4}\pi)$. $(1, \frac{5}{4}\pi)$, $(-1, \frac{9}{4}\pi)$

In Exercises 11 and 12, convert from polar coordinates to rectangular cartesian coordinates.

11. (a) $(1, \frac{1}{2}\pi) = (0, 1)^c$ (b) $(2, -\frac{1}{3}\pi) = (2\cos(-\frac{1}{3}\pi), 2\sin(-\frac{1}{3}\pi))^c = (2\cdot\frac{1}{2}, 2(-\frac{1}{2}\sqrt{3}))^c = (1, -\sqrt{3})^c$
(c) $(4, \frac{5}{4}\pi) = (4\cos\frac{5}{4}\pi, 4\sin\frac{5}{4}\pi)^c = (4(-\frac{1}{2}\sqrt{2}), 4(-\frac{1}{2}\sqrt{2}))^c = (-2\sqrt{2}, -2\sqrt{2})^c$
(d) $(-3, \frac{1}{6}\pi) = (-3\cos\frac{1}{6}\pi, -3\sin\frac{1}{6}\pi)^c = (-3\cdot\frac{1}{2}\sqrt{3}, -3\cdot\frac{1}{2})^c = (-\frac{3}{2}\sqrt{3}, -\frac{3}{2})^c$

12. (a) $(5, \pi) = (-5, 0)^c$ (b) $(-2, \frac{5}{6}\pi) = (-2\cos\frac{5}{6}\pi, -2\sin\frac{5}{6}\pi)^c = (-2\cdot -\frac{1}{2}\sqrt{3}, -2\cdot\frac{1}{2})^c = (\sqrt{3}, -1)^c$
(c) $(-\sqrt{2}, \frac{1}{4}\pi) = (-\sqrt{2}\cos\frac{1}{4}\pi, -\sqrt{2}\sin\frac{1}{4}\pi)^c = (-\sqrt{2}\cdot\frac{1}{2}\sqrt{2}, -\sqrt{2}\cdot\frac{1}{2}\sqrt{2})^c = (-1, -1)^c$
(d) $(1, \frac{4}{3}\pi) = (\cos\frac{4}{3}\pi, \sin\frac{4}{3}\pi)^c = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^c$

In Exercises 13 and 14, convert from rectangular cartesian to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

13. (a) $(-4, 4)^c$. $r = 4\sqrt{2}$. Q2, $\theta = \tan^{-1}(-4/4) + \pi = \frac{3}{4}\pi$. $(4\sqrt{2}, \frac{3}{4}\pi)$
(b) $(1, -\sqrt{3})^c$. $r = \sqrt{1+3} = 2$. Q4, $\theta = \tan(-\sqrt{3}/1) + 2\pi = \frac{5}{3}\pi$. $(2, \frac{5}{3}\pi)$
(c) $(0, 6)^c = (6, \frac{1}{2}\pi)$ (d) $(-2\sqrt{3}, -2)^c$. $r = \sqrt{12+4} = 4$. Q3, $\theta = \tan(-2/-2\sqrt{3}) + \pi = \frac{7}{6}\pi$. $(4, \frac{7}{6}\pi)$

14. (a) $(-4, 0)^c = (4, \pi)$ (b) $(\sqrt{3}, 1)^c$. $r = \sqrt{3+1} = 2$. Q1, $\theta = \tan^{-1}(1/\sqrt{3}) = \frac{1}{6}\pi$. $(2, \frac{1}{6}\pi)$
(c) $(-2, -2)^c$. $r = 2\sqrt{2}$. Q3, $\theta = \tan(-2/-2) + \pi = \frac{3}{4}\pi$. $(2\sqrt{2}, \frac{3}{4}\pi)$
(d) $(3, -3\sqrt{3})^c$. $r = \sqrt{9+27} = 6$. Q4, $\theta = \tan^{-1}(-3\sqrt{3}/3) + 2\pi = \frac{5}{3}\pi$. $(6, \frac{5}{3}\pi)$

In Exercises 15–18, find a polar equation of the graph having the given cartesian equation.

15. $4x^2 - 9y^2 = 36$ ▸ $4(r\cos\theta)^2 - 9(r\sin\theta)^2 = 36$. $r^2(4\cos^2\theta - 9\sin^2\theta) = 36$

16. $2xy = 1$ ▸ $2(r\cos\theta)(r\cos\theta) = r^2(2\sin\theta\cos\theta) = 1$, $r^2\sin 2\theta = 1$

17. $x^2 + y^2 - 9x + 8y = 0$ ▸ $r^2 - 9r\cos\theta + 8r\sin\theta = 0$. $r = 9\cos\theta - 8\sin\theta$ (pole at $\tan^{-1}\frac{9}{8}$)

18. $y^4 = x^2(a^2 - y^2)$

▹ $y^2(y^2 + x^2) = a^2x^2$, $r^2\sin^2\theta \cdot r^2 = a^2r^2\cos^2\theta$, $r^2\sin^2\theta = a^2\cos^2\theta$ (pole at $\frac{1}{2}\pi$), $r = a\cot\theta$

In Exercises 19–22, find a cartesian equation of the graph having the given polar equation.

19. $r^2\sin 2\theta = 4$ ▹ $2(r\sin\theta)(r\cos\theta) = 4$, $2xy = 4$, $xy = 2$

20. $r(1 - \cos\theta) = 2$ ▹ $r = 2 + r\cos\theta$, $r^2 = (2 + r\cos\theta)^2$, $x^2 + y^2 = (2 + x)^2$, $y^2 = 4x + 4$

21. $r^2 = \sin^2\theta$ ▹ $(r^2)^2 = (r\sin\theta)^2$, $(x^2 + y^2)^2 = y^2$. $x^4 + 2x^2y^2 + y^4 - y^2 = 0$

22. $r = a\tan^2\theta$

▹ $r\cos^2\theta = a\sin^2\theta$, $r \cdot r^2\cos^2\theta = ar^2\sin^2\theta$ (pole at 0), $r^2(r\cos\theta)^4 = a^2(r\sin\theta)^4$, $(x^2 + y^2)x^4 = a^2y^4$

In Exercises 23–26, sketch the graph of the equation.

23. (a) $\theta = \frac{1}{4}\pi$ ▹ line through the pole with direction 45°

(b) $r = 4$ ▹ circle centered at the pole of radius 4

24. (a) $\theta = \frac{2}{3}$ ▹ line through the pole with direction $\frac{2}{3} \cdot 180°/\pi = 38.2°$

(b) $r = \frac{3}{2}$ ▹ circle centered at the pole of radius $\frac{3}{2}$

25. (a) $r\cos\theta = 3$ ▹ line $x = 3$

(b) $r = 3\cos\theta$ ▹ circle centered at $(\frac{3}{2}, 0)$ of radius $\frac{3}{2}$

26. (a) $r\sin\theta = 6$ ▹ line $y = 6$

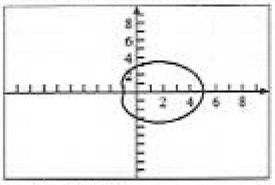
(b) $r = 6\sin\theta$ ▹ circle centered at $(3, \frac{1}{2}\pi)$ of radius 3

In Exercises 27–32, determine the type of limaçon, its symmetry S, and the direction it points. Plot it.

27. $r = 3 + 2\cos\theta$

▹ $\frac{a}{b} = \frac{3}{2} \in (1, 2)$, dented

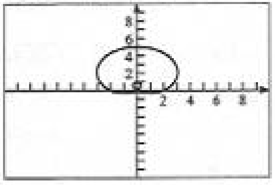
S: polar axis, points right



28. $r = 2 + 3\sin\theta$

▹ $\frac{a}{b} = \frac{2}{3} \in (0, 1)$, with a loop

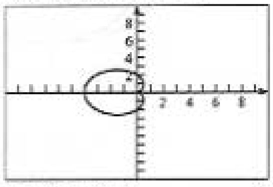
S: $\frac{1}{2}\pi$ axis, points upward



29. $r = 2(1 - \cos\theta)$

▹ $\frac{a}{b} = \frac{2}{2} = 1$, a cardioid

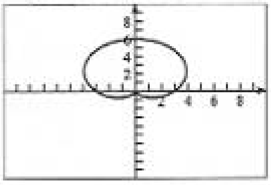
S: polar axis, points left



30. $r = 3(1 + \sin\theta)$

▹ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid

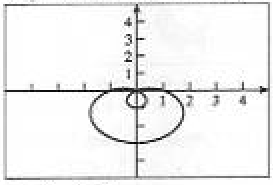
S: $\frac{1}{2}\pi$ axis, points upward



31. $r = 1 - 2\sin\theta$

▹ $\frac{a}{b} = \frac{1}{2} \in (0, 1)$, with a loop

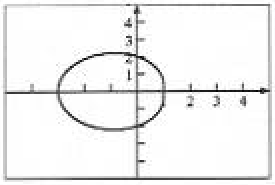
S: $\frac{1}{2}\pi$ axis, points downward



32. $r = 2 - \cos\theta$

▹ $\frac{a}{b} = \frac{2}{1} \geq 2$, 1 convex

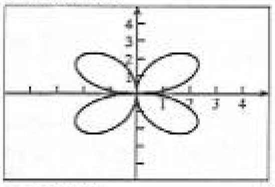
S: polar axis, points left



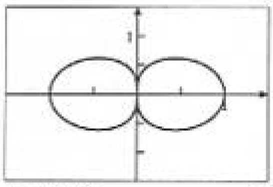
In Exercises 33–38, describe and plot the graph of the equation.

33. $r = 3\sin 2\theta$

▹ 4-leafed rose

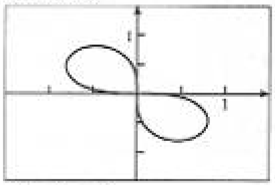


35. $r = \sqrt{|\cos\theta|}$



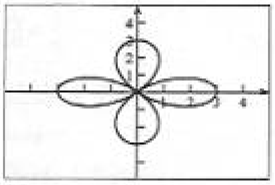
37. $r^2 = -\sin 2\theta$

▹ lemniscate



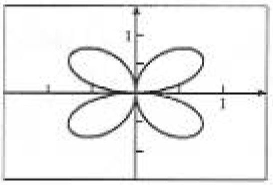
34. $r = 3\cos 2\theta$

▹ 4-leafed rose



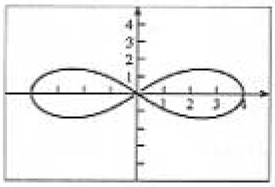
36. $r = |\sin 2\theta|$

▹ 4-leafed rose



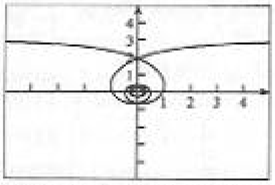
38. $r^2 = 16\cos\theta$

▹ lemniscate



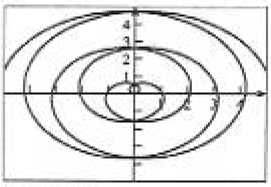
39. (a) $r\theta = 3$

▹ reciprocal spiral



(b) $3r = \theta$

▹ spiral of Archimedes



40. Show that $r = 1 + \sin\theta$ and $r = \sin\theta - 1$ have the same graph.

▹ $r = 1 + \sin\theta \Leftrightarrow -r = 1 + \sin(\theta + \pi) \Rightarrow r = \sin\theta - 1$

In Exercises 41–44, find the exact length of arc of the parametric or polar graph.

41. $x = 2 - t$, $y = t^2$. $L = \int_0^3 \sqrt{x'^2 + y'^2}\,dt = \int_0^3 \sqrt{(-1)^2 + (2t)^2} = \int_0^3 \sqrt{1 + 4t^2}\,dt = 2\int_0^3 \sqrt{\tfrac{1}{4} + t^2}\,dt$
$= \Big[t\sqrt{\tfrac{1}{4} + t^2} + \tfrac{1}{4}\ln(t + \sqrt{\tfrac{1}{4} + t^2})\Big]_0^3 = 3\sqrt{\tfrac{1}{4} + 9} + \tfrac{1}{4}\ln(3 + \sqrt{\tfrac{1}{4} + 9}) - \tfrac{1}{4}\ln\sqrt{\tfrac{1}{4}} = \tfrac{3}{2}\sqrt{37} + \tfrac{1}{4}\ln(6 + \sqrt{37})$

42. $x = t^2$, $y = t^3$. $L = \int_1^2 \sqrt{x'^2 + y'^2}\,dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \tfrac{1}{27}(4 + 9t^2)^{3/2}\Big]_1^2$
$= \tfrac{1}{27}(40^{3/2} - 13^{3/2})$

43. $r = 4(1 - \sin\theta)$. $L = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-4\cos\theta)^2 + [4(1 - \sin\theta)]^2}\,d\theta = 8\int_{-\pi/2}^{\pi/2} \sqrt{2 - 2\sin\theta}\,d\theta$
$= 16\int_{-\pi/2}^{\pi/2} \sqrt{\tfrac{1}{2}[1 - \cos(\theta - \tfrac{1}{2}\pi)]}\,d\theta = 16\int_{-\pi/2}^{\pi/2} -\sin(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi)\,d\theta = 32\cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi)\Big]_{-\pi/2}^{\pi/2} = 32$

44. $r = 3\sec\theta$ from $\theta = 0$ to $\theta = \tfrac{1}{4}\pi$.

▹ We have

$$L = \int_0^{\pi/4} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^{\pi/4} \sqrt{(3\sec\theta\tan\theta)^2 + (3\sec\theta)^2}\,d\theta = \int_0^{\pi/4} 3\sqrt{\sec^2\theta(\tan^2\theta + 1)}\,d\theta$$
$$= \int_0^{\pi/4} 3\sqrt{\sec^4\theta}\,d\theta = \int_0^{\pi/4} 3\sec^2\theta\,d\theta = 3\tan\theta\Big]_0^{\pi/4} = 3$$

45. The given limaçon has the equation $r = 4(1 + 2\cos\theta)$.

(a) Half of the region enclosed by the loop is obtained when $\theta \in [\frac{2}{3}\pi, \pi]$.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[4(1 + 2\cos w_i)]^2 \Delta_i\theta = 16\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2 d\theta = 16\int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= 16\int_{2\pi/3}^{\pi} [1 + 4\cos\theta + 2(1 + \cos 2\theta)] d\theta = 16\int_{2\pi/3}^{\pi} (3 + 4\cos\theta + 2\cos 2\theta) d\theta$$

$$= 16\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{\pi} = 16[3\pi - (2\pi + 2\sqrt{3} - \tfrac{1}{2}\sqrt{3})] = 16\pi - 24\sqrt{3}$$

(b) One half of the region enclosed by the outer part of the limaçon is obtained when $\theta \in [0, \frac{2}{3}\pi]$.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[4(1 + 2\cos w_i)]^2 \Delta_i\theta = 16\int_{0}^{2\pi/3} (1 + 2\cos\theta)^2 d\theta$$

$$= 16\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{0}^{2\pi/3} = 16(2\pi + 2\sqrt{3} - \tfrac{1}{2}\sqrt{3}) = 32\pi + 24\sqrt{3}$$

46. See Exercise 9.4.17

47. The given equations are $r = 2a\sin\theta$ and $r = a$. The points of intersection of the two graphs are $(a, \frac{1}{6}\pi)$ and $(a, \frac{5}{6}\pi)$. The required region is symmetric with respect to the $\frac{1}{2}\pi$ axis. Thus, if we take $\theta \in [\frac{1}{6}\pi, \frac{1}{2}\pi]$, we obtain one-half of the required area. Therefore,

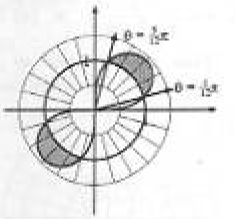
$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\tfrac{1}{2}(2a\sin w_i)^2 - \tfrac{1}{2}a^2]\Delta_i\theta = a^2\int_{\pi/6}^{\pi/2} (4\sin^2\theta - 1) d\theta = \int_{\pi/6}^{\pi/2} (1 - 2\cos 2\theta) d\theta$$

$$= \Big[\theta - \sin 2\theta\Big]_{\pi/3}^{\pi/2} = a^2(\tfrac{1}{2}\pi - 0 - \tfrac{1}{6}\pi - \tfrac{1}{2}\sqrt{3}) = a^2(\tfrac{1}{3}\pi + \tfrac{1}{2}\sqrt{3})$$

48. Find the area of the region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$.

▹ The figure shows the region. Solving the equations simultaneously, we get

$$2\sin 2\theta = 1$$
$$\sin 2\theta = \tfrac{1}{2}$$
$$2\theta = \tfrac{1}{6}\pi \quad \text{or} \quad 2\theta = \tfrac{5}{6}\pi$$
$$\theta = \tfrac{1}{12}\pi \quad \text{or} \quad \theta = \tfrac{5}{12}\pi$$



By symmetry, the region bounded by the lines $\theta = \frac{1}{12}\pi$ and $\theta = \frac{1}{4}\pi$ and the lemniscate and the circle is one-fourth the entire area. Therefore,

$$A = 4 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[2\sin 2w_i - 1]\Delta_i\theta$$

$$= 2\int_{\pi/12}^{\pi/4} (2\sin 2\theta - 1) d\theta = 2\Big[-\cos 2\theta - \theta\Big]_{\pi/12}^{\pi/4} = 2[(-\tfrac{1}{4}\pi) - (-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{12}\pi)] = \sqrt{3} - \tfrac{1}{3}\pi$$

Thus, the area is $\sqrt{3} - \frac{1}{3}\pi$ square units.

49. $A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}\left(e^{kw_i}\right)^2 \Delta_i\theta = \frac{1}{2}\int_{0}^{2\pi} e^{2k\theta}\, d\theta = \frac{1}{4k} e^{2k\theta}\Big]_{0}^{2\pi} = \frac{e^{4k\pi} - 1}{4k}$

50. $r = a\cos\theta$ (circle center $(\frac{1}{2}a, 0)$), $r = a(1 - \cos\theta)$ (cardioid pointing left). $a\cos\theta = a(1 - \cos\theta)$, $2a\cos\theta = a$, $\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{1}{3}\pi$. $A = 2\left[\frac{1}{2}\int_{0}^{\pi/3} [a(1 - \cos\theta)]^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} [a\cos\theta]^2 d\theta\right]$

$$= a^2\int_{0}^{\pi/3} [1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)] d\theta + a^2\int_{\pi/3}^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta) d\theta$$

$$= a^2\Big[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\Big]_{0}^{\pi/3} + \tfrac{1}{2}a^2\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_{\pi/3}^{\pi/2} = a^2(\tfrac{1}{2}\pi - \sqrt{3} + \tfrac{1}{8}\sqrt{3}) + \tfrac{1}{2}a^2(\tfrac{1}{6}\pi - \tfrac{1}{4}\sqrt{3}) = a^2(\tfrac{7}{12}\pi - \sqrt{3})$$

51. We apply the law of cosines to the triangle with sides r, r_0, and a where the angle opposite a is $\theta - \theta_0$. Thus, $a^2 = {r_0}^2 + r^2 - 2r_0 r\cos(\theta - \theta_0)$

52. Find the area enclosed by one loop of the curve $r = a \sin n\theta$.

▹ The graph is a $2n$-leafed rose if n is even, a circle if $n = 1$, and an n-leafed rose if n is odd and greater than 1. We take the first loop. Because $r = 0$ when $n\theta = \pi$, or equivalently, $\theta = \frac{1}{n}\pi$, we take the lines $\theta = 0$ and $\theta = \frac{1}{n}\pi$ as boundaries. Thus, the area of the loop is given by

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[a \sin nw_i]^2 \Delta_i\theta = \tfrac{1}{2}\int_0^{\pi/n} a \sin n\theta]^2 d\theta = \tfrac{1}{2}a^2 \int_0^{\pi/n} \sin^2 n\theta \, d\theta$$
$$= \frac{1}{4}a^2 \int_0^{\pi/n} (1 - \cos 2n\theta) d\theta = \frac{1}{4}a^2\Big[\theta - \frac{1}{2n}\sin 2n\theta\Big]_0^{\pi/n} = \frac{1}{4n}\pi a^2$$

■ The area of the loop is $\frac{1}{4n}\pi a^2$ square units.

In Exercises 53–56, the equation is of a conic having a focus at the pole. (a) Find the eccentricity e; (b) identify the conic; (c) write an equation of the directrix corresponding to the focus at the pole; (d) sketch the curve.

53. $r = \dfrac{2}{2 - \sin\theta} = \dfrac{1}{1 - \frac{1}{2}\sin\theta}$ is of the form $r = \dfrac{ed}{1 - e\sin\theta}$ with (a) $e = \frac{1}{2} < 1$; (b) the conic is an ellipse.

(c) $d = \frac{2}{1} = 2$ and the directrix is below the focus at the pole; so its equation is $r \sin\theta = -2$.

54. $r = \dfrac{5}{3 + 3\sin\theta} = \dfrac{\frac{5}{3}}{1 + \sin\theta}$ is of the form $r = \dfrac{ed}{1 + e\sin\theta}$ with (a) $e = 1$; (b) the conic is a parabola.

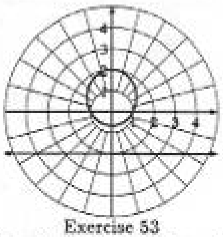
(c) $d = \dfrac{\frac{5}{3}}{1} = \frac{5}{3}$ and the equation of the directrix is $r \sin\theta = \frac{5}{3}$.

55. $r = \dfrac{4}{2 + 3\cos\theta} = \dfrac{2}{1 + \frac{3}{2}\cos\theta}$ is of the form $r = \dfrac{ed}{1 + e\cos\theta}$ with (a) $e = \frac{3}{2} > 1$; (b) the conic is a hyperbola.

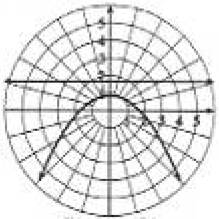
(c) $d = \frac{4}{3}$ and the directrix is to the right of the focus, so its equation is $r \cos\theta = \frac{4}{3}$.

56. $r = \dfrac{4}{3 - 2\cos\theta}$ ▹ (a) Dividing the numerator and denominator by 3, we obtain $r = \dfrac{\frac{4}{3}}{1 - \frac{2}{3}\cos\theta}$.

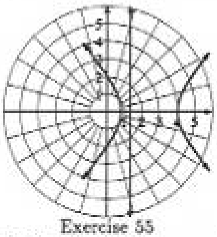
Thus $e = \frac{2}{3} < 1$ and so (b) the conic is an ellipse. (c) We get the equation of the directrix by setting the constant of the denominator to 0. Thus $r \cos\theta = -\frac{4}{2}$. $r \cos\theta = -2$. (d) A sketch is shown below right.



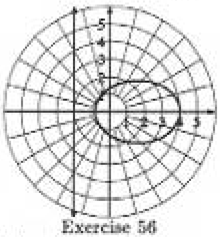
Exercise 53



Exercise 54



Exercise 55



Exercise 56

In Exercises 57–60, find a polar equation of the conic satisfying the conditions and sketch the graph.

57. The directrix corresponding to the focus at the pole is to the left of the focus. Thus an equation of the conic is of the form $r = \dfrac{ed}{1 - e\cos\theta}$. Because a vertex is at $(2, \pi)$, $2 = \dfrac{ed}{1 - e\cos\pi}$; $2 = \dfrac{ed}{1 + e}$; $ed = 2 + 2e$

Because a vertex is at $(4, \pi)$ or, equivalently $(-4, 0)$, then $-4 = \dfrac{ed}{1 - e\cos 0}$; $-4 = \dfrac{ed}{1 - e}$; $ed = -4 + 4e$

Solving simultaneously, we get $e = 3$ and $ed = 8$. The conic is a hyperbola; an equation is $r = \dfrac{8}{1 - 3\cos\theta}$.

58. A focus at the pole, a vertex at $(6, \frac{1}{2}\pi)$, and $e = \frac{3}{4}$.

▹ There are two ellipses possible. Because the focus is at the pole and a vertex is above the pole, one polar equation is of the form. $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{ed}{1 + \frac{3}{4}\sin\theta}$. We let $(r, \theta) = (6, \frac{1}{2}\pi)$ to obtain $6 = \dfrac{ed}{1 + \frac{3}{4}} = \dfrac{ed}{\frac{7}{4}}$

and so $ed = \frac{21}{2}$. Thus, an equation is $r = \dfrac{\frac{21}{2}}{1 + \frac{3}{4}\sin\theta} = \dfrac{42}{4 + 3\sin\theta}$.

Another polar equation is of the form $r = \dfrac{ed}{1 - \frac{3}{4}\sin\theta}$. We let $(r, \theta) = (6, \frac{1}{2}\pi)$ to get $6 = \dfrac{ed}{1 - \frac{3}{4}} = 4ed$

and so $ed = \frac{3}{2}$. Thus an equation is $r = \dfrac{\frac{3}{2}}{1 - \frac{3}{4}\sin\theta} = \dfrac{6}{4 - 3\sin\theta}$

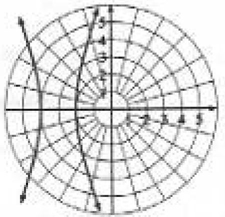
59. Because $e = 1$, the conic is a parabola. Because a focus is at the pole and the vertex is at $(3, \frac{3}{2}\pi)$, the directrix is below the focus. Thus an equation of the parabola is of the form $r = \frac{d}{1 - \cos\theta}$. Because d is the undirected distance between the focus and the directrix $d = 2(3) = 6$. Hence an equation of the parabola is $r = \frac{6}{1 - \sin\theta}$.

60. The line $r \sin\theta = 6$ is the directrix corresponding to the focus at the pole and $e = \frac{5}{3}$.

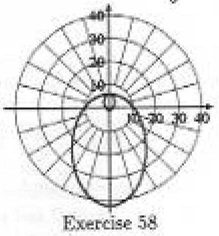
▹ Because the directrix is obtained by setting the constant to 0, the equation is of the form $r = \frac{6}{a + \sin\theta}$

Dividing numerator and denominator by a, we get the form $r = \frac{6/a}{1 + (1/a)\sin\theta}$

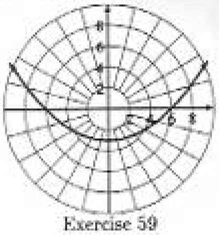
Therefore $\frac{1}{a} = e = \frac{5}{3}$ and so $a = \frac{3}{5}$. Hence, $r = \frac{6}{\frac{3}{5} + \sin\theta} = \frac{30}{3 + 5\sin\theta}$. A sketch appears below right.



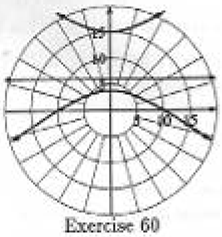
Exercise 57



Exercise 58



Exercise 59



Exercise 60

In Exercises 61–64, use NINT to approximate to four digits the length L of the parametric or polar curve.

61. $x = 2t^3$, $y = t - 2$. $L = \int_2^3 \sqrt{x'^2 + y'^2}\,dt = \int_2^3 \sqrt{(6t^2)^2 + 1^2}\,dt = \int_2^3 \sqrt{36t^4 + 1}\,dt = 38.014 \approx 38.02$

62. $x = e^t$, $y = \cos t$. $L = \int_{-\pi/2}^{\pi/2} \sqrt{x'^2 + y'^2}\,dt = \int_{-\pi/2}^{\pi/2} \sqrt{e^{2t} + \sin^2 t}\,dt = 5.3476 \approx 5.348$

63. $r = 4 - 2\sin\theta$.

$$L = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-2\cos\theta)^2 + (4 - 2\sin\theta)^2}\,d\theta = 4\int_{-\pi/2}^{\pi/2} \sqrt{5 - 4\sin\theta}\,d\theta = 26.730 \approx 26.73$$

64. One leaf of the rose $r = 4\cos 3\theta$.

▹ Setting $r = 0$, we get
$4\cos 3\theta = 0$, $3\theta = \pm\frac{1}{2}\pi$, $\theta = \pm\frac{1}{6}\pi$
By symmetry, and with the help of NINT, we get

$$L = 2\int_0^{\pi/6} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 2\int_0^{\pi/6} \sqrt{(-12\sin 3\theta)^2 + (4\cos 3\theta)^2}\,d\theta = 8\int_0^{\pi/6} \sqrt{9\sin^2 3\theta + \cos^2 3\theta}\,d\theta = 8.910$$

▪ The length of arc is 9.910 units.

65. The distance from the center to the focus is $ae = 36(0.206) = 7.4$. Thus Mercury gets as near as $36.0 - 7.4 = 28.6$ and as far as $36.0 + 7.4 = 43.4$ million miles from the sun.

66. From the center, $200 + 4000 = 4200 = a - c = 2c - c = c$. $a + c = 3c = 12600$, $12600 - 4000 = 8600$. The greatest distance from the surface is 8600 miles.

67. Choose the coordinate system with the pole at the focus. The conic is a parabola; so $e = 1$ and an equation is $r = \frac{d}{1 - \cos\theta}$. Let $(15, \theta)$ and $(5, \theta + \frac{1}{2}\pi)$ be the given points. Then $15 = \frac{d}{1 - \cos\theta}$; $15 - 15\cos\theta = d$;

$\cos\theta = \frac{15 - d}{15}$; $5 = \frac{d}{1 - \cos(\theta + \frac{1}{2}\pi)} = \frac{d}{1 + \sin\theta}$; $5 + 5\sin\theta = d$; $\sin\theta = \frac{d - 5}{5}$

$\sin^2\theta + \cos^2\theta = \left(\frac{d - 5}{5}\right)^2 + \left(\frac{15 - d}{15}\right)^2$; $1 = \frac{d^2 - 10d + 25}{25} + \frac{d^2 - 30d + 25}{225}$

$225 = 9d^2 - 90d + 225 + d^2 - 30d + 225$; $10d^2 - 120d + 225 = 0$; $d^2 - 12d + 2.5 = 0$

$d = \frac{1}{2}(12 \pm \sqrt{144 - 90}) = \frac{1}{2}(12 \pm \sqrt{54}) = 6 \pm \frac{3}{2}\sqrt{6}$

▪ The closest distances of the two orbits are $\frac{1}{2}d = (3 \pm \frac{3}{4}\sqrt{6})$ million miles.

68. If the distance between the two directrices of an ellipse is three times the distance between the foci, find the eccentricity e.

▹ The distance from the center to the focus is ae, and to the directrix is a/e.
$2(a/e) = 3 \cdot 2ae, \qquad e^2 = \frac{1}{3}, \qquad e = \frac{1}{3}\sqrt{3}$

69. Because the conic is a parabola, $e = 1$. Because the directrix is to the left of the focus, an equation of the parabola is of the form $r = \frac{d}{1 - \cos\theta}$. The point $(2, \frac{1}{3}\pi)$ is on the parabola; so $2 = \frac{d}{1 - \cos\frac{1}{3}\pi}$; $2 = \frac{d}{1 - \frac{1}{2}}$; $d = 1$. An equation of the parabola is $r = \frac{1}{1 - \cos\theta}$.

70. $\mathrm{A} = 4 \cdot \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{[\frac{1}{2}(1 + \cos\theta)]^2} = 2\int_0^{\pi/2} \frac{d\theta}{\cos^4\frac{1}{2}\theta} = 2\int_0^{\pi/2} \sec^4\frac{1}{2}\theta\, d\theta = 2\int_0^{\pi/2} (\tan^2\frac{1}{2}\theta + 1)\sec^2\frac{1}{2}\theta\, d\theta$
$= 4\Big[\frac{1}{3}\tan^3\frac{1}{2}\theta + \tan\frac{1}{2}\theta\Big]_0^{\pi/2} = \frac{16}{3}$

In Exercises 71 and 72, a focal chord of a conic is a segment through a focus with endpoints on the conic.

▹ A focal chord of a conic is divided by the focus into two segments whose directions differ by π.

71. Show: the sum of the reciprocals of the lengths s of perpendicular focal chords of a parabola is a constant.

▹ $s(\theta) = \frac{d}{1 - \cos\theta} + \frac{d}{1 + \cos\theta} = \frac{2d}{1 - \cos^2\theta} = \frac{2d}{\sin^2\theta}$. $\frac{1}{s(\theta)} + \frac{1}{s(\theta + \frac{1}{2}\pi)} = \frac{\sin^2\theta}{2d} + \frac{\cos^2\theta}{2d} = \frac{1}{2d}$

72. Show that the sum of the reciprocals of their length is the same for any chord.

▹ An equation of the conic is

$r(\theta) = \frac{ed}{1 - e\cos\theta}$

Because $\cos(\theta + \pi) = -\cos\theta$, the sum of the reciprocals of the lengths is

$\frac{1}{r(\theta)} + \frac{1}{r(\theta + \pi)} = \frac{1 - e\cos\theta}{ed} + \frac{1 + e\cos\theta}{ed} = \frac{2}{ed}$

73. Derive the formula for an epicycloid. (Section 9.1)

▹ Let the centers of the fixed and moving circles be the origin and C. Let $\mathrm{A}(a, 0)$ and B be the original and moving points of contact. t is the $\mathrm{m}(\angle\mathrm{AOB})$. Let $\alpha = \mathrm{m}(\angle\mathrm{BCP})$. Then $\alpha + t$ is the angle between CP and the horizontal. Let a vertical line through C meet a horizontal line through $\mathrm{P}(x, y)$ in D and the x axis in E. Rolling means arc BP = arc BA, $b\alpha = at$, $\alpha = \frac{a}{b}t$, $\alpha + t = \left(\frac{a}{b} + 1\right)t = \frac{a + b}{b}t$.

$x = \mathrm{OE} - \mathrm{PD} = (a + b)\cos t - b\cos(\alpha + t) = (a + b)\cos\theta - b\cos\frac{a + b}{b}t$

$y = \mathrm{CE} - \mathrm{CD} = (a + b)\sin t + \sin(\alpha + t) = (a + b)\sin t - b\sin\frac{a + b}{b}t$. If $b = a$, then

$x - a = a[2\cos t - (\cos 2t + 1)] = a(2\cos t - 2\cos^2 t) = 2a\cos t(1 - \cos t)$
$y = a(2\sin t - \sin 2t) = a(2\sin t - 2\sin t\cos t) = 2a\sin t(1 - \cos t)$

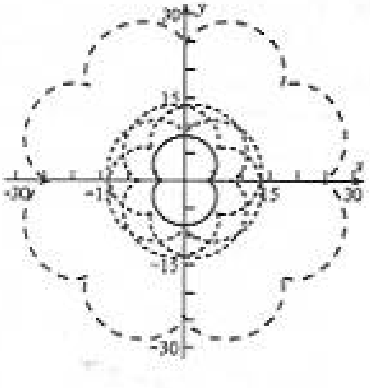
which is the cardioid $r = 2a(1 - \cos t)$. Similarly, a hypocycloid with $b = a$ is a limaçon with a loop.

74. Plot ans sketch the epicycloids.

(a) $a = 4$, $b = 2$, $t \in [-\pi, \pi]$ solid

(b) $a = 24$, $b = 3$, $t \in [-\pi, \pi]$ long dashes

(c) $a = 8$, $b = 3$, $t \in [-3\pi, 3\pi]$ short dashes



TEN

VECTORS AND PLANES, LINES, AND SURFACES IN SPACE

10.1 VECTORS IN THE PLANE

10.1.1 Definition A *vector in the plane* is an ordered pair of real numbers (x, y). The numbers x and y are called the *components* of the vector (x, y). Vectors (x, y) and (u, v) are *equal* if and only if $x = u$ and $y = v$. $\mathbf{0} = (0, 0)$ is the *zero* vector. In contrast, we call a number a *scalar*.

If $\mathbf{A} = (a_1, a_2)$, then the directed line segment $\overrightarrow{OA}$, where O is the origin and A is the point (a_1, a_2), is called the *position representation* of the vector $\mathbf{A}$. If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then the directed line segment $\overrightarrow{PQ}$ is a *representation* of the vector $\langle q_1 - p_1, q_2 - p_2 \rangle$. We use the abbreviation $\mathbf{a} = \mathbf{V}(\overrightarrow{OA})$.

10.1.2 Definition The *magnitude* of a vector is the length of any of its representations, and the *direction* of a nonzero vector is the direction of any of its representations.

The magnitude of the vector $\mathbf{A}$ is denoted by $\|\mathbf{A}\|$. $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

10.1.3 Theorem If $\mathbf{A}$ is the vector (a_1, a_2), then $\|\mathbf{A}\| = \sqrt{{a_1}^2 + {a_2}^2}$.
If $\mathbf{A} \neq \mathbf{0}$, the direction of $\mathbf{A}$ is the radian measure of the angle $\theta \in [0, 2\pi)$ such that

$\cos\theta = \dfrac{a_1}{\|\mathbf{A}\|}$ and $\sin\theta = \dfrac{a_2}{\|\mathbf{A}\|}$

Thus, if $a_1 \neq 0$, we have

$\tan\theta = \dfrac{a_2}{a_1}$

In navigation, the *course* or *heading* ϕ of a ship or airplane is the angle measured in degrees clockwise from the north to the direction in which the vessel is traveling. The angle is considered positive. Thus $\theta = 90^\circ - \phi$ if $0 \leq \phi \leq 90^\circ$, or $450^\circ - \phi$ if $\phi > 90^\circ$.

10.1.4 Definition The *sum* of two vectors $\mathbf{A} = (a_1, a_2)$ and $\mathbf{B} = (b_1, b_2)$ is the vector $\mathbf{A} + \mathbf{B}$, defined by
$\mathbf{A} + \mathbf{B} = (a_1 + b_1, a_2 + b_2)$
The rule for addition of vectors is sometimes referred to as the *parallelogram law*.

Force Force is a vector quantity. Any set of forces acting at a single point can be replaced by a single force, the *resultant*, which is their vector sum.

10.1.5 Definition If $\mathbf{A} = (a_1, a_2)$, then the *negative* of $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined to be the vector $(-a_1, -a_2)$.

10.1.6 Definition The *difference* of the two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} - \mathbf{B}$, is the vector obtained by adding $\mathbf{A}$ to the negative of $\mathbf{B}$; that is

$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

By Definition 10.1.6, it follows that the vector represented by the directed line segment $\overrightarrow{PQ}$ is the difference of the vectors whose position representations are $\overrightarrow{OQ}$ and $\overrightarrow{OP}$. In symbols
$\mathbf{V}(\overrightarrow{PQ}) = \mathbf{q} - \mathbf{p} = \langle q_1 - p_1, q_2 - p_2 \rangle$. Furthermore, ABCD is a parallelogram if and only if $\mathbf{V}(\overrightarrow{AB}) = \mathbf{V}(\overrightarrow{DC})$.

10.1.7 Definition If c is a scalar and $\mathbf{A}$ is the vector (a_1, a_2), then the *product* of c and $\mathbf{A}$, denoted by $c\mathbf{A}$, is the vector given by $c\mathbf{A} = c(a_1, a_2) = (ca_1, ca_2)$.

If $c \neq 0$, then $\dfrac{\mathbf{A}}{c}$ is defined to be $\frac{1}{c}\mathbf{A}$.

V_2 is the set of vectors together with addition and scalar multiplication.
A sum of terms of the form $c_i\mathbf{A}_i$ is called a *linear combination* of the vectors $\mathbf{A}_i$. The linear combination is *trivial* if all the scalars c_i are zero. The set of vectors is *dependent* if some nontrivial combination is the zero vector.

10.1.8 Theorem If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any vectors in V_2, and c and d are any scalars, then vector addition and scalar multiplication satisfy the following properties:

(i) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ (commutative law)
(ii) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$ (associative law)
(iii) There is a vector $\mathbf{0}$ in V_2 for which $\mathbf{A}+\mathbf{0}=\mathbf{A}$ (existence of additive identity)
(iv) There is a vector $-\mathbf{A}$ in V_2 such that $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$ (existence of negative)
(v) $(cd)\mathbf{A}=c(d\mathbf{A})$ (associative law)
(vi) $c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B}$ (distributive law)
(vii) $(c+d)\mathbf{A}=c\mathbf{A}+d\mathbf{A}$ (distributive law)
(viii) $1(\mathbf{A})=\mathbf{A}$ (existence of scalar multiplicative identity)

10.1.9 Definition A *real vector space* V is a set of elements, called *vectors*, together with the set of real numbers, called *scalars*, with two operations called *vector addition* and *scalar multiplication* such that for every pair of vectors **A** and **B** in V and for ever scalar c, a vector $\mathbf{A}+\mathbf{B}$ and a vector $c\mathbf{A}$ are defined so that properties (i)–(viii) of Theorem 10.1.8 are satisfied.

Any vector that has magnitude one is called a *unit* vector. We define the vectors **i** and **j** which form a basis for the vector space V_2 as follows:

$$\mathbf{i}=\langle 1,0\rangle \quad \text{and} \quad \mathbf{j}=\langle 0,1\rangle$$

Thus, for any vector $\langle a_1,a_2\rangle$ in V_2, we have

$$\langle a_1,a_2\rangle = a_1\mathbf{i}+a_2\mathbf{j}$$

10.1.10 Theorem If the nonzero vector $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}$, then the unit vector **U** having the same direction as **A** is given by

$$\mathbf{U}=\frac{\mathbf{A}}{\|\mathbf{A}\|}=\frac{a_1}{\|\mathbf{A}\|}\mathbf{i}+\frac{a_2}{\|\mathbf{A}\|}\mathbf{j}$$

Following are two important inequalities. See Exercise 56.

The Cauchy-Schwarz Inequality $(a_1b_1+a_2b_2)^2 \le (a_1{}^2+a_2{}^2)(b_1{}^2+b_2{}^2)$

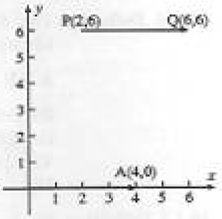
The Triangle Inequality for Vectors $\|\mathbf{A}+\mathbf{B}\| \le \|\mathbf{A}\|+\|\mathbf{B}\|$

Equality holds if and only if **A** and **B** have the same direction or either is the zero vector.

Exercises 10.1

In Exercises 1–4, (a) draw the position representation of the vector **A** and also the particular representation $\overrightarrow{PQ}$ through the given point P. (b) Find the magnitude of **A**.

1. $\mathbf{A}=\langle 3,4\rangle$; $P=(2,1)$. $\mathbf{q}=\mathbf{p}+\mathbf{a}=\langle 2+3,1+4\rangle=\langle 5,5\rangle$. $\|\langle 3,4\rangle\|=\sqrt{3^2+4^2}=\sqrt{25}=5$
2. $\mathbf{A}=\langle -2,5\rangle$; $P=(-3,4)$. $\mathbf{q}=\mathbf{p}+\mathbf{a}=\langle -3-2,4+5\rangle=\langle -5,9\rangle$. $\|\langle -2,5\rangle\|=\sqrt{2^2+5^2}=\sqrt{29}$
3. $\mathbf{A}=\langle c,-\frac{1}{2}\rangle$; $P=(-2,-c)$. $\mathbf{q}=\mathbf{p}+\mathbf{a}=\langle c-2,-c-\frac{1}{2}\rangle$. $\|\langle c,-\frac{1}{2}\rangle\|=\sqrt{c^2+(-\frac{1}{2})^2}=\sqrt{c^2+\frac{1}{4}}$
4. $\mathbf{A}=\langle 4,0\rangle$; $P=(2,6)$.



▸ Let A be the point (4,0). The directed line segment $\overline{OA}$ is the position representation of vector **A**, as shown in the figure. Let $\overrightarrow{PQ}$ be the particular representation of vector **A** through point P. Then

$$\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}=\mathbf{a}$$

and so

$$\mathbf{q}=\mathbf{p}+\mathbf{a}=\langle 2+4,6+0\rangle=\langle 6,6\rangle$$

Thus $Q=(6,6)$. The representation $\overrightarrow{PQ}$ is also shown in the figure. By Theorem 10.1.3, the magnitude of vector **A** is given by

$$\|\mathbf{A}\|=\sqrt{4^2+0^2}=4$$

In Exercises 5 and 6, find the exact (in (c) also approximate) radian measure of the direction angle of the vector.

5. (a) $\langle 1,-1\rangle$. $\tan\theta=\frac{-1}{1}=-1$; $\theta=\frac{7}{4}\pi$
   (b) $\langle -3,0\rangle$. $\tan\theta=\frac{0}{-3}=0$; $\theta=\pi$
   (c) $\langle 5,2\rangle$. $\tan\theta=\frac{2}{5}=0.4$; $\theta=\tan^{-1}(\frac{2}{5})\approx 0.38$
6. (a) $\langle\sqrt{3},1\rangle$. $\tan\theta=\frac{1}{\sqrt{3}}$; $\theta=\frac{1}{6}\pi$
   (b) $\langle 0,4\rangle$. $\tan\theta=\frac{4}{0}$, undefined. $\theta=\frac{1}{2}\pi$
   (c) $\langle -3,2\rangle$. $\tan\theta=\frac{2}{-3}$; $\theta=\pi-\tan^{-1}\frac{2}{3}\approx 2.55$

In Exercises 7–10, find the vector **A** having $\overrightarrow{PQ}$ as a representation. Draw $\overrightarrow{PQ}$ and the position representation of **A**.

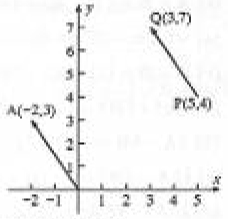
7. $P = (3,7)$; $Q = (5,4)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 5-3, 4-7\rangle = \langle 2,-3\rangle$.

8. $P = (5,4)$; $Q = (3,7)$

▷ Since $\overrightarrow{PQ}$ is a representation of **A**, we have

$\mathbf{A} = \mathbf{q} - \mathbf{p} = \langle 3-5, 7-4\rangle = \langle -2,3\rangle$

The representation $\overrightarrow{PQ}$ and the position representation **a** of the vector **A** are shown in the figure.



9. $P = (-5,-3)$; $Q = (0,3)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 0+5, 3+3\rangle = \langle 5,6\rangle$.

10. $P = (-\sqrt{2},0)$; $Q = (0,0)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 0+\sqrt{2}, 0-0\rangle = \langle\sqrt{2},0\rangle$

In Exercises 11–14, find the point S so that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are each representations of the same vector.

▷ We have $\mathbf{q} - \mathbf{p} = \mathbf{s} - \mathbf{r}$ and so $\mathbf{s} = \mathbf{q} - \mathbf{p} + \mathbf{r}$.

11. $P = (2,5)$; $Q = (1,6)$; $R = (-3,2)$; $\mathbf{s} = \langle 1,6\rangle - \langle 2,5\rangle + \langle -3,2\rangle = \langle -4,3\rangle$

12. $P = (-2,0)$; $Q = (-3,-4)$; $R = (4,2)$

▷ Because $\overrightarrow{PQ} = \overrightarrow{RS}$ then $\mathbf{q} - \mathbf{p} = \mathbf{s} - \mathbf{r}$ and so

$\mathbf{s} = \mathbf{q} - \mathbf{p} + \mathbf{r} = \langle -3-(-2)+4, -4-0+2\rangle = \langle 3,-2\rangle$

Thus $S = -(3,-2)$.

13. $P = (0,3)$; $Q = (5,-2)$; $R = (7,0)$; $\mathbf{s} = \langle 5,-2\rangle - \langle 0,3\rangle + \langle 7,0\rangle = \langle 12,-5\rangle$

14. $P = (-1,4)$; $Q = (2,-3)$; $R = (-5,-2)$. $\mathbf{s} = \langle 2,-3\rangle - \langle -1,4\rangle + \langle -5,-2\rangle = \langle -2,-9\rangle$

In Exercises 15 and 16, find the sum of the pair of vectors and illustrate geometrically.

15. (a) $\langle 2,4\rangle + \langle -3,5\rangle = \langle 2-3, 4+5\rangle = \langle -1,9\rangle$ (b) $\langle -3,0\rangle + \langle 4,-5\rangle = \langle -3+4, 0-5\rangle = \langle 1,-5\rangle$

16. (a) $\langle 0,3\rangle, \langle -2,3\rangle$; (b) $\langle 2,3\rangle, \langle -\sqrt{2},-1\rangle$

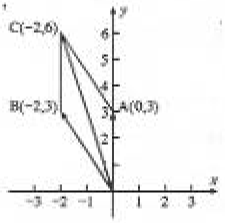
▷ (a) Let $\mathbf{A} = \langle 0,3\rangle$ and $\mathbf{B} = \langle -2,3\rangle$. By Definition 10.1.1,

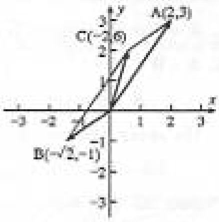
$\mathbf{A} + \mathbf{B} = \langle 0,3\rangle + \langle -2,3\rangle = \langle 0+(-2), 3+3\rangle = \langle -2,6\rangle$

(b) Let $\mathbf{A} = \langle 2,3\rangle$ and $\mathbf{B} = \langle -\sqrt{2},-1\rangle$. Then

$\mathbf{A} + \mathbf{B} = \langle 2,3\rangle + \langle -\sqrt{2},-1\rangle = \langle 2-\sqrt{2},2\rangle$

In the figures, OACB is a parallelogram, $\overrightarrow{OA}$ is the position representation of vector **A** and $\overrightarrow{OB}$ is the position representation of vector **B**. The diagonal $\overrightarrow{OC}$ is the position representation of $\mathbf{A} + \mathbf{B}$.





In Exercises 17 and 18, subtract the second vector from the first and illustrate geometrically.

17. (a) $\langle -3,-4\rangle - \langle 6,0\rangle = \langle -3-6, -4-0\rangle = \langle -9,-4\rangle$ (b) $\langle 1,e\rangle - \langle -3,2e\rangle = \langle 1+3, e-2e\rangle = \langle 4,-e\rangle$

18. (a) $\langle 0,5\rangle - \langle 2,8\rangle = \langle 0-2, 5-8\rangle = \langle -2,-3\rangle$ (b) $\langle 3,7\rangle - \langle 3,7\rangle = \mathbf{0}$

In Exercises 19 and 20, find the vector or scalar if $\mathbf{A} = \langle 2,4\rangle$, $\mathbf{B} = \langle 4,-3\rangle$, and $\mathbf{C} = \langle -3,2\rangle$.

19. $\mathbf{A} = \langle 2,4\rangle$, $\mathbf{B} = \langle 4,-3\rangle$, $\mathbf{C} = \langle -3,2\rangle$. (a) $\mathbf{A} + \mathbf{B} = \langle 2,4\rangle + \langle 4,-3\rangle = \langle 6,1\rangle$

(b) $\|\mathbf{C} - \mathbf{B}\| = \|\langle -3,2\rangle - \langle 4,-3\rangle\| = \|\langle -7,5\rangle\| = \sqrt{(-7)^2 + 5^2} = \sqrt{49+25} = \sqrt{74}$

(c) $\|7\mathbf{A} - \mathbf{B}\| = \|7\langle 2,4\rangle - \langle 4,-3\rangle\| = \|\langle 14,28\rangle - \langle 4,-3\rangle\| = \|\langle 10,31\rangle\| = \sqrt{10^2 + 31^2} = \sqrt{100+961} = \sqrt{1061}$

20. (a) $\mathbf{A} - \mathbf{B}$; (b) $\|\mathbf{C}\|$; (c) $2\mathbf{A} + 3\mathbf{B}$

▷ (a) $\mathbf{A} - \mathbf{B} = \langle 2,4\rangle - \langle 4,-3\rangle = \langle 2-4, 4-(-3)\rangle = \langle -2,7\rangle$

(b) $\|\mathbf{C}\| = \|\langle -3,2\rangle\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

(c) $2\mathbf{A} + 3\mathbf{B} = 2\langle 2,4\rangle + 3\langle 4,-3\rangle = \langle 4,8\rangle + \langle 12,-9\rangle = \langle 16,-1\rangle$

In Exercises 21–24, find the vector or scalar if $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} - \mathbf{j}$.

21. (a) $5\mathbf{A} = 5(2\mathbf{i} + 3\mathbf{j}) = 10\mathbf{i} + 15\mathbf{j}$ (b) $-6\mathbf{B} = -6(4\mathbf{i} - \mathbf{j}) = -24\mathbf{i} + 6\mathbf{j}$

(c) $\mathbf{A} + \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} - \mathbf{j}) = 6\mathbf{i} + 2\mathbf{j}$ (d) $\|\mathbf{A} + \mathbf{B}\| = \|6\mathbf{i} + 2\mathbf{j}\| = \sqrt{6^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$

22. (a) $-2\mathbf{A} = -2(2\mathbf{i} + 3\mathbf{j}) = -4\mathbf{i} - 6\mathbf{j}$ (b) $3\mathbf{B} = 3(4\mathbf{i} - \mathbf{j}) = 12\mathbf{i} - 3\mathbf{j}$

(c) $\mathbf{A} - \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) - (4\mathbf{i} - \mathbf{j}) = -2\mathbf{i} + 4\mathbf{j}$ (d) $\|\mathbf{A} - \mathbf{B}\| = \|-2\mathbf{i} + 4\mathbf{j}\| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

23. (a) $\|\mathbf{A}\| + \|\mathbf{B}\| = \|2\mathbf{i} + 3\mathbf{j}\| + \|4\mathbf{i} - \mathbf{j}\| = \sqrt{2^2 + 3^2} + \sqrt{4^2 + (-1)^2} = \sqrt{13} + \sqrt{17}$

(b) $5\mathbf{A} - 6\mathbf{B} = 5(2\mathbf{i} + 3\mathbf{j}) - 6(4\mathbf{i} - \mathbf{j}) = 10\mathbf{i} + 15\mathbf{j} - 24\mathbf{i} + 6\mathbf{j} = -14\mathbf{i} + 21\mathbf{j}$

(c) $\|5\mathbf{A} - 6\mathbf{B}\| = \|-14\mathbf{i} + 21\mathbf{j}\| = \sqrt{(-14)^2 + 21^2} = \sqrt{96 + 441} = \sqrt{637} = 7\sqrt{13}$

(d) $\|5\mathbf{A}\| - \|6\mathbf{B}\| = 5\|\mathbf{A}\| - 6\|\mathbf{A}\| = 5\sqrt{2^2 + 3^2} - 6\sqrt{4^2 + (-1)^2} = 5\sqrt{13} - 6\sqrt{17}$

24. (a) $\|\mathbf{A}\| - \|\mathbf{B}\|$; (b) $3\mathbf{B} - 2\mathbf{A}$; (c) $\|3\mathbf{B} - 2\mathbf{A}\|$; (d) $\|3\mathbf{B}\| - \|2\mathbf{A}\|$

▷ (a) $\|\mathbf{A}\| - \|\mathbf{B}\| = \|2\mathbf{i} + 3\mathbf{j}\| - \|4\mathbf{i} - \mathbf{j}\| = \sqrt{2^2 + 3^2} - \sqrt{4^2 + (-1)^2} = \sqrt{13} - \sqrt{17}$

(b) $3\mathbf{B} - 2\mathbf{A} = 3(4\mathbf{i} - \mathbf{j}) - 2(2\mathbf{i} + 3\mathbf{j}) = (12\mathbf{i} - 3\mathbf{j}) - (4\mathbf{i} + 6\mathbf{j}) = 8\mathbf{i} - 9\mathbf{j}$

(c) $\|3\mathbf{B} - 2\mathbf{A}\| = \|8\mathbf{i} - 9\mathbf{j}\| = \sqrt{8^2 + 9^2} = \sqrt{145}$

(d) $\|3\mathbf{B}\| - \|2\mathbf{A}\| = \|12\mathbf{i} - 3\mathbf{j}\| - \|4\mathbf{i} + 6\mathbf{j}\| = \sqrt{12^2 + 3^2} - \sqrt{4^2 + 6^2} = \sqrt{153} - \sqrt{52} = 3\sqrt{17} - 2\sqrt{13}$

In Exercises 25 and 26, $\mathbf{A} = -4\mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = -\mathbf{i} + 3\mathbf{j}$, and $\mathbf{C} = 5\mathbf{i} - \mathbf{j}$.

25. (a) $5\mathbf{A} - 2\mathbf{B} - 2\mathbf{C} = 5(-4\mathbf{i} + 2\mathbf{j}) - 2(-\mathbf{i} + 3\mathbf{j}) - 2(5\mathbf{i} - \mathbf{j}) = (-20\mathbf{i} + 10\mathbf{j}) + (2\mathbf{i} - 6\mathbf{j}) + (-10\mathbf{i} + 2\mathbf{j})$
$= (-20 + 2 - 10)\mathbf{i} + (10 - 6 + 2)\mathbf{j} = -28\mathbf{i} + 6\mathbf{j}$

(b) $\|5\mathbf{A} - 2\mathbf{B} - 2\mathbf{C}\| = \sqrt{(-28)^2 + 6^2} = \sqrt{784 + 36} = \sqrt{820} = 2\sqrt{205}$

26. (a) $3\mathbf{B} - 2\mathbf{A} - \mathbf{C} = 3(-\mathbf{i} + 3\mathbf{j}) - 2(-4\mathbf{i} + 2\mathbf{j}) - (5\mathbf{i} - \mathbf{j}) = (-3\mathbf{i} + 9\mathbf{j}) + (8\mathbf{i} - 4\mathbf{j}) + (-5\mathbf{i} + \mathbf{j}) = 6\mathbf{j}$

(b) $\|3\mathbf{B} - 2\mathbf{A} - \mathbf{C}\| = \|6\mathbf{j}\| = 6$

In Exercises 27 and 28, let $\mathbf{A} = 8\mathbf{i} + 5\mathbf{j}$ and $\mathbf{B} = 3\mathbf{i} - \mathbf{j}$. find a unit vector $\mathbf{U}$ having the same direction as:

27. $\mathbf{A} + \mathbf{B} = (8\mathbf{i} + 5\mathbf{j}) + (3\mathbf{i} - \mathbf{j}) = 11\mathbf{i} + 4\mathbf{j}; \|\mathbf{A} + \mathbf{B}\| = \sqrt{11^2 + 4^2} = \sqrt{121 + 16} = \sqrt{137}; \mathbf{U} = \dfrac{11}{\sqrt{137}}\mathbf{i} + \dfrac{4}{\sqrt{137}}\mathbf{j}$

28. $\mathbf{A} - \mathbf{B}$

▷ $\mathbf{A} - \mathbf{B} = (8\mathbf{i} + 5\mathbf{j}) - (3\mathbf{i} - \mathbf{j}) = 5\mathbf{i} + 6\mathbf{j}$

We apply Theorem 10.1.10. Because

$$\|\mathbf{A} - \mathbf{B}\| = \|5\mathbf{i} + 6\mathbf{j}\| = \sqrt{5^2 + 6^2} = \sqrt{61}$$

then the unit vector having the same direction as $\mathbf{A} - \mathbf{B}$ is

$$\mathbf{U} = \frac{5}{\sqrt{61}}\mathbf{i} + \frac{6}{\sqrt{61}}\mathbf{j}$$

In Exercises 29–32, write the given vector in the form $r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, where r is the magnitude and θ is the direction angle. Also find a unit vector having the same direction.

29. (a) $3\mathbf{i} - 4\mathbf{j}$. $r = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$; $3\mathbf{i} - 4\mathbf{j} = 5(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j})$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

(b) $2\mathbf{i} + 2\mathbf{j}$. $r = 2\|\mathbf{i} + \mathbf{j}\| = 2\sqrt{1^2 + 1^2} = 2\sqrt{2}$; $2\mathbf{i} + 2\mathbf{j} = 2\sqrt{2}(\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}) = 2\sqrt{2}(\cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j})$.

$\mathbf{U} = \frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}$

30. (a) $8\mathbf{i} + 6\mathbf{j}$. $r = \sqrt{8^2 + 6^2} = 10$. $8\mathbf{i} + 6\mathbf{j} = 10(\frac{8}{10}\mathbf{i} + \frac{6}{10}\mathbf{j})$. $\mathbf{U} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

(b) $2\sqrt{5}\mathbf{i} + 4\mathbf{j}$. $r = \sqrt{20 + 16} = 6$. $2\sqrt{5}\mathbf{i} + 4\mathbf{j} = 6(\frac{1}{3}\sqrt{5}\mathbf{i} + \frac{2}{3}\mathbf{j})$. $\mathbf{U} = \frac{1}{3}\sqrt{5}\mathbf{i} + \frac{2}{3}\mathbf{j}$

31. (a) $r = \|-4\mathbf{i} + 4\sqrt{3}\mathbf{j}\| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$

$-4\mathbf{i} + 4\sqrt{3}\mathbf{j} = 8(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = 8(\cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j})$; $\mathbf{U} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$

(b) $r = \|-16\mathbf{i}\| = 16\|\mathbf{i}\| = 16$; $-16\mathbf{i} = 16(-\mathbf{i} + 0\mathbf{j}) = 16(\cos\pi\mathbf{i} + \sin\pi\mathbf{j})$; $\mathbf{U} = -\mathbf{i}$

32. (a) $3\mathbf{i}-3\mathbf{j}$; (b) $2\mathbf{j}$

▸ $r=\|3\mathbf{i}-3\mathbf{j}\|=\sqrt{3^2+(-3)^2}=3\sqrt{2}$

Because $\cos\theta=\frac{a_1}{r}=\frac{3}{3\sqrt{2}}=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{a_2}{r}=\frac{-3}{3\sqrt{2}}=-\frac{1}{\sqrt{2}}$ then $\theta=\frac{7}{4}\pi$. Therefore, we have

$3\mathbf{i}-3\mathbf{j}=3\sqrt{2}(\cos\frac{7}{4}\pi\mathbf{i}+\sin\frac{7}{4}\pi\mathbf{j})$

A unit vector having the same direction as the given vector is $\cos\frac{7}{4}\pi\mathbf{i}+\sin\frac{7}{4}\pi\mathbf{j}$ or, equivalently, $\frac{1}{2}\sqrt{2}\mathbf{i}-\frac{1}{2}\sqrt{2}\mathbf{j}$

(b) $r=\|2\mathbf{j}\|=\sqrt{2^2}=2$

$\cos\theta=\frac{a_1}{r}=0$ and $\sin\theta=\frac{a_2}{r}=\frac{2}{2}=1$

Thus, $\theta=\frac{1}{2}\pi$ and

$2\mathbf{j}=2(\cos\frac{1}{2}\pi\mathbf{i}+\sin\frac{1}{2}\pi\mathbf{j})$

A unit vector having the same direction as the given vector $2\mathbf{j}$ is $\mathbf{j}$. Note that this result is also given by $\cos\frac{1}{2}\pi\mathbf{i}+\sin\frac{1}{2}\pi\mathbf{j}$.

33. $\mathbf{A}=-2\mathbf{i}+\mathbf{j}$; $\mathbf{B}=3\mathbf{i}-2\mathbf{j}$; $\mathbf{C}=5\mathbf{i}-4\mathbf{j}$; $\mathbf{C}=h\mathbf{A}+k\mathbf{B}\Rightarrow 5\mathbf{i}-4\mathbf{j}=h(-2\mathbf{i}+\mathbf{j})+k(3\mathbf{i}-2\mathbf{j})=(-2h+3k)\mathbf{i}+(h-2k)\mathbf{j}$

Thus $\begin{matrix}-2h+3k=5\\ h-2k=-5\end{matrix}\Rightarrow\begin{matrix}h=2\\ k=3\end{matrix}$.

34. $\mathbf{A}=5\mathbf{i}-2\mathbf{j}$; $\mathbf{B}=-4\mathbf{i}+3\mathbf{j}$; $\mathbf{C}=-6\mathbf{i}+8\mathbf{j}$. $\mathbf{B}=h\mathbf{C}-k\mathbf{A}\Rightarrow -4\mathbf{i}+3\mathbf{j}=h(-6\mathbf{i}+8\mathbf{j})-k(5\mathbf{i}-2\mathbf{j})$. Therefore,

$\begin{matrix}-6h-5k=-4\\ 8h+2k=3\end{matrix}\Rightarrow\begin{matrix}h=\frac{1}{4}\\ k=\frac{1}{2}\end{matrix}$

35. $\mathbf{A}=\mathbf{i}-2\mathbf{j}$; $\mathbf{B}=-2\mathbf{i}+4\mathbf{j}$; $\mathbf{C}=7\mathbf{i}-5\mathbf{j}$. $\mathbf{C}=h\mathbf{A}+k\mathbf{B}\Rightarrow 7\mathbf{i}-5\mathbf{j}=h(\mathbf{i}-2\mathbf{j})+k(-2\mathbf{i}+4\mathbf{j})=(h-2k)\mathbf{i}+(-2h+4k)\mathbf{j}$

Therefore $\begin{matrix}h-2k=7\\ -2h+4k=-5\end{matrix}\Rightarrow\begin{matrix}-2h+4k=-14\\ -2h+4k=-5\end{matrix}$.

Because this system has no solution, $\mathbf{C}$ cannot be written in the form $h\mathbf{A}+k\mathbf{B}$.

36. Two forces of magnitudes 340 lb and 475 lb make an angle of 34.6° with each other and are applied to an object at the same point. Find (a) the magnitude of the resultant force and (b) to the nearest tenth of a degree the angle it makes with the force of 475 lb.

▸ (a) Refer to the figure. Let $\mathbf{A}=\langle 475,0\rangle$ represent the force of magnitude 475 lb. If $\mathbf{B}=\langle b_1,b_2\rangle$ represents the force of magnitude 340 lb, then because the angle between $\mathbf{A}$ and $\mathbf{B}$ is 34.6°, we have

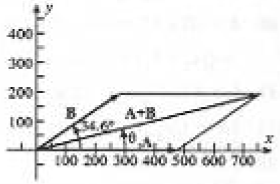
$b_1=340\cos 34.6^\circ=280$ and $b_2=340\sin 34.6^\circ=193$

Thus, $\mathbf{B}=\langle 280,193\rangle$. The resultant force is $\mathbf{A}+\mathbf{B}$ and

$\mathbf{A}+\mathbf{B}=\langle 475,0\rangle+\langle 280,193\rangle=\langle 755,193\rangle$

Then

$\|\mathbf{A}+\mathbf{B}\|=\sqrt{755^2+193^2}=779$



The magnitude of the resultant force is 779 lb.

(b) If θ is the angle the vector $\mathbf{A}+\mathbf{B}$ makes with $\mathbf{A}$, then

$\tan\theta=\frac{193}{755}=0.2556 \qquad \theta=14.3^\circ$

The angle that the resultant force makes with the force of 475 lb is 14.3°.

37. Choose the axes so that the position representation of the 60 lb force is along the positive x axis. Then the vector $\mathbf{A}=\langle 60,0\rangle$ represents this force. Let the vector $\mathbf{B}=\langle b_1,b_2\rangle$ represent the 80 lb. force. Then $b_1=80\cos 30^\circ\approx 69.3$ and $b_2=80\sin 30^\circ=40$. The resultant force is $\mathbf{A}+\mathbf{B}=\langle 60,0\rangle+\langle 69.3,40\rangle=\langle 129.3,40\rangle$.

(a) $\|\mathbf{A}+\mathbf{B}\|=\sqrt{(129.3)^2+(40)^2}\approx 135.3$

(b) If θ is the angle between $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ then $\tan\theta=\frac{40}{129.3}\approx 0.309$; $\theta\approx 17^\circ$.

38. Let $\mathbf{A}=\langle 34,0\rangle$. Then $\mathbf{B}=\langle 22\cos\theta,22\sin\theta\rangle$. $46^2=\|\mathbf{A}+\mathbf{B}\|^2=(34+22\cos\theta)^2+(22\sin\theta)^2$

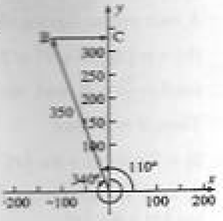
$=34^2+2\cdot 34\cdot 22\cos\theta+22^2$, $\cos\theta=\frac{46^2-34^2-22^2}{2\cdot 34\cdot 22}=0.3182$, $\theta=71.4^\circ\approx 71^\circ$

39. Draw a figure similar to Figure 14 in the text. Let $a = \| \mathbf{A} \| = 112$, $b = \| \mathbf{B} \| = 84$, $c = \| \mathbf{A} + \mathbf{B} \| = 162$. From the law of cosines, $\cos\theta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{112^2 + 162^2 - 84^2}{2 \cdot 112 \cdot 162} = 0.874$; $\theta \approx 29.0°$.

40. A plane has an air speed of 350 mi/h. In order for the actual course of the plane to be due north, the compass heading is 340°. If the wind is blowing from the west, (a) what is its speed? (b) What is the plane's ground speed?

▹ Refer to the Figure. Let $\overrightarrow{OB} = \langle b_1, b_2 \rangle$ be the plane's velocity. Because the compass heading is 340°, the direction of $\overrightarrow{OB}$ is $450° - 340° = 110°$. Thus

$$b_1 = 350 \cos 110° = -119.7 \quad b_2 = 350 \sin 110° = 119.7$$



Therefore C is the point $(0, 328.9)$.

(a) The magnitude of the wind's velocity is $\| \overrightarrow{BC} \| = |b_1| = 119.7$.
(b) The number of miles per hour in the plane's ground speed is

$$\| \overrightarrow{OC} \| = |b_2| = 328.9$$

41. (a) heading: $360° - \sin^{-1}(60/250) = 360° - 13.9° = 346.1°$ (b) $v = \sqrt{250^2 - 60^2} = 242.7$ (mi/hr)

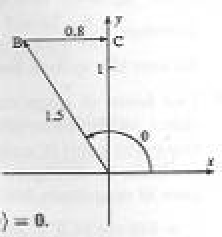
42. $v = \sqrt{15^2 + 3^2} = 15.30$ knots, course $= 180° + \tan^{-1}\frac{3}{15} = 191.31°$

43. (a) heading: $\tan^{-1}\frac{0.8}{1.5} = 28.1°$ (b) $v = \sqrt{1.5^2 + .8^2} = 1.7$ (c) in $\frac{1}{1.5}$hr he moves $\frac{0.8}{1.5} = 0.53$ mi

44. A swimmer who can swim at a speed of 1.5 mi/h relative to the water leaves the south bank of a river whose current is toward the east at 0.8 mi/h. He wishes to reach a point directly north across the river. (a) In what direction should he head? (b) What will be his speed relative to the land if this direction is taken?

▹ Refer to the figure. Let $\overrightarrow{OB} = \langle b_1, b_2 \rangle$ be his 's velocity and θ his direction. Then

$$b_1 = -0.8 = 1.5 \cos\theta$$
$$\cos\theta = -\frac{0.8}{1.5} = -0.5333$$
$$b_2 = 1.5 \sin 122.2 = 1.27$$



(a) The swimmer's heading should be $450° - 122.2° = 327.8°$.

(b) His speed relative to the land will be 1.27 mi/h.

45. Let $\mathbf{A} = \langle a_1, a_2 \rangle$. The $0\mathbf{A} = 0\langle a_1, a_2 \rangle = \langle 0a_1, 0a_2 \rangle = \mathbf{0}$ and $c\mathbf{0} = c\langle 0,0 \rangle = \langle c0, c0 \rangle = \langle 0,0 \rangle = \mathbf{0}$.

46. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle = \langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2) \rangle$
$= \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

47. Let $\mathbf{A} = \langle a_1, a_2 \rangle$. Then $\mathbf{A} + \mathbf{0} = \langle a_1, a_2 \rangle + \langle 0,0 \rangle = \langle a_1 + 0, a_2 + 0 \rangle = \langle a_1, a_2 \rangle = \mathbf{A}$. Also
$1\mathbf{A} = 1\langle a_1, a_2 \rangle = \langle 1a_1, 1a_2 \rangle = \langle a_1, a_2 \rangle = \mathbf{A}$

48. Prove Theorem 10.1.8(iv).

▹ We are asked to prove that if $\mathbf{A}$ is any vector then there is a vector $-\mathbf{A}$ such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$. Let $\mathbf{A} = \langle a_1, a_2 \rangle$ and let $-\mathbf{A} = \langle -a_1, -a_2 \rangle$ be the vector of Definition 10.1.5. Then, by Definition 10.1.4

$$\mathbf{A} + (-\mathbf{A}) = \langle a_1 + (-a_1), a_2 + (-a_2) \rangle = \langle 0,0 \rangle = \mathbf{0}$$

49. Let $\mathbf{A} = \langle a_1, a_2 \rangle$. Then $(cd)\mathbf{A} = (cd)\langle a_1, a_2 \rangle = \langle (cd)a_1, (cd)a_2 \rangle = \langle c(da_1), c(da_2) \rangle = c\langle da_1, da_2 \rangle = c(d\mathbf{A})$

50. $(c + d)\mathbf{A} = (c + d)\langle a_1, a_2 \rangle = \langle (c + d)a_1, (c + d)a_2 \rangle = \langle ca_1 + da_1, ca_2 + da_2 \rangle = \langle ca_1, ca_2 \rangle + \langle da_1, da_2 \rangle$
$= c\langle a_1, a_2 \rangle + d\langle a_1, a_2 \rangle = c\mathbf{A} + d\mathbf{A}$

51. (a) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \langle 2, -5 \rangle + (\langle 3,1 \rangle + \langle -4,2 \rangle) = \langle 2, -5 \rangle + \langle -1,3 \rangle = \langle 1, -2 \rangle$
(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = (\langle 2, -5 \rangle + \langle 3,1 \rangle) + \langle -4,2 \rangle = \langle 5, -4 \rangle + \langle -4,2 \rangle = \langle 1, -2 \rangle$

52. Two vectors are said to be *independent* if and only if their position representations are not collinear. Furthermore, two vectors **A** and **B** are said to form a *basis* for the vector space V_2 if and only if any vector in V_2 can be written as a linear combination of **A** and **B**. A theorem can be proved which states that two vectors form a basis for the vector space V_2 if they are independent. Show that this theorem holds for the two vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$ by doing the following: (a) Verify that the vectors are independent by showing that the position representations are not collinear; (b) verify that the vectors form a basis by showing that any vector $a_1\mathbf{i} + a_2\mathbf{j}$ can be written as $c(2\mathbf{i} + 5\mathbf{j}) + d(3\mathbf{i} - \mathbf{j})$, where c and d are scalars. (Hint. Find c and d in terms of a_1 and a_2.)

▸ (a) If $A = (2,5)$ then $\overrightarrow{OA}$ is the position representation of vector $\langle 2,5\rangle$. If $B = (3,-1)$, then $\overrightarrow{OB}$ is the position representation of vector $\langle 3,-1\rangle$. Because the slope of $\overrightarrow{OA}$ is $\frac{5}{2}$ and the slope of $\overrightarrow{OB}$ is $-\frac{1}{3} \neq \frac{5}{2}$, the position representations are not collinear and thus the vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$ are independent.
(b) If $a_1\mathbf{i} + b_1\mathbf{j} = c(2\mathbf{i} + 5\mathbf{j}) + d(3\mathbf{i} - \mathbf{j}) = (2c + 3d)\mathbf{i} + (5c - d)\mathbf{j}$ then

$$a_1 = 2c + 3d$$
$$a_2 = 5c - d$$

Therefore,

$$a_1 + 3a_2 = 2c + 3(5c) = 17c \qquad\qquad 5a_1 - 2a_2 = 5(3d) + 2d = 17d$$
$$c = \tfrac{1}{17}(a_1 + 3a_2 \qquad\qquad d = \tfrac{1}{17}(5a_1 - 2a_2)$$

Thus any vector can be written as a linear combination of the vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$. That is, with the values of c and d above,
$\langle a_1, a_2\rangle = c\langle 2,5\rangle + d\langle 3,-1\rangle$

53. (a) Let $\mathbf{A} = \langle 3,-2\rangle$ and $\mathbf{B} = \langle -6,4\rangle$. Because $\mathbf{B} = -2\mathbf{A}$, **B** is a scalar multiple of **A**. Hence representations of the two vectors are parallel. Thus the position representations of the two vectors are collinear.
(b) Let $\mathbf{C} = \mathbf{i} + \mathbf{j}$. If $\mathbf{C} = c\mathbf{A} + d\mathbf{B}$ then $\mathbf{i} + \mathbf{j} = c(3\mathbf{i} - 2\mathbf{j}) + d(-6\mathbf{i} + 4\mathbf{j}) = (3c - 6d)\mathbf{i} + (-2c + 4d)\mathbf{j}$.
$\begin{matrix} 3c - 6d = 1 \\ -2c + 4d = 1 \end{matrix} \Rightarrow \begin{matrix} 6c - 12d = 2 \\ 6c - 12d = -3 \end{matrix}$. Since this system has no solution, **A** and **B** are not a basis.

54. $a(3\mathbf{i} - 2\mathbf{j}) + b(\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} + 5\mathbf{j}) = (3a + b + 2)\mathbf{i} + (-2a + 4b + 5)\mathbf{j} = \mathbf{0} \Rightarrow \begin{matrix} 3a + b = -2 \\ -2a + 4b = -5 \end{matrix} \Rightarrow a = -\frac{3}{14}, b = -\frac{19}{14}$

55. We are given $\mathbf{A} = \mathbf{V}(\overrightarrow{PQ})$, $\mathbf{B} = \mathbf{V}(\overrightarrow{QR})$, $\mathbf{C} = \mathbf{V}(\overrightarrow{RS})$. Because $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ are sides of a triangle, S is the same as point P. From Figure 13 of the text we see that $\mathbf{V}(\overrightarrow{PQ}) + \mathbf{V}(\overrightarrow{QR}) = \mathbf{V}(\overrightarrow{PR})$. Hence

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{V}(\overrightarrow{PQ}) + \mathbf{V}(\overrightarrow{QR}) + \mathbf{V}(\overrightarrow{RS}) = \mathbf{V}(\overrightarrow{PR}) + \mathbf{V}(\overrightarrow{RS}) = \mathbf{V}(\overrightarrow{PS}) = \mathbf{0}$$

56. Prove analytically the triangle inequality for vectors $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.

▸ Let $\mathbf{A} = \langle a_1, a_2\rangle$ and $\mathbf{B} = \langle b_1, b_2\rangle$. Then $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2\rangle$.

$$\begin{aligned} \|\mathbf{A} + \mathbf{B}\|^2 &= (a_1 + b_1)^2 + (a_2 + b_2)^2 = (a_1{}^2 + 2a_1b_1 + b_1{}^2) + (a_2{}^2 + 2a_2b_2 + b_2{}^2) \\ &= (a_1{}^2 + a_2{}^2) + (b_1{}^2 + b_2{}^2) + 2(a_1b_1 + a_2b_2) = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2(a_1b_1 + a_2b_2) \\ &\leq \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2|a_1b_1 + a_2b_2| \end{aligned} \tag{1}$$

We prove the Cauchy-Schwarz inequality. Consider the quadratic equation

$$(a_1x - b)^2 + (a_2x - b_2)^2 = 0 \tag{2}$$

or, equivalently,

$$(a_1{}^2 + a_2{}^2)x^2 - 2(a_1b_1 + a_2b_2)x + (b_1{}^2 + b_2{}^2) = 0 \tag{3}$$

From (2) we see that if $x = k$ is a solution then we must have

$$a_1k - b_1 = 0 \quad a_2k - b_2 = 0$$

or, equivalently,

$$b_1 = ka_1 \quad b_2 = ka_2 \tag{4}$$

that is, **B** is a scalar multiple of **A**, and otherwise there is no real solution. Therefore, the discriminant of the quadratic equation (3) is either negative or 0, that is

$$[-2(a_1b_1 + a_2b_2)]^2 - 4(a_1{}^2 + a_2{}^2)(b_1{}^2 + b_2{}^2) \leq 0$$

or, equivalently,

$$(a_1b_1 + a_2b_2)^2 \le (a_1{}^2 + a_2{}^2)(b_1{}^2 + b_2{}^2)$$

with equality if and only if (4) holds. Taking the square root of both sides, we have

$$|a_1b_1 + a_2b_2| \le \sqrt{a_1{}^2 + a_2{}^2}\sqrt{b_1{}^2 + b_2{}^2} \tag{5}$$

Substituting from (5) into (1), we obtain

$$\|\mathbf{A} + \mathbf{B}\|^2 \le \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2\sqrt{a_1{}^2 + a_2{}^2}\sqrt{b_1{}^2 + b_2{}^2} = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2\|\mathbf{A}\|\|\mathbf{B}\| = (\|\mathbf{A}\| + \|\mathbf{B}\|)^2$$

Taking the square root of both sides gives

$$\|\mathbf{A} + \mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$$

with equality if and only if (4) holds for some $k \ge 0$, that is, if and only if **A** and **B** have the same direction or either is the zero vector.

10.2 VECTORS IN THREE DIMENSIONAL SPACE

10.2.1 Definition The set of all ordered triples of real numbers is called the *three-dimensional number space* and is denoted by R^3. Each ordered triple (x, y, z) is called a point in the three-dimensional number space. The x axis is $\{(x, y, z) \mid y = 0,\ z = 0\}$; similarly for the y and z axes. The *first octant* is $\{(x, y, z) \mid x > 0,\ y > 0,\ z > 0\}$.

We use a *right-handed system*: if the x axis is pointing to us, the positive y axis rotates in the positive direction (counterclockwise) to coincide with the positive z axis.

10.2.2-3 Theorem

A line is parallel to	if and only if all points on the line have
the yz plane	equal x coordinates
the xz plane	equal y coordinates
the xy plane	equal z coordinates
the x axis	equal y coordinates and equal z coordinates
the y axis	equal x coordinates and equal z coordinates
the z axis	equal x coordinates and equal y coordinates

10.2.5 Theorem The undirected distance between the two points $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$ is given by

$$|\mathrm{P}_1\mathrm{P}_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

10.2.7 Definition The *graph of an equation* in R^3 is the set of all points (x, y, z) whose coordinates are numbers satisfying the equation.

A surface is the graph of an equation in R^3. (A graph like $[\![z]\!] = 1$ is not a surface. In §12.7 we formally define "surface" similar to "curve" in §11.1.) One particular surface is the sphere.

10.2.8 Definition A *sphere* is the set of all points in three-dimensional space equidistant from a fixed point. The fixed point is called the *center* of the sphere and the measure of the constant distance is called the *radius* of the sphere.

10.2.9 Theorem An equation of the sphere of radius r and center at (h, k, l) is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

An equation of the sphere with diameter endpoints $\mathrm{A}(h_1, k_1, l_1)$ and $\mathrm{B}(h_2, k_2, l_2)$ is (Ex. 26)

$$(x - h_1)(x - h_2) + (y - k_1)(y - k_2) + (z - l_1)(z - l_2) = 0$$

10.2.10 Theorem The graph of a second-degree equation in x, y, and z, of the form

$$x^2 + y^2 + z^2 + \mathrm{G}x + \mathrm{H}y + \mathrm{I}z + \mathrm{J} = 0$$

is either a sphere, a point, or the empty set.

10.2.11 Definition A *vector in three-dimensional space* is an ordered triple of real numbers $\langle x, y, z\rangle$. The numbers x, y, and z are called the *components* of the vector $\langle x, y, z\rangle$.

Vectors $\langle x, y, z\rangle$ and $\langle u, v, w\rangle$ are *equal* if and only if $x = u, y = v$, and $z = w$. The *position representation* of the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ is the directed line segment from the origin to the point (a_1, a_2, a_3). The *zero vector* is the vector $\langle 0, 0, 0\rangle$ and is denoted by $\mathbf{0}$.

The magnitude of the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ is given by $\|\mathbf{A}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$.
$\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

10.2.12 Definition The *direction angles* of a nonzero vector are the three angles that have the smallest nonnegative radian measure α, β, γ measured from the positive x, y, and z axes, respectively, to the position representation of the vector.
Thus the direction angles for the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ are α, β and γ with $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$, and $0 \le \gamma \le \pi$, such that

$$\cos\alpha = \frac{a_1}{\|\mathbf{A}\|} \quad \cos\beta = \frac{a_2}{\|\mathbf{A}\|} \quad \cos\gamma = \frac{a_3}{\|\mathbf{A}\|}$$

The three numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of vector $\mathbf{A}$.

10.2.13 Theorem If $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of a vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

The remaining definitions and theorems in this section are extensions of the corresponding definitions for vectors in V_2.

Definition If $\mathbf{A} = \langle a_1, a_2, a_3\rangle$, $\mathbf{B} = \langle b_1, b_2, b_3\rangle$, and c is a scalar, then

$$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$

$$-\mathbf{A} = \langle -a_1, -a_2, -a_3\rangle$$

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$$

$$c\mathbf{A} = c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Theorem $c\mathbf{A}$ has the same direction as $\mathbf{A}$ if $c > 0$; the opposite direction if $c < 0$. Also, $\|c\mathbf{A}\| = |c|\|\mathbf{A}\|$.

10.2.6 Theorem The coordinates of the midpoint of the line segment having endpoints $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by

$$\bar{x} = \tfrac{1}{2}(x_1 + x_2) \quad \bar{y} = \tfrac{1}{2}(y_1 + y_2) \quad \bar{z} = \tfrac{1}{2}(z_1 + z_2)$$

Theorem 10.2.6 says that the position representation of the midpoint P of segment P_1P_2 is given by $\mathbf{p} = \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2$. More generally, if P is the fraction w of the way from P_1 to P_2, then $\mathbf{p} = (1 - w)\mathbf{p}_1 + w\mathbf{p}_2$.

V_3 is the set of vectors together with addition and scalar multiplication and satisfies the properties given in Theorem 10.2.8. Thus V_3 is a real vector space. The three unit vectors
$\mathbf{i} = \langle 1, 0, 0\rangle \quad \mathbf{j} = \langle 0, 1, 0\rangle \quad \mathbf{k} = \langle 0, 0, 1\rangle$
form a basis for V_3 because any vector can be expressed in terms of them as follows:
$\mathbf{A} = \langle a_1, a_2, a_3\rangle = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

10.2.14 Theorem If the nonzero vector $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then the unit vector $\mathbf{U}$ having the same direction as $\mathbf{A}$ is given by

$$\mathbf{U} = \frac{a_1}{\|\mathbf{A}\|}\mathbf{i} + \frac{a_2}{\|\mathbf{A}\|}\mathbf{j} + \frac{a_3}{\|\mathbf{A}\|}\mathbf{k}$$

Exercises 10.2

In Exercises 1–5, points A and B are opposite vertices of a rectangular parallelopiped, having its faces parallel to the coordinate planes. In each exercise, (a) sketch the figure, (b) find the coordinates of the other six vertices, (c) Find the length of the diagonal AB.

▸ If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are opposite vertices of a parallepiped having its faces parallel to the coordinate planes, then any vertex has the form (c_1, c_2, c_3) where $c_i = a_i$ or b_i, $i = 1, 2, 3$.

1. $A(0, 0, 0)$, $B(7, 2, 3)$ (b) $(7, 2, 0)$, $(0, 0, 3)$, $(0, 2, 0)$, $(0, 2, 3)$, $(7, 0, 3)$, $(7, 0, 0)$
(c) $|\overline{AB}| = \sqrt{(7-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{49 + 4 + 9} = \sqrt{62}$

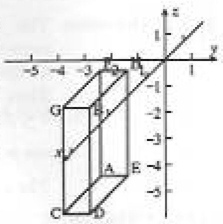
2. $A(1, 1, 1)$; $B(3, 4, 2)$ (b) $(1, 1, 2)$, $(1, 4, 1)$, $(1, 4, 2)$, $(3, 1, 1)$, $(3, 1, 2)$, $(3, 4, 1)$
(c) $|\overline{AB}| = \sqrt{(3-1)^2 + (4-1)^2 + (2-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

3. $A = (-1,1,2)$, $B = (2,3,5)$. (b) $(2,1,2)$, $(-1,3,2)$, $(-1,1,5)$, $(2,3,2)$, $(-1,3,5)$, $(2,1,5)$

(c) $|\overline{AB}| = \sqrt{(2+1)^2+(3-1)^2+(5-2)^2} = \sqrt{9+4+9} = \sqrt{22}$

4. $A(2,-1,-3)$; $B(4,0,1)$

▹ (a) The figure shows the parallelopiped. Rectangles ACDE and FGBH are parallel faces.
(b) We use Theorems 10.2.2-3 to find the coordinates of vertex C. Because line BC is parallel to the yz plane, then points B and C have equal x coordinates. Because line AC is parallel to the x axis, then points A and C have equal y coordinates and equal z coordinates. Thus $C = (4,-1,-3)$. By similar reasoning, we find the coordinates of the remaining vertices. Therefore, $D = (4,0,3)$; $E = (2,0,-3)$; $F = (2,-1,-1)$; $G = (4,-1,-1)$; and $H = (2,0,-1)$.



(c) We use the distance formula. Thus,

$$|\overline{AB}| = \sqrt{(4-2)^2+(0+1)^2+(-1+3)^2} = 3$$

5. $A = (1,-1,0)$, $B = (3,3,5)$. (b) $(3,-1,0)$, $(3,3,0)$, $(1,3,0)$, $(1,3,5)$, $(1,-1,5)$, $(3,-1,5)$

(c) $|\overline{AB}| = \sqrt{(3-1)^2+(3+1)^2+(5-0)^2} = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$

6. (b) $d = \sqrt{18^2+15^2+12^2} = 3\sqrt{6^2+5^2+4^2} = 3\sqrt{77}$

In Exercises 7–11, find (a) the undirected distance between points A and B and (b) the midpoint of segment AB.

7. $A(3,4,2)$, $B(1,6,3)$. (a) $|\overline{AB}| = \sqrt{(3-1)^2+(4-6)^2+(2-3)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(3+1), \frac{1}{2}(4+6), \frac{1}{2}(2+3)\right) = \left(2,5,\frac{5}{2}\right)$.

8. $A(4,-3,2)$; $B(-2,3,-5)$

▹ (a) Applying the distance formula, we obtain

$$|AB| = \sqrt{(-2-4)^2+(3+3)^3+(-5-2)^2} = \sqrt{6^2+6^2+7^2} = 11$$

(b) We apply Theorem 10.2.6 to find $(\bar{x},\bar{y},\bar{z})$, the midpoint of segment AB. Thus,

$$\bar{x} = \tfrac{1}{2}(4-2) = 1 \quad \bar{y} = \tfrac{1}{2}(-3+3) = 0 \quad \bar{z} = \tfrac{1}{2}(2-5) = -\tfrac{3}{2}$$

Hence the midpoint of segment AB is $\left(1,0,-\frac{3}{2}\right)$.

9. $A(2,-4,1)$, $B\left(\frac{1}{2},2,3\right)$. (a) $|\overline{AB}| = \sqrt{\left(2-\frac{1}{2}\right)^2+(-4-2)^2+(1-3)^2} = \sqrt{\frac{9}{4}+36+4} = \frac{1}{2}\sqrt{169} = \frac{13}{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}\left(2+\frac{1}{2}\right), \frac{1}{2}(-4+2), \frac{1}{2}(1+3)\right) = \left(\frac{5}{4},-1,2\right)$.

10. $A\left(-2,-\frac{1}{2},5\right)$; $B(5,1,-4)$. (a) $|\overline{AB}| = \sqrt{(5+2)^2+\left(1+\frac{1}{2}\right)^2+(-4-5)^2} = \sqrt{49+\frac{9}{4}+81} = \frac{1}{2}\sqrt{529} = \frac{23}{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(-2+5), \frac{1}{2}\left(-\frac{1}{2}+1\right), \frac{1}{2}(5-4)\right) = \left(\frac{3}{2},\frac{1}{4},\frac{1}{2}\right)$

11. $A(-5,2,1)$, $B(3,7,-2)$. (a) $|\overline{AB}| = \sqrt{(-5-3)^2+(2-7)^2+(1+2)^2} = \sqrt{64+25+9} = \sqrt{98} = 7\sqrt{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(-5+3), \frac{1}{2}(2+7), \frac{1}{2}(1-2)\right) = \left(-1,\frac{9}{2},-\frac{1}{2}\right)$.

12. Prove that the three points $A(1,-1,3)$, $B(2,1,7)$, $C(4,2,6)$ are the vertices of a right triangle. Find its area.

▹ Applying the distance formula, we get

$$|AB| = \sqrt{(2-1)^2+(1+1)^2+(7-3)^2} = \sqrt{21}$$
$$|BC| = \sqrt{(4-2)^2+(2-1)^2+(6-7)^2} = \sqrt{6}$$
$$|AC| = \sqrt{(4-1)^2+(2+1)^2+(6-3)^2} = \sqrt{27}$$

Because $|AB|^2 + |BC|^2 = 21 + 6 = 27 = |AC|^2$, then triangle ABC is a right triangle with hypotenuse side AC. To find the area, we use the area formula $\frac{1}{2}bh$. We take $b = |AB| = \sqrt{21}$ and $h = |BC| = \sqrt{6}$. Thus, the number of square units in the area is $\frac{1}{2}\sqrt{21}\sqrt{6} = \frac{3}{2}\sqrt{14}$.

13. Because the line is perpendicular to the yz plane, it is parallel to the x axis. Thus any point on this line differs from $(6,4,2)$ only in its x coordinate and so has the form $(x,4,2)$. Because the distance from $(x,4,2)$ to the point $(0,4,0)$ is 10 units then $\sqrt{(x-0)^2+(4-4)^2+(2-0)^2} = 10$; $x^2+4 = 100$, $x^2 = 96$; $x = \pm 4\sqrt{6}$
Therefore the required points are $(4\sqrt{6},4,2)$ and $(-4\sqrt{6},4,2)$.

14. The point $(6,4,z)$ is 10 units from $(0,4,0)$. Thus, $\sqrt{(6-0)^2+(4-4)^3+(z-0)^2}=10^2$; $36+z^2=100$; $z^2=64$; $z=\pm 8$. The points are $(6,4,8)$ and $(6,4,-8)$.

15. Let $A=(-3,2,4)$, $B=(6,1,2)$, $C=(-12,3,6)$. Then

$$|\overline{AB}|=\sqrt{(6+3)^2+(1-2)^2+(2-4)^2}=\sqrt{81+1+4}=\sqrt{86}$$
$$|\overline{AC}|=\sqrt{(-12+3)^2+(3-2)^2+(6-4)^2}=\sqrt{81+1+4}=\sqrt{86}$$
$$|\overline{BC}|=\sqrt{(-12-6)^2+(3-1)^2+(6-2)^2}=\sqrt{324+4+16}=\sqrt{344}=2\sqrt{86}$$

If A, B, C are vertices of a triangle then $|AB|+|AC|>|BC|$. But $|AB|+|AC|=\sqrt{86}+\sqrt{86}=|BC|$. Hence A, B, and C are not the vertices of a triangle, and so they are collinear.

16. Find the vertices of the triangle whose sides have midpoints at $D(3,2,3)$, $E(-1,1,5)$, and $F(0,3,4)$.

▷ Let A, B, C be the vertices of the triangle, with D the midpoint of BC, E the midpoint of AC and F the midpoint of AB. Then DEAF is a parallelogram, and and using position vectors we have

$$\overrightarrow{FA}=\overrightarrow{DE}$$
$$\mathbf{a}-\mathbf{f}=\mathbf{e}-\mathbf{d}$$
$$\mathbf{a}=-\mathbf{d}+\mathbf{e}+\mathbf{f}=-\langle 3,2,3\rangle+\langle 0,3,4\rangle+\langle -1,1,5\rangle=\langle -4,2,6\rangle$$

Similarly,

$$\mathbf{b}=\mathbf{d}-\mathbf{e}+\mathbf{f}=\langle 3,2,3\rangle-\langle 0,3,4\rangle+\langle -1,1,5\rangle=\langle 2,0,4\rangle$$
$$\mathbf{c}=\mathbf{d}+\mathbf{e}-\mathbf{f}=\langle 3,2,3\rangle+\langle 0,3,4\rangle-\langle -1,1,5\rangle=\langle 4,4,2\rangle$$

Thus, $A=(-4,2,6)$ $B=(2,0,4)$ $C=(4,4,2)$

17. $A=(2,-5,3)$, $B=(-1,7,0)$, $C=(-4,9,7)$

(a) $|\overline{AB}|=\sqrt{(2+1)^2+(-5-7)^2+(3-0)^2}=\sqrt{9+144+9}=\sqrt{162}=9\sqrt{2}$

$|\overline{AC}|=\sqrt{(2+4)^2+(-5-9)^2+(3-7)^2}=\sqrt{36+196+16}=\sqrt{248}=2\sqrt{62}$

$|\overline{BC}|=\sqrt{(-1+4)^2+(7-9(^2+(0-7)^2}=\sqrt{9+4+49}=\sqrt{62}$

(b) The midpoint of side AB is $(\frac{1}{2}(2-1),\frac{1}{2}(-5+7),\frac{1}{2}(3+0))=(\frac{1}{2},1,\frac{3}{2})$.

The midpoint of side AC is $(\frac{1}{2}(2-4),\frac{1}{2}(-5+9),\frac{1}{2}(3+7))=(-1,2,5)$.

The midpoint of side BC is $(\frac{1}{2}(-1-4),\frac{1}{2}(7+9),\frac{1}{2}(0+7))=(-\frac{5}{2},8,\frac{7}{2})$.

18. If P is the fraction w of the way from P_1 to P_2 then $\overrightarrow{P_1P}=w\overrightarrow{P_1P_2}$ and so $\mathbf{p}-\mathbf{p}_1=w(\mathbf{p}_2-\mathbf{p}_1)$ and so $\mathbf{p}=(1-w)\mathbf{p}_1+w\mathbf{p}_2$. If $w=\frac{1}{2}$, we have $\mathbf{p}=\frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2)=(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2),\frac{1}{2}(z_1,z_2))$

19.
$$x^2+y^2+z^2+Gx+Hy+Iz+J=0$$
$$(x^2+Gx+\tfrac{1}{4}G^2)+(y^2+Hy+\tfrac{1}{4}H^2)+(z^2+Iz+\tfrac{1}{4}I^2)=\tfrac{1}{4}(G^2+H^2+I^2-4J)$$
$$(x+\tfrac{1}{2}G)^2+(y+\tfrac{1}{2}H)^2+(z+\tfrac{1}{2}I)^2=\tfrac{1}{4}(G^2+H^2+I^2-4J)$$
$$(x-h)^2+(y-k)^2+(z-l)^2=K$$

where $h=-\frac{1}{2}G$, $k=-\frac{1}{2}H$, $l=-\frac{1}{2}I$, and $K=\frac{1}{4}(G^2+H^2+I^2-4J)$.

In Exercises 20–25, determine the graph of the equation.

20. $x^2+y^2+z^2-8y+6z-25=0$

▷ We complete the squares on the terms in y and z. Thus,

$$x^2+(y^2-8y+\underline{16})+(z^2+6z+\underline{9})=25+\underline{16}+\underline{9}$$
$$x^2+(y-4)^2+(z+3)^2=50$$

By Theorem 10.2.9, we find that the graph is a sphere with center at $(0,4,-3)$ and radius $\sqrt{50}=5\sqrt{2}$ units.

21.
$$x^2+y^2+z^2-8x+4y+2z-4=0$$
$$(x^2-8x+16)+(y^2+4y+4)+(z^2+2z+1)=4+16+4+1$$
$$(x-4)^2+(y+2)^2+(z+1)^2=25$$

The graph is the sphere with center at $(4,-2,-1)$ and $r=5$.

22.
$$x^2+y^2+z^2-x-y-3z+2=0$$
$$(x^2-x+\tfrac{1}{4})+(y^2-y+\tfrac{1}{4})+(z^2-3z+\tfrac{9}{4})=-2+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{9}{4}$$
$$(x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2+(z-\tfrac{3}{2})^2=\tfrac{3}{4}$$
The graph is the sphere with center at $(\frac{1}{2},\frac{1}{2},\frac{3}{2})$ and $r=\frac{1}{2}\sqrt{3}$.

23. $x^2+y^2+z^2-6z+9=0$; $x^2+y^2+(z-3)^2=0$. The graph is the point $(0,0,3)$.

24. $x^2+y^2+z^2-8x+10y-4z+13=0$

▹ We complete the squares.
$$(x^2-8x+\underline{16})+(y^2+10y+\underline{25})+(z^2-4z+\underline{4})=-13+\underline{16}+\underline{4}$$
$$(x-4)^2+(y+5)^2+(z-2)^2=32$$
By Theorem 10.2.9, we find that the graph is a sphere with center at $(4,-5,2)$ and radius $\sqrt{32}=4\sqrt{2}$ units.

25.
$$x^2+y^2+z^2-6x+2y-4z+19=0$$
$$(x^2-6x+9)+(y^2+2y+1)+(z^2-4z+4)=-19+9+1+4$$
$(x-3)^2+(y+1)^2+(z-2)^2=-5$. The graph is the empty set.

In Exercises 26–28, find an equation of the sphere satisfying the conditions.

26. (a) A diameter has endpoints $A(h_1,k_1,l_1)$ and $B(h_2,k_2,l_2)$.

▹ If $P(x,y,z)$ is on the sphere, then APB is a right triangle and so $AP^2+BP^2=AB^2$, that is
$$(x-h_1)^2+(y-k_1)^2+(z-l_1)^2+(x-h_2)^2+(y-k_2)^2+(z-l_2)^2=(h_2-h_1)^2+(k_2-k_1)^2+(l_2-l_1)^2$$
$$2\{[x^2-(h_1+h_2)x+h_1h_2]+[y^2-(k_1+k_2)y+k_1k_2]+[z^2-(l_1+l_2)z+l_1l_2]\}=0$$
(b) A diameter has endpoints $(6,2,-5)$ and $(-4,0,7)$. $(x-6)(x+4)+(y-2)(y-0)+(z+5)(z-7)=0$

27.
$$x^2+y^2+z^2-2y+8z-9=0$$
$$x^2+(y^2-2y+1)+(z^2+8z+16)=9+1+16$$
$$x^2+(y-1)^2+(z+4)^2=26$$
The center of the given sphere is $(0,1,-4)$ so an equation of the required sphere is
$x^2+(y-1)^2+(z+4)^2=9$

28. It contains the points $(0,0,4)$, $(2,1,3)$, and $(0,2,6)$ and has its center in the yz plane.

▹ From Theorem 10.2.10 we have
$$Gx+Hy+Iz+J=-(x^2+y^2+z^2)$$
Because the center is in the yz plane, we have $G=0$. Thus
$$Hy+Iz+J=-(x^2+y^2+z^2)$$
We substitute the coordinates of the given points:

$(0,0,4)$: $4I+J=-16$ (1)
$(2,1,6)$: $H+3I+J=-14$ (2)
$(0,2,6)$: $2H+6I+J=-40$ (3)

$2(2)-(3)$: $J=12$
From (1): $4I=-16-12=-28$
$I=-7$
From (2): $H=-14-12-3(-7)=-5$

The equation is

$x^2+y^2+z^2=-5y-7z+12=0$

In Exercises 29–34, $\mathbf{A}=\langle 1,2,3\rangle$, $\mathbf{B}=\langle 4,-3,-1\rangle$, $\mathbf{C}=\langle -5,-3,5\rangle$, $\mathbf{D}=\langle -2,1,6\rangle$

29. (a) $\mathbf{A}+5\mathbf{B}=\langle 1,2,3\rangle+5\langle 4,-3,-1\rangle=\langle 1,2,3\rangle+\langle 20,-15,-5\rangle=\langle 21,-13,-2\rangle$

(b) $7\mathbf{C}-5\mathbf{D}=7\langle -5,-3,5\rangle-5\langle -2,1,6\rangle=\langle -35,-21,35\rangle+\langle 10,-5,-30\rangle=\langle -25,-26,5\rangle$

(c) $\|7\mathbf{C}\|-\|5\mathbf{D}\|=7\|\langle -5,-3,5\rangle\|-5\|\langle -2,1,6\rangle\|=7\sqrt{25+9+25}-5\sqrt{4+1+36}=7\sqrt{59}-5\sqrt{41}$

(d) $\|7\mathbf{C}-5\mathbf{D}\|=\|\langle -25,-26,5\rangle\|=\sqrt{625+676+25}=\sqrt{1326}$

30. (a) $2\mathbf{A} - \mathbf{C} = 2\langle 1,2,3\rangle - \langle -5,-3,5\rangle = \langle 2,4,6\rangle + \langle 5,3,-5\rangle = \langle 7,7,1\rangle$

(b) $\|2\mathbf{A}\| - \|\mathbf{C}\| = 2\sqrt{1^2+2^2+3^2} - \sqrt{5^2+3^2+5^2} = 2\sqrt{14} - \sqrt{59}$. (c) $4\mathbf{B} + 6\mathbf{C} - 2\mathbf{D}$

$= 4\langle 4,-3,-1\rangle + 6\langle -5,-3,5\rangle - 2\langle -2,1,6\rangle = \langle 16,-12,-4\rangle + \langle -30,-18,30\rangle + \langle 4,-2,-12\rangle = \langle -10,-32,14\rangle$

(d) $\|4\mathbf{B}\| + \|6\mathbf{C}\| - \|2\mathbf{D}\| = 4\sqrt{4^2+3^2+1^2} + 6\sqrt{5^2+3^2+5^2} - 2\sqrt{2^2+1^2+6^2} = 4\sqrt{26} + 6\sqrt{59} - 2\sqrt{41}$

31. (a) $\mathbf{C} + 3\mathbf{D} - 8\mathbf{A} = \langle -5,-3,5\rangle + 3\langle -2,1,6\rangle - 8\langle 1,2,3\rangle$
$= \langle -5,-3,5\rangle + \langle -6,3,18\rangle + \langle -8,-16,-24\rangle = \langle -19,-16,-1\rangle$

(b) $\|\mathbf{A}\|\|\mathbf{B}\|(\mathbf{C}-\mathbf{D}) = \|\langle 1,2,3\rangle\|\|\langle 4,-3,-1\rangle\|(\langle -5,-3,5\rangle - \langle -2,1,6\rangle) = \sqrt{1+4+9}\sqrt{16+9+1}\langle -3,-4,-1\rangle$

$= \sqrt{14}\sqrt{26}\langle -3,-4,-1\rangle = \sqrt{2^2 \cdot 7 \cdot 13}\langle -3,-4,-1\rangle = 2\sqrt{91}\langle -3,-4,-1\rangle = \langle -6\sqrt{91}, -8\sqrt{91}, -2\sqrt{91}\rangle$

32. Find: (a) $3\mathbf{A} - 2\mathbf{B} + \mathbf{C} - 12\mathbf{D}$; (b) $\|\mathbf{A}\|\mathbf{C} - \|\mathbf{B}\|\mathbf{D}$

(a)$3\mathbf{A} - 2\mathbf{B} + \mathbf{C} - 12\mathbf{D} = 3\langle 1,2,3\rangle - 2\langle 4,-3,-1\rangle + \langle -5,-4,5\rangle - 12\langle -2,1,6\rangle$
$= \langle 3,6,9\rangle + \langle -8,6,2\rangle + \langle -5,-3,5\rangle + \langle 24,-12,-72\rangle = \langle 14,-3,-56\rangle$

(b) $\|\mathbf{A}\| = \|\langle 1,2,3\rangle\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $\|\mathbf{B}\| = \|\langle 4,-3,-1\rangle\| = \sqrt{4^2+(-3)^2+(-1)^2} = \sqrt{26}$

Thus,

$$\|\mathbf{A}\|\mathbf{C} - \|\mathbf{B}\|\mathbf{D} = \sqrt{14}\langle -5,-3,5\rangle - \sqrt{26}\langle -2,1,6\rangle = \langle -5\sqrt{14} + 2\sqrt{26}, -3\sqrt{14} - \sqrt{26}, 5\sqrt{15} - 6\sqrt{26}\rangle$$

33. $a(\mathbf{A}+\mathbf{B}) + b(\mathbf{C}+\mathbf{D}) = \mathbf{0}$; $a(\langle 1,2,3\rangle + \langle 4,-3,-1\rangle) + b(\langle -5,-3,5\rangle + \langle -2,1,6\rangle) = \langle 0,0,0\rangle$
$a\langle 5,-1,2\rangle + b\langle -7,-2,11\rangle = \langle 0,0,0\rangle$. Hence $5a - 7b = 0$, $-a - 2b = 0$, and $2a + 11b = 0$.
The only solution to this system of three equations is $a = 0$ and $b = 0$.

34. $a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = \mathbf{D}$. $a\langle 1,2,3\rangle + b\langle 4,-3,-1\rangle + c\langle -5,-3,5\rangle = \langle -2,1,6\rangle$
$a + 4b - 5c = -2$, $2a - 3b - 3c = 1$, $3a - 3b + 5c = 6$. $a = \frac{141}{129}$, $b = -\frac{16}{129}$, $c = \frac{67}{129}$

In Exercises 35–38, find the direction cosines of $\mathbf{V}(\overrightarrow{P_1P_2})$ and check by verifying that the sum of their squares is 1.

35. $P_1(3,-1,-4)$; $P_2(7,2,4)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 7-3, 2+1, 4+4\rangle = \langle 4,3,8\rangle$; $\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{16+9+64} = \sqrt{89}$.
Therefore $\cos\alpha = \dfrac{4}{\sqrt{89}}$, $\cos\beta = \dfrac{3}{\sqrt{89}}$, $\cos\gamma = \dfrac{8}{\sqrt{89}}$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{16}{89} + \frac{9}{89} + \frac{64}{89} = 1$.

36. $P_1(-2,6,5)$; $P_2(2,4,1)$

▷ We have

$$\mathbf{V} = \langle 2+2, 4-6, 1-5\rangle = \langle 4,-2,-4\rangle \qquad \|\mathbf{V}\| = \sqrt{4^2+(-2)^2+(-4)^2} = 6$$

Thus,

$$\cos\alpha = \frac{a_1}{\|\mathbf{V}\|} = \frac{4}{6} = \frac{2}{3} \qquad \cos\beta = \frac{a_2}{\|\mathbf{V}\|} = \frac{-2}{6} = -\frac{1}{3} \qquad \cos\gamma = \frac{a_3}{\|\mathbf{V}\|} = \frac{-4}{6} = -\frac{2}{3}$$

Furthermore,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (\tfrac{2}{3})^2 + (-\tfrac{1}{3})^2 + (-\tfrac{2}{3})^2 = 1$$

37. $P_1(4,-3,-1)$; $P_2(-2,-4,-8)$. $\mathbf{V} = \langle -2-4, -4+3, -8+1\rangle = \langle -6,-1,-7\rangle$; $\|\mathbf{V}\| = \sqrt{36+1+49} = \sqrt{86}$.
Thus $\cos\alpha = \dfrac{-6}{\sqrt{86}}$, $\cos\beta = \dfrac{-1}{\sqrt{86}}$, $\cos\gamma = \dfrac{-7}{\sqrt{86}}$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{36}{86} + \frac{1}{86} + \frac{49}{86} = 1$.

38. $P_1(1,3,5)$; $P_2(2,-1,4)$. $\mathbf{V} = \langle 2-1, -1-3, 4-5\rangle = \langle 1,-4,-1\rangle$; $\|\mathbf{V}\| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$
$\cos\alpha = \dfrac{1}{3\sqrt{2}} = \frac{1}{6}\sqrt{2}$, $\cos\beta = \dfrac{-4}{3\sqrt{2}} = -\frac{2}{3}\sqrt{2}$, $\cos\gamma = \dfrac{-1}{3\sqrt{2}} = -\frac{1}{6}\sqrt{2}$; $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{2}{36} + \frac{32}{36} + \frac{2}{36} = 1$

39. $P_1 = (3,-1,-4)$, $P_2 = (7,2,4)$. Let $Q = (x,y,z)$. Then $\mathbf{V}(\overrightarrow{P_1P_2}) = 3\mathbf{V}(\overrightarrow{P_1Q})$. $\langle 4,3,8\rangle = 3\langle x-3, y+1, z+4\rangle$;
$3x - 9 = 4$, $x = \frac{13}{3}$; $3y + 3 = 3$, $y = 0$; and $3z + 2 = 8$, $z = -\frac{4}{3}$. Therefore, Q is the point $(\frac{13}{3}, 0, -\frac{4}{3})$.

40. Given $P_1(1,3,5)$ and $P_2(2,-1,4)$ find the point R such that $\mathbf{V}(\overrightarrow{P_1R}) = -2\mathbf{V}(\overrightarrow{P_2R})$.

▸ Expressing the vectors in terms of position vectors, we have

$$\mathbf{r}-\mathbf{p}_1 = -2(\mathbf{r}-\mathbf{p}_2) = -2\mathbf{r}+2\mathbf{p}_2$$

$$3\mathbf{r} = \mathbf{p}_1 + 2\mathbf{p}_2 = \langle 1,3,5\rangle + 2\langle 2,-1,4\rangle = \langle 5,1,13\rangle$$

$$\mathbf{r} = \langle \tfrac{5}{3}, \tfrac{1}{3}, \tfrac{13}{3}\rangle$$

Thus,

$$R = (\tfrac{5}{3}, \tfrac{1}{3}, \tfrac{13}{3})$$

41. $P_1 = (3,2,-4)$, $P_2 = (-5,4,2)$. Let $P_3 = (x,y,z)$. Then $4\mathbf{V}(\overrightarrow{P_1P_2}) = -3\mathbf{V}(\overrightarrow{P_2P_3})$.
$4\langle -5-3, 4-2, 2+4\rangle = -3\langle x+5, y-4, z-2\rangle$; $\langle -32, 8, 24\rangle = \langle -3x-15, -3y+2, -3z+6\rangle$
Thus $-3x-15 = -32$, $x = \frac{17}{3}$; $-3y+12 = 8$, $y = \frac{4}{3}$; $-3z+6 = 24$, $z = -6$. Hence, $P_3 = (\frac{17}{3}, \frac{4}{3}, -6)$.

42. $P_1(7,0,-2)$, $P_2(2,-3,5)$. $\overrightarrow{P_1P_3} = 5\overrightarrow{P_2P_3}$, $\mathbf{p}_3 - \mathbf{p}_1 = 5(\mathbf{p}_3 - \mathbf{p}_2)$, $4\mathbf{p}_3 = 5\mathbf{p}_2 - \mathbf{p}_1$, $\mathbf{p}_3 = \frac{1}{4}(5\mathbf{p}_2 - \mathbf{p}_1) = \frac{1}{4}\langle 3,-15,27\rangle$

$P_3 = (\frac{3}{4}, -\frac{15}{4}, \frac{27}{4})$

In Exercises 43 and 44, express the vector in terms of its magnitude and direction cosines.

43. (a) Let $\mathbf{A} = -6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Then $\|\mathbf{A}\| = \sqrt{36+4+9} = \sqrt{49} = 7$. $\mathbf{A} = 7(-\frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k})$

(b) Let $\mathbf{A} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Then $\|\mathbf{A}\| = \sqrt{4+1+9} = \sqrt{14}$. $\mathbf{A} = \sqrt{14}\left(-\dfrac{2}{\sqrt{14}}\mathbf{i} + \dfrac{1}{\sqrt{14}}\mathbf{j} - \dfrac{3}{\sqrt{14}}\mathbf{k}\right)$

44. (a) $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$; (b) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

▸ If α, β, and γ are the direction angles for the vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\cos\alpha = \frac{a}{\|\mathbf{V}\|} \quad \cos\beta = \frac{b}{\|\mathbf{V}\|} \quad \cos\gamma = \frac{c}{\|\mathbf{V}\|} \quad \text{and} \quad \mathbf{V} = \|\mathbf{V}\|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$$

(a) $\|2\mathbf{i} - 2\mathbf{j} + \mathbf{k}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

$\cos\alpha = \frac{2}{3} \quad \cos\beta = \frac{-2}{3} \quad \cos\gamma = \frac{1}{3}$

Therefore,

$2\mathbf{i} - 2\mathbf{j} + \mathbf{k} = 3(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k})$

(b) $\|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}\| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$

$\cos\alpha = \dfrac{3}{5\sqrt{2}} = \dfrac{3}{10}\sqrt{2} \quad \cos\beta = \dfrac{4}{5\sqrt{2}} = \dfrac{2}{5}\sqrt{2} \quad \cos\gamma = \dfrac{-5}{5\sqrt{2}} = -\dfrac{1}{2}\sqrt{2}$

Therefore,

$3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}(\frac{3}{10}\sqrt{2}\mathbf{i} + \frac{2}{5}\sqrt{2}\mathbf{j} - \frac{1}{2}\sqrt{2}\mathbf{k})$

In Exercises 45 and 46, find the unit vector $\mathbf{U}$ having the same direction as $\mathbf{V}(\overrightarrow{P_1P_2})$.

45. (a) $P_1 = (4,-1,-6)$, $P_2 = (5,7,-2)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 5-4, 7+1, -2+6\rangle = \langle 1,8,4\rangle$.
$\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{1+64+16} = \sqrt{81} = 9$. $\mathbf{U} = \langle \frac{1}{9}, \frac{8}{9}, \frac{4}{9}\rangle$.

(b) $P_1 = (-2,5,3)$, $P_2 = (-4,7,5)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle -4+2, 7-5, 5-3\rangle = \langle -2,2,2\rangle$.
$\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$. $\mathbf{U} = \langle -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\rangle$.

46. (a) $P_1(3,0,-1)$, $P_2(-3,8,-1)$. $\mathbf{V} = \langle -3-3, 8-0, -1+1\rangle = \langle -6,8,0\rangle$. $\|\mathbf{V}\| = \sqrt{6^2+8^2} = 10$. $\mathbf{U} = \langle -\frac{3}{5}, \frac{4}{5}, 0\rangle$

(b) $P_1(-8,-5,2)$, $P_2(-3,-9,4)$. $\mathbf{V} = \langle -3+8, -9+5, 4-2\rangle = \langle 5,-4,2\rangle$. $\|\mathbf{V}\| = \sqrt{5^2+4^2+2^2} = \sqrt{45} = 15/\sqrt{5}$
$\mathbf{U} = \langle \frac{1}{3}\sqrt{5}, -\frac{4}{15}\sqrt{5}, \frac{2}{15}\sqrt{5}\rangle$

In Exercises 47 and 48, prove the property if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any vectors of V_3 and c and d are any scalars.

47. Let $\mathbf{A} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{B} = \langle b_1,b_2,b_3\rangle$. Use the same property of the real numbers.
(a) $\mathbf{A} + \mathbf{B} = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle = \langle b_1+a_1, b_2+a_2, b_3+a_3\rangle = \mathbf{B} + \mathbf{A}$
(b) Let $\mathbf{0} = \langle 0,0,0\rangle$. Then $\mathbf{A} + \mathbf{0} = \langle a_1+0, a_2+0, a_3+0\rangle = \langle a_1,a_2,a_3\rangle = \mathbf{A}$.
(c) Let $-\mathbf{A} = \langle -a_1,-a_2,-a_3\rangle$. Then $\mathbf{A} + (-\mathbf{A}) = \langle a_1+(-a_1), a_2+(-a_2), a_3+(-a_3)\rangle = \langle 0,0,0\rangle = \mathbf{0}$.
(d) $c(\mathbf{A}+\mathbf{B}) = c\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle = \langle c(a_1+b_1), c(a_2+b_2), c(a_3+b_3)\rangle$
$= \langle ca_1+cb_1, ca_2+cb_2, ca_3+cb_3\rangle = \langle ca_1,ca_2,ca_3\rangle + \langle cb_1,cb_2,cb_3\rangle = c\mathbf{A} + c\mathbf{B}$

48. (a) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$ (associative law)
(b) $(cd)\mathbf{A}=c(d\mathbf{A})$ (associative law)
(c) $(c+d)\mathbf{A}=c\mathbf{A}+d\mathbf{A}$ (distributive law)

▹ Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$, $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, and $\mathbf{C}=\langle c_1,c_2,c_3\rangle$. We apply the definitions for addition of vectors and scalar multiplication and the associative and distributive laws for real numbers.

$$
\begin{aligned}
\text{(a)}\ \mathbf{A}+(\mathbf{B}+\mathbf{C}) &= \langle a_1,a_2,a_3\rangle+(\langle b_1,b_2,b_3\rangle+\langle c_1,c_2,c_3\rangle)=\langle a_1,a_2,a_3\rangle+\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle\\
&=\langle a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3)\rangle=\langle (a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3\rangle\\
&=\langle a_1+b_1,a_2+b_2,a_3+b_3\rangle+\langle c_1,c_2,c_3\rangle=(\langle a_1,a_2,a_3\rangle+\langle b_1,b_2,b_3\rangle)+\langle c_1,c_2,c_3\rangle\\
&=(\mathbf{A}+\mathbf{B})+\mathbf{C}
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)}\qquad (cd)\mathbf{A} &= (cd)\langle a_1,a_2,a_3\rangle=\langle (cd)a_1,(cd)a_2,(cd)a_3\rangle=\langle c(da_1),c(da_2),c(da_3)\rangle=c\langle da_1,da_2,da_3\rangle\\
&=c(d\mathbf{A})
\end{aligned}
$$

$$
\begin{aligned}
\text{(c)}\qquad (c+d)\mathbf{A} &= (c+d)\langle a_1,a_2,a_3\rangle=\langle (c+d)a_1,(c+d)a_2,(c+d)a_3\rangle=\langle ca_1+da_1,ca_2+da_2,ca_3+da_3\rangle\\
&=\langle ca_1,ca_2,ca_3\rangle+\langle da_1,da_2,da_3\rangle=c\langle a_1,a_2,a_3\rangle+d\langle a_1,a_2,a_3\rangle=c\mathbf{A}+d\mathbf{A}
\end{aligned}
$$

In Exercises 49–51, prove the theorem by analytic geometry.

49. The four diagonals of a rectangular parallelpiped bisect each other.

▹ Choose the coordinate planes as in Figure 3 of the text so the vertices of the parallelepiped are A$(0,0,0)$, B(a,b,c); C$(a,b,0)$, H$(0,0,c)$; D$(a,0,0)$, G$(0,b,c)$; E$(a,0,c)$ and F$(0,b,0)$. Because each of the diagonals AB, CH, DG, EF have $(\frac{1}{2}a,\frac{1}{2}b,\frac{1}{2}c)$ as midpoint, they bisect each other.

50. If P, Q, R, S are four points of R^3, then the midpoints A, B, C, D of PQ, QR, RS, SP form a parallelogram.

▹ We have $\mathbf{a}=\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\mathbf{b}=\frac{1}{2}(\mathbf{q}+\mathbf{r})$, $\mathbf{c}=\frac{1}{2}(\mathbf{r}+\mathbf{s})$, $\mathbf{d}=\frac{1}{2}(\mathbf{s}+\mathbf{p})$. Therefore, $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\frac{1}{2}(\mathbf{r}-\mathbf{p})=\mathbf{c}-\mathbf{d}=\overrightarrow{DC}$.

51. The four diagonals of a rectangular parallelopiped have the same length.

▹ Choose the coordinate planes as in Figure 3 of the text so the vertices of the rectangular parallepiped are at A$(0,0,0)$, B(a,b,c); C$(a,b,0)$, H$(0,0,c)$; D$(a,0,0)$, G$(0,b,c)$; E$(a,0,c)$ and F$(0,b,0)$. The four diagonals have the same length:

$$
\begin{aligned}
|\overline{AB}| &= \sqrt{(a-0)^2+(b-0)^2+(c-0)^2}=\sqrt{a^2+b^2+c^2}\\
|\overline{CH}| &= \sqrt{(a-0)^2+(0-b)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}\\
|\overline{DG}| &= \sqrt{(a-0)^2+(b-0)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}\\
|\overline{EF}| &= \sqrt{(0-a)^2+(b-0)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}
\end{aligned}
$$

52. Three vectors in V_3 are said to be *independent* if and only if their position representations do not lie in a plane, and three vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are said to form a *basis* for the vector space V_3 if and only if any vector in V_3 can be written as a linear combination of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. A theorem can be proved which states that two vectors form a basis for the vector space V_3 if they are independent. Show that this theorem holds for the three vectors $\langle 1,0,0\rangle$, $\langle 1,1,0\rangle$ and $\langle 1,1,1\rangle$ by doing the following: (a) Verify that the vectors are independent by showing that the position representations are not coplanar; (b) verify that the vectors form a basis by showing that any vector $\mathbf{A}$ can be written

$$\mathbf{A}=r\langle 1,0,0\rangle+s\langle 1,1,0\rangle+t\langle 1,1,1\rangle \qquad (1)$$

where r, s, and t are scalars. (c) If $\mathbf{A}=\langle 6,-2,5\rangle$, find the particular values of r, s, and t, so that (1) holds.

▹ (a) The position representations of the vectors $\langle 1,0,0\rangle$ and $\langle 1,1,0\rangle$ are in the xy plane because their z components are both zero; because they are not collinear, they do not lie in any other plane. The position representation of the vector $\langle 1,1,1\rangle$ is not in the xy plane because its z component is not zero. Therefore, the position representations are not coplanar and the vectors are independent.

(b) Let $\mathbf{A}=\langle a,b,c\rangle$. We show that there exist scalars r, s, and t such that

$$\langle a,b,c\rangle=r\langle 1,0,0\rangle+s\langle 1,1,0\rangle+t\langle 1,1,1\rangle=\langle r,0,0\rangle+\langle s,s,0\rangle+\langle t,t,t\rangle=\langle r+s+t,s+t,t\rangle$$

Corresponding components of equal vectors are equal numbers. Thus,

$$a=r+s+t \quad b=s+t \quad c=t$$

Solving for r, s, and t in terms of a, b, and c, we obtain

$$r = a - b \quad s = b - c \quad t = c \tag{2}$$

Thus, any vector **A** can be written

$$\mathbf{A} = r\langle 1,0,0\rangle + s\langle 1,1,0\rangle + t\langle 1,1,1\rangle$$

and the vectors $\langle 1,0,0\rangle$, $\langle 1,1,0\rangle$ and $\langle 1,1,1\rangle$ form a basis for V_3.
(c) If $\mathbf{A} = \langle 6,-2,5\rangle$ then we have $a = 6$, $b = -2$, and $c = 5$. Substituting these values for a, b, and c into Eqs. (2), we obtain

$$r = 8 \quad s = -7 \quad t = 5$$

53. (a) Let $\mathbf{A} = \langle a,b,c\rangle$. We solve $r\langle 2,0,1\rangle + s\langle 0,-1,0\rangle + t\langle 1,-1,0\rangle = \langle a,b,c\rangle$ for r, s, t.
$2r + t = a$, $-s - t = b$, $r = c$; $t = a - 2c$, $s = 2c - a - b$.
Thus $\langle a,b,c\rangle = c\langle 2,0,1\rangle + (2c - a - b)\langle 0,-1,0\rangle + (a - 2c)\langle 1,-1,0\rangle$ so the vectors form a basis.
(b) Set $a = -2$, $b = 3$, $c = 5$ in (a). Then $r = 5$, $s = 2(5) - (-2) - 3 = 9$, $t = (-2) - 2(5) = -12$.

54. Let $\mathbf{F}_1 = \langle 1,0,1\rangle$, $\mathbf{F}_2 = \langle 1,1,1\rangle$, $\mathbf{F}_3 = \langle 2,1,2\rangle$. (a) Because $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$, the vectors represent the sides of a triangle and so are coplanar. (b) If $r\langle 1,0,1\rangle + s\langle 1,1,1\rangle + t\langle 2,1,2\rangle = \langle a,b,c\rangle$, then $r + s + 2t = a$, $s + t = b$, and $r + s + 2t = c$. Hence, if $a \neq c$ there is no solution.

55. $\mathbf{U} = \dfrac{a_1}{\|\mathbf{A}\|}\mathbf{i} + \dfrac{a_2}{\|\mathbf{A}\|}\mathbf{j} + \dfrac{a_3}{\|\mathbf{A}\|}\mathbf{k} = \dfrac{1}{\|\mathbf{A}\|}\mathbf{A}$. $\|\mathbf{U}\| = \dfrac{1}{\|\mathbf{A}\|}\|\mathbf{A}\| = 1$. Because $\dfrac{1}{\|\mathbf{A}\|} > 0$, then **U** has the same direction as **A**.

56. If the radian measure of each direction angle is the same, what is it?

▸ Let α be the radian measure of each direction angle of a vector. By Theorem 10.2.13 we have
$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$; $\cos^2\alpha = \frac{1}{3}$; $\cos\alpha = \pm\dfrac{1}{\sqrt{3}}$; $\alpha = \cos^{-1}\dfrac{1}{\sqrt{3}}$ or $\alpha = \cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

10.3 DOT PRODUCT

10.3.1 Definition The *dot product* of two vectors **A** and **B**, denoted by $\mathbf{A}\cdot\mathbf{B}$ is defined as follows:
If $\mathbf{A} = \langle a_1, a_2\rangle = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = \langle b_1, b_2\rangle = b_1\mathbf{i} + b_2\mathbf{j}$ are two vectors in V_2
$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2$
If $\mathbf{A} = \langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors in V_3
$\mathbf{A}\cdot\mathbf{B} = a_2b_1 + a_2b_2 + a_3b_3$

We note that the dot product, or *inner product*, of two vectors is a scalar.

Work We use the dot product to find the work W done by a force **F** that causes a displacement **D** of an object. We have $W = \mathbf{F}\cdot\mathbf{D}$.

10.3.2 Theorem If **A**, **B**, and **C** are any vectors in V_2 or V_3, then

(i) $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ (commutative law)
(ii) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ (distributive law)

10.3.3 Theorem If **A** and **B** are any vectors in V_2 or V_3 and c is any scalar, then

(i) $c(\mathbf{A}\cdot\mathbf{B}) = (c\mathbf{A})\cdot\mathbf{B}$
(ii) $\mathbf{0}\cdot\mathbf{A} = 0$
(iii) $\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2$

10.3.4 Definition Let **A** and **B** be two nonzero vectors such that **A** is not a scalar multiple of **B**. If $\overrightarrow{OP}$ is the position representation of **A** and $\overrightarrow{OQ}$ is the position representation of **B**, then *the angle between the vectors* **A** and **B**, denoted by $\angle(\mathbf{A}, \mathbf{B})$, is defined to be the angle of positive measure between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ interior to the triangle determined by the points O, P, and Q. If $\mathbf{A} = c\mathbf{B}$, where c is a scalar, then if $c > 0$, the angle between the vectors has radian measure 0; if $c < 0$, the angle between the vectors has radian measure π.

10.3.5 Theorem If α is the radian measure of the angle between the two nonzero vectors **A** and **B**, then
$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\|\cos\alpha$
Thus,
$\mathbf{A}\cdot\mathbf{B} > 0$ if $0 \le \alpha < \frac{1}{2}\pi$ $\quad \mathbf{A}\cdot\mathbf{B} = 0$ if $\alpha = \frac{1}{2}\pi$ $\quad \mathbf{A}\cdot\mathbf{B} < 0$ if $\frac{1}{2}\pi < \alpha \le \pi$
We see that the dot product of two vectors does not depend on the coordinate system.

10.3.6 Definition Two vectors called parallel if and only if one of the vectors is a scalar multiple of the other.

10.3.7 Definition Two vectors $\mathbf{A}\cdot\mathbf{B}$ are said to be *orthogonal* (*perpendicular*) if and only if $\mathbf{A}\cdot\mathbf{B}=0$.

The *scalar* and *vector projection* of the vector **A** onto the nonzero vector **B** are given by

$$A_B=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} \quad\text{and}\quad \mathbf{A}_B=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B}$$

$\mathbf{A}-\mathbf{A}_B$ is orthogonal to **B**. In fact, $(\mathbf{A}-\mathbf{A}_B)\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}-\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}(\mathbf{B}\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{B}=0$

If **U** is a unit vector, then $A_U=\mathbf{A}\cdot\mathbf{U}$, and the scalar projection of **A** onto **U** is called the *component* of the vector **A** in the direction of **U**. As a special case, for the vector $a_1\mathbf{i}+a_2\mathbf{j}$ the numbers a_1 and a_2 are the components of the vector **A** in the directions of **i** and **j**, respectively.

Exercises 10.3

In Exercises 1-4, find $\mathbf{A}\cdot\mathbf{B}$.

1. (a) $\langle -1,2\rangle\cdot\langle -4,3\rangle=(-1)(-4)+2(3)=10$ (b) $(2\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})=2(1)+(-1)3=-1$

2. (a) $\langle \frac{1}{3},-\frac{1}{2}\rangle\cdot\langle \frac{5}{2},\frac{4}{3}\rangle=\frac{1}{3}(\frac{5}{2})+(-\frac{1}{2})\frac{4}{3}=\frac{1}{6}$ (b) $(-2\mathbf{i})\cdot(-\mathbf{i}+\mathbf{j})=(-2)(-1)+0(1)=2$

3. (a) $\langle \frac{2}{5},\frac{1}{4},-\frac{3}{2}\rangle\cdot\langle \frac{1}{2},\frac{3}{5},\frac{1}{2}\rangle=\frac{2}{5}(\frac{1}{2})+\frac{1}{4}(\frac{3}{5})+(-\frac{3}{2})\frac{1}{2}=-\frac{2}{5}$ (b) $(3\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-3\mathbf{k})=0(1)+3(1)+(-2)(-3)=9$

4. (a) $\mathbf{A}=\langle 4,0,2\rangle$, $\mathbf{B}=\langle 5,2,-1\rangle$; (b) $\mathbf{A}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{B}=6\mathbf{i}+7\mathbf{j}+2\mathbf{k}$

▹ We apply Definition 10.3.1.
(a) $\mathbf{A}\cdot\mathbf{B}=\langle 4,0,2\rangle\cdot\langle 5,2,-1\rangle=4(5)+0(2)+2(-1)=18$
(b) $\mathbf{A}\cdot\mathbf{B}=(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(6\mathbf{i}+7\mathbf{j}+2\mathbf{k})=3(6)+(-2)7+1(2)=6$

5. $\mathbf{i}\cdot\mathbf{i}=\langle 1,0,0\rangle\cdot\langle 1,0,0\rangle=1(1)+0(0)+0(0)=1$
$\mathbf{i}\cdot\mathbf{k}=\langle 1,0,0\rangle\cdot\langle 0,0,1\rangle=1(0)+0(0)+0(1)=0$
$\mathbf{j}\cdot\mathbf{k}=\langle 0,1,0\rangle\cdot\langle 0,0,1\rangle=0(0)+1(0)+0(1)=0$

6. $\mathbf{j}\cdot\mathbf{j}=\langle 0,1,0\rangle\cdot\langle 0,1,0\rangle=0(0)+1(1)+0(0)=1$
$\mathbf{k}\cdot\mathbf{k}=\langle 0,0,1\rangle\cdot\langle 0,0,1\rangle=0(0)+0(0)+1(1)=1$
$\mathbf{i}\cdot\mathbf{j}=\langle 1,0,0\rangle\cdot\langle 0,1,0\rangle=1(0)+0(1)+0(0)=0$

In Exercises 7–10, prove the theorem for vectors in V_3.

▹ Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$, $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, $\mathbf{C}=\langle c_1,c_2,c_3\rangle$.

7. $\mathbf{A}\cdot\mathbf{B}=\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle=\mathbf{B}\cdot\mathbf{A}$

8. Theorem 10.3.2(ii)

▹
$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})&=\langle a_1,a_2,a_3\rangle\cdot\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle\\&=a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3)\\&=(a_1b_1+a_1c_1)+(a_2b_2+a_2c_2)+(a_3b_3+a_3c_3) &&\text{(distributive law for real numbers)}\\&=(a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3)\\&=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

9.
$$\begin{aligned}c(\mathbf{A}\cdot\mathbf{B})&=c(\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle)=c(a_1b_1+a_2b_2+a_3b_3)=c(a_1b_1)+c(a_2b_2)+c(a_3b_3)\\&\overset{\text{associative for numbers}}{=}(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=\langle ca_1,ca_2,ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle=(c\mathbf{A})\cdot\mathbf{B}\end{aligned}$$

10. $\mathbf{0}\cdot\mathbf{A}=\langle 0,0,0\rangle\cdot\langle a_1,a_2,a_3\rangle=0a_1+0a_2+0a_3=0$ $\quad\mathbf{A}\cdot\mathbf{A}=\langle a_1,a_2,a_3\rangle\cdot\langle a_1,a_2,a_3\rangle={a_1}^2+{a_2}^2+{a_3}^2=\|\mathbf{A}\|^2$

In Exercises 11 and 12, if θ is the angle between **A** and **B**, find $\cos\theta$.

11. (a) $\mathbf{A}=\langle 4,3\rangle$; $\mathbf{B}=\langle -1,-1\rangle$. $\mathbf{A}\cdot\mathbf{B}=4-3=1$
$\|\mathbf{A}\|=\sqrt{16+9}=5$; $\|\mathbf{B}\|=\sqrt{1+1}=\sqrt{2}$
$\cos\theta=\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\dfrac{1}{5\sqrt{2}}=\dfrac{1}{10}\sqrt{2}$

(b) $\mathbf{A}=5\mathbf{i}-12\mathbf{j}$; $\mathbf{B}=4\mathbf{i}+3\mathbf{j}$. $\mathbf{A}\cdot\mathbf{B}=20-36=-16$
$\|\mathbf{A}\|=\sqrt{25+144}=13$; $\|\mathbf{B}\|=\sqrt{16+9}=5$
$\cos\theta=\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\dfrac{-16}{13\cdot 5}=-\dfrac{16}{65}$

12. (a) $\mathbf{A}=\langle -2,-3\rangle$, $\mathbf{B}=\langle 3,2\rangle$; (b) $\mathbf{A}=2\mathbf{i}+4\mathbf{j}$, $\mathbf{B}=-5\mathbf{j}$

▹ (a) $\mathbf{A}\cdot\mathbf{B}=\langle -2,-3\rangle\cdot\langle 3,2\rangle=(-2)3+(-3)2=-6$
$\|\mathbf{A}\|=\sqrt{2^2+3^2}=\sqrt{13}$, $\|\mathbf{B}\|=\sqrt{3^2+2^2}=\sqrt{13}$
$\cos\theta=\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\dfrac{-6}{\sqrt{13}\sqrt{13}}=-\dfrac{6}{13}$

(b) $\mathbf{A}\cdot\mathbf{B}=(2\mathbf{i}+4\mathbf{j})\cdot(-5\mathbf{j})=2(0)+4(-5)=-20$
$\|\mathbf{A}\|=\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}$, $\|\mathbf{B}\|=\sqrt{0^2+5^2}=5$
$\cos\theta=\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\dfrac{-20}{2\sqrt{5}(5)}=-\dfrac{2}{5}\sqrt{5}$

13. $\mathbf{A}=3\mathbf{i}+2\mathbf{j}$; $\mathbf{B}=2\mathbf{i}+k\mathbf{j}$. We wish to find k so that $\mathbf{A}\cdot\mathbf{B}=\|\mathbf{A}\|\|\mathbf{B}\|\cos\frac{1}{4}\pi$; $6+2k=\sqrt{13}\sqrt{4+k^2}(\frac{1}{2}\sqrt{2})$; $72+48k+8k^2=52+3k^2$; $5k^2-48k-20=0$; $(5k+2)(k-10)=0$; $k=-\frac{2}{5}$ or $k=10$

14. **A** and **B** are orthogonal $\Leftrightarrow 0=(k\mathbf{i}-2\mathbf{j})\cdot(k\mathbf{i}+6\mathbf{j})=k^2-12$, $k=\pm\sqrt{12}=\pm 2\sqrt{3}$

15. $\mathbf{A} = 5\mathbf{i} - k\mathbf{j}$ and $\mathbf{B} = k\mathbf{i} + 6\mathbf{j}$, where k is a scalar.
(a) $\mathbf{A}$ and $\mathbf{B}$ are orthogonal $\Leftrightarrow \mathbf{A}\cdot\mathbf{B} = 0 \Leftrightarrow 5k - 6k = 0 \Leftrightarrow k = 0$
(b) $\mathbf{A}$ and $\mathbf{B}$ are parallel if and only if there is a nonzero scalar a such that
$a\mathbf{A} = \mathbf{B}$; $a(5\mathbf{i} - k\mathbf{j}) = k\mathbf{i} + 6\mathbf{j}$; $5a = k$ and $-ak = 6$; $-5a^2 = 6$. This equation has no real solution.

16. Find k such that $\mathbf{A} = k\mathbf{i} - 2\mathbf{j}$ and $\mathbf{B} = k\mathbf{i} + 6\mathbf{j}$ have opposite directions.
▸ $\mathbf{A}$ and $\mathbf{B}$ have opposite directions if $\mathbf{A} = c\mathbf{B}$ for some negative scalar c.
$k\mathbf{i} - 2\mathbf{j} = c(k\mathbf{i} + 6\mathbf{j}) = ck\mathbf{i} + 6c\mathbf{j}$
Because $6c = -2$, then $c = -\frac{1}{3}$. Because $k = ck$ then $(c-1)k = 0$. Because $c - 1 \neq 0$, then $k = 0$.

17-18. $\mathbf{A} = -8\mathbf{i} + 4\mathbf{j}$; $\mathbf{B} = 7\mathbf{i} - 6\mathbf{j}$. $\mathbf{A}\cdot\mathbf{B} = (-8)(7) + 4(-6) = -80$, $\|\mathbf{A}\| = \sqrt{8^2+4^2} = 4\sqrt{5}$, $\|\mathbf{B}\| = \sqrt{7^2+6^2} = \sqrt{85}$.

$$\mathrm{A}_B = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = -\frac{80}{\sqrt{85}},\ \mathbf{A}_B = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = -\frac{80}{85}(7\mathbf{i} - 6\mathbf{j}) = -\frac{112}{17}\,\mathbf{i} + \frac{96}{17}\,\mathbf{j}.$$

$$\mathrm{B}_A = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|} = \frac{-80}{4\sqrt{5}} = -4\sqrt{5},\ \mathbf{B}_A = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{-80}{80}(-8\mathbf{i} + 4\mathbf{j}) = 8\mathbf{i} - 4\mathbf{j}$$

19. The component of $\mathbf{A} = 5\mathbf{i} - 6\mathbf{j}$ in the direction of $\mathbf{B} = 7\mathbf{i} + \mathbf{j}$ is $\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = \frac{35-6}{\sqrt{49+1}} = \frac{29}{\sqrt{50}} = \frac{29}{10}\sqrt{2}$.

20. For the vectors $\mathbf{A} = 5\mathbf{i} - 6\mathbf{j}$ and $\mathbf{B} = 7\mathbf{i} + \mathbf{j}$, find the component of the vector $\mathbf{B}$ in the direction of vector $\mathbf{A}$.
▸ he component of the vector $\mathbf{B}$ in the direction of vector $\mathbf{B}$ is B_A, the scalar projection of $\mathbf{B}$ onto $\mathbf{A}$. Thus,

$$\mathrm{B}_A = \frac{\mathbf{B}\cdot\mathbf{A}}{\|\mathbf{A}\|} = \frac{(5\mathbf{i} - 6\mathbf{j})\cdot(7\mathbf{i} + \mathbf{j})}{\|5\mathbf{i} - 6\mathbf{j}\|} = \frac{5(7) - (-6)1}{\sqrt{5^2+6^2}} = \frac{29}{\sqrt{61}}$$

In Exercises 21–26, $\mathbf{A} = \langle -4,-2,4\rangle$; $\mathbf{B} = \langle 2,7,-1\rangle$; $\mathbf{C} = \langle 6,-3,0\rangle$, and $\mathbf{D} = \langle 5,4,-3\rangle$.

21. (a) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \langle -4,-2,4\rangle\cdot(\langle 2,7,-1\rangle + \langle 6,-3,0\rangle) = \langle -4,-2,4\rangle\cdot\langle 8,4,-1\rangle = -32 - 8 - 4 = -44$
(b) $(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D}) = (\langle -4,-2,4\rangle\cdot\langle 2,7,-1\rangle)(\langle 6,-3,0\rangle\cdot\langle 5,4,-3\rangle) = (8 - 14 - 4)(30 - 12 + 0) = (-26)(18) = -468$
(c) $\mathbf{A}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{C} = (\langle -4,-2,4\rangle\cdot\langle 5,4,-3\rangle) - (\langle 2,7,-1\rangle\cdot\langle 6,-3,0\rangle) = (-20 - 8 - 12) - (12 - 21 + 0)$
$= -40 - (-9) = -31$
(d) $(\mathbf{B}\cdot\mathbf{D})\mathbf{A} - (\mathbf{D}\cdot\mathbf{A})\mathbf{B} = (\langle 2,7,-1\rangle\cdot\langle 5,4,-3\rangle)\langle -4,-2,4\rangle - (\langle 5,4,-3\rangle\cdot\langle -4,-2,4\rangle)\langle 2,7,-1\rangle$
$= (10 + 28 + 3)\langle -4,-2,4\rangle - (-20 - 8 - 12)\langle 2,7,-1\rangle = 41\langle -4,-2,4\rangle + 40\langle 2,7,-1\rangle =$
$= \langle -164,-82,164\rangle + \langle 80,280,-40\rangle = \langle -84,198,124\rangle$

22. (a) $\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} = \langle -4,-2,4\rangle\cdot\langle 2,7,-1\rangle + \langle -4,-2,4\rangle\cdot\langle 6,-3,0\rangle = -8 - 14 - 4 - 24 + 6 + 0 = -44$
(b) $(\mathbf{A}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{C}) = (\langle -4,-2,4\rangle\cdot\langle 2,7,-1\rangle)(\langle 2,7,-1\rangle\cdot\langle 6,-3,0\rangle) = (-8 - 14 - 4)(12 - 21 + 0) = (-26)(-9) = 234$
(c) $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} + (\mathbf{B}\cdot\mathbf{C})\mathbf{D} = (\langle -4,-2,4\rangle\cdot\langle 2,7,-1\rangle)\mathbf{C} + (\langle 2,7,-1\rangle\cdot\langle 6,-3,0\rangle)\mathbf{D} = -26\langle 6,3,0\rangle - 9\langle 5,4,-3\rangle$
$= \langle -201,42,27\rangle$
(d) $(2\mathbf{A} + 3\mathbf{B})\cdot(4\mathbf{C} - \mathbf{D}) = (\langle -8,-4,8\rangle + \langle 6,21,-3\rangle)\cdot(\langle 24,-12,0\rangle - \langle 5,4,-3\rangle) = \langle -2,17,5\rangle\cdot\langle 19,-16,3\rangle = -295$

23. $\mathbf{A}\cdot\mathbf{C} = \langle -4,-2,4\rangle\cdot\langle 6,-3,0\rangle = -24 + 6 + 0 = -18$, $\|\mathbf{A}\| = \sqrt{16+4+16} = 6$, $\|\mathbf{C}\| = \sqrt{36+9+0} = \sqrt{45} = 3\sqrt{5}$
(a) $\cos\theta = \frac{\mathbf{A}\cdot\mathbf{C}}{\|\mathbf{A}\|\|\mathbf{C}\|} = \frac{-18}{6(3\sqrt{5})} = -\frac{1}{\sqrt{5}}$ (b) component of $\mathbf{C}$ in the direction of $\mathbf{A}$ is $\|\mathbf{C}\|\cos\theta = 3\sqrt{5}\left(-\frac{1}{\sqrt{5}}\right) = -3$
(c) The vector projection of $\mathbf{C}$ onto $\mathbf{A}$ is $\frac{\mathbf{A}\cdot\mathbf{C}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{-18}{36}\langle -4,-2,4\rangle = \langle 2,1,-2\rangle$.

24. Find: (a) $\cos\theta$ if θ is the angle between $\mathbf{B}$ and $\mathbf{D}$;
(b) the component of $\mathbf{B}$ in the direction of $\mathbf{D}$;
(c) the vector projection of $\mathbf{B}$ onto $\mathbf{D}$.

▸

$$\|\mathbf{B}\| = \|\langle 2,7,-1\rangle\| = \sqrt{2^2+7^2+(-1)^2} = \sqrt{54}$$
$$\|\mathbf{D}\| = \|\langle 5,4,-3\rangle\| = \sqrt{5^2+4^2+(-3)^2} = \sqrt{50}$$
$$\mathbf{B}\cdot\mathbf{D} = \langle 2,7,-1\rangle\cdot\langle 5,4,-3\rangle = 2(5) + 7(4) + 2(-1)(-1) = 41$$

(a) $\cos\theta = \frac{\mathbf{B}\cdot\mathbf{D}}{\|\mathbf{B}\|\|\mathbf{D}\|} = \frac{41}{\sqrt{54}\sqrt{50}} = \frac{41}{90}\sqrt{3}$

(b) The component of $\mathbf{B}$ in the direction of $\mathbf{D}$ is $\mathrm{B}_D = \frac{\mathbf{B}\cdot\mathbf{D}}{\|\mathbf{D}\|} = \frac{41}{\sqrt{50}} = \frac{41}{10}\sqrt{2}$

(c) $\mathbf{B}_D = \frac{\mathbf{B}\cdot\mathbf{D}}{\|\mathbf{D}\|^2}\mathbf{D} = \frac{41}{50}\langle 5,4,-3\rangle = \langle \frac{41}{10}, \frac{82}{25}, -\frac{123}{50}\rangle$

25. $\mathbf{A}\cdot\mathbf{B} = \langle -4,-2,4\rangle \cdot \langle 2,7,-1\rangle = -8-14-4 = -26$, $\|\mathbf{B}\| = \sqrt{4+49+1} = \sqrt{54} = 3\sqrt{6}$.

(a) The scalar projection of **A** onto **B** is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = \dfrac{-26}{3\sqrt{6}} = -\frac{13}{9}\sqrt{6}$.

(b) The vector projection of **A** onto **B** is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = \frac{-26}{54}\langle 2,7,-1\rangle = \langle -\frac{26}{27}, -\frac{91}{27}, \frac{13}{27}\rangle$.

26. $\mathbf{C}\cdot\mathbf{D} = \langle 6,-3,0\rangle \cdot \langle 5,4,-3\rangle = 30-12+0 = 18$, $\|\mathbf{C}\| = \sqrt{6^2+3^2+0^2} = 3\sqrt{5}$

(a) $\mathrm{D}_C = \dfrac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{C}\|} = \dfrac{18}{3\sqrt{5}} = \frac{6}{5}\sqrt{5}$ (b) $\mathbf{D}_C = \dfrac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{C}\|^2}\mathbf{C} = \frac{18}{45}\langle 6,-3,0\rangle = \langle \frac{12}{5}, -\frac{6}{5}, 0\rangle$

27. d units is the distance from $P(2,-1,-4)$ to the line through $A(3,-2,2)$ and $B(-9,-6,6)$.

$\mathbf{V}(\overrightarrow{AP}) = \langle -1,1,-6\rangle$, $\left\|\mathbf{V}(\overrightarrow{AP})\right\|^2 = 1+1+36 = 38$, $\mathbf{V}(\overrightarrow{AB}) = \langle -12,-4,4\rangle$, $\left\|\mathbf{V}(\overrightarrow{AB}\right\|^2 = 144+16+16 = 176$

$\mathbf{V}(\overrightarrow{AP})\cdot\mathbf{V}(\overrightarrow{AB}) = 12-4-24 = -16$. $d = \sqrt{\left\|\mathbf{V}(\overrightarrow{AP})\right\|^2 - \dfrac{[\mathbf{V}(\overrightarrow{AP})\cdot\mathbf{V}(\overrightarrow{AB})]}{\left\|\mathbf{V}(\overrightarrow{AB})\right\|^2}} = \sqrt{38-\frac{256}{176}} = \sqrt{\frac{402}{11}} = \frac{1}{11}\sqrt{4422}$

28. Find the distance d from the point $P(3,2,1)$ to the line through the points $A(1,2,9)$ and $B(-3,-6,-3)$.

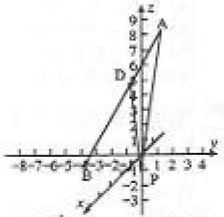
▹ Let D be the foot of the perpendicular from P on AB. See the figure.

$\overrightarrow{AP} = \langle 3,2,1\rangle - \langle 1,2,9\rangle = \langle 2,0,-8\rangle$. Let $c = \|\overrightarrow{AP}\| = \sqrt{2^2+0^2+8^2} = \sqrt{68}$.

$\overrightarrow{AB} = \langle -3,-6,-3\rangle - \langle 1,2,9\rangle = \langle -4,-8,-12\rangle$.

Let $a = \|\overrightarrow{AD}\| = |AP_{AB}| = \dfrac{|\overrightarrow{AP}\cdot\overrightarrow{AB}|}{\|\overrightarrow{AB}\|} = \dfrac{|-8+0+96|}{\sqrt{4^2+8^2+12^2}} = \dfrac{88}{4\sqrt{14}} = \dfrac{22}{\sqrt{14}}$. Then

$d = \sqrt{c^2-a^2} = \sqrt{68-\frac{484}{14}} = \sqrt{\frac{234}{7}} = \frac{1}{7}\sqrt{1638}$



29. Let $A = (2,2,2)$, $B = (2,0,1)$, $C = (4,1,-1)$, $D = (4,3,0)$. $\mathbf{V}(\overrightarrow{BC}) = \langle 2,1,-2\rangle = \mathbf{V}(\overrightarrow{AD})$ so ABCD is a parallelogram. $\mathbf{V}(\overrightarrow{AB})\cdot\mathbf{V}(\overrightarrow{AD}) = \langle 0,-2,-1\rangle\cdot\langle 2,1,-2\rangle = 0$ so there is a right angle at A. Therefore the parallelogram is a rectangle.

30. $A = (2,2,2)$, $B = (0,1,2)$, $C = (-1,3,3)$, $D = (3,0,1)$. $\overrightarrow{AD} = \langle 1,-2,-1\rangle = \overrightarrow{CB} \Rightarrow$ ADBC is a parallelogram.

31. Let $A = (-2,3,1)$, $B = (1,2,3)$, $P = (3,-1,2)$. $\left\|\mathbf{V}(\overrightarrow{AB})\right\|^2 = \|\langle 3,-1,2\rangle\|^2 = 9+1+4 = 14$, $\left\|\mathbf{V}(\overrightarrow{AP})\right\|^2 = \|\langle 5,-4,1\rangle\|^2 = 25+16+1 = 42$, $\mathbf{V}(AP)\cdot\mathbf{V}(AB) = 15+4+2 = 21$. If A square units is the area,

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}\left\|\mathbf{V}(\overrightarrow{AB})\right\|\sqrt{\left\|\mathbf{V}(\overrightarrow{AP})\right\|^2 - \frac{[\mathbf{V}(\overrightarrow{AP})\cdot\mathbf{V}(\overrightarrow{AB})]^2}{\left\|\mathbf{V}(\overrightarrow{AB})\right\|^2}} = \tfrac{1}{2}\sqrt{\left\|\mathbf{V}(\overrightarrow{AP})\right\|^2\left\|\mathbf{V}(\overrightarrow{AB})\right\|^2 - [\mathbf{V}(\overrightarrow{AP})\cdot\mathbf{V}(\overrightarrow{AB})]^2}$$

$$= \tfrac{1}{2}\sqrt{(14)(42)-(21)^2} = \tfrac{7}{2}\sqrt{(2)(6)-9} = \tfrac{7}{2}\sqrt{3}$$

32. Prove using vectors that the points $A(-2,1,6)$, $B(2,4,5)$, and $C(-1,-2,1)$ form a right triangle. Find its area.

▹ We have $\overrightarrow{AB} = \langle 2,4,5\rangle - \langle -2,1,6\rangle = \langle 4,3,-1\rangle$ and $\overrightarrow{AC} = \langle -1,-2,1\rangle - \langle -2,1,6\rangle = \langle 1,-3,-5\rangle$.

Because $\overrightarrow{AB}\cdot\overrightarrow{AC} = 4(1)+3(-3)+(-1)(-5) = 0$, then ABC is a right angle.

$\text{Area} = \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| = \frac{1}{2}\sqrt{4^2+3^2+1^2}\sqrt{1^2+3^2+5^2} = \frac{1}{2}\sqrt{26}\sqrt{35} = \frac{1}{2}\sqrt{910}$

33. $y = x^2$, $y' = 2x$, $y'(2) = 4$. Thus $\langle 1,4\rangle$ is a tangent vector. $\mathbf{U} = \pm\langle 1,4\rangle/\sqrt{1^2+4^2} = \pm\langle 1,4\rangle/\sqrt{17}$

34. For the parabola $y = x^2$, $y' = 2x$. Hence a normal to the parabola at the point $(2,4)$ has a slope of $-\frac{1}{4}$. Thus $\tan\theta = -\frac{1}{4}$. The required unit vectors are $\pm(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, that is

$-\dfrac{4}{\sqrt{17}}\mathbf{i} + \dfrac{1}{\sqrt{17}}\mathbf{j}$ and $\dfrac{4}{\sqrt{17}}\mathbf{i} - \dfrac{1}{\sqrt{17}}\mathbf{j}$ or equivalently $-\frac{4}{17}\sqrt{17}\,\mathbf{i} + \frac{1}{17}\sqrt{17}\,\mathbf{j}$ and $\frac{4}{17}\sqrt{17}\,\mathbf{i} - \frac{1}{17}\sqrt{17}\,\mathbf{j}$.

35. $\mathbf{A} = 3\mathbf{i}+5\mathbf{j}-3\mathbf{k}$, $\mathbf{B} = -\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $\mathbf{C} = 2\mathbf{i}-\mathbf{j}+4\mathbf{k}$. $\mathbf{A}-2\mathbf{C} = 3\mathbf{i}+5\mathbf{j}-3\mathbf{k}-2(2\mathbf{i}-\mathbf{j}+4\mathbf{k}) = -\mathbf{i}+7\mathbf{j}-11\mathbf{k}$.

The component of **B** in the direction of $\mathbf{A}-2\mathbf{C}$ is $\dfrac{\mathbf{B}\cdot(\mathbf{A}-2\mathbf{C})}{\|\mathbf{A}-2\mathbf{C}\|} = \dfrac{1-14-33}{\sqrt{1+49+121}} = \dfrac{-46}{\sqrt{171}} = -\frac{46}{57}\sqrt{19}$.

36. Find the cosines of the angles of the triangle having vertices at A(0, 0, 0), B(4, −1, 3), and C(1, 2, 3).

▹ Theorem 10.3.5 requires the calculation of three magnitudes *and* three dot products, while a direct application of the law of cosines needs only the three magnitudes. We find

$$a = \|\overrightarrow{\mathrm{BC}}\| = \|\langle -3, 3, 0\rangle\| = \sqrt{3^2+3^2+0^2} = 3\sqrt{2}$$
$$b = \|\overrightarrow{\mathrm{AC}}\| = \|\langle 1, 2, 3\rangle\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$
$$c = \|\overrightarrow{\mathrm{AB}}\| = \|\langle 4, 1, -3\rangle\| = \sqrt{4^2+1^2+3^2} = \sqrt{26}$$
$$\cos \mathrm{A} = \frac{b^2+c^2-a^2}{2bc} = \frac{14+26-18}{2\sqrt{14}\sqrt{26}} = \frac{11}{\sqrt{14}\sqrt{26}} = \frac{11}{182}\sqrt{91}$$
$$\cos \mathrm{B} = \frac{a^2+c^2-b^2}{2ac} = \frac{18+26-14}{2(3\sqrt{2})\sqrt{26}} = \frac{5}{2\sqrt{13}} = \frac{5}{26}\sqrt{13}$$
$$\cos \mathrm{C} = \frac{a^2+b^2-c^2}{2ab} = \frac{18+14-26}{2(3\sqrt{2})\sqrt{14}} = \frac{1}{2\sqrt{7}} = \frac{1}{14}\sqrt{7}$$

37. $\mathbf{F} = 8(\cos \frac{1}{3}\pi\mathbf{i} + \sin \frac{1}{3}\pi\mathbf{j}) = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.
(a) If W_1 ft-lb is the work done by **F** in moving an object along the x axis from the origin to the point (6,0), then $\mathrm{W}_1 = \mathbf{F}\cdot\langle 6,0\rangle = \langle 4, 4\sqrt{3}\rangle\cdot\langle 6,0\rangle = 24$.
(b) If W_2 ft-lb is the work done by **F** in moving an object along the y axis from the origin to the point (0,6), then $\mathrm{W}_2 = \mathbf{F}\cdot\langle 0,6\rangle = \langle 4, 4\sqrt{3}\rangle\cdot\langle 0,6\rangle = 24\sqrt{3}$.

38. $\mathrm{W} = \mathbf{F}\cdot\mathbf{D} = 10(\cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j})\cdot[5-(-2)]\mathbf{j} = 10(\frac{1}{2}\sqrt{2})7 = 35\sqrt{2}$. The work done is $35\sqrt{2}$ foot-pounds.

39. $\mathbf{F} = 9(\cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j}) = -\frac{9}{2}\mathbf{i} + \frac{9}{2}\sqrt{3}\,\mathbf{j}$.
If W ft-lb is the work done by **F** in moving an object from the origin to (−4, −2), then
$\mathrm{W} = \mathbf{F}\cdot\langle -4,-2\rangle = \langle -\frac{9}{2}, \frac{9}{2}\sqrt{3}\rangle\cdot\langle -4,-2\rangle = 18 - 9\sqrt{3} \approx 2.41$

40. Two forces represented by the vectors $\mathbf{F}_1 = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{F}_2 = -4\mathbf{i} + 5\mathbf{j}$ act on a particle and cause it to move along a straight line from point A(2,5) to point B(7,3). If the magnitudes of the forces are measured in pounds and distance is measured in feet, find the work done by the two forces acting together.

▹ The displacement vector is given by
$\mathbf{D} = \mathbf{b} - \mathbf{a} = \langle 7,3\rangle - \langle 2,5\rangle = 5\mathbf{i} - 2\mathbf{j}$
The number of foot-pounds in the work done by the two forces is given by
$$\mathrm{W} = (\mathbf{F}_1 + \mathbf{F}_2)\cdot\mathbf{D} = [(3\mathbf{i}-\mathbf{j}) + (-4\mathbf{i}+5\mathbf{j})]\cdot\mathbf{D} = (-\mathbf{i}+4\mathbf{j})\cdot(5\mathbf{i}-2\mathbf{j}) = (-1)5 + 4(-2) = -13$$
The work done is −13 foot-pounds.

41. If W ft-lb is the work done by the force $\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in moving an object from $\mathrm{P}_1(-2,4,3)$ to $\mathrm{P}_2(1,-3,5)$, then $\mathrm{W} = \mathbf{F}\cdot\mathbf{V}(\overrightarrow{\mathrm{P}_1\mathrm{P}_2})(3\mathbf{i}-2\mathbf{j}+\mathbf{k})(3\mathbf{i}-7\mathbf{j}+2\mathbf{k}) = 9 + 14 + 2 = 25$.

42. $\mathrm{W} = \mathbf{F}\cdot\mathbf{D} = (5\mathbf{i} - 3\mathbf{k})\cdot(\langle -5,6,2\rangle - \langle 4,1,3\rangle) = \langle 5,0,-3\rangle\cdot\langle -9,5,-1\rangle = -42$. The work done is −42 foot-pounds.

43. For force **F**, $\cos\alpha = \frac{1}{6}\sqrt{6}$ and $\cos\beta = \frac{1}{3}\sqrt{6}$. Therefore
$(\frac{1}{6}\sqrt{6})^2 + (\frac{1}{3}\sqrt{6})^2 + \cos^2\gamma = 1$; $\frac{1}{6} + \frac{2}{3} + \cos^2\gamma = 1$; $\cos^2\gamma = \frac{1}{6}$; $\cos\gamma = \frac{1}{6}\sqrt{6}$. $\mathbf{F} = 10(\frac{1}{6}\sqrt{6}\,\mathbf{i} + \frac{1}{3}\sqrt{6}\,\mathbf{j} + \frac{1}{6}\sqrt{6}\,\mathbf{k})$
If W ft-lb is the work done by **F** in moving an object from the origin to (7, −4, 2), then
$$\mathrm{W} = \mathbf{F}\cdot\mathbf{V}(\overrightarrow{\mathrm{OP}}) = (\tfrac{5}{3}\sqrt{6}\,\mathbf{i} + \tfrac{10}{3}\sqrt{6}\,\mathbf{j} + \tfrac{5}{3}\sqrt{6}\,\mathbf{k})\cdot(7\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \tfrac{35}{3}\sqrt{6} - \tfrac{40}{3}\sqrt{6} + \tfrac{10}{3}\sqrt{6} = \tfrac{5}{3}\sqrt{6}$$

44. If **A** and **B** are nonzero vectors, prove that the vector $\mathbf{A} - c\mathbf{B}$ is orthogonal to **B** if $c = \mathbf{A}\cdot\mathbf{B}/\|\mathbf{B}\|^2$.

▹ The vectors $\mathbf{A} - c\mathbf{B}$ and **B** are orthogonal if
$$(\mathbf{A} - c\mathbf{B})\cdot\mathbf{B} = 0 \tag{1}$$
Equation (1) is true if and only if
$$\mathbf{A}\cdot\mathbf{B} - c\mathbf{B}\cdot\mathbf{B} = 0$$
$$\mathbf{A}\cdot\mathbf{B} - c\|\mathbf{B}\|^2 = 0 \tag{2}$$
Because $\|\mathbf{B}\| \neq 0$, we may solve Eq. (2), which contains all scalars, for c. Thus, (1) is true if
$$c = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}.$$

45. $\mathbf{A} = 12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. From Exercise 44, $\mathbf{B} - c\mathbf{A}$ is orthogonal to $\mathbf{A}$ if

$$c = \frac{\mathbf{B}\cdot\mathbf{A}}{\|\mathbf{A}\|^2} = \frac{(4\mathbf{i}+3\mathbf{j}-5\mathbf{k})\cdot(12\mathbf{i}+9\mathbf{j}-5\mathbf{k})}{144+81+25} = \frac{48+27+25}{250} = \frac{100}{250} = \frac{2}{5}$$

46. $\mathbf{A} = 12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. From Exercise 44, $\mathbf{A} - d\mathbf{B}$ is orthogonal to $\mathbf{B}$ if

$$c = \frac{\mathbf{B}\cdot\mathbf{A}}{\|\mathbf{B}\|^2} = \frac{(4\mathbf{i}+3\mathbf{j}-5\mathbf{k})\cdot(12\mathbf{i}+9\mathbf{j}-5\mathbf{k})}{16+9+25} = \frac{48+27+25}{50} = \frac{100}{50} = 2$$

47. $(\|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B})\cdot(\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}) = \|\mathbf{B}\|\mathbf{A}\cdot\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}\cdot\|\mathbf{A}\|\mathbf{B} = \|\mathbf{B}\|^2\|\mathbf{A}\|^2 - \|\mathbf{A}\|^2\|\mathbf{B}\|^2 = 0$
Therefore, the vectors $\|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B}$ and $\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}$ are orthogonal.

48. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are any nonzero vectors and $\mathbf{C} = \|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B}$, then the angle θ_1 between $\mathbf{A}$ and $\mathbf{C}$ has the same measure as the angle θ_2 between $\mathbf{B}$ and $\mathbf{C}$.

▸ Let $\mathbf{U} = \dfrac{\mathbf{A}}{\|\mathbf{A}\|}$ and $\mathbf{V} = \dfrac{\mathbf{B}}{\|\mathbf{B}\|}$ be the directions of $\mathbf{A}$ and $\mathbf{B}$. Then

$$\mathbf{D} = \frac{\mathbf{C}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \frac{\mathbf{A}}{\|\mathbf{A}\|} + \frac{\mathbf{B}}{\|\mathbf{B}\|}$$

has the same direction as $\mathbf{C}$. Thus,

$$\cos\theta_1 = \frac{\mathbf{U}\cdot\mathbf{D}}{\|\mathbf{U}\|\|\mathbf{D}\|} = \frac{\mathbf{U}\cdot(\mathbf{U}+\mathbf{V})}{1\|\mathbf{D}\|} = \frac{1+\mathbf{U}\cdot\mathbf{V}}{\|\mathbf{D}\|}$$

$$\cos\theta_2 = \frac{\mathbf{V}\cdot\mathbf{D}}{\|\mathbf{V}\|\|\mathbf{D}\|} = \frac{\mathbf{V}\cdot(\mathbf{U}+\mathbf{V})}{1\|\mathbf{D}\|} = \frac{1+\mathbf{U}\cdot\mathbf{V}}{\|\mathbf{D}\|}$$

Because $\cos\theta_1 = \cos\theta_2$, the angle between $\mathbf{A}$ and $\mathbf{C}$ has the same measure as the angle between $\mathbf{B}$ and $\mathbf{C}$.

49. If $\mathbf{A}$ and $\mathbf{B}$ are two nonzero parallel vectors, then $\mathbf{B} = k\mathbf{A}$. If α is the radian measure of the angle between them, $\cos\alpha = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{\mathbf{A}\cdot(k\mathbf{A})}{\|\mathbf{A}\|\|k\mathbf{A}\|} = \dfrac{k(\mathbf{A}\cdot\mathbf{A})}{|k|\|\mathbf{A}\|^2} = \dfrac{k\|\mathbf{A}\|^2}{|k|\|\mathbf{A}\|^2} = \pm 1$. Thus $\alpha = 0$ or π.

Conversely, if $\alpha = 0$ then $\cos\alpha = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = 1$. Therefore

$$\left(\frac{\mathbf{A}}{\|\mathbf{A}\|} - \frac{\mathbf{B}}{\|\mathbf{B}\|}\right)^2 = \frac{\mathbf{A}\cdot\mathbf{A}}{\|\mathbf{A}\|^2} - 2\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} + \frac{\mathbf{B}\cdot\mathbf{B}}{\|\mathbf{B}\|^2} = 1 - 2 + 1 = 0;\ \text{so}\ \frac{\mathbf{A}}{\|\mathbf{A}\|} - \frac{\mathbf{B}}{\|\mathbf{B}\|} = \mathbf{0};\ \mathbf{B} = \frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}\mathbf{A}.$$

Thus there is a scalar $k = \|\mathbf{B}\|/\|\mathbf{A}\|$ such that $\mathbf{B} = k\mathbf{A}$ and so $\mathbf{A}$ and $\mathbf{B}$ are parallel.
Similarly, if $\alpha = \pi$, then $\cos\alpha = -1$ and $\mathbf{B} = -(\|\mathbf{B}\|/\|\mathbf{A}\|)\mathbf{A}$ and again $\mathbf{A}$ and $\mathbf{B}$ are parallel.

In Exercises 50–52, prove the theorem by vector analysis.

50. The medians of triangle ABC meet in a point.

▸ Let G be the point $\frac{2}{3}$ of the way from A to the midpoint of BC. Then $\mathbf{g} = \left(1 - \frac{2}{3}\right)\mathbf{a} + \frac{2}{3}\left(\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}\right) = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$.
By symmetry, we see that G also lies on the other two medians.

51. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

▸ If P is the midpoint of AB and Q the midpoint of AC, then $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{a}+\mathbf{c}) - \frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}(\mathbf{c}-\mathbf{b}) = \frac{1}{2}\overrightarrow{BC}$
Thus PQ is parallel to AB and half its length.

52. The line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the parallel sides and that its length is one-half the sum of the lengths of the parallel sides.

▸ ABCD is a trapezoid with parallel sides AB and DC. Let E be the midpoint of side AD, and F the midpoint of side BC. Refer to the figure. To simplify the notation, we let $\overrightarrow{AB}$ denote the vector having the directed line segment $\overrightarrow{AB}$ as a representation, and similarly for other directed line segments. The origin O is not shown. Because E is the midpoint of segment AD, and F is the midpoint of BC, then

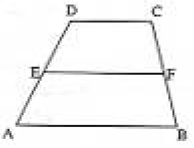
$$\overrightarrow{OE} = \tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{OD}$$

and

$$\overrightarrow{OF} = \tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC}$$

It follows that

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \left(\tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC}\right) - \left(\tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{OD}\right) = \tfrac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OD}\right)$$



$$= \tfrac{1}{2}\overrightarrow{\mathrm{AB}} + \tfrac{1}{2}\overrightarrow{\mathrm{DC}} \tag{3}$$

Because $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{DC}}$ are parallel and the vectors have the same direction, there is a positive scalar k such that

$$\overrightarrow{\mathrm{DC}} = k\overrightarrow{\mathrm{AB}} \tag{4}$$

Substituting from (4) into (3) we have

$$\overrightarrow{\mathrm{EF}} = \tfrac{1}{2}\overrightarrow{\mathrm{AB}} + \tfrac{1}{2}k\overrightarrow{\mathrm{AB}} = \tfrac{1}{2}(1+k)\overrightarrow{\mathrm{AB}}$$

Because $\overrightarrow{\mathrm{EF}}$ is a scalar multiple of $\overrightarrow{\mathrm{AB}}$, then $\overrightarrow{\mathrm{EF}}$ is parallel to $\overrightarrow{\mathrm{AB}}$. Furthermore,

$$\begin{aligned}\|\overrightarrow{\mathrm{EF}}\| &= \left\|\tfrac{1}{2}(1+k)\overrightarrow{\mathrm{AB}}\right\| = \tfrac{1}{2}(1+k)\|\overrightarrow{\mathrm{AB}}\| && (1+k>0)\\ &= \tfrac{1}{2}(\|\overrightarrow{\mathrm{AB}}\| + k\|\overrightarrow{\mathrm{AB}}\|) = \tfrac{1}{2}(\|\overrightarrow{\mathrm{AB}}\| + \|k\overrightarrow{\mathrm{AB}}\|) && (k>0)\\ &= \tfrac{1}{2}(\|\overrightarrow{\mathrm{AB}}\| + \|\overrightarrow{\mathrm{DC}}\|) && (\text{from } (4))\end{aligned}$$

53. Let **F** be a unit vector in the interface pointing to the left, let α_1 be the complement of θ_1 and let α_2 be the supplement of θ_2. Then $\mathbf{A}\cdot\mathbf{F} + \mu\mathbf{B}\cdot\mathbf{F} = \cos\alpha_1 + \mu\cos\alpha_2 = \sin\theta_1 - \mu\sin\theta_2 = 0$

54. We have $\|x\mathbf{A} - \mathbf{B}\| > 0$ unless $\mathbf{B} = x\mathbf{A}$. Then
$0 < \|x\mathbf{A} - \mathbf{B}\|^2 = (x\mathbf{A} - \mathbf{B})\cdot(x\mathbf{A} - \mathbf{B}) = x^2\mathbf{A}\cdot\mathbf{A} - 2x\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} = x^2\|\mathbf{A}\|^2 - 2x\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2$
Because the quadratic has no roots, its discriminant is negative, that is,
$(2\mathbf{A}\cdot\mathbf{B})^2 - 4\|\mathbf{A}\|^2\|\mathbf{B}\|^2 < 0 \Leftrightarrow (\mathbf{A}\cdot\mathbf{B})^2 < \|\mathbf{A}\|^2\|\mathbf{B}\|^2 \Leftrightarrow |\mathbf{A}\cdot\mathbf{B}| < \|\mathbf{A}\|\|\mathbf{B}\|$

55. $\|\mathbf{A}+\mathbf{B}\|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} = \|\mathbf{A}\|^2 + 2\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2$

56. **A** and **B** are orthogonal $\Leftrightarrow \mathbf{A}\cdot\mathbf{B} = 0 \Leftrightarrow \|\mathbf{A}\|^2 + 2\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 \Leftrightarrow \|\mathbf{A}+\mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2$

57. Prove the *Parallelogram law.* $\|\mathbf{A}+\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{B}\|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) + (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$
$= (\mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) + (\mathbf{A}\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) = 2\mathbf{A}\cdot\mathbf{A} + 2\mathbf{B}\cdot\mathbf{B} = 2\|\mathbf{A}\|^2 + 2\|\mathbf{B}\|^2$.
The geometric interpretation is that a parallelogram's perimeter equals the sum of the length of its diagonals. The converse is also true: If a quadrilateral Q in E^3 is such that its perimeter equals the sum of the lengths of its diagonals, then Q is a plane parallelogram.

58. Prove the *polarization identity:* $\tfrac{1}{4}(\|\mathbf{A}+\mathbf{B}\|^2 - \|\mathbf{A}-\mathbf{B}\|^2) = \tfrac{1}{4}[(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) - (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})]$
$= \tfrac{1}{4}[(\mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B})] = \tfrac{1}{4}(4\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B}$

59. $\mathbf{E}_1 = \mathbf{E}_H = \dfrac{\mathbf{E}\cdot\mathbf{H}}{\|\mathbf{H}\|^2}\mathbf{H}$ is parallel to **H** and $\mathbf{E}_2 = \mathbf{E} - \mathbf{E}_H$ is orthogonal to **H**.

60. Find the total revenue $\mathrm{R} = \mathbf{A}\cdot\mathbf{S}$ for each day of the week.
▷ Mon. $\langle 250, 180, 310\rangle\cdot\langle 25.50, 16.80, 54.55\rangle = \$26{,}309.50$
Tues. $\langle 185, 210, 215\rangle\cdot\langle 27.50, 14.60, 61.25\rangle = \$21{,}322.25$
Wed. $\langle 400, 120, 180\rangle\cdot\langle 21.20, 21.50, 66.50\rangle = \$23{,}030.00$
Thurs. $\langle 355, 165, 200\rangle\cdot\langle 23.40, 18.50, 62.30\rangle = \$23{,}819.50$
Fri. $\langle 370, 145, 240\rangle\cdot\langle 22.60, 19.10, 61.75\rangle = \$25{,}951.50$

10.4 PLANES AND LINES IN R^3

10.4.1 Definition If **N** is a given nonzero vector and P_0 is a given point, then the set of all points P for which $\mathbf{V}(\overrightarrow{\mathrm{P}_0\mathrm{P}})$ and **N** are orthogonal is defined to be a plane through P_0 having **N** as a *normal vector.*

10.4.2 Definition If $\mathrm{P}_0(x_0, y_0, z_0)$ is a point in a plane and a normal vector to the plane is $\mathbf{N} = \langle a, b, c\rangle$, then an equation of the plane is
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Theorem 10.4.3 If a, b, and c are not all zero, the graph of an equation of the form $ax + by + cz + d = 0$ is a plane, and $\langle a, b, c\rangle$ is a normal vector to the plane.

If one of the coefficients a, b, and c in Theorem 10.4.3 is zero, then the plane is parallel to a coordinate axis. If two of the coefficients a, b, and c are zero, then the plane is parallel to two coordinate axes and, hence, perpendicular to one of the coordinate axes. We summarize the possible cases.

$a = 0$	Plane is parallel to the x axis.
$b = 0$	Plane is parallel to the y axis.
$c = 0$	Plane is parallel to the z axis.
$a = 0$ and $b = 0$	Plane is perpendicular to the z axis.
$a = 0$ and $c = 0$	Plane is perpendicular to the y axis.
$b = 0$ and $c = 0$	Plane is perpendicular to the x axis.

10.4.4 Definition An *angle between two planes* is defined to be the angle between normal vectors of the planes. There are two angles between two planes. If one of those angles is θ, the other is the supplement of θ.

10.4.5 Definition Two planes are *parallel* if and only if their normal vectors are parallel.

10.4.6 Definition Two planes are *perpendicular* if and only if their normal vectors are orthogonal.

Line A line L that contains the point (x_0, y_0, z_0) and is parallel to representations of the vector $\langle a, b, c\rangle$ has *parametric equations*

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct \tag{I}$$

It has the *symmetric equations*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{if } a \neq 0,\ b \neq 0,\ c \neq 0 \tag{II}$$

$\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$ and $z = z_0$ if $a \neq 0$, $b \neq 0$, $c = 0$, and similar equations if $a = 0$ or $b = 0$.

The vector $\mathbf{V} = \langle a, b, c\rangle$ is called a *direction vector* of the line L; we also say that $[a, b, c]$ is a set of *direction numbers* for L. $k\mathbf{V}$, $k \neq 0$, is also a direction vector of L.

Let $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$ be direction vectors for lines L_1 and L_2. L_1 and L_2 are parallel if $\langle a_1, b_1, c_1\rangle = k\langle a_2, b_2, c_2\rangle$ and L_1 and L_2 are perpendicular if $\langle a_1, b_1, c_1\rangle \cdot \langle a_2, b_2, c_2\rangle = 0$.

Let M be the plane $ax + by + cz + d = 0$ with normal vector $\langle a, b, c\rangle$. If line L is perpendicular to plane M, then $\langle a, b, c\rangle$ is a direction vector for line L. If line L lies in plane M or if L is parallel to M, then $\langle a', b', c'\rangle$ is a direction vector of line L if and only if $\langle a, b, c\rangle \cdot \langle a', b', c'\rangle = 0$.

Two lines that do not lie in one plane are *skew*. See Exercises 57 and 58.

Determinant A second order determinant is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

A third-order determinant may defined by expanding along the first row:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Similarly, a fourth-order determinant is defined by expanding along its first row:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_3 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_3 \end{vmatrix} - a_4\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

A determinant of any order has following properties:

(i) A determinant is zero if and only if some row is a linear combination of the others. In particular, a determinant is zero if two rows are equal.

(ii) Multiplying the elements of a row by a number c multiplies the value of the determinant by c.

The following result is useful because after expanding along the first row we can use our graphics calculators to evaluate the four numerical determinants. See Exercise 8.

Determinant Form An equation of the plane through the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Proof Expanding by the first row, we see from Theorem 10.4.3 that this is the equation of a plane. Furthermore, if $(x, y, z) = (x_1, y_1, z_1)$, then two rows of the determinant are equal. Thus, P_1 satisfies the equation. similarly, P_2 and P_3 satisfy the equation.

Intercept Form If a, b, and c are nonzero and are the x intercept, y intercept, and z intercept, respectively of a plane, then an equation of the plane is $x/a + y/b + z/c = 1$. See Exercise 28.

Distance to a Plane The distance from the plane $ax + by + cz + d = 0$ to the point $P_0(x_0, y_0, z_0)$ is given by $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$. See Exercise 60.

Exercises 10.4

In Exercises 1–6, find an equation of the plane containing the point P and having the vector **N** as a normal vector.

1. $P(3,1,2)$; $\mathbf{N} = \langle 1,2,-3\rangle$: $1(x-3) + 2(y-1) - 3(z-2) = 0$; $x - 3 + 2y - 2 - 3z + 6 = 0$; $x + 2y - 3z + 1 = 0$

2. $P(-3,2,5)$; $\mathbf{N} = \langle 6,-3,-2\rangle$: $6(x+3) - 3(y-2) - 2(z-5) = 6x + 18 - 3y + 6 - 2z + 10 = 0$; $6x - 3y - 2z + 34 = 0$

3. $P(0,-1,2)$; $\mathbf{N} = \langle 0,1,-1\rangle$: $0(x-0) + 1(y+1) - 1(z-2) = 0$; $y + 1 - z + 2 = 0$; $y - z + 3 = 0$

4. $P(-1,8,3)$; $\mathbf{N} = \langle -7,-1,1\rangle$

▹ We apply Theorem 10.4.2. Thus, an equation of the plane is

$$-7(x+1) - (y-8) + (z-3) = 0 \qquad 7x + y - z + 2 = 0$$

5. $P(2,1,-1)$; $\mathbf{N} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$: $-1(x-2) + 3(y-1) + 4(z+1) = 0$; $x + 2 + 3y - 3 + 4z + 4 = 0$; $x - 3y - 4z - 3 = 0$

6. $P(1,0,0)$; $\mathbf{N} = \mathbf{i} + \mathbf{k}$: $1(x-1) + 0(y-0) + 1(z-0) = 0$; $x + z - 1 = 0$

In Exercises 7 and 8, find an equation of the plane containing the three given points.

7. By Theorem 10.4.3 an equation of the plane is of the form $ax + by + cz + d = 0$ (1)

Because the plane contains $(3,4,1)$, $(1,7,1)$, and $(-1,-2,5)$, we have the three equations

E_1: $3a + 4b + c = -d$

$F_1 = E_1 + 3E_3$: $-2b + 16c = -4d$

E_2: $a + 7b + c = -d$ $\quad 5F_1 + 2F_2$: $92c = -24d$

$F_2 = E_2 + E_3$: $5b + c = -2d$

E_3: $-a - 2b + 5c = -d$

Hence $c = -\frac{6d}{23}$, $b = -\frac{2d}{23}$, $a = -\frac{3d}{23}$. Substituting the values into (1) we get

$-\frac{3d}{23}x - \frac{2d}{23}y - \frac{6d}{23}z + d = 0$; $3x + 2y + 6z - 23 = 0$ (2)

$$\text{Alternatively, }\begin{vmatrix} x & y & z & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ -1 & -2 & 5 & 1 \end{vmatrix} = x\begin{vmatrix} 4 & 1 & 1 \\ 7 & 1 & 1 \\ -2 & 5 & 1 \end{vmatrix} - y\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & 1 \end{vmatrix} + z\begin{vmatrix} 3 & 4 & 1 \\ 1 & 7 & 1 \\ -1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 4 & 1 \\ 1 & 7 & 1 \\ -1 & -2 & 5 \end{vmatrix}$$

Evaluating the determinants, we get the equation $12x + 8y + 24z - 92 = 0$, which is equivalent to (2).

8. $(0,0,2)$, $(2,4,1)$, $(-2,3,3)$

▹ By Theorem 10.4.3, an equation of the plane is of the form

$$ax + by + cz = -d$$

We substitute the coordinates of the points and solve the resulting system for a, b, and c in terms of d.

$A(0,0,2)$: $2c = -d$

$B(2,4,1)$: $2a + 4b + c = -d$

$C(-2,3,3)$: $-2a + 3b + 3c = -d$

$D = B + C$: $7b + 4c = -2d$

$E = D - 2A$: $7b = 0$

$F = \frac{1}{7}E$: $b = 0$

$G = \frac{1}{2}A$: $c = -\frac{1}{2}d$

$\frac{1}{2}(B - 4F - C)$: $a = -\frac{1}{4}d$

Setting $d = -4$, we get $a = 1$, $b = 0$, $c = 2$, and so an equation is

$$x + 2z - 4 = 0 \qquad (1)$$

Alternatively, we apply the determinant form to the given points.

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 4 & 1 & 1 \\ -2 & 3 & 3 & 1 \end{vmatrix} = x\begin{vmatrix} 0 & 2 & 1 \\ 4 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} - y\begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ -2 & 3 & 1 \end{vmatrix} + z\begin{vmatrix} 0 & 0 & 1 \\ 2 & 4 & 1 \\ -2 & 3 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 2 \\ 2 & 4 & 1 \\ -2 & 3 & 3 \end{vmatrix}$$

Evaluating the determinants, we get the equation $7x + 14z - 28 = 0$ which is equivalent to (1).

In Exercises 9–14, sketch the plane and find two unit vectors normal to the plane.

9. A normal vector to the plane $2x - y + 2z - 6 = 0$ is $\langle 2, -1, 2\rangle$. $\|\langle 2, -1, 2\rangle\| = \sqrt{4+1+4} = 3$. Hence two unit vectors normal to the plane are $\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$ and $\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$.

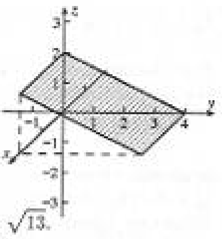
10. A normal vector to the plane $4x - 4y + 2z - 9 = 0$ is $\langle 4, -4, 2\rangle$. $\|\langle 4, -4, 2\rangle\| = \sqrt{4^2+4^2+2^2} = 6$ Hence two unit vectors normal to the plane are $\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$ and $\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$.

11. A normal vector to the plane $4x + 3y - 12z = 0$ is $\langle 4, 3, -12\rangle$. $\|\langle 4, 3, -12\rangle\| = \sqrt{16+9+144} = 13$. Hence two unit vectors normal to the plane are $\langle \frac{4}{13}, \frac{3}{13}, -\frac{12}{13}\rangle$ and $\langle -\frac{4}{13}, -\frac{3}{13}, \frac{12}{13}\rangle$.

12. $y + 2z - 4 = 0$

▹ Because the coefficient of x is zero, the plane is parallel to the x axis and perpendicular to the yz plane. The yz trace is the line in the yz plane determined by $y + 2z - 4 = 0$. This line intersects the y axis at $(0, 4, 0)$ and the z axis at $(0, 0, 2)$. The figure shows a sketch of the plane. By Theorem 10.4.3, a normal vector of the plane is $\mathbf{N} = \langle 0, 1, 2\rangle$. Because $\|\mathbf{N}\| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$, unit normal vectors are

$$\pm\frac{1}{\sqrt{5}}\langle 0, 1, 2\rangle = \pm\langle 0, \tfrac{1}{5}\sqrt{5}, \tfrac{2}{5}\sqrt{5}\rangle$$



13. A normal vector to the plane $3x + 2z - 6 = 0$ is $\langle 3, 0, 2\rangle$. $\|\langle 3, 0, 2\rangle\| = \sqrt{9+0+4} = \sqrt{13}$. Hence two unit vectors normal to the plane are $\langle \frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\rangle$ and $\langle -\frac{3}{\sqrt{13}}, 0, -\frac{2}{\sqrt{13}}\rangle$.

14. Unit normals to the plane $z = 5$ are $\langle 0, 0, 1\rangle$ and $\langle 0, 0, -1\rangle$.

In Exercises 15–20, find an equation of the plane satisfying the conditions.

15. Let $A = (2, 2, -4)$, $B = (7, -1, 3)$. Then $\mathbf{V}(\overrightarrow{AB}) = \langle 5, -3, 7\rangle$ is a normal vector to the plane. Because the plane contains the point $(-5, 1, 2)$, an equation of the plane is
$(x+5) - 3(y-1) + 7(z-2) = 0$; $5x - 3y + 7z + 14 = 0$

16. Parallel to the plane $4x - 2y + z - 1 = 0$ and containing the point $(2, 6, -1)$.

▹ Any plane parallel to the given plane is of the form
$4x - 2y + z + d = 0$
Substituting the coordinates of the given point, we find
$4(2) - 2(6) + (-1) + d = 0$
$d = 5$
Thus
$4x - 2y + z + 5 = 0$

17. A normal vector to the plane $x + 3y - z - 7 = 0$ is $\langle 1, 3, -1\rangle$. A normal vector $\langle a, b, c\rangle$ to the required plane will be orthogonal to $\langle 1, 3, -1\rangle$ so $\langle a, b, c\rangle \cdot \langle 1, 3, -1\rangle = 0$; $a + 3b - c = 0$. (1) Because $A(2, 0, 5)$ and $B(0, 2, -1)$ lie in the required plane, $\mathbf{V}(\overrightarrow{AB}) = \langle -2, 2, -6\rangle$ will be orthogonal to $\langle a, b, c\rangle$ so $\langle a, b, c\rangle \cdot \langle -2, 2, -6\rangle = 0$; $-2a + 2b - 6c = 0$ (2). Solving equations (1) and (2) simultaneously we get $a = -2c$ and $b = c$. An equation of the required plane is
$-2c(x-2) + c(y-0) + c(z-5) = 0$; $2x - y - z + 1 = 0$

18. Perpendicular to $x - y + z = 0$ and $2x + y - 4z - 5 = 0$ and containing $(4, 0, -2)$. Let $\mathbf{N} = \langle a, b, c\rangle$. $\langle 1, -1, 1\rangle \cdot \langle a, b, c\rangle = 0 \Rightarrow$ A: $a - b + c = 0$. $\langle 2, 1, -4\rangle \cdot \langle a, b, c\rangle = 0 \Rightarrow$ B: $2a + b - 4c = 0$. A + B: $3a - 3c = 0$, B − 2A: $3b - 6c = 0$. Choose $c = 1$ so $a = 1$, $b = 2$. $1(x-4) + 2(y-0) + 1(z+2) = 0$, $x + 2y + z - 2 = 0$

19. Because the required plane is perpendicular to the yz plane, a normal vector has 0 as its first component. Hence it has the form $\mathbf{N} = \langle 0, b, c\rangle$. Because the required plane makes an angle of measure $\cos^{-1}(\frac{2}{3})$ with the plane $2x - y + 2z - 3$ their normal vectors make the same angle. Therefore

$$\frac{\langle 0, b, c\rangle \cdot \langle 2, -1, 2\rangle}{\|\langle 0, b, c\rangle\| \|\langle 2, -1, 2\rangle\|} = \frac{2}{3}; \ -b + 2c = 2\sqrt{b^2+c^2}; \ b^2 - 4bc + 4c^2 = 4b^2 + 4c^2; \ 0 = 3b^2 + 4bc$$

from which we obtain $b = 0$ and $b = -\frac{4}{3}c$. The plane contains the point $(2, 1, 1)$.
When $b = 0$, $\mathbf{N} = \langle 0, 0, c\rangle$ and an equation is $0(x-2) + 0(y-1) + c(z-1) = 0$; $z = 1$.
When $b = -\frac{4}{3}c$, $\mathbf{N} = \langle 0, -\frac{4}{3}c, c\rangle$. An equation is $0(x-2) - \frac{4}{3}c(y-1) + c(z-1) = 0$; $4y - 3z - 1 = 0$.

20. Containing the point $P(-3,5,-2)$ and perpendicular to the vector $\mathbf{V}(\overrightarrow{OP})$.

▹ We apply Theorem 10.4.2 with the point $P(-3,5,-2)$ and the normal vector $\mathbf{N} = \langle -3,5,-2\rangle$. An equation of the plane is

$$-3(x+3)+5(y-5)-2(z+2)=0 \qquad 3x-5y+2z+38=0$$

In Exercises 21–23, find the acute angle θ between the two planes to the nearest tenth of a degree.

21. $2x-y-2z-5,\ 6x-2y+3z+8=0.\ \cos\theta = \dfrac{\langle 2,-1,-2\rangle\cdot\langle 6,-2,3\rangle}{\sqrt{4+1+4}\sqrt{36+4+9}} = \dfrac{12+2-6}{\sqrt{9}\sqrt{49}} = \dfrac{8}{3\cdot 7} = \dfrac{8}{21};\ \theta = 67.6°$

22. $2x-5y+3z-1=0,\ y-5z+3=0.\ \cos\theta = \dfrac{|\langle 2,-5,3\rangle\cdot\langle 0,1,-5\rangle|}{\sqrt{2^2+5^2+3^2}\sqrt{0^2+1^2+5^2}} = \dfrac{20}{\sqrt{38}\sqrt{26}};\ \theta = 50.5°$

23. $3x+4y=0$ and $4x-7y+4z-6=0.\ \cos\theta = \dfrac{\langle 3,4,0\rangle\cdot\langle 4,-7,4\rangle}{\sqrt{9+16+0}\sqrt{16+49+16}} = \dfrac{12-28+0}{\sqrt{25}\sqrt{81}} = \dfrac{16}{5\cdot 9} = \dfrac{16}{45};\ \theta = 69.2°$

24. Find the distance from the plane $2x+2y-z-6=0$ to the point $(2,2,-4)$.

▹ We use the formula of Exercise 60.

$$\text{distance} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2(2)+2(2)-(-4)-6|}{\sqrt{2^2+2^2+1^2}} = \frac{6}{3} = 2$$

25. A normal vector to the plane $5x+11y+2z-30=0$ is $\mathbf{N} = \langle 5,11,2\rangle$. Choose any point in the given plane, say $Q=(6,0,0)$. If $P=(-2,6,3)$ then $\mathbf{V}(\overrightarrow{PQ}) = \langle -8,6,3\rangle$. If d units is the distance from the plane to point P, it follows by Example 10.4.5 that

$$d = \frac{\mathbf{N}\cdot\mathbf{V}(\overrightarrow{QP})}{\|\mathbf{N}\|} = \frac{\langle 5,11,2\rangle\cdot\langle -8,6,3\rangle}{\|\langle 5,11,23\rangle\|} = \frac{-40+66+6}{\sqrt{25+121+4}} = \frac{32}{\sqrt{150}} = \frac{16}{15}\sqrt{6}$$

26. The point $(0,0,9)$ lies in $4x-8y-z+9=0$. Distance to $4x-8y-z-6=0$ is $\dfrac{|4(0)-8(0)-9-6|}{\sqrt{4^2+8^2+1^2}} = \dfrac{15}{9} = \dfrac{5}{3}$

27. Choose a point in the plane $8y-6z-27=0$, say $P=(0,0,-\frac{9}{2})$, and a point in the plane $4y-3z-6=0$, say $Q=(0,0,-2)$. $\mathbf{N}=\langle 0,4,-3\rangle$ is normal to the parallel planes. If d units is the distance between the planes then

$$d = \frac{\mathbf{N}\cdot\mathbf{V}(\text{QP})}{\|\mathbf{N}\|} = \frac{\langle 0,4,-3\rangle\cdot\langle 0,0,\frac{5}{2}\rangle}{\|\langle 0,4,-3\rangle\|} = \frac{15/2}{5} = \frac{3}{2}$$

28. Prove the intercept form of the equation of a plane.

▹ We are given that the x, y, and z intercepts of a plane are a, b, and c. Thus the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ lie on the plane. Substituting these points into the given equation $x/a+y/b+z/c=1$, we get the identities $1+0+0=1$, $0+1+0=1$ and $0+0+1=1$. Thus the given equation is that of the plane with the given intercepts.

In Exercises 29–36, find parametric and symmetric equations for the line L satisfying the conditions.

▹ $\mathbf{V}$ is a direction vector.

29. Let $P_1=(1,2,1)$ and $P_2=(5,-1,1)$. Then $\mathbf{V}=\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 4,-3,0\rangle$. Choosing $P_0=P_1$ we get parametric equations $x=1+4t$, $y=2-3t$, $z=1$. Symmetric equations of L are $\dfrac{x-1}{4} = \dfrac{y-2}{-3}$ and $z=1$.

30. $P(5,3,2)$, $\mathbf{V}=\langle 4,1,-1\rangle$. $x=5+4t$, $y=3+t$, $z=2-t$; $\dfrac{x-5}{4} = \dfrac{y-3}{1} = \dfrac{z-2}{-1}$

31. Through the origin and perpendicular to to the line $\frac{1}{4}(x-10) = \frac{1}{3}y = \frac{1}{2}z = t$ at their intersection. A set of parametric equations for the given line is $x=10+4t$, $y=3t$, $z=2t$. Let $P=(4t+10,3t,2t)$ on the given line be the point of intersection. Because L contains the origin and the two lines are perpendicular, it follows that the two vectors $\langle 4,3,2\rangle$ and $\mathbf{V}(\overrightarrow{OP}) = \langle 4t+10,3t,2t\rangle$ are orthogonal. Therefore $\langle 4,3,2\rangle\cdot\langle 4t+10,3t,2t\rangle = 0$; $16t+10+9t+4t=0$; $t=-\frac{40}{29}$; $P=(\frac{130}{29},-\frac{120}{29},-\frac{80}{29})$. Hence $\left[\frac{130}{29},-\frac{120}{29},-\frac{80}{29}\right]$ is a set of direction numbers of L, and so is the proportional set $[13,-12,-8]$. Therefore symmetric equations of L are $\dfrac{x}{13} = \dfrac{y}{-12} = \dfrac{z}{-8}$ and parametric equations are $x=13t$, $y=-12t$, $z=-8t$.

32. Through the origin and perpendicular to the lines having direction numbers $[4,2,1]$ and $[-3,-2,1]$.

▹ Let $[a,b,c]$ be a set of direction numbers of the required line L. Because line L is perpendicular to the line with direction numbers $[4,2,1]$, the vector $\langle a,b,c\rangle$ is orthogonal to the vector $\langle 4,2,1\rangle$. Thus

$$\langle a,b,c\rangle \cdot \langle 4,2,1\rangle = 0$$
$$4a+2b+c=0 \tag{1}$$

Because line L is perpendicular to the line with direction numbers $[-3,-2,1]$, we have

$$\langle a,b,c\rangle \cdot \langle -3,-2,1\rangle = 0$$
$$-3a-2b+c=0 \tag{2}$$

$$\begin{aligned} (1)+(2)\text{:} &\quad a+2c=0 \\ 3(1)+4(2)\text{:} &\quad -2b+7c=0 \end{aligned}$$

Thus, $a=-2c$, $b=\frac{7}{2}c$. We take $c=2$ to eliminate the fraction and obtain $[-4,7,2]$ as a set of direction numbers for L. Because L contains the origin, we use parametric equations (I) with $(x_0,y_0,z_0)=(0,0,0)$. Thus, parametric equations for L are

$x=-4t \quad y=7t \quad z=2t$

We use symmetric equations (II) to obtain

$$\frac{x}{-4}=\frac{y}{7}=\frac{z}{2}$$

33. Let $[a,b,c]$ be a set of direction numbers of L. Because L is perpendicular to the lines having direction numbers $[-5,1,2]$ and $[2,-3,-4]$, then

$\langle a,b,c\rangle \cdot \langle -5,1,2\rangle = 0$; $-5\mathrm{a}+b+2\mathrm{c}=0$; $b+2c=5\mathrm{a}$

$\langle a,b,c\rangle \cdot \langle 2,-3,-4\rangle = 0$; $2\mathrm{a}-3\mathrm{b}-4\mathrm{c}=0$; $3b+4c=2\mathrm{a}$

Solving for b and c, we obtain $b=-8a$, $c=\frac{13}{2}a$. Thus $\left[\mathrm{a},-8\mathrm{a},\frac{13}{2}\mathrm{a}\right]$ is a set of direction numbers of L, and so is the proportional set $[2,-16,13]$. Because L contains the point $(-2,0,3)$, parametric equations of L are $x=-2+2\mathrm{t}$, $y=-16\mathrm{t}$, $z=3+13\mathrm{t}$ and symmetric equations are $\frac{x+2}{2}=\frac{y}{-16}=\frac{z-3}{13}$.

34. $P(-3,1,-5)$, $\mathbf{V}=\langle 4,-2,1\rangle$. $x=-3+4t$, $y=1-2t$, $z=-5+t$; $\frac{x+3}{4}=\frac{y-1}{-2}=\frac{z+5}{1}$

35. A normal vector to the plane $x+3y-6z-8=0$ is $\langle 1,3,-6\rangle$. Hence a set of direction numbers of L is $[1,3,-6]$. Because $(4,-5,20)$ is on L, a set of parametric equations of L is $x=4+\mathrm{t}$, $y=-5+3\mathrm{t}$, $z=20-6\mathrm{t}$. Symmetric equations are $\frac{x-4}{1}=\frac{y+5}{3}=\frac{z-20}{-6}$.

36. Through the point $(2,0,-4)$ and parallel to each of the planes $2x+y-z=0$ and $x+3y+5z=0$.

▹ Let $\langle a,b,c\rangle$ be a direction vector of the line. Because the vector $\langle 2,1,-1\rangle$ is normal to the plane $2x+y-z=0$, and the vector $\langle a,b,c\rangle$ is parallel to the plane $2x+y-z=0$, then $\langle 2,1,-1\rangle$ is orthogonal to $\langle a,b,c\rangle$. Thus

$$\langle 2,1,-1\rangle \cdot \langle a,b,c\rangle = 0$$
$$2a+b-c=0 \tag{1}$$

Similarly, the vector $\langle 1,3,5\rangle$ is normal to the plane $x+3y+5z=0$, and thus $\langle 1,3,5\rangle$ is orthogonal to $\langle a,b,c\rangle$. Hence

$$\langle 1,3,5\rangle \cdot \langle a,b,c\rangle = 0$$
$$a+3b+5c=0 \tag{2}$$

$$\begin{aligned} 3(1)-2\text{:} &\quad 5a-8c=0 \\ (1)-2(2)\text{:} &\quad -5b-11c=0 \end{aligned}$$

If $c=5$, then $a=8$ and $b=-11$. Thus $\langle a,b,c\rangle=\langle 8,-11,5\rangle$ is a direction vector for the line. We use parametric equations (I) with the given point $(2,0,-4)$ and the direction vector $\langle 8,-11,5\rangle$ to obtain

$x=2t+8 \quad y=-11t \quad z=-4+5t$

which are parametric equations of the line. We use symmetric equations (II) to obtain

$$\frac{x-2}{8}=\frac{y}{-11}=\frac{z+4}{5}$$

37. We wish to find symmetric equations for the line $\begin{cases} 4x-3y+z-2=0 \\ 2x+5y-3z+4=0 \end{cases}$; $\begin{cases} 4x+z=3y+2 \\ 2x-3z=-5y-4 \end{cases}$. Solving simultaneously for x and z in terms of y, we get $x=\frac{2}{7}y+\frac{1}{7}$, $z=\frac{13}{7}y+\frac{10}{7}$. Hence symmetric equations are $\frac{x-\frac{1}{7}}{\frac{2}{7}}=y=\frac{z-\frac{10}{7}}{\frac{13}{7}}$; $\frac{x-\frac{1}{7}}{2}=\frac{y}{7}=\frac{z-\frac{10}{7}}{13}$; $\frac{x-1}{2}=\frac{y-3}{7}=\frac{z-7}{13}$, subtracting $\frac{3}{7}$ from each.

38. $\frac{x+1}{2}=\frac{y+4}{-5}=\frac{z-2}{3}=s,\ \frac{x-3}{-2}=\frac{y+14}{5}=\frac{z-8-3}{2}=t$

$x=2s-1=-2t+3,\ 2s+2t=4;\ y=-5s-4=5t-14,\ 5s+5t=10;\ z=3s+2=-3t+8,\ 3s+3t=6$

The lines coincide because replacing t with $2-s$ yields the same parametric equations.

39. The line $\frac{1}{2}(x-3)=\frac{1}{3}(y+2)=\frac{1}{4}(z+1)=t$ has parametric equations $x=3+2t,\ y=-2+3t,\ z=-1+4t$.
Substituting in the equation of the plane $x-2y+z=6$, we get the identity
$(3+2t)-2(-2+3t)+(-1+4t)\equiv 6$. Therefore the given line lies in the plane.

40. Prove that the line $x+1=-\frac{1}{2}(y-6)=z$ lies in the plane $3x+y-z=3$.

▸ We write the equations of the line in parametric form by setting $z=t$. Thus

$$x=t-1 \qquad y=-2t+6 \qquad z=t$$

Substituting into the equation of the plane, we have

$$3(t-1)+(-2t+6)-t=3 \qquad 3=3$$

Because the equation of the plane is satisfied for all points of the line, the line lies in the plane.

In Exercises 41–44, the planes through a line that are perpendicular to the coordinate planes are called the *projecting planes* of the line. Find equations of the projecting planes of the given line and sketch the line.

41. The given line has equations $\begin{cases}3x-2y+5z-30=0\\2x+3y-10z-6=0\end{cases}$.

Eliminating x between the two equations we get $13y-40z+42=0$.
Eliminating y between the two equations we get $13x-5z-102=0$.
Eliminating z between the two equations we get $8x-y-66=0$.
These are equations of the projecting planes.

42. The given line has equations $\begin{cases}\text{A: } x+y-3z+1=0\\\text{B: } 2x-y-3z+14=0\end{cases}$. The equations of the projecting planes are
$2\text{A}-\text{B}$: $3y-3z-12=0$; $y-z-4=0$. $\text{A}+\text{B}$: $3x-6z+15=0$, $x-2z+5=0$; $\text{A}-\text{B}$: $-x+2y-13=0$

43. Equations of the given line are $\begin{cases}x-2y-3z+6=0\\x+y+z-1=0\end{cases}$.

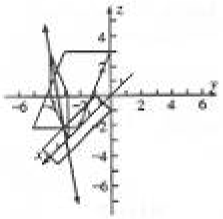
Eliminating x between the two equations we obtain $3y+4z-7=0$.
Eliminating y between the two equations we obtain $3x-z+4=0$.
Eliminating z between the two equations we obtain $4x+y+3=0$.
These are equations of the projecting planes.

44. A: $2x-y+z-7=0$
B: $4x-y+3z-13=0$

▸ Let L be the given line. If point P lies on L, (the coordinates of) P satisfies (1) and P satisfies (2). Thus, P satisfies a linear combination of (1) and (2). We form combinations of (1) and (2) to eliminate each of x, y, and z to get the projecting planes perpendicular to the yz-, xz- and xy-planes, respectively.

$\text{B}-2\text{A}$: $y+z+1=0$
$\text{B}-\text{A}$: $2x+2z-6=0$ or, equivalently, $x+z-3=0$
$3\text{A}-\text{B}$: $2x-2y-8=0$ or, equivalently $x-y-4=0$

The figure shows the line and the three projecting plane.



45. Choosing y as parameter, the lines $x=2y+4,\ z=-y+4$ and $x=y+7,\ 2z=y+2$ have parametric equations $x=2y+4,\ y=y,\ z=-y+4$ and $x=y+7,\ y=y,\ z=\frac{1}{2}y+1$. Hence vectors whose representations are parallel to these lines are $\langle 2,1,-1\rangle$ and $\langle 1,1,\frac{1}{2}\rangle$. The angle θ between the two vectors is the angle between the two lines. Therefore

$$\cos\theta=\frac{\langle 2,1,-1\rangle\cdot\langle 1,1,\frac{1}{2}\rangle}{\left\|\langle 2,1,-1\rangle\cdot\langle 1,1,\frac{1}{2}\rangle\right\|}=\frac{2+1-\frac{1}{2}}{\sqrt{4+1+1}\sqrt{1+1+\frac{1}{4}}}=\frac{\frac{5}{2}}{\sqrt{6}\cdot\frac{3}{2}}=\frac{5}{18}\sqrt{6}$$

46. We have $x=1+5t,\ y=-2+6t,\ z=3+7t$. With $t=0$, $t=1$, and the given $(6,2,4)$, we get

$$\begin{vmatrix}x&y&z&1\\1&-2&3&1\\6&4&10&1\\6&2&4&1\end{vmatrix}=-22x+30y-10z+112=0 \text{ or } 11x-15y+5z-56=0$$

In Exercises 47 and 48, find an equation of the plane containing the given intersecting lines.

47. $\frac{x-2}{4}=\frac{y+3}{-1}=\frac{z+2}{3}=t$ and $\begin{cases}3x+2y+z+2=0\\ x-y+2z-1=0\end{cases}$

Parametric equations of the first line are $x=4t+2$, $y=-t-3$, $z=3t-2$. $t=0$ gives the point $(2,-3,-2)$ which also belongs to the second line; $t=1$ gives the point $(6,-4,1)$ which does not. For any value of k, $(3x+2y+z+2)+k(x-y+2z-1)=0$ is a plane containing the second line. Because plane C contains the point $(6,-4,1)$ of the second line, we have $[3(6)+2(-4)+1+2]+k[6-(-4)+2(1)-1]=0$; $13-11k=0$; $k=-\frac{13}{11}$. Hence an equation of C is $(3x+2y+z+2)-\frac{13}{11}(x-y+2z-1)=0$; $4x+7y-3z+7=0$.

48. $\frac{x}{2}=\frac{y-2}{3}=\frac{z-1}{1}$ and $\frac{x}{1}=\frac{y-2}{-1}=\frac{z-1}{1}$

▹ Because the numerators are same in both lines, we see that $P(0,2,1)$ is on both lines. Let $\mathbf{N}=\langle a,b,c\rangle$ be a normal vector of the required plane. A direction vector of the first line is $\mathbf{A}=\langle 2,3,1\rangle$ and a direction vector of the second line is $\mathbf{B}=\langle 1,-1,1\rangle$. Because the plane contains the lines, $\mathbf{N}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$. Thus,

$$\begin{aligned}\langle 2,3,1\rangle\cdot\langle a,b,c\rangle&=0\\ 2a+3b+c&=0 \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\langle 1,-1,1\rangle\cdot\langle a,b,c\rangle&=0\\ a-b+c&=0 \qquad (2)\\ (1)+3(2):\quad 5a+4c&=0\\ (1)-2(2):\quad 5b-c&=0\end{aligned}$$

Choosing $c=5$, we obtain $\mathbf{N}=\langle -4,1,5\rangle$. Applying Theorem 10.4.3, with $\mathbf{N}$ and the point $P(0,2,1)$, we get an equation of the plane

$$\begin{aligned}-4x+(y-2)+5(z-1)&=0\\ 4x-y-5z&=0\end{aligned}$$

49. The given lines have equations $\begin{cases}3x-y-z=0\\ 8x-2y-3z+1=0\end{cases}$ and $\begin{cases}x-3y+z+3=0\\ 3x-y-z+5=0\end{cases}$. Solving the equations of each line for x and y in terms of z we get the symmetric equations $\frac{x+\frac12}{\frac12}=\frac{y+\frac32}{\frac12}=\frac{z-0}{1}$ and $\frac{x+\frac32}{\frac12}=\frac{y-\frac12}{\frac12}=\frac{z-0}{1}$.

Because $[\frac12,\frac12,1]$ is a set of direction numbers for each line, the lines are parallel. For any value of k, $(8x-2y-3z+1)+k(3x-y-z)=0$ is a plane containing the first line. Because the required plane must contain the point $(-\frac32,\frac12,0)$ of the second line, we must have

$[8(-\frac32)-2(\frac12)-3(0)+1]+k[3(-\frac32-\frac12-0]=0$; $-12-5k=0$; $k=-\frac{12}{5}$.

Hence an equation of the plane is $(8x-2y-3z+1)-\frac{12}{5}(3x-y-z)=0$; $4x+2y-3z+5=0$.

50. The lines $\frac{x+2}{5}=\frac{y-1}{-2}=\frac{z+4}{1}$ and $\frac{x-3}{-5}=\frac{y+4}{2}=\frac{z-3}{-1}$ have $[5,-2,1]$ for direction numbers and are parallel. Planes containing the first line are $x+2=5z+20$; $x-5z-18=0$ and $y-1=-2z-8$, $y+2z+7=0$ and hence $(x-5z-18)+k(y+2z+7)=0$. Because the required plane C must contain the point $(3,-4,3)$ of the second line, we have $[3-5(3)-18]+k[-4+2(3)+7]=0$; $-30+9k=0$; $k=\frac{10}{3}$. Hence an equation of C is $(x-5z-18)+\frac{10}{3}(y+2z+7)=0$; $3x+10y+5z+16=0$

51. We wish to find the intersection of the plane $5x-y+2z-12=0$ and the line $\frac{x-2}{4}=\frac{y+3}{-2}=\frac{z-1}{7}=t$;

$x=4t+2$, $y=-2t-3$, $z=7t+1$. Substituting into the equation of the plane we get $5(4t+2)-(-2t-3)+2(7t+1)-12=0$; $36t+3=0$; $t=-\frac{1}{12}$. The point of intersection is $(-\frac13+2,\frac16-3,-\frac{7}{12}+1)=(\frac53,-\frac{17}{6},\frac{5}{12})$.

52. Find equations of the line through the point $(1,-1,1)$ perpendicular to the line $3x=2y=z$, and parallel to the plane $x+y-z=0$.

▹ Let $\langle a,b,c\rangle$ be a direction vector of the required line. Because symmetric equations of the given line are

$$\frac{x}{\frac13}=\frac{y}{\frac12}=\frac{z}{1}$$

then the vector $\langle\frac13,\frac12,1\rangle$ is a direction vector for the given line. Because $\langle\frac13,\frac12,1\rangle$ is orthogonal to $\langle a,b,c\rangle$, then

$$\langle\tfrac13,\tfrac12,1\rangle\cdot\langle a,b,c\rangle=0$$

$$\tfrac{1}{3}a + \tfrac{1}{2}b + c = 0$$

$$2a + 3b + 6c = 0 \qquad (1)$$

Because the vector $\langle 1, 1, -1\rangle$ is normal to the plane $x + y - z = 0$, then $\langle 1, 1, -1\rangle$ is orthogonal to $\langle a, b, c\rangle$ and thus

$$\begin{aligned} \langle 1, 1, -1\rangle \cdot \langle a, b, c\rangle &= 0 \\ a + b - c &= 0 \qquad (2) \end{aligned}$$

$$\begin{aligned} (1) - 3(2){:}\quad & -a + 9c = 0 \\ (1) - 2(2){:}\quad & b + 8c = 0 \end{aligned}$$

Choosing $c = 1$, we find $\langle a, b, c\rangle = \langle 9, -8, 1\rangle$ is a direction vector of the required line. Parametric equations of the line are obtained from (1) with the given point $(1, -1, 1)$ and the direction vector $\langle 9, -8, 1\rangle$. Thus

$x = 1 + 9t \quad y = -1 - 8t \quad z = 1 + t$

53. Because the line L intersects the z axis it contains the point $(0, 0, z_0)$. Because the point $(3, 6, 4)$ is also on L a set of direction numbers of L is $[3, 6, 4 - z_0]$. A normal vector for the plane $x - 3y + 5z - 6 = 0$ is $\langle 1, -3, 5\rangle$. Because L is parallel to this plane, $\langle 3, 6, 4 - z_0\rangle \cdot \langle 1, -3, 5\rangle = 0$; $3 - 18 + 5(4 - z_0) = 0$; $z_0 = 1$. A set of direction numbers of L is then $[3, 6, 3]$, and another set is $[1, 2, 1]$. Because $(3, 6, 4)$ is on L, symmetric equations of L are $\frac{x-3}{1} = \frac{y-6}{2} = \frac{z-4}{1}$.

54. $Q(-2, 7, 4)$ is a point of the line. $\mathbf{V} = \langle 6, -2, 3\rangle$ is a vector in the line. $\|\overrightarrow{OQ}\| = c = \sqrt{2^2 + 7^2 + 4^2} = \sqrt{69}$.

$a = OQ_V = \dfrac{\overrightarrow{OQ} \cdot \mathbf{V}}{\|\mathbf{V}\|} = \dfrac{\langle -2, 7, 4\rangle \cdot \langle 6, -2, 3\rangle}{\sqrt{6^2 + 2^2 + 3^2}} = \dfrac{-14}{7} = -2$. $d = \sqrt{c^2 - a^2} = \sqrt{69 - 4} = \sqrt{65}$

55. $Q(7, 1, 0)$ is a point of the line $\frac{x-7}{2} = \frac{z}{1}$. $\mathbf{V} = \langle 2, 0, 1\rangle$ is a vector in the line. $P(-1, 3, -1)$ is the given point.

$c = \|\overrightarrow{PQ}\| = \|\langle 8, -2, 1\rangle\| = \sqrt{64 + 4 + 1} = \sqrt{69}$. $a = PQ_V = \dfrac{\overrightarrow{PQ} \cdot \mathbf{V}}{\|\mathbf{V}\|} = \dfrac{\langle 8, -2, 1\rangle \cdot \langle 2, 0, 1\rangle}{\sqrt{4 + 0 + 1}} = \dfrac{17}{\sqrt{5}}$

$d = \sqrt{c^2 - a^2} = \sqrt{69 - \frac{289}{5}} = \sqrt{\frac{56}{5}} = \frac{2}{5}\sqrt{70}$

56. Find equations of the line through the origin, perpendicular to the line $x = y - 5$, $z = 2y - 3$, and intersecting the line $y = 2x + 1$, $z = x + 2$.

▸ Let L be the required line and let L_1 be the line $y = 2x + 1$, $z = x + 2$ that intersects L. If $P(a, b, c)$ is the point of intersection, then the coordinates of P satisfy the equations of L_1. Thus, we let $x = a$, $y = b$, and $z = x$ in the equations of L_1, obtaining

$$b = 2a + 1 \qquad (1)$$

$$c = a + 2 \qquad (2)$$

Because line L contains the origin and point P, then the vector $\mathbf{V}(\overrightarrow{OP}) = \langle a, b, c\rangle$ is a direction vector of line L. Let L_2 be the line $x = y - 5$, $z = 2y - 3$. We find symmetric equations of L_2. Solving each equation for y, we have

$x + 5 = y$ and $y = \frac{1}{2}(z + 3)$

Thus, symmetric equations of L_2 are

$$\frac{x+5}{1} = \frac{y}{1} = \frac{z+3}{2}$$

Hence, a direction vector of line L_2 is $\langle 1, 1, 2\rangle$. Because line L is perpendicular to line L_2, any direction vector of L is orthogonal to a direction vector of L_2. Thus,

$$\begin{aligned} \langle 1, 1, 2\rangle \cdot \langle a, b, c\rangle &= 0 \\ a + b + 2c &= 0 \qquad (3) \end{aligned}$$

Substituting from (1) and (2) into (3) we obtain

$$a + (2a + 1) + 2(a + 2) = 0$$

$a = -1 \quad b = -1 \quad c = 1$

We use the direction vector $\langle -1, -1, 1\rangle$ and the origin to write the symmetric equations of L, which are

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$$

57. The given lines are $\frac{x-1}{5}=\frac{y-2}{-2}=\frac{z+1}{-3}=t$ and $\frac{x-2}{1}=\frac{y+1}{-3}=\frac{z+3}{2}=s$. The two sets of direction numbers $[5,-2,-3]$ and $[1,-3,2]$ are not proportional, and so the lines are not parallel. Parametric equations of the lines are L_1: $x=5t+1$, $y=-2t+2$, $z=-3t-1$ and L_2: $x=s+2$, $y=-3s-1$, $z=2s-3$
If (x,y,z) is a point of intersection, we must have $5t+1=s+2$, $-2t+2=-3s-1$, $-3t-1=2s-3$
The first two equations have the unique solution $t=0$ and $s=-1$. These values do not satisfy the third equation, so the two lines do not intersect. Thus they are skew.

58. With $t=0$ and 1 in L_1, and the given, we get the plane $\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & -1 & 1 \\ 6 & 0 & -4 & 1 \\ 3 & -4 & -5 & 1 \end{vmatrix} = -10x+14y-26z-44=0$ or
$5x-7y+13z+22=0$. Substituting from L_2, $5(s+2)-7(-3s-1)+13(2s-3)+22=0$, $52s=0$, $s=0$
and so the plane meets L_2 in $Q(2,-1,-3)$. $\mathbf{V}=\overrightarrow{PQ}=\langle -1,3,2\rangle$. The line is $\frac{x-3}{-1}=\frac{y+4}{5}=\frac{z+5}{2}$.

59. Choose a point in the plane $ax+by+cz+d_1=0$, say $P=(0,0,-d_1/c)$ and a point in plane $ax+by+cz+d_2=0$, say $Q=(0,0,d_2/c)$. $\mathbf{N}=\langle a,b,c\rangle$ is normal to the parallel planes. If d units is the distance between the planes then

$$d=\frac{\mathbf{N}\cdot\mathbf{V}(QP)}{\|\mathbf{N}\|}=\frac{\langle a,b,c\rangle\cdot\langle 0,0,(d_1-d_2)/c\rangle}{\|\langle a,b,c\rangle\|}=\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}$$

60. Prove that the undirected distance from the plane $ax+by+cz+d=0$ to the point $P_0(x_0,y_0,z_0)$ is given by

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

▸ If $P_1(x_1,y_1,z_1)$ is any point in the plane, then $ax_1+by_1+cz_1=-d$. The required distance is the absolute value of the scalar projection

$$|PQ_N|=\frac{|\mathbf{N}\cdot\overrightarrow{P_0P_1}|}{\|\mathbf{N}\|}=\frac{|\langle a,b,c\rangle\cdot\langle x_1-x_0,y_1-y_0,z_1-z_0\rangle|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_1+by_1+cz_1-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}}$$
$$=\frac{|-d-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

61. If the two direction numbers a and b are zero, then $x=x_0$ and $y=y_0$.

10.5 CROSS PRODUCT

10.5.1 Definition If $\mathbf{A}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, then the *cross product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\times\mathbf{B}$, is given by

$$\begin{aligned}\mathbf{A}\times\mathbf{B}&=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle\\&=\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\mathbf{k}\\&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}\end{aligned}$$

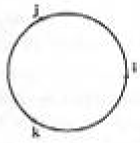
The operation of cross product for vectors is neither commutative nor associative, and depends on the orientation of the coordinate system. However, cross multiplication of vectors is distributive with respect to vector addition. Furthermore, scalar multiplication and cross multiplication are associative. We formally state these properties in the following theorems.

10.5.2 Theorem If $\mathbf{A}$ is any vector in V_3, then
(i) $\mathbf{A}\times\mathbf{A}=\mathbf{0}$
(ii) $\mathbf{0}\times\mathbf{A}=\mathbf{0}$
(iii) $\mathbf{A}\times\mathbf{0}=\mathbf{0}$

10.5.3 Theorem If $\mathbf{A}$ and $\mathbf{B}$ are any vectors in V_3,
$\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$

10.5.4 Theorem If **A**, **B**, and **C** are any vectors in V_3, then
$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$ and $(\mathbf{B}+\mathbf{C})\times\mathbf{A}=\mathbf{B}\times\mathbf{A}+\mathbf{C}\times\mathbf{A}$

10.5.5 Theorem If **A** and **B** are any two scalars in V_3 and c is a scalar, then
(i) $(c\mathbf{A})\times\mathbf{B}=\mathbf{A}\times(c\mathbf{B})$
(ii) $(c\mathbf{A})\times\mathbf{B}=c(\mathbf{A}\times\mathbf{B})$



$\mathbf{i}\times\mathbf{i}=\mathbf{0}$ $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
$\mathbf{j}\times\mathbf{i}=-\mathbf{k}$ $\mathbf{j}\times\mathbf{j}=\mathbf{0}$ $\mathbf{j}\times\mathbf{k}=\mathbf{i}$
$\mathbf{k}\times\mathbf{i}=\mathbf{j}$ $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ $\mathbf{k}\times\mathbf{k}=\mathbf{0}$

The cross product of each unit vector with itself is the vector **0**. The figure illustrates a way to remember the signs of the remaining six cross products of the unit vectors. The cross product of two consecutive vectors in the positive (counterclockwise) direction, is the next vector. The cross product of two vectors in the negative direction is the negative of the next vector.

10.5.6 Theorem (Scalar Triple Product) If **A**, **B**, and **C** are vectors in V_3, then
$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$
If $\mathbf{A}=\langle a_1,a_2,a_3\rangle$, $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, and $\mathbf{C}=\langle c_1,c_2,c_3\rangle$, then the scalar triple product is given by

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

10.5.7 Theorem (Vector Triple Product) If **A**, **B**, and **C** are vectors in V_3, then
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{C}\cdot\mathbf{A})\mathbf{B}-(\mathbf{C}\cdot\mathbf{B})\mathbf{A}$
Remember both as : (outer dot remote)adjacent − (outer dot adjacent)remote

10.5.8 Theorem If **A** and **B** are two vectors in V_3 and θ is the measure of the angle between **A** and **B**, then
$\|\mathbf{A}\times\mathbf{B}\|=\|\mathbf{A}\|\|\mathbf{B}\|\sin\theta$

10.5.9 Theorem If **A** and **B** are two vectors in V_3, **A** and **B** are parallel if and only if $\mathbf{A}\times\mathbf{B}=\mathbf{0}$.

10.5.10 Theorem If **A** and **B** are two vectors in V_3, then the vector $\mathbf{A}\times\mathbf{B}$ is orthogonal to both **A** and **B**. Thus,
$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{A}=0$ and $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{B}=0$

Torque of a force **F** applied at a point P with respect to O has magnitude $\|\overrightarrow{OP}\times\mathbf{F}\|$.

Area and Volume If representations of the vectors **A** and **B** are two adjacent sides of a parallelogram P or a triangle T then the measure of the area of P is $\|\mathbf{A}\times\mathbf{B}\|$ and the measure of the area of T is $\frac{1}{2}\|\mathbf{A}\times\mathbf{B}\|$. If representations of the vectors **A**, **B**, and **C** are three coterminal edges of a parallelopiped P or a tetrahedron T, then the measure of the volume of P is $|\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}|$ and the measure of the volume of T is $\frac{1}{6}|\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}|$.
Let L be a line with direction vector **L**, and let P be a point. If Q is any point on the line L, then the undirected distance between the point P and the line L is given by

$$d=\frac{\|\mathbf{L}\times\mathbf{V}(\overrightarrow{PQ})\|}{\|\mathbf{L}\|}$$

Exercises 10.5

In Exercises 1–12, let $\mathbf{A}=\langle 1,2,3\rangle$, $\mathbf{B}=\langle 4,-3,-1\rangle$, $\mathbf{C}=\langle -5,-3,5\rangle$, $\mathbf{D}=\langle -2,1,6\rangle$, $\mathbf{E}=\langle 4,0,-7\rangle$, and $\mathbf{F}=\langle 0,2,1\rangle$.

1. $\mathbf{A}\times\mathbf{B}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix}=[-2-(-9)]\mathbf{i}-(-1-12)\mathbf{j}+(-3-8)\mathbf{k}=\langle 7,13,-11\rangle$

2. $\mathbf{D}\times\mathbf{E}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 6 \\ 4 & 0 & -7 \end{vmatrix}=-7\mathbf{i}-(14-24)\mathbf{j}+(-4)\mathbf{k}=\langle -7,10,-4\rangle$

3. $(\mathbf{C}\times\mathbf{D})\cdot(\mathbf{E}\times\mathbf{F})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-5&-3&5\\-2&1&6\end{vmatrix}\cdot\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&0&-7\\0&2&1\end{vmatrix}$

$=[(-18-5)\mathbf{i}-(-30+10)\mathbf{j}+(-5-6)\mathbf{k}]\cdot(14\mathbf{i}-4\mathbf{j}+8\mathbf{k})=\langle-23,20,11\rangle\cdot\langle14,-4,8\rangle=-322-80-88=-490$

4. Find $(\mathbf{C}\times\mathbf{E})\cdot(\mathbf{D}\times\mathbf{F})$

▷ $$\mathbf{C}\times\mathbf{E}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-5&-3&5\\4&0&-7\end{vmatrix}=\begin{vmatrix}-3&5\\0&-7\end{vmatrix}\mathbf{i}-\begin{vmatrix}-5&5\\4&-7\end{vmatrix}\mathbf{j}+\begin{vmatrix}-5&-3\\4&0\end{vmatrix}\mathbf{k}$$

$$=[(-3)(-7)-0(5)]\mathbf{i}-[(-5)(-7)-4(5)]\mathbf{j}+[(-5)0-4(-3)]\mathbf{k}=21\mathbf{i}-15\mathbf{j}+12\mathbf{k}$$

$$\mathbf{D}\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&1&6\\0&2&1\end{vmatrix}=\begin{vmatrix}1&6\\2&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-2&6\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-2&1\\0&2\end{vmatrix}\mathbf{k}=-11\mathbf{i}+2\mathbf{j}-4\mathbf{k}$$

Thus,

$(\mathbf{C}\times\mathbf{E})\cdot(\mathbf{D}\times\mathbf{F})=(21\mathbf{i}-15\mathbf{j}+12\mathbf{k})\cdot(-11\mathbf{i}+2\mathbf{j}-4\mathbf{k})=21(-11)+(-15)2+12(-4)=-309$

5. $-(\mathbf{B}\times\mathbf{A})=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-3&-1\\1&2&3\end{vmatrix}=-\langle-7,-13,11\rangle=\langle7,13,-11\rangle=\mathbf{A}\times\mathbf{B}$, from the result of Exercise 1.

6. $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\langle1,2,3\rangle\times\langle-1,-6,4\rangle=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-1&-6&4\end{vmatrix}=26\mathbf{i}-7\mathbf{j}-4\mathbf{k}$

$\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\4&-3&-1\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-5&-3&5\end{vmatrix}=(7\mathbf{i}+13\mathbf{j}-11\mathbf{k})+(19\mathbf{i}-20\mathbf{j}+7\mathbf{k})=26\mathbf{i}-7\mathbf{j}-4\mathbf{k}$

7. If $c=3$, $(c\mathbf{A})\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&6&9\\4&-3&-1\end{vmatrix}=(-6+27)\mathbf{i}-(-3-36)\mathbf{j}+(-9-24)\mathbf{k}=\langle21,39,-33\rangle$

$\mathbf{A}\times(c\mathbf{B})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\12&-9&-3\end{vmatrix}=(-6+27)\mathbf{i}-(-3-36)\mathbf{j}+(-9-24)\mathbf{k}=\langle21,39,-33\rangle$

8. Verify Theorem 10.5.5(ii) for vectors **A** and **B** and $c=3$.

▷ $(c\mathbf{A})\times\mathbf{B}=3\langle1,2,3\rangle\times\langle4,-3,-1\rangle=\langle3,6,9\rangle\times\langle4,-3,-1\rangle$

$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&6&9\\4&-3&-1\end{vmatrix}=\begin{vmatrix}6&9\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&9\\4&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&6\\4&-3\end{vmatrix}\mathbf{k}=21\mathbf{i}+39\mathbf{j}-33\mathbf{k}$$

$c(\mathbf{A}\times\mathbf{B})=3(\langle1,2,3\rangle\times\langle4,-3,-1\rangle)$

$$=3\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\4&-3&-1\end{vmatrix}=3\left(\begin{vmatrix}2&3\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&3\\4&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\4&-3\end{vmatrix}\mathbf{k}\right)$$

$$=3(7\mathbf{i}+13\mathbf{j}-11\mathbf{k})=21\mathbf{i}+39\mathbf{j}-33\mathbf{k}=(c\mathbf{A})\times\mathbf{B}$$

9. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\langle1,2,3\rangle\cdot\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-3&-1\\-5&-3&5\end{vmatrix}=\langle1,2,3\rangle\cdot\langle-15-3,-(20-5),-12-15\rangle=\langle1,2,3\rangle\cdot\langle-18,-15,-27\rangle$

$=-18-30-81=-129$. From Exercise 1, $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=\langle7,13,-11\rangle\cdot\langle-5,-3,5\rangle=-36-39-55=-129$

10. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{A}\times\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-3&-1\\-5&-3&5\end{vmatrix}=\langle1,2,3\rangle\times\langle-18,-15,-27\rangle=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-18&-15&-27\end{vmatrix}=\langle-9,-27,21\rangle$

$(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}=(\langle1,2,3\rangle\cdot\langle-5,-3,5\rangle)\mathbf{B}-(\langle1,2,3\rangle\cdot\langle4,-3,-1\rangle)=4\langle4,-3,-1\rangle+5\langle-5,-3,5\rangle=\langle-9,-27,21\rangle$

11. $(\mathbf{A}+\mathbf{B})\times(\mathbf{C}-\mathbf{D}) = (\langle 1,2,3\rangle + \langle 4,-3,-1\rangle)\times(\langle -5,-3,5\rangle - \langle -2,1,6\rangle)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 2 \\ -3 & -4 & -1 \end{vmatrix} = \langle 1+8, -(-5+6), -20-3\rangle = \langle 9,-1,-23\rangle$$

$$(\mathbf{D}-\mathbf{C})\times(\mathbf{A}+\mathbf{B}) = (\langle -2,1,6\rangle - \langle -5,-3,5\rangle)\times\langle 5,-1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 5 & -1 & 2 \end{vmatrix} = \langle 8+1, -(6-5), -3-20\rangle = \langle 9,-1,-23\rangle$$

12. Find $\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{C}\times\mathbf{D}\|$

▹
$$\mathbf{A}\times\mathbf{B} = \langle 1,2,3\rangle\times\langle 4,-3,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix}\mathbf{k} = 7\mathbf{i}+13\mathbf{j}-11\mathbf{k}$$

$$\mathbf{C}\times\mathbf{D} = \langle -5,-3,5\rangle\times\langle -2,1,6\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & 5 \\ -2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 1 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 5 \\ -2 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & -3 \\ -2 & 1 \end{vmatrix}\mathbf{k} = -23\mathbf{i}+20\mathbf{j}-11\mathbf{k}$$

$$\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{C}\times\mathbf{D}\| = \sqrt{7^2+13^2+11^2}\sqrt{23^2+20^2+11^2} = \sqrt{339}\sqrt{1050} = 15\sqrt{1582}$$

13. Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$. Then
$\mathbf{0}\times\mathbf{A} = \langle 0,0,0\rangle\times\langle a_1,a_2,a_3\rangle = \langle 0\cdot a_3 - 0\cdot a_2, 0\cdot a_1 - 0\cdot a_3, 0\cdot a_2 - 0\cdot a_1\rangle = \langle 0,0,0\rangle = \mathbf{0}$
$\mathbf{A}\times\mathbf{0} = \langle a_1,a_2,a_3\rangle\times\langle 0,0,0\rangle = \langle a_2\cdot 0 - a_3\cdot 0, a_3\cdot 0 - a_1\cdot 0, a_1\cdot 0 - a_2\cdot 0\rangle = \langle 0,0,0\rangle = \mathbf{0}$

14. (a) $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{4}{9} & \frac{7}{9} & -\frac{4}{9} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{15}{27}\mathbf{i} + \frac{4}{27}\mathbf{j} + \frac{22}{27}\mathbf{k}$. $\sin(\mathbf{A},\mathbf{B}) = \|\mathbf{A}\times\mathbf{B}\| = \frac{1}{27}\sqrt{225+16+484} = \frac{1}{27}\sqrt{725} = \frac{5}{27}\sqrt{29}$

(b) $\cos(\mathbf{A},\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} = \frac{1}{27}[4(-2)+7(2)-4(1)] = \frac{2}{27}$. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(\frac{2}{27})^2} = \frac{5}{27}\sqrt{29}$

15. $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and $\mathbf{B} = \frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}$. Because **A** and **B** are unit vectors

(a) $\sin\theta = \dfrac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\|\mathbf{B}\|} = \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & \frac{5}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}\right\| = \left\|-\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{k}\right\| = \sqrt{\frac{4}{9}+\frac{4}{9}} = \frac{2}{3}\sqrt{2}|$

(b) $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \mathbf{A}\cdot\mathbf{B} = \frac{1}{9} - \frac{5}{9} + \frac{1}{9} = -\frac{1}{3}$. Because $0 \le \theta \le \pi$, $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \frac{2}{3}\sqrt{2}$

16. Show that the quadrilateral having vertices at $P(1,1,3)$, $Q(-2,1,-1)$, $R(-5,4,0)$, and $S(-8,4,-4)$ is a parallelogram and find its area.

▹ We use position representations.

$$\overrightarrow{PQ} = \mathbf{q}-\mathbf{p} = \langle -2,1,-1\rangle - \langle 1,1,3\rangle = \langle -3,0,4\rangle$$
$$\overrightarrow{PR} = \mathbf{p}-\mathbf{r} = \langle -5,4,0\rangle - \langle 1,1,3\rangle = \langle -6,3,-3\rangle$$
$$\overrightarrow{RS} = \mathbf{s}-\mathbf{r} = \langle -8,4,-4\rangle - \langle -5,4,0\rangle = \langle -3,0,-4\rangle$$

Because $\overrightarrow{PQ} = \overrightarrow{RS}$, then PQSR is a parallelogram. Moreover

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 12\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}$$

and

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \|12\mathbf{i}+15\mathbf{j}-9\mathbf{k}\| = \sqrt{144+225+81} = \sqrt{450} = 15\sqrt{2}$$

Therefore the area of parallelogram PQSR is $15\sqrt{2}$ square units.

17. Let $P = (1,-2,3)$, $Q = (4,3,-1)$, $R = (2,2,1)$, $S = (5,7,-3)$. Because $\mathbf{V}(\overrightarrow{PQ}) = \langle 3,5,4\rangle = \mathbf{V}(\overrightarrow{RS})$, PQRS is a parallelogram. The measure of the area of PQRS is

$$\left\|\mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR})\right\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -4 \\ 1 & 4 & -2 \end{matrix} \right\| = \|6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}\| = \sqrt{36+4+49} = \sqrt{89}$$

18. $$\text{area} = \left\|\overrightarrow{PQ} \times \overrightarrow{PS}\right\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 0 & 3 & 4 \end{matrix} \right\| = \|-8\mathbf{i} - 12\mathbf{j} + 9\mathbf{k}\| = \sqrt{64+144+81} = 17$$

19. Let $A = (0,2,2)$, $B = (8,8,-2)$, $C = (9,12,6)$. The measure of the area of triangle ABC is

$$\tfrac{1}{2}\left\|\mathbf{V}(\overrightarrow{AB}) \times \mathbf{V}(\overrightarrow{BC})\right\| = \tfrac{1}{2}\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -4 \\ 1 & 4 & -2 \end{matrix} \right\| = \tfrac{1}{2}\|\langle 64,-68,26\rangle\| = \|\langle 32,-34,13\rangle\| = \sqrt{1024+1156+169} = \sqrt{2349} = 9\sqrt{29}$$

20. Find the area of the triangle having vertices at $P(4,5,6)$, $Q(4,4,5)$, and $R(3,5,5)$.

▹ The area of triangle PQR is $\frac{1}{2}\left\|\overrightarrow{QP} \times \overrightarrow{QR}\right\|$

$$\overrightarrow{QP} \times \overrightarrow{QR} = (\mathbf{p} - \mathbf{q}) \times (\mathbf{r} - \mathbf{q}) = \langle 0,1,1\rangle \times \langle -1,1,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\left\|\overrightarrow{QP} \times \overrightarrow{QR}\right\| = \sqrt{1+1+1} = \sqrt{3}$$

Hence the area of triangle PQR is $\frac{1}{2}\sqrt{3}$ square units.

In Exercises 21 and 22, use the cross product to find an equation of the plane containing the three points.

21. Let $A = (-2,2,2)$, $B = (-8,1,6)$, $C = (3,4,-1)$. A normal vector for the plane of A, B, C is

$\mathbf{V}(\overrightarrow{AB}) \times \mathbf{V}(\overrightarrow{AC}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -1 & 4 \\ 5 & 2 & -3 \end{vmatrix} = \langle -5,2,-7\rangle$. Because point A lies in the plane, an equation is

$-5(x+2) + 2(y-2) - 7(z-2) = 0$; $5x - 2y + 7z = 0$

22. $A(2,3,0)$, $B(2,0,4)$, $C(0,3,4)$. $\mathbf{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 4 \\ -2 & 0 & 4 \end{vmatrix} = \langle -12,-8,-6\rangle$. $-\frac{1}{2}\mathbf{N} = \langle 6,4,3\rangle$

$6(x-2) + 4(y-3) + 3(z-0) = 0$; $6x + 4y + 3z - 24 = 0$

23. The given planes $x - y + z = 0$ and $2x + y - 4z - 5 = 0$ have normal vectors $\mathbf{A} = \langle 1,-1,1\rangle$ and $\mathbf{B} = \langle 2,1,-4\rangle$. The required plane C has normal vector orthogonal to these. Such a vector is

$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -4 \end{vmatrix} = \langle 3,6,3\rangle$. Because C contains the point $(4,0,-2)$, an equation is

$3(x-4) + 6(y-0) + 3(z+2) = 0$; $3x + 6y + 3z - 6 = 0$; $x + 2y + z - 2 = 0$

24. Find a unit vector perpendicular to the plane containing $\overrightarrow{PQ} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{PR} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

▹ Let

$$\mathbf{N} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -1 & -1 \end{vmatrix} = -5\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$$

By Theorem 10.5.10 the vector $\mathbf{N}$ is normal to the plane containing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Unit vectors are

$$\mathbf{U} = \pm\frac{\mathbf{N}}{\|\mathbf{N}\|} = \pm\frac{1}{\sqrt{25+9+49}}(5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}) = \pm\frac{1}{\sqrt{83}}(5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k})$$

In Exercises 25–27, find a unit vector perpendicular to the plane through the points P, Q, and R.

25. $P(5,2,-1)$, $Q(2,4,-2)$, $R(11,1,4)$. A normal vector to the plane of P, Q, and R is

$$\mathbf{N} = \mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 6 & -1 & 5 \end{vmatrix} = 9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k};\ \|\mathbf{N}\| = 9\|\mathbf{i} + \mathbf{j} - \mathbf{k}\| = 9\sqrt{1++1} = 9\sqrt{3}$$

Hence unit normal vectors are $\pm\frac{1}{9\sqrt{3}}(9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}) = \pm\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

26. $P(-2,1,0)$, $Q(2,-2,-1)$, $R(-5,0,2)$. $\mathbf{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ -3 & -1 & 2 \end{vmatrix} = \langle -7,-5,-13 \rangle$.

$\|\mathbf{N}\| = \sqrt{49+25+169} = \sqrt{243} = 9\sqrt{3}$. $\mathbf{U} = \pm\langle \frac{7}{27}\sqrt{3}, \frac{5}{27}\sqrt{3}, \frac{13}{27}\sqrt{3} \rangle$

27. $P = (1,4,2)$, $Q = (3,2,4)$, $R = (4,3,1)$. A normal vector to the plane of P, Q, and R is

$\mathbf{N} = \mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 2 \\ 3 & -1 & -1 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$; $\|\mathbf{N}\| = 4\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\| = 4\sqrt{1+4+1} = 4\sqrt{6}$

Hence unit normal vectors are $\pm\frac{1}{4\sqrt{6}}(4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}) = \pm\frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

28. Find the volume of the parallelopiped having edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ for points $P(1,3,4)$, $Q(3,5,3)$, $R(2,1,6)$ and $S(2,2,5)$.

▹
$$\begin{aligned} \mathbf{A} &= \overrightarrow{PQ} = \langle 3,5,3 \rangle - \langle 1,3,4 \rangle = \langle 2,2,-1 \rangle \\ \mathbf{B} &= \overrightarrow{PR} = \langle 2,1,6 \rangle - \langle 1,3,4 \rangle = \langle 1,-2,2 \rangle \\ \mathbf{C} &= \overrightarrow{PS} = \langle 2,2,5 \rangle - \langle 1,3,4 \rangle = \langle 1,-1,1 \rangle \end{aligned}$$

The number of cubic units is the volume of the parallelepiped is $|\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}|$.

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 1$$

Thus the volume of the parallelepiped is 1 cubic unit.

29. Let $\mathbf{A} = \mathbf{V}(\overrightarrow{PQ}) = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = \mathbf{V}(\overrightarrow{PR}) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. $\mathbf{C} = \mathbf{V}(\overrightarrow{PS}) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The measure of the volume of parallelepiped PQRS is

$$|\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 2 & 1 & -1 \end{vmatrix} \cdot \mathbf{C} \right| = |(-5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + \mathbf{k})| = |-5 - 10 - 5| = 20$$

30. The plane has equation $\begin{vmatrix} x & y & z & 1 \\ 2 & -1 & 3 & 1 \\ -1 & 1 & 2 & 1 \\ 5 & 1 & -1 & 1 \end{vmatrix} = -6x - 15y - 12z + 33 = 0$; $2x + 5y + 4z - 11 = 0$

In Exercises 31 and 32, find the perpendicular distance between the two given skew lines.

31. The given lines are $\frac{x-1}{5} = \frac{y-2}{3} = \frac{z+1}{2}$ and $\frac{x+2}{4} = \frac{y+1}{2} = \frac{z-3}{-3}$. A vector normal to them is

$\mathbf{N} = \langle 5,3,2 \rangle \times \langle 4,2,-3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 2 \\ 4 & 2 & -3 \end{vmatrix} = \langle -13,23,-2 \rangle$. $\|\mathbf{N}\| = \sqrt{169+529+4} = \sqrt{702} = 3\sqrt{78}$

$A = (1,2,-1)$ is a point of the first line and $B = (-2,-1,3)$ is a point of the second.
The measure of the distance between the lines is the scalar projection of $\mathbf{V}(\overrightarrow{AB})$ on $\mathbf{N}$

$$= \left| \frac{\mathbf{V}(\overrightarrow{AB}) \cdot \mathbf{N}}{\|\mathbf{N}\|} \right| = \left| \frac{\langle -3,-3,4 \rangle \cdot \langle -13,23,-2 \rangle}{3\sqrt{78}} \right| = \frac{38}{3\sqrt{78}}$$

32. $\frac{x+1}{2} = \frac{y+2}{-4} = \frac{z-1}{-3}$ and $\frac{x-1}{5} = \frac{y-1}{3} = \frac{z+1}{2}$

▹ $\mathbf{L}_1 = \langle 2,-4,-3 \rangle$ and $\mathbf{L}_2 = \langle 5,3,2 \rangle$ are direction vectors of the first and second lines. The vector

$$\mathbf{N} = \mathbf{L}_1 \times \mathbf{L}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 5 & 3 & 2 \end{vmatrix} = \langle 1,-19,26 \rangle$$

is perpendicular to each of the lines. The first line contains the point $P_1(-1,-2,1)$ and the second line contains the point $P_2(1,1,-1)$. The measure of the perpendicular distance between the lines is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ on $\mathbf{N}$. Because $\overrightarrow{P_1P_2} = \langle 2,3,-2 \rangle$, we take

$$\frac{|\mathbf{N} \cdot \overrightarrow{P_1P_2}|}{\|\mathbf{N}\|} = \frac{|\langle 1,-19,26 \rangle \cdot \langle 2,3,-2 \rangle|}{\sqrt{1^2 + 19^2 + 26^2}} = \frac{107}{\sqrt{1038}}$$

Thus, the perpendicular distance between the lines is $107/\sqrt{1038}$ units.

33. $25(8)\sin 70^\circ = 187.9$ in-lb **34.** $30(6)\sin 80^\circ = 177.3$ in-lb **35.** $\dfrac{\|\mathbf{A}\times\mathbf{B}\|}{\mathbf{A}\cdot\mathbf{B}} = \dfrac{\|\mathbf{A}\|\|\mathbf{B}\|\sin\theta}{\|\mathbf{A}\|\|\mathbf{B}\|\cos\theta} = \tan\theta$

36. If $\mathbf{A}$ and $\mathbf{B}$ are vectors in V_3, prove that $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$.

▸ Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$. Because a determinant is 0 if two rows are equal, then

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

37. By Th 10.5.4, $(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B}) = (\mathbf{A}-\mathbf{B})\times\mathbf{A} + (\mathbf{A}-\mathbf{B})\times\mathbf{B} = \mathbf{A}\times\mathbf{A} - \mathbf{B}\times\mathbf{A} + \mathbf{A}\times\mathbf{B} - \mathbf{B}\times\mathbf{B}$
$= \mathbf{0} + \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{B} + \mathbf{0} = 2(\mathbf{A}\times\mathbf{B})$

38. $\mathbf{A} = \mathbf{V}(\overrightarrow{OP})$, $\mathbf{B} = \mathbf{V}(\overrightarrow{OQ})$, and $\mathbf{C} = \mathbf{V}(\overrightarrow{OR})$. Then $\mathbf{V}(\overrightarrow{PQ}) = \mathbf{B} - \mathbf{A}$ and $\mathbf{V}(\overrightarrow{PR}) = \mathbf{C} - \mathbf{A}$. A normal vector to the plane of P, Q and R is $\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})$
$= (\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A}) = (\mathbf{B}-\mathbf{A})\times\mathbf{C} - (\mathbf{B}-\mathbf{A})\times\mathbf{A} = \mathbf{B}\times\mathbf{C} - \mathbf{A}\times\mathbf{C} - \mathbf{B}\times\mathbf{A} + \mathbf{A}\times\mathbf{A} = \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{A}\times\mathbf{B}$

In Exercises 39–42, let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$, $\mathbf{B} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{C} = \langle c_1, c_2, c_3\rangle$.

39. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \langle a_1, a_2, a_3\rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3\rangle$
$= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1)\rangle$
$= \langle a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 + a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 + a_2c_1\rangle$
$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1\rangle$
$= (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$ and
$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = -[\mathbf{A}\times(\mathbf{B}+\mathbf{C})] = -[\mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}] = -(\mathbf{A}\times\mathbf{B}) - (\mathbf{A}\times\mathbf{C}) = \mathbf{B}\times\mathbf{B} + \mathbf{C}\times\mathbf{A}$

40. Prove Theorem 10.5.5.

▸ Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$ be any two vectors in V_3 and let c be any scalar. We use the fact that multiplying the elements of a row by a number c multiplies the value of the determinant by c. Thus,

$$(c\mathbf{A})\times\mathbf{B} = \langle ca_1, ca_2, ca_3\rangle \times \langle b_1, b_2, b_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ca_1 & ca_2 & ca_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1}$$

From (1) we get immediately

$$(c\mathbf{A})\times\mathbf{B} = c(\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle) = c(\mathbf{A}\times\mathbf{B})$$

Applying the determinant rule to the third row of (1), we get

$$(c\mathbf{A})\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{vmatrix} = \langle a_1, a_2, a_3\rangle \times \langle cb_1, cb_2, cb_3\rangle = \mathbf{A}\times(c\mathbf{B})$$

41. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle \times \langle c_1, c_2, c_3\rangle = \langle a_1, a_2, a_3\rangle \cdot \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\rangle$
$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$
$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle \cdot \langle c_1, c_2, c_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \cdot \langle c_1, c_2, c_3\rangle$
$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$
$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$

42. Choose the axes so that $\mathbf{B} = b_1\mathbf{i}$, $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Then $\mathbf{B}\times\mathbf{C} = b_1c_2\mathbf{k}$ and
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times b_1c_2\mathbf{k} = -a_1b_1c_2\mathbf{j} + a_2b_1c_2\mathbf{i}$ while
$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (a_1c_1 + a_2c_2)b_1\mathbf{i} - (a_1b_1)(c_1\mathbf{i} + c_2\mathbf{j}) = a_2b_1c_2\mathbf{i} - a_1b_1c_2\mathbf{j}$, proving (i).

43. For a parallelpiped, volume = area of base × height and so height $= \dfrac{\text{volume}}{\text{area of base}} = \dfrac{|\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}|}{\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\|}$

44. $\mathbf{A} = \mathbf{V}(\overrightarrow{OP})$, $\mathbf{B} = \mathbf{V}(\overrightarrow{OQ})$, $\mathbf{C} = \mathbf{V}(\overrightarrow{OR})$. The area of triangle PQR is half the area of a parallelogram with PQ and PR as adjacent sides, that is $\frac{1}{2}\left\|\mathbf{V}(\overrightarrow{PR})\times\mathbf{V}(\overrightarrow{PQ})\right\| = \frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\|$.

45. $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -[\mathbf{C}\times(\mathbf{A}\times\mathbf{B})] = -[(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}] = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}$

46. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}] + [(\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}] + [(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}]$
$= (-\mathbf{B}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{B})\mathbf{A} + (\mathbf{A}\cdot\mathbf{C} - \mathbf{C}\cdot\mathbf{A})\mathbf{B} + (-\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A})\mathbf{C} = 0\mathbf{A} + 0\mathbf{B} + 0\mathbf{C} = \mathbf{0}$, proving Jacobi's identity.

47. From Ex. 47, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -[\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B})] = (\mathbf{A}\times\mathbf{B})\times\mathbf{C} - \mathbf{B}\times(\mathbf{C}\times\mathbf{A})$
The given identity is true if and only if the last term is the zero vector.

10.6 SURFACES

10.6.1 Definition A *cylinder* (*cone*) is a surface that is generated by a line moving along a given plane curve $\mathcal{C}$ in such a way that it always remains parallel to a fixed line (passes through a fixed point, the *vertex*) not lying in the plane of $\mathcal{C}$. The moving line is called a *generator* of the surface and $\mathcal{C}$ is called a *directrix*. Any position of a generator is a *ruling*. If $\mathcal{C}$ has a center C, the line through C parallel to the fixed line (passing through the vertex) is the *axis* of the surface.

10.6.2 Theorem In three-dimensional space, the graph of an equation in two of the three variables x, y, and z is a cylinder whose rulings are parallel to the axis associated with the missing variable and whose directrix is a curve in the plane associated with the two variables appearing in the equation. The graph of a homogeneous equation is a cone with vertex at the origin.

The intersection of a plane with a surface is called a *cross section* of the surface.

10.6.3 Definition If a plane curve is revolved about a fixed line lying in the plane of the curve, the surface generated is called a *surface of revolution*. The fixed line is called the *axis* of the surface of revolution, and the plane curve is called the *generating curve*.

The following table describes three types of surfaces of revolution.

Curve in the	revolved about	replace	by
xy plane	x axis	y^2	$y^2 + z^2$
	y axis	x^2	$x^2 + z^2$
xz plane	x axis	z^2	$y^2 + z^2$
	z axis	x^2	$x^2 + y^2$
yz plane	y axis	z^2	$x^2 + z^2$
	z axis	y^2	$x^2 + y^2$

Quadric Surfaces If $A \neq 0$, $B \neq 0$, and $C \neq 0$, the graph of the equation

$$Ax^2 + By^2 + Cz^2 = 1 \tag{I}$$

is symmetric with respect to each of the coordinate planes and is called a *central quadric*. If A, B, and C are all positive, the graph of Eq. (I) is an *ellipsoid*. If two of the numbers A, B, and C are positive and one is negative, the graph of (I) is an *elliptic hyperboloid of one sheet* with axis corresponding to the variable whose coefficient is negative. If one of the numbers A, B, and C is positive and two are negative, the graph of (I) is an *elliptic hyperboloid of two sheets* with axis corresponding to the variable whose coefficient is positive. If A, B, and C are all negative, the graph of (I) is the empty set.

If $A \neq 0$ and $B \neq 0$, the graphs of the equations

$$z = Ax^2 + By^2 \quad y = Ax^2 + Bz^2 \quad x = Ay^2 + Bz^2 \tag{II}$$

contain the origin and are symmetric with respect to two of the coordinate plane but are not symmetric with respect to the coordinate plane corresponding to the two squared variables. If A and B are either both positive or both negative, the graphs of Eqs. (II) are *elliptic paraboloids* with axis corresponding to the variable that is not squared. If A and B have opposite signs, the graphs of (II) are *hyperbolic paraboloids* with axis corresponding to the variable that is not squared.

If $A > 0$ and $B > 0$, the graphs of the homogeneous equations

$$z^2 = Ax^2 + By^2 \quad y^2 = Ax^2 + Bz^2 \quad x^2 = Ay^2 + Bz^2 \tag{III}$$

are *elliptic cones* with axis corresponding to the isolated variable. If $A = B$, we have a right-circular cone and sections perpendicular to the axis are circles. If $A \neq B$, there are two directions with circular sections.

The hyperboloid of one sheet and the hyperbolic paraboloid are doubly ruled surfaces. See Exercise 48. The volume of an ellipsoid of semiaxes a, b, and c is $\frac{4}{3}\pi abc$; thus, the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. See Exercise 50.

Exercises 10.6

In Exercises 1–4, sketch the cross section of the given cylinder in the indicated plane

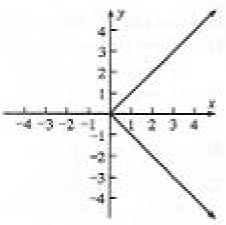
1. The cross section of the cylinder $4x^2 + y^2 = 16$ in the xy plane is the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$.
2. The cross section of the cylinder $4z^2 - y^2 = 4$ in the yz plane is the hyperbola $z^2 - y^2/4 = 1$.
3. The cross section of $z = e^x$ in the xz plane is the graph of the exponential function.

4. $x = |y|$; xy plane

▹ Because

$$x = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$$

the cross section of the cylinder in the xy plane consists of two half-lines. See the figure. The cylinder itself consists of two half-planes perpendicular to the xy plane.



In Exercises 5–12, sketch the cylinder having the given equation.

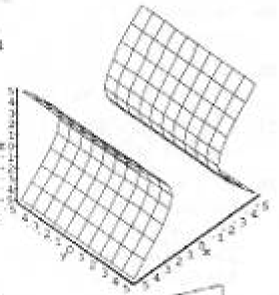
5. $4x^2 + 9y^2 = 36$ has rulings parallel to the z axis; the directrix in the xy plane is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

6. The cylinder $z = \sin y$ has rulings parallel to the x axis; the directrix in the yz plane is the sine curve.

7. $y = |z|$ has rulings parallel to the x axis; the directrix in the yz plane consists of the lines $y = z$ and $y = -z$.

8. $x^2 - z^2 = 4$

▹ The directrix in the xz plane is the hyperbola whose equation is $x^2 - z^2 = 4$. The rulings of the cylinder are parallel to the y axis. The figure shows a sketch of the cylinder.



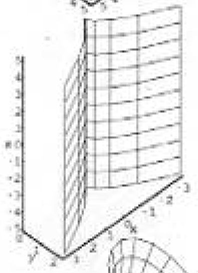
9. The cylinder $z = 2x^2$ has its rulings parallel to the y axis, and its directrix in the xz plane is the parabola $x^2 = \frac{1}{2}z$.

10. The cylinder $z^2 = 4y^2$ has its rulings parallel to the x axis, and its directrix in the yz plane consists of the lines $z = \pm 2y$.

11. The cylinder $y = \cosh x$ has its rulings parallel to the z axis, and the directrix in the xy plane is the graph of the hyperbolic cosine function.

12. $x^2 = y^3$

▹ The rulings are parallel to the z axis. The directrix is the curve $x^2 = y^3$ in the xy plane. The figure shows a sketch of the cylinder.



In Exercises 13–20, find an equation of the surface of revolution generated by revolving the given plane curve about the indicated axis. Sketch the surface.

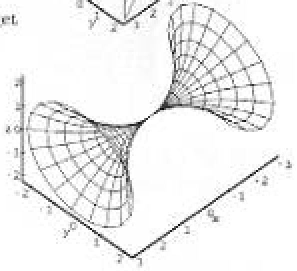
13. $x^2 = 4y$, about the y axis. We replace x^2 with $x^2 + z^2$: $x^2 + z^2 = 4y$

14. $x^2 + 4z^2 = 16$, about the z axis. We replace x^2 with $x^2 + y^2$: $x^2 + y^2 + 4z^2 = 16$

15. $x^2 + 4z^2 = 16$, about the x axis. We replace z^2 with $z^2 + y^2$: $x^2 + 4y^2 + 4z^2 = 16$

16. $x^2 = 4y$ in the xy plane, about the x axis.

▹ Because we wish to replace y^2 with $y^2 + z^2$, we first square the equation to get
$x^4 = 16y^2$
The equation of the surface of revolution is
$x^4 = 16(y^2 + z^2)$
The figure shows a sketch.



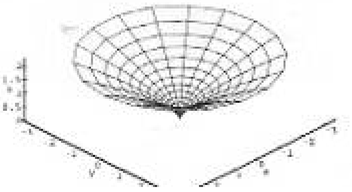
17. $y = 3z$, about the y axis. We square to get $y^2 = 9z^2$, then replace z with $z^2 + x^2$ to get $y^2 = 9x^2 + 9z^2$

18. $9y^2 - z^2 = 144$, about the z axis. Replace y^2 with $x^2 + y^2$ to get $9x^2 + 9y^2 - z^2 = 144$

19. $y = \sin x$, about the x axis. Square to get $y^2 = \sin^2 x$, then replace y^2 with $y^2 + z^2$ to get $y^2 + z^2 = \sin^2 x$

20. $y^2 = z^3$ in the yz plane, about the z axis.

▹ We replace y^2 by $x^2 + y^2$. Thus,
$x^2 + y^2 = z^3$
is an equation of the surface of revolution. The figure shows a sketch of the surface.



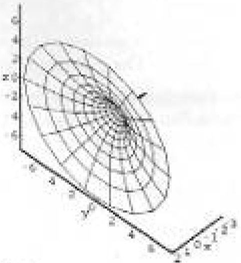
In Exercises 21–28, find a generating curve and the axis for the surface of revolution. Sketch the surface.

21. The sphere $x^2 + y^2 + z^2 = 16$ can be obtained as a surface of revolution in six ways:
by revolving $x^2 + y^2 = 16$ in the xy plane about the x axis or the y axis,
by revolving $x^2 + z^2 = 16$ in the xz plane about the x axis or the z axis, and
by revolving $y^2 + z^2 = 16$ in the yz plane about the y axis or the z axis.

22. $x^2 + z^2 = y$. Revolve $x^2 = y$ in xy plane or $z^2 = y$ in zy plane about y axis.

23. $x^2 + y^2 - z^2 = 4$
▹ Revolve $x^2 - z^2 = 4$ in xz plane or $y^2 - z^2 = 4$ in yz plane, about z axis

24. $y^2 + z^2 = e^{2x}$
▹ The axis of revolution is the x axis. We can start with either $y^2 = e^{2x}$ or $y = e^x$ or $y = -e^x$ in the xy plane; or $z^2 = e^{2x}$ or $z = e^x$ or $z = -e^x$ in the xz plane. The figure shows a sketch of the surface.

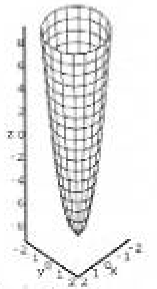


25. The equation $x^2 + z^2 = |y|$ has the y axis as its axis of revolution and as a generating curve either $x^2 = |y|$ or in the xy plane or $z^2 = |y|$ in the yz plane.

26. $4x^2 + 9y^2 + 4z^2 = 9$. The axis of revolution is the y axis. We can start with $4x^2 + 9y^2 = 9$ or $9y^2 + 4z^2 = 9$.

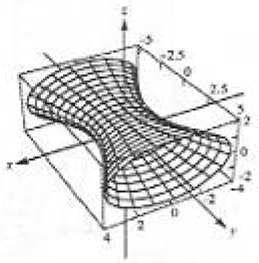
27. $x^2 + z^2 = \frac{1}{9}y^2$. The axis of revolution is the y axis. The generating curve is $x^2 = \frac{1}{9}y^2$ or $y = 3x$ or $y = -3x$ in the xy plane; or $z^2 = \frac{1}{9}y^2$ or $y = 3z$ or $y = -3z$ in the yz plane.

28. $4x^2 + 4y^2 - z = 9$
▹ We can write the equation as
$4(x^2 + y^2) - z = 9$
The axis of revolution is the z axis. We can start with $4x^2 - z = 9$ or $4y^2 - z = 9$.
The figure shows a sketch of the surface.

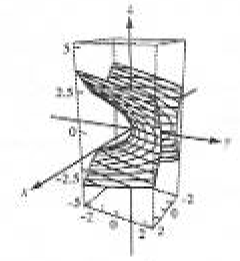


In Exercises 29 and 30, match the equation with one of the figures and name the surface.

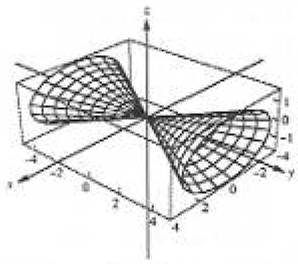
29. (a) $9x^2 - 4y^2 + 36z^2 = 36$
(v) hyperboloid of one sheet



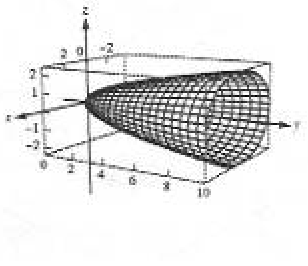
(b) $5x^2 - 2z^2 = 3y$
(iii) hyperbolic paraboloid



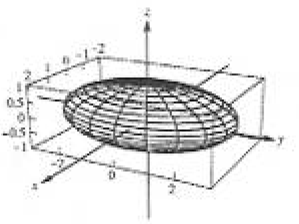
(c) $9x^2 - 4y^2 + 36z^2 = 0$
(vi) elliptic cone



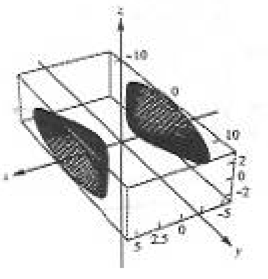
(d) $5x^2 + 2z^2 = 3y$
(i) elliptic paraboloid



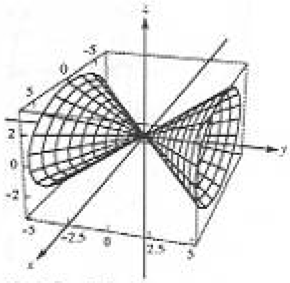
(e) $9x^2 + 4y^2 + 36z^2 = 36$
(ii) ellipsoid



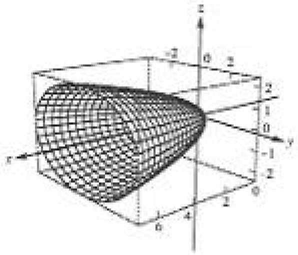
(f) $9x^2 - 4y^2 - 36z^2 = 36$
(iv) hyperboloid of two sheets



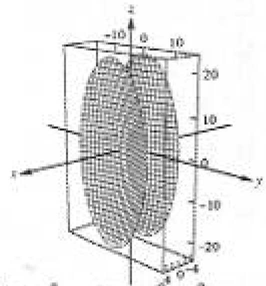
30. (a) $4x^2 - 16y^2 + 9z^2 = 0$
(iii) elliptic cone



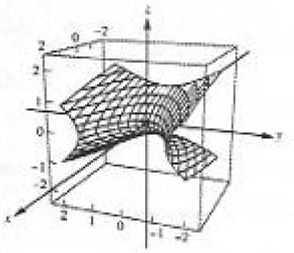
(b) $3y^2 + 7z^2 = 6x$
(v) elliptic paraboloid



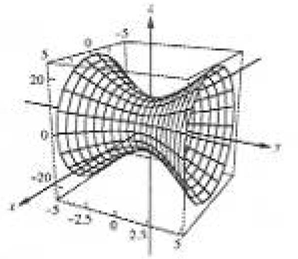
(c) $25x^2 = 4y^2 + z^2 + 100$
(i) hyperboloid of two sheets



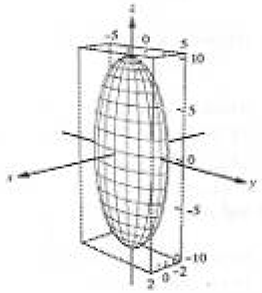
(d) $3y^2 - 7z^2 = 6x$
(vi) hyperbolic paraboloid



(e) $25x^2 = 4y^2 - z^2 + 100$
(iv) hyperboloid of one sheet



(f) $25x^2 = 100 - 4y^2 - z^2$
(ii) ellipsoid



In Exercises 31–42, sketch the graph of the equation and name the surface.

31. $4x^2 + 9y^2 + z^2 = 36$; $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} = 1$. The surface is an ellipsoid with semi-axes 3, 2, and 6.

32. $4x^2 - 9y^2 - z^2 = 36$

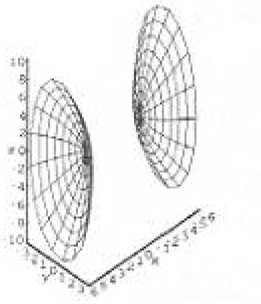
▷ Dividing on both sides of the given equation by 36, we obtain

$$\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{36} = 1$$

Comparing the equation with (I), we conclude the graph is an elliptic hyperboloid of two sheets with the x axis the transverse axis. Because $x = \pm 3$ when $(y, z) = (0, 0)$, the vertices of the hyperboloid are $(\pm 3, 0, 0)$. If $x = k$, $|k| > 3$, we have

$$\frac{y^2}{4} + \frac{z^2}{36} = \frac{k^2}{9} - 1; \quad \frac{y^2}{4(k^2/9 - 1)} + \frac{z^2}{36(k^2/9 - 1)} = 1$$

an ellipse of semiaxes $2\sqrt{k^2/9 - 1}$ and $6\sqrt{k^2/9 - 1}$. The figure shows the graph.

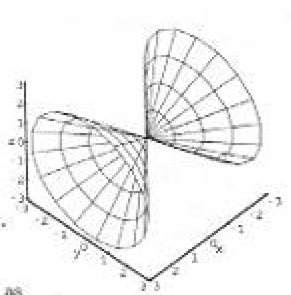


33. $4x^2 + 9y^2 - z^2 = 36$; $\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{36} = 1$

Comparing with (I), we conclude the surface is an elliptic hyperboloid of one sheet whose axis is the z axis.

34. $4x^2 - 9y^2 + z^2 = 36$. From (I), the surface is an elliptic hyperboloid of one sheet whose axis is the y axis.

35. $x^2 - y^2 + z^2 = 0$ is of type (III) with A = B = 1. The surface is a right-circular cone whose axis is the y axis.



36. $x^2 = y^2 + z^2$

▹ The equation is of type (III) with A = B = 1 and hence, its graph is a right-circular cone. Each cross section in the plane $x = k$ with $k \neq 0$ is the circle $y^2 + z^2 = k^2$. The figure shows the graph.

37. $\frac{x^2}{36} + \frac{z^2}{25} = 4y$ is of type (II), an elliptic paraboloid whose axis is the y axis.

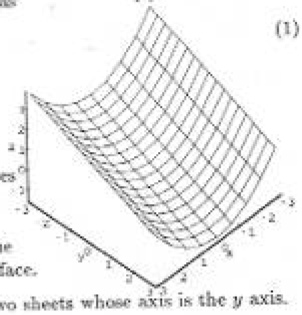
38. $\frac{y^2}{25} + \frac{x^2}{36} = 4z$ is of type (II), an elliptic paraboloid whose axis is the z axis.

39. $\frac{x^2}{36} - \frac{z^2}{25} = 9y$ is of type (II), a hyperbolic paraboloid whose axis is the y axis.

40. $x^2 = 2y + 4z$

▹ Because $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$, we may write the equation of the surface as

$$x^2 = 2\sqrt{5}\left(\frac{1}{\sqrt{5}}y + \frac{2}{\sqrt{5}}z\right) \tag{1}$$



If $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$, the substitutions

$$\bar{y} = \frac{1}{\sqrt{5}}y + \frac{2}{\sqrt{5}}z \qquad \bar{z} = -\frac{2}{\sqrt{5}}y + \frac{1}{\sqrt{5}}z$$

represent a rotation of the y and z axes in the yz plane of 63°. Eq (1) becomes

$$x^2 = 2\sqrt{5}\bar{y}$$

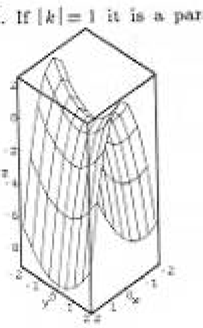
which is a parabolic cylinder. The directrix is the parabola $x^2 = 2\sqrt{5}\bar{y}$ in the $x\bar{y}$ plane and the rulings are parallel to the z axis. The figure shows the surface.

41. $x^2 + 16z^2 = 4y^2 - 1$; $\frac{y^2}{4} - \frac{x^2}{16} - \frac{z^2}{1} = 1$. Type (I), an elliptic hyperboloid of two sheets whose axis is the y axis.

42. $9y^2 - 4z^2 + 18x = 0$; $\frac{z^2}{9/2} - \frac{y^2}{2} = x$ is of type (II), a hyperbolic paraboloid whose axis is the x axis.

43. Substituting the equation $x = 1 - ky$ of the plane into the equation $y^2 - x^2 - z^2 = 1$ of the hyperboloid we obtain the projection of the intersection on the zy plane:
$y^2 - (1 - ky)^2 - z^2 = 1$; $(k^2 - 1)y^2 - 2ky + z^2 = -2$
(a) The intersection will be an ellipse if its projection is, that is, if $k^2 - 1 > 0$. Then an equivalent equation is

$(k^2 - 1)\left[y^2 - \frac{2k}{k^2 - 1}y + \frac{k^2}{(k^2 - 1)^2}\right] + z^2 = \frac{k^2}{k^2 - 1} - 2$. This graph will have real points if $\frac{k^2}{k^2 - 1} - 2 > 0$;
$k^2 > 2k^2 - 2$; $k^2(2$; $|k| < \sqrt{2}$. Hence the intersection is an ellipse if $1 < |k| < \sqrt{2}$. If $|k| = 1$ it is a parabola; if $|k| = \sqrt{2}$, it is a single point. If $|k| > \sqrt{2}$, the intersection is empty.
(b) The intersection is a hyperbola if $k^2 - 1 < 0$; $k^2 < 1$; $|k| < 1$.



44. Find the vertex and the focus of the parabola that is the intersection of the plane $y = 2$ with the hyperbolic paraboloid $y^2/16 - x^2/4 = z/9$.

▹ We let $y = 2$ in the given equation and write the resulting equation in standard form. Thus, we have

$$\frac{1}{4} - \frac{x^2}{4} = \frac{z}{9}; \qquad x^2 = 4(-\tfrac{1}{9})(z - \tfrac{9}{4})$$

The cross section of the given surface in the plane $y = 2$ is a parabola with vertex $(0, 2, \frac{9}{4})$. The parabola has axis parallel to the z axis and opens downward. Because $p = -\frac{1}{9}$, then the focus of the parabola is $(0, 2, \frac{9}{4} - \frac{1}{9}) = (0, 2, \frac{77}{36})$. The figure shows the given hyperbolic paraboloid. The right face is the cross section in the plane $y = 2$.

45. Substituting the equation of the plane $x = 1$ into the equation of the hyperbolic paraboloid $\frac{z^2}{4} - \frac{x^2}{9} = \frac{y}{3}$, we get $\frac{z^2}{4} - \frac{1}{9} = \frac{y}{3}$; $z^2 = 4\left(\frac{1}{3}\right)\left(y + \frac{1}{3}\right)$. Because $x = 1$ is parallel to the the projection, we find the vertex is at $(1, -\frac{1}{3}, 0)$. The axis is parallel to the y axis and the parabola opens in the direction of the positive side of the y axis. $p = \frac{1}{3}$ so the focus is at $(1, 0, 0)$.

46. $y = 3$ meets $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$ in $\frac{x^2}{9} + \frac{9}{25} + \frac{z^2}{4} = 1$, $\frac{x^2}{9} + \frac{z^2}{4} = \frac{16}{25}$, $\frac{x^2}{\frac{144}{25}} + \frac{z^2}{\frac{64}{25}} = 1$. Area $= \pi(\frac{12}{5})\frac{8}{5} = \frac{96}{25}\pi$

47. Any point on $x^2 - 4y^2 - 9z^2 = 36$; $4x^2 + 4y^2 + 4z^2 = 5x^2 - 5z^2 - 36 = 5(x+z)(x-z) - 36$ and a plane of the form $x + z = k$ must lie on the sphere $4x^2 + 4y^2 + 4z^2 = 5k(x - y)$, and the intersection of a plane and a sphere is a circle. Similarly for planes $x - z = k$.

48. Show that each point of (a) the hyperbolic paraboloid $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$ and (b) the elliptic hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is the intersection of two lines lying entirely in the surface.

▸ (a) We factor the given equation.

$$\left(\frac{y}{b} - \frac{x}{a}\right)\left(\frac{y}{b} + \frac{x}{a}\right) = \frac{z}{c}$$

For any value of k, each point of the line of intersection of the planes

$$\frac{y}{b} - \frac{x}{a} = k\frac{z}{c} \quad \text{and} \quad \frac{y}{b} + \frac{x}{a} = \frac{1}{k} \tag{1}$$

satisfies the equation of the hyperbolic paraboloid, and so the line lies entirely in it. Conversely, each point of the paraboloid determines a value of k. Similarly, for any k, each point of the line of intersection of the planes

$$\frac{y}{b} + \frac{x}{a} = m\frac{z}{c} \quad \text{and} \quad \frac{y}{b} - \frac{x}{a} = \frac{1}{m} \tag{2}$$

lies in the hyperbolic paraboloid.

(b) We arrange the given equation so that each side is the difference of two squares and then factor. Thus,

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2};\ \left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right)$$

For any value of k, each point of the line of intersection of the planes

$$\frac{x}{a} - \frac{z}{c} = k\left(1 - \frac{y}{b}\right) \text{ and } \frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 + \frac{y}{b}\right)$$

satisfies the equation of the hyperboloid of one sheet, and so the line lies entirely in it. Conversely, each point of the hyperboloid determines a value of k. Similarly, for any value of m, each point of the line of intersection of the planes

$$\frac{x}{a} + \frac{z}{c} = m\left(1 - \frac{y}{b}\right) \text{ and } \frac{x}{a} - \frac{z}{c} = \frac{1}{m}\left(1 + \frac{y}{b}\right)$$

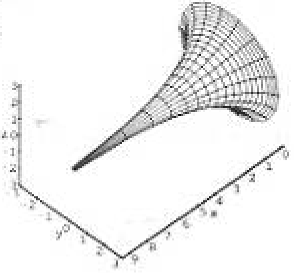
lies in the paraboloid.

49. The ellipsoid is $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$; $\frac{y^2}{4} + \frac{z^2}{9} = 1 - x^2$. A plane section of the ellipsoid at $x = w_i$, $0 \le w_i \le 1$, is a region enclosed by an ellipse whose semiaxes have lengths $2\sqrt{1 - w_i^2}$ and $3\sqrt{1 - w_i^2}$. Therefore if V cubic units is the volume of the ellipsoid,

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(2\sqrt{1 - w_i^2})(3\sqrt{1 - w_i^2})\Delta_i x = 12\pi \int_0^1 (1 - x^2)\,dx = 12\pi\left[1 - \frac{1}{2}y^3\right]_0^1 = 12\pi\left(1 - \frac{1}{2}\right) = 8\pi$$

50. Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$. A z-slice is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2} \Rightarrow \frac{x^2}{a^2(1 - z^2/c^2)} + \frac{y^2}{b^2(1 - z^2/c^2)} = 1$, ellipse of semiaxes $a' = a\sqrt{1 - z^2/c^2}$ and $b' = b\sqrt{1 - z^2/c^2}$ and area $A = \pi a'b' = \pi ab(1 - z^2/c^2)$.

$$V = 2\pi ab \int_0^c \left(1 - \frac{z^2}{c^2}\right)dz = 2\pi ab\left[z - \frac{z^3}{3c^2}\right]_0^c = \frac{4}{3}\pi abc$$



51. The elliptic paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ $(c > 0)$. A plane section of the surface at $z = w_i$, $0 \le w_i \le h$, is an ellipse having semiaxes of lengths $a\sqrt{\frac{w_i}{c}}$ and $b\sqrt{\frac{w_i}{c}}$.

If V cubic units is the volume of the solid,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(a\sqrt{\frac{w_i}{c}}\right)\left(b\sqrt{\frac{w_i}{c}}\right)\Delta_i z = \frac{\pi ab}{c}\int_0^h z\,dz = \frac{\pi ab}{c}\left[\frac{1}{2}\right]_0^h = \frac{abh^2}{c}\pi.$$

52. Sketch the surface of revolution generated by revolving the tractrix

$$x = 3\ln\left(\frac{3 + \sqrt{9 - y^2}}{y}\right) - \sqrt{9 - y^2} \text{ about the } x \text{ axis.}$$

▸ When $y = 3$, $x = 0$ and as $y \to 0^+$, $x \to +\infty$. The figure shows the surface, called a pseudosphere, which has application in noneuclidean geometry.

Miscellaneous Exercises for Chapter 10

In Exercises 1–18, $\mathbf{A} = 4\mathbf{i} - 6\mathbf{j}$, $\mathbf{B} = \mathbf{i} + 7\mathbf{j}$, and $\mathbf{C} = 9\mathbf{i} - 5\mathbf{j}$

1. $3\mathbf{B} - 7\mathbf{A} = 3(\mathbf{i} + 7\mathbf{j}) - 7(4\mathbf{i} - 6\mathbf{j}) = 3\mathbf{i} + 21\mathbf{j} - 28\mathbf{i} + 42\mathbf{j} = -25\mathbf{i} + 63\mathbf{j}$

2. $5\mathbf{B} - 3\mathbf{C} = 5(\mathbf{i} + 7\mathbf{j}) - 3(9\mathbf{i} - 5\mathbf{j}) = 5\mathbf{i} + 35\mathbf{j} - 27\mathbf{i} + 15\mathbf{j} = -22\mathbf{i} + 50\mathbf{j}$

3. $\|3\mathbf{B} - 7\mathbf{A}\| = \|-25\mathbf{i} + 63\mathbf{j}\| = \sqrt{(-25)^2 + (63)^2} = \sqrt{625 + 3969} = \sqrt{4594}$

4. Find $\|5\mathbf{B} - 3\mathbf{C}\|$

▹ $$5\mathbf{B} - 3\mathbf{C} = 5(\mathbf{i} + 7\mathbf{j}) - 3(9\mathbf{i} - 5\mathbf{j}) = 5\mathbf{i} + 35\mathbf{j} - 27\mathbf{i} + 15\mathbf{j} = -22\mathbf{i} + 50\mathbf{j} = 2(-11\mathbf{i} + 25\mathbf{j})$$

Then

$$\|5\mathbf{B} - 3\mathbf{C}\| = 2\sqrt{(-11)^2 + 25^2} = 2\sqrt{746}$$

5. $\|3\mathbf{B}\| - \|7\mathbf{A}\| = 3\|\mathbf{B}\| - 7\|\mathbf{A}\| = 3\sqrt{1 + 49} - 7\sqrt{16 + 36} = 3\sqrt{50} - 7\sqrt{52} = 15\sqrt{2} - 14\sqrt{13}$

6. $\|5\mathbf{B}\| - \|3\mathbf{C}\| = 5\|\mathbf{B}\| - 3\|\mathbf{C}\| = 5\sqrt{1 + 49} - 3\sqrt{81 + 25} = 5\sqrt{50} - 3\sqrt{106} = 25\sqrt{2} - 3\sqrt{106}$

7. $(\mathbf{A} - \mathbf{B}) \cdot \mathbf{C} = [(4\mathbf{i} - 6\mathbf{j}) - (\mathbf{i} + 7\mathbf{j})] \cdot (9\mathbf{i} - 5\mathbf{j}) = (3\mathbf{i} - 3\mathbf{j}) \cdot (9\mathbf{i} - 5\mathbf{j}) = 27 + 65 = 92$

8. Find $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$.

▹ $$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = [(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})]\mathbf{C} = [4(1) + (-6)7]\mathbf{C} = -38(9\mathbf{i} - 5\mathbf{j}) = -342\mathbf{i} + 190\mathbf{j}$$

9. $2\mathbf{A} + \mathbf{B} = 9\mathbf{i} - 5\mathbf{j}$. $\|2\mathbf{A} + \mathbf{B}\| = \sqrt{9^2 + (-5)^2} = \sqrt{106}$. The required unit vector is $\frac{9}{\sqrt{106}}\mathbf{i} - \frac{5}{\sqrt{106}}\mathbf{j}$.

10. Swap components, change a sign. $\mathbf{N} = 7\mathbf{i} - \mathbf{j}$. $\|\mathbf{N}\| = \sqrt{49 + 1} = \sqrt{50}$. $\mathbf{U} = \pm\frac{1}{\sqrt{50}}(7\mathbf{i} - \mathbf{j}) = \pm(\frac{7}{10}\sqrt{2}\mathbf{i} - \frac{1}{10}\sqrt{2}\mathbf{j})$

11. $\mathbf{A} = h\mathbf{B} + k\mathbf{C}$. $4\mathbf{i} - 6\mathbf{j} = h(\mathbf{i} + 7\mathbf{j}) + k(9\mathbf{i} - 5\mathbf{j}) = (h + 9k)\mathbf{i} + (7h - 5k)\mathbf{j}$; $h + 9k = 4$ and $7h - 5k = -6$. Thus, $h = -\frac{1}{2}$ and $k = \frac{1}{2}$.

12. Find scalars h and k such that $h\mathbf{A} + k\mathbf{B} = -\mathbf{C}$.

▹ Substituting the given vectors **A**, **B**, and **C** into the equation $h\mathbf{A} + k\mathbf{B} = -\mathbf{C}$, we have

$$h(4\mathbf{i} - 6\mathbf{j}) + k(\mathbf{i} + 7\mathbf{j}) = -(9\mathbf{i} - 5\mathbf{j})$$
$$(4h + k)\mathbf{i} + (-6h + 7k)\mathbf{i} = -9\mathbf{i} + 5\mathbf{j}$$

Therefore

$$4h + k = -9 \quad (1)$$
$$-6h + 7k = 5 \quad (2)$$

Solving for h and k, we obtain

$7(1) - (2)$: $34h = -68$, $h = -2$

$3(1) + 2(2)$: $17h = -17$, $k = -1$

13. The scalar projection of **A** onto **B** is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{\sqrt{1 + 49}} = \frac{-38}{\sqrt{50}} = -\frac{19}{5}\sqrt{2}$.

14. $C_A = \frac{\mathbf{C} \cdot \mathbf{A}}{\|\mathbf{A}\|} = \frac{(9\mathbf{i} - 5\mathbf{j}) \cdot (4\mathbf{i} - 6\mathbf{j})}{\|4\mathbf{i} - 6\mathbf{j}\|} = \frac{66}{\sqrt{16 + 36}} = \frac{66}{\sqrt{52}} = \frac{33}{\sqrt{13}} = \frac{33}{13}\sqrt{13}$

15. The vector projection of **A** onto **B** is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{1 + 49}(\mathbf{i} + 7\mathbf{j}) = \frac{-38}{50}(\mathbf{i} + 7\mathbf{j}) = -\frac{19}{25}\mathbf{i} - \frac{133}{25}\mathbf{j}$

16. Find the vector projection of **C** onto **A**.

▹ The the vector projection of **C** onto **A** is given by

$$\mathbf{C}_A = \frac{\mathbf{C} \cdot \mathbf{A}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{(9\mathbf{i} - 5\mathbf{j}) \cdot (4\mathbf{i} - 6\mathbf{j})}{4^2 + 6^2}\mathbf{A} = \frac{66}{52}(4\mathbf{i} - 6\mathbf{j}) = \frac{66}{13}\mathbf{i} - \frac{99}{13}\mathbf{j}$$

17. The component of **B** in the direction of **A** is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{\sqrt{4^2 + 6^2}} = \frac{-38}{\sqrt{52}} = -\frac{19}{13}\sqrt{13}$.

18. $\cos(\mathbf{A}, \mathbf{C}) = \frac{\mathbf{A} \cdot \mathbf{C}}{\|\mathbf{A}\|\|\mathbf{C}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (9\mathbf{i} - 5\mathbf{j})}{\sqrt{4^2 + 6^2}\sqrt{9^2 + 5^2}} = \frac{66}{\sqrt{52}\sqrt{106}} = \frac{33}{\sqrt{1378}} = \frac{33}{1378}\sqrt{1378}$

19–20. Let $\mathbf{A} = -2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{B} = h\mathbf{i} - 2\mathbf{j}$ and let θ be the angle between **A** and **B**. Determine h so that $\theta = \frac{2}{3}\pi$ and show that no h exists such that $\theta = \frac{1}{3}\pi$.

▹ We have
$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \frac{-2h-10}{\sqrt{29}\sqrt{h^2+4}}$$
Furthermore,
$$(\cos \tfrac{2}{3}\pi)^2 = (-\tfrac{1}{2})^2 = \tfrac{1}{4} \text{ and } (\cos \tfrac{1}{3}\pi)^2 = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$
Thus, in either case,
$$\cos^2\theta = \frac{1}{4}$$
$$\frac{4h^2-40h+100}{29(h^2+4)} = \frac{1}{4}$$
$$4(4h^2-40h+100) = 29(h^2+4)$$
$$13h^2-160h-284 = 0$$
$$h = \tfrac{1}{13}(80 \pm 58\sqrt{3}) \approx 13.88 \text{ or } -1.57$$
Because both values of h make $\cos\theta$ negative, both are solutions for $\theta = \frac{2}{3}\pi$ and $\theta = \frac{1}{3}\pi$ has no solution.

In Exercises 21–30 $\mathbf{A} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{C} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{D} = 3\mathbf{j} - \mathbf{k}$, $\mathbf{E} = 5\mathbf{i} - 2\mathbf{j}$. Find the vector or scalar.

21. $6\mathbf{C} + 4\mathbf{D} - \mathbf{E} = 6(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + 4(3\mathbf{j} - \mathbf{k}) - 5\mathbf{i} - 2\mathbf{j} = 6\mathbf{i} + 2\mathbf{j} - 12\mathbf{k} + 12\mathbf{j} - 4\mathbf{k} - 5\mathbf{i} + 2\mathbf{j} = \mathbf{i} + 26\mathbf{j} - 16\mathbf{k}$

22. $3\mathbf{A} - 2\mathbf{B} + \mathbf{C} = 3(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) - 2(2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k} - 4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
$= -6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}$

23. $\mathbf{D}\cdot\mathbf{B}\times\mathbf{C} = (3\mathbf{j} - \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})\times(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = (3\mathbf{j} - \mathbf{k})\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -4 \\ 1 & 2 & -2 \end{vmatrix} = (3\mathbf{j} - \mathbf{k})\cdot(6\mathbf{i} + 3\mathbf{k}) = -3$

24. $(\mathbf{A}\times\mathbf{C}) - (\mathbf{D}\times\mathbf{E})$

▹ $\mathbf{A}\times\mathbf{C} = (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\times(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{k} = -10\mathbf{i} - 5\mathbf{k}$

$\mathbf{D}\times\mathbf{E} = (3\mathbf{j} - \mathbf{k})\times(5\mathbf{i} - 2\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 5 & -2 & 0 \end{vmatrix} = -2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}$

Thus
$$(\mathbf{A}\times\mathbf{C}) - (\mathbf{D}\times\mathbf{E}) = (-10\mathbf{i} - 5\mathbf{k}) - (-2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}) = -8\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$$

25. $\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{D}\times\mathbf{E}\| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ 2 & 1 & -4 \end{matrix}\right\|\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 5 & -2 & 0 \end{matrix}\right\| = \|-14\mathbf{i} - 7\mathbf{k}\|\|-2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}\| = \sqrt{14^2+7^2}\sqrt{2^2+5^2+15^2}$
$= 7\sqrt{5}\sqrt{254} = 7\sqrt{1270}$

26. $2\mathbf{B}\cdot\mathbf{C} + 3\mathbf{D}\cdot\mathbf{E} = 2(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})\cdot(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + 3(3\mathbf{j} - \mathbf{k})\cdot(5\mathbf{i} - 2\mathbf{j}) = 2(12) + 3(-6) = 6$

27. $\mathrm{A}_B = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = \dfrac{(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{2^2+1^2+4^2}} = \dfrac{-2+3-8}{\sqrt{21}} = \dfrac{-7}{\sqrt{21}} = -\tfrac{1}{3}\sqrt{21}.$

28. Find the scalar projection of **C** onto **D**.

▹ The scalar projection of **C** onto **D** is given by
$$\mathrm{C}_D = \frac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{D}\|} = \frac{\langle 1,2,-2\rangle\cdot\langle 0,3,-1\rangle}{\|\langle 0,3,1\rangle\|} = \frac{8}{\sqrt{10}} = \tfrac{4}{5}\sqrt{10}$$

29. $\mathrm{E}_C = \dfrac{\mathbf{E}\cdot\mathbf{C}}{\|\mathbf{C}\|^2}\mathbf{C} = \dfrac{(5\mathbf{i} - 2\mathbf{j})\cdot(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})}{1^2+2^2+2^2}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \dfrac{5-4+0}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \tfrac{1}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

30. $\mathrm{D}_E = \dfrac{\mathbf{D}\cdot\mathbf{E}}{\|\mathbf{E}\|^2}\mathbf{E} = \dfrac{(3\mathbf{j} - \mathbf{k})\cdot(5\mathbf{i} - 2\mathbf{j})}{\|5\mathbf{i} - 2\mathbf{j}\|^2}(5\mathbf{i} - 2\mathbf{j}) = \dfrac{-6}{25+4}(5\mathbf{i} - 2\mathbf{j}) = -\tfrac{30}{29}\mathbf{i} + \tfrac{12}{29}\mathbf{j}$

In Exercises 31–36, there is only one way that a meaningful expression can be obtained by inserting parentheses. Insert the parentheses and find the indicated vector or scalar if $\mathbf{A} = \langle 3, -2, 4\rangle$, $\mathbf{B} = \langle -5, 7, 2\rangle$, $\mathbf{C} = \langle 4, 6, -1\rangle$.

31. $\mathbf{AB}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = \langle 3,-2,4\rangle(\langle -5,7,2\rangle\cdot\langle 4,6,-1\rangle) = \langle 3,-2,4\rangle(-20+42-2) = \langle 60,-40,80\rangle$

32. $\mathbf{A}\cdot\mathbf{BC}$

▸ Because $\mathbf{B}$ and $\mathbf{C}$ are vectors, $\mathbf{BC}$ is not defined. On the other hand, $\mathbf{A}\cdot\mathbf{B}$ is a scalar whose product with the vector $\mathbf{C}$ is defined. Thus, we take

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (\langle 3,-2,4\rangle\cdot\langle -5,7,2\rangle)\mathbf{C} = -21\langle 4,6,1\rangle = \langle -84,-126,21\rangle$$

33. $\mathbf{A}+\mathbf{B}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = (\langle 3,-2,4\rangle + \langle -5,7,2\rangle)\cdot\langle 4,6,-1\rangle = \langle -2,5,6\rangle\cdot\langle 4,6,-1\rangle = -8+30-6 = 16$

34. $\mathbf{B}\cdot\mathbf{A}-\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}-\mathbf{C}) = \langle -5,7,2\rangle\cdot(\langle 3,-2,4\rangle - \langle 4,6,-1\rangle) = \langle -5,7,2\rangle\cdot\langle -1,-8,5\rangle = -41$

35. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{A}+\mathbf{B}-\mathbf{C} = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}-\mathbf{C}) = (\langle 3,-2,4\rangle\times\langle -5,7,2\rangle)\cdot(\langle 3,-2,4\rangle + \langle -5,7,2\rangle - \langle 4,6,-1\rangle)$
$= \langle -32,-26,11\rangle\cdot\langle -6,-1,7\rangle = 192+26+77 = 295$

36. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$

▸ Both $\mathbf{A}\times\mathbf{B}$ and $\mathbf{C}\times\mathbf{A}$ are vectors, and the dot product of two vectors is defined. Thus, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ -5 & 7 & 2 \end{vmatrix} = -32\mathbf{i} - 26\mathbf{j} + 11\mathbf{k} \qquad \mathbf{C}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 22\mathbf{i} - 19\mathbf{j} - 26\mathbf{k}$$

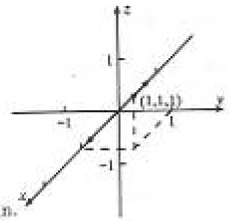
$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{A}) = \langle -32,-26,11\rangle\cdot\langle 22,-19,-26\rangle = -496$

37. The graph of $x = 3$ in R is a point; in R^2 it is a straight line; in R^3 it is a plane.

38. The graph of $x = 6$ and $y = 3$ in R^2 is a point; in R^3 it is a line.

Exercises 39–48, describe in words the set in R^3 satisfying the equation or pair of equations. Sketch the graph.

39. The graph of $\begin{cases} y = 0 \\ z = 0 \end{cases}$ is the x axis.



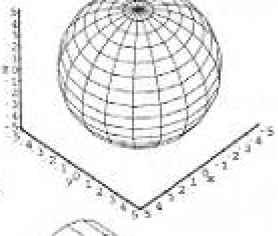
40. $\begin{cases} x = z \\ y = z \end{cases}$

▸ Because
$x = y = z$
these are symmetric equations of the line that contains the origin and representations of the vector $\langle 1,1,1\rangle$. The figure shows the line.

41. The graph of $\begin{cases} x^2 + z^2 = 4 \\ y = 0 \end{cases}$ is a circle in the xz plane of radius 2 centered at the origin.

42. $y^2 - z^2 = 0$ is a cylinder with rulings parallel to the x axis. Its generator is a hyperbola in the yz plane.

43. The graph of $x = y$ is the plane perpendicular to the xy plane and intersecting the xy plane in the line $x = y$.



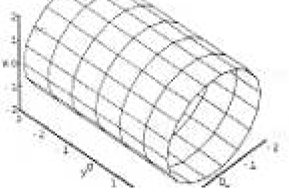
44. $x^2 + y^2 + z^2 = 25$

▸ The graph is a sphere with center at the origin and radius 5. See the figure.

45. The graph of $x^2 + y^2 = 9z$ is the paraboloid of revolution generated by revolving $x^2 = 9z$ or $y^2 = 9z$ about the z axis.

46. $x^2 + y^2 = z^2$ is a right-circular cone whose axis is the z axis.

47. The graph of $x^2 - y^2 = z^2$; $x^2 = y^2 + z^2$ is the right circular cone generated by revolving about the x axis the lines $x = y$ or $x = -y$ in the xy plane or the lines $x = z$ or $x = -z$ in the xz plane.



48. $x^2 + z^2 = 4$

▸ Because there is no term containing y, the graph is a cylinder with rulings parallel to the y axis. The directrix is a circle in the xz plane with center at the origin and radius 2. See the figure.

49. $c = \sqrt{50^2 + 70^2 - 2\cdot 50\cdot 70\cos 120^\circ} = 104.40$
(b) $\sin\alpha = 70\sin 120^\circ/104.4 = .5807$, $\alpha = 35.50^\circ$

50. $\cos\theta = \dfrac{168^2 - 112^2 - 136^2}{2\cdot 112\cdot 136} = 95.3^\circ$

51. If W ft-lb is the work, $W = \langle 30\cos\frac{3}{4}\pi, 30\sin\frac{3}{4}\pi\rangle \cdot \langle -2-3, 7-6\rangle = -150\cos\frac{3}{4}\pi + 30\sin\frac{3}{4}\pi$
$= -150(-\frac{1}{2}\sqrt{2}) + 30(\frac{1}{2}\sqrt{2}) = 90\sqrt{2} \approx 127.28$

52. The compass heading of an airplane is 107° and its airspeed is 210 mi/hr. If a wind is blowing from the west at 36 mi/hr, what are (a) the plane's ground speed and (b) its course?

▹ Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be the plane's air velocity. Because the compass heading is 107°, the direction of $\overrightarrow{OB}$ is $90° - 107° = -17°$. Thus
$b_1 = 210\cos(-17°) = 200.82 \quad b_2 = 210\sin(-17°) = -61.40$
The wind vector is $\langle 36, 0\rangle$. Thus the plane's ground velocity is $\mathbf{V} = \langle 236.82, -61.40\rangle$
(a) Because
$\|\mathbf{V}\| = \sqrt{236.82^2 + 61.40^2} = 244.65$
the plane's ground speed is about 245 mi/hr
(b) Because
$90° - \tan^{-1}(-61.40/236.82) = 104.53°$
the plane's course is about 105°.

53. $P(-3, y, 1)$ is on the line through $(-3, 5, 1)$ perpendicular to the xz plane. If $A = (-2, 0, 0)$, $\|\overline{PA}\| = 13$;
$\sqrt{(-3+2)^2 + (y-0)^2 + (1-0)^2} = 13$; $1 + y^2 + 1 = 169$; $y^2 = 167$; $y = \pm\sqrt{167}$
Thus, the required points are $(-3, \sqrt{167}, 1)$ and $(-3, -\sqrt{167}, 1)$.

54. Sphere whose diameter has endpoints $(3, 5, -4)$ $(-1, 7, 4)$: $(x-3)(x+1) + (y-5)(y-7) + (z+4)(z-4) = 0$.

55. The given sphere has the equation $x^2 + y^2 + z^2 + 4x + 2y - 6z + 10 = 0$, that is
$(x^2 + 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$; $(x+2)^2 + (y+1)^2 + (z-3)^2 = 4$
Thus the required sphere S is $(x+2)^2 + (y+1)^2 + (z-3)^2 = r^2$. Because $(-4, 2, 5)$ is on S,
$(-4+2)^2 + (2+1)^2 + (5-3)^2 = r^2$; $r^2 = 17$. An equation is $(x+2)^2 + (y+1)^2 + (z-3)^2 = 17$.

56. Prove that the points $P(4, 1, -1)$, $Q(2, 0, 1)$, $R(4, 3, 0)$ are the vertices of a right triangle and find the area of the triangle.

▹
$$\mathbf{V}(\overrightarrow{PQ}) = \langle 2, 0, 1\rangle - \langle 4, 1, -1\rangle = \langle -2, -1, 2\rangle$$
$$\mathbf{V}(\overrightarrow{PR}) = \langle 4, 3, 0\rangle - \langle 4, 1, -1\rangle = \langle 0, 2, 1\rangle$$
Thus,
$$\mathbf{V}(\overrightarrow{PQ}) \cdot \mathbf{V}(\overrightarrow{PR}) = \langle -2, -1, 2\rangle \cdot \langle 0, 2, 1\rangle = 0$$
Thus vector $\mathbf{V}(\overrightarrow{PQ})$ is orthogonal to vector $\mathbf{V}(\overrightarrow{PR})$, and hence the triangle has a right angle at P. Also,
$$\left\|\mathbf{V}(\overrightarrow{PQ})\right\| = \|\langle -2, -1, 2\rangle\| = \sqrt{2^2 + 1^2 + 2^2} = 3$$
$$\left\|\mathbf{V}(\overrightarrow{PR})\right\| = \|\langle 0, 2, 1\rangle\| = \sqrt{5}$$
Applying the formula $\frac{1}{2}bh$ for the area of a triangle, we find the area to be $\frac{1}{2}(3)\sqrt{5} = \frac{3}{2}\sqrt{5}$ square units.

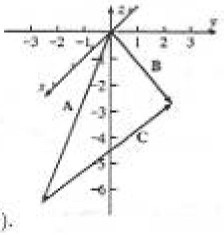
57. The surface $x^2 + z^2 = e^{4y}$ has the y axis as its axis of revolution and a generating curve is either $x^2 = e^{4y}$ or $x = e^{2y}$ or $x = -e^{2y}$ in the xy plane or $z^2 = e^{4y}$ or $z = e^{2y}$ or $z = -e^{2y}$ in the yz plane.

58. Revolve $9x^2 + 4z^2 = 36$ about x axis: replace z^2 with $y^2 + z^2$ to get $9x^2 + 4y^2 + 4z^2 = 36$.

59. If $\langle 3, c, -3\rangle$ and $\langle 5, -4, 1\rangle$ are orthogonal then $\langle 3, c, -3\rangle \cdot \langle 5, -4, 1\rangle = 0$; $15 - 4c - 3 = 0$; $c = 3$.

60. Show that there are representations of the three vectors $\mathbf{A} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{C} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which form a triangle.

▹ Representations of the vectors form a triangle if their sum is the zero vector or if one of the vectors is the sum of the other two. Because
$$\mathbf{A} + \mathbf{C} = (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (-4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \mathbf{B}$$
there are representations of the vectors that form a triangle. See the figure.



61. $A = (5, 9, -3)$, $B = (-2, 4, -5)$. $\mathbf{V}(\overrightarrow{AB}) = -7\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$; $\left\|\mathbf{V}(\overrightarrow{AB})\right\| = \sqrt{49 + 25 + 4}$
$= \sqrt{78}$. (a) $\cos\alpha = -\frac{7}{\sqrt{78}}$, $\cos\beta = -\frac{5}{\sqrt{78}}$, $\cos\gamma = -\frac{2}{\sqrt{78}}$
(b) $-\frac{7}{\sqrt{78}}\mathbf{i} - \frac{5}{\sqrt{78}}\mathbf{j} - \frac{2}{\sqrt{78}}\mathbf{k}$ is the unit vector having the same direction as $\mathbf{V}(\overrightarrow{AB})$.

In Exercises 31–36, there is only one way that a meaningful expression can be obtained by inserting parentheses. Insert the parentheses and find the indicated vector or scalar if $\mathbf{A} = \langle 3,-2,4\rangle$, $\mathbf{B} = \langle -5,7,2\rangle$, $\mathbf{C} = \langle 4,6,-1\rangle$.

31. $\mathbf{AB}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = \langle 3,-2,4\rangle(\langle -5,7,2\rangle\cdot\langle 4,6,-1\rangle) = \langle 3,-2,4\rangle(-20+42-2) = \langle 60,-40,80\rangle$

32. $\mathbf{A}\cdot\mathbf{BC}$

▹ Because **B** and **C** are vectors, **BC** is not defined. On the other hand, $\mathbf{A}\cdot\mathbf{B}$ is a scalar whose product with the vector **C** is defined. Thus, we take

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (\langle 3,-2,4\rangle\cdot\langle -5,7,2\rangle)\mathbf{C} = -21\langle 4,6,1\rangle = \langle -84,-126,21\rangle$$

33. $\mathbf{A}+\mathbf{B}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = (\langle 3,-2,4\rangle+\langle -5,7,2\rangle)\cdot\langle 4,6,-1\rangle = \langle -2,5,6\rangle\cdot\langle 4,6,-1\rangle = -8+30-6 = 16$

34. $\mathbf{B}\cdot\mathbf{A}-\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}-\mathbf{C}) = \langle -5,7,2\rangle\cdot(\langle 3,-2,4\rangle-\langle 4,6,-1\rangle) = \langle -5,7,2\rangle\cdot\langle -1,-8,5\rangle = -41$

35. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{A}+\mathbf{B}-\mathbf{C} = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}-\mathbf{C}) = (\langle 3,-2,4\rangle\times\langle -5,7,2\rangle)\cdot(\langle 3,-2,4\rangle+\langle -5,7,2\rangle-\langle 4,6,-1\rangle)$
$= \langle -32,-26,11\rangle\cdot\langle -6,-1,7\rangle = 192+26+77 = 295$

36. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$

▹ Both $\mathbf{A}\times\mathbf{B}$ and $\mathbf{C}\times\mathbf{A}$ are vectors, and the dot product of two vectors is defined. Thus, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ -5 & 7 & 2 \end{vmatrix} = -32\mathbf{i}-26\mathbf{j}+11\mathbf{k} \qquad \mathbf{C}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 22\mathbf{i}-19\mathbf{j}-26\mathbf{k}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{A}) = \langle -32,-26,11\rangle\cdot\langle 22,-19,-26\rangle = -496$$

37. The graph of $x = 3$ in R is a point; in R^2 it is a straight line; in R^3 it is a plane.

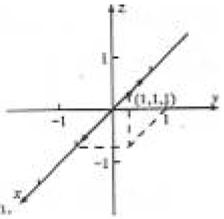
38. The graph of $x = 6$ and $y = 3$ in R^2 is a point; in R^3 it is a line.

Exercises 39–48, describe in words the set in R^3 satisfying the equation or pair of equations. Sketch the graph.

39. The graph of $\begin{cases} y = 0 \\ x = 0 \end{cases}$ is the z axis.

40. $\begin{cases} x = z \\ y = z \end{cases}$

▹ Because
$x = y = z$
these are symmetric equations of the line that contains the origin and representations of the vector $\langle 1,1,1\rangle$. The figure shows the line.



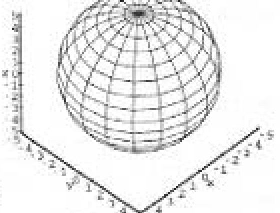
41. The graph of $\begin{cases} x^2+z^2 = 4 \\ y = 0 \end{cases}$ is a circle in the xz plane of radius 2 centered at the origin.

42. $y^2 - z^2 = 0$ is a cylinder with rulings parallel to the x axis. Its generator is a hyperbola in the yz plane.

43. The graph of $x = y$ is the plane perpendicular to the xy plane and intersecting the xy plane in the line $x = y$.

44. $x^2+y^2+z^2 = 25$

▹ The graph is a sphere with center at the origin and radius 5. See the figure.



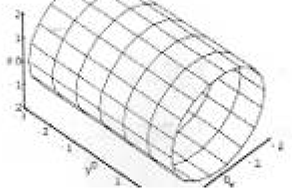
45. The graph of $x^2+y^2 = 9z$ is the paraboloid of revolution generated by revolving $x^2 = 9z$ or $y^2 = 9z$ about the z axis.

46. $x^2+y^2 = z^2$ is a right-circular cone whose axis is the z axis.

47. The graph of $x^2-y^2 = z^2$; $x^2 = y^2+z^2$ is the right circular cone generated by revolving about the x axis the lines $x = y$ or $x = -y$ in the xy plane or the lines $x = z$ or $x = -z$ in the xz plane.

48. $x^2+z^2 = 4$

▹ Because there is no term containing y, the graph is a cylinder with rulings parallel to the y axis. The directrix is a circle in the xz plane with center at the origin and radius 2. See the figure.



49. $c = \sqrt{50^2+70^2-2\cdot 50\cdot 70\cos 120^\circ} = 104.40$
(b) $\sin\alpha = 70\sin 120^\circ/104.4 = .5807$, $\alpha = 35.50^\circ$

50. $\cos\theta = \dfrac{168^2-112^2-136^2}{2\cdot 112\cdot 136} = 95.3^\circ$

51. If W ft-lb is the work, $W = \langle 30\cos\frac{3}{4}\pi, 30\sin\frac{3}{4}\pi\rangle \cdot \langle -2-3, 7-6\rangle = -150\cos\frac{3}{4}\pi + 30\sin\frac{3}{4}\pi$
$= -150(-\frac{1}{2}\sqrt{2}) + 30(\frac{1}{2}\sqrt{2}) = 90\sqrt{2} \approx 127.28$

52. The compass heading of an airplane is 107° and its airspeed is 210 mi/hr. If a wind is blowing from the west at 36 mi/hr, what are (a) the plane's ground speed and (b) its course?

▹ Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be the plane's air velocity. Because the compass heading is 107°, the direction of $\overrightarrow{OB}$ is $90° - 107° = -17°$. Thus
$b_1 = 210\cos(-17°) = 200.82 \quad b_2 = 210\sin(-17°) = -61.40$
The wind vector is $\langle 36, 0\rangle$. Thus the plane's ground velocity is $\mathbf{V} = \langle 236.82, -61.40\rangle$
(a) Because
$\|\mathbf{V}\| = \sqrt{236.82^2 + 61.40^2} = 244.65$
the plane's ground speed is about 245 mi/hr
(b) Because
$90° - \tan^{-1}(-61.40/236.82) = 104.53°$
the plane's course is about 105°.

53. $P(-3, y, 1)$ is on the line through $(-3, 5, 1)$ perpendicular to the xz plane. If $A = (-2, 0, 0)$, $\|\overline{PA}\| = 13$;
$\sqrt{(-3+2)^2 + (y-0)^2 + (1-0)^2} = 13$; $1 + y^2 + 1 = 169$; $y^2 = 167$; $y = \pm\sqrt{167}$
Thus, the required points are $(-3, \sqrt{167}, 1)$ and $(-3, -\sqrt{167}, 1)$.

54. Sphere whose diameter has endpoints $(3, 5, -4)$ $(-1, 7, 4)$: $(x-3)(x+1) + (y-5)(y-7) + (z+4)(z-4) = 0$.

55. The given sphere has the equation $x^2 + y^2 + z^2 + 4x + 2y - 6z + 10 = 0$, that is
$(x^2 + 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$; $(x+2)^2 + (y+1)^2 + (z-3)^2 = 4$
Thus the required sphere S is $(x+2)^2 + (y+1)^2 + (z-3)^2 = r^2$. Because $(-4, 2, 5)$ is on S,
$(-4+2)^2 + (2+1)^2 + (5-3)^2 = r^2$; $r^2 = 17$. An equation is $(x+2)^2 + (y+1)^2 + (z-3)^2 = 17$.

56. Prove that the points $P(4, 1, -1)$, $Q(2, 0, 1)$, $R(4, 3, 0)$ are the vertices of a right triangle and find the area of the triangle.

▹
$$\mathbf{V}(\overrightarrow{PQ}) = \langle 2, 0, 1\rangle - \langle 4, 1, -1\rangle = \langle -2, -1, 2\rangle$$
$$\mathbf{V}(\overrightarrow{PR}) = \langle 4, 3, 0\rangle - \langle 4, 1, -1\rangle = \langle 0, 2, 1\rangle$$
Thus,
$$\mathbf{V}(\overrightarrow{PQ}) \cdot \mathbf{V}(\overrightarrow{PR}) = \langle -2, -1, 2\rangle \cdot \langle 0, 2, 1\rangle = 0$$
Thus vector $\mathbf{V}(\overrightarrow{PQ})$ is orthogonal to vector $\mathbf{V}(\overrightarrow{PR})$, and hence the triangle has a right angle at P. Also,
$$\left\|\mathbf{V}(\overrightarrow{PQ})\right\| = \|\langle -2, -1, 2\rangle\| = \sqrt{2^2 + 1^2 + 2^2} = 3$$
$$\left\|\mathbf{V}(\overrightarrow{PR})\right\| = \|\langle 0, 2, 1\rangle\| = \sqrt{5}$$
Applying the formula $\frac{1}{2}bh$ for the area of a triangle, we find the area to be $\frac{1}{2}(3)\sqrt{5} = \frac{3}{2}\sqrt{5}$ square units.

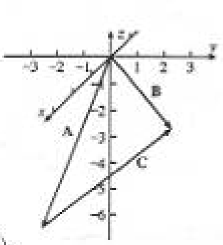
57. The surface $x^2 + z^2 = e^{4y}$ has the y axis as its axis of revolution and a generating curve is either $x^2 = e^{4y}$ or $x = e^{2y}$ or $x = -e^{2y}$ in the xy plane or $z^2 = e^{4y}$ or $z = e^{2y}$ or $z = -e^{2y}$ in the yz plane.

58. Revolve $9x^2 + 4z^2 = 36$ about x axis: replace z^2 with $y^2 + z^2$ to get $9x^2 + 4y^2 + 4z^2 = 36$.

59. If $\langle 3, c, -3\rangle$ and $\langle 5, -4, 1\rangle$ are orthogonal then $\langle 3, c, -3\rangle \cdot \langle 5, -4, 1\rangle = 0$; $15 - 4c - 3 = 0$; $c = 3$.

60. Show that there are representations of the three vectors $\mathbf{A} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{C} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which form a triangle.

▹ Representations of the vectors form a triangle if their sum is the zero vector or if one of the vectors is the sum of the other two. Because
$$\mathbf{A} + \mathbf{C} = (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (-4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \mathbf{B}$$
there are representations of the vectors that form a triangle. See the figure.



61. $A = (5, 9, -3)$, $B = (-2, 4, -5)$. $\mathbf{V}(\overrightarrow{AB}) = -7\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$; $\left\|\mathbf{V}(\overrightarrow{AB})\right\| = \sqrt{49 + 25 + 4}$ $= \sqrt{78}$. (a) $\cos\alpha = -\frac{7}{\sqrt{78}}$, $\cos\beta = -\frac{5}{\sqrt{78}}$, $\cos\gamma = -\frac{2}{\sqrt{78}}$
(b) $-\frac{7}{\sqrt{78}}\mathbf{i} - \frac{5}{\sqrt{78}}\mathbf{j} - \frac{2}{\sqrt{78}}\mathbf{k}$ is the unit vector having the same direction as $\mathbf{V}(\overrightarrow{AB})$.

62. $a(\mathbf{i}+\mathbf{j}-\mathbf{k})+b(2\mathbf{i}-\mathbf{j}+\mathbf{k})+c(3\mathbf{i}-2\mathbf{j}+4\mathbf{k})=5\mathbf{i}+6\mathbf{j}-8\mathbf{k}$

$$\begin{aligned}
\text{I:}\quad & a+2b+3c=5\\
\text{J:}\quad & a-b-2c=6\\
\text{K:}\quad & -a+b+4c=-8
\end{aligned}$$

$$\begin{aligned}
\text{L}=\text{I}-\text{J:}\quad & 3b+5c=-1\\
\text{M}=\text{I}+\text{K:}\quad & 3b+7c=-3\\
\text{M}-\text{L:}\quad & 2c=-2
\end{aligned}$$

Then $c=-1$, $b=\frac{4}{3}$, $a=\frac{16}{3}$

63. $\mathbf{A}=\langle 7,-1,5\rangle$, $\mathbf{B}=\langle -2,3,1\rangle$. $\mathbf{A}\cdot\mathbf{B}=-14-3+5=-12$; $\|\mathbf{A}\|=\sqrt{49+1+25}=\sqrt{75}=5\sqrt{3}$.

(a) The scalar projection of $\mathbf{B}$ onto $\mathbf{A}$ is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|}=\dfrac{-12}{5\sqrt{3}}=-\frac{4}{5}\sqrt{3}$.

(b) The vector projection of $\mathbf{B}$ onto $\mathbf{A}$ is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|^2}\mathbf{A}=\dfrac{-12}{75}\langle 7,-1,5\rangle=\langle -\frac{28}{25},\frac{4}{25},-\frac{4}{5}\rangle$.

64. Find an equation of the plane containing points $(1,7,-3)$ and $(3,1,2)$ and which does not intersect the x axis.

▹ Let $P=(1,7,-3)$ and $Q=(3,1,2)$. We have the vector

$\mathbf{V}(\overrightarrow{PQ})=2\mathbf{i}-6\mathbf{j}+5\mathbf{k}$

whose representations are parallel to the plane. Because the plane is parallel to the x axis, representations of the unit vector $\mathbf{i}$ are parallel to the plane. Therefore $\mathbf{i}\times\mathbf{V}(\overrightarrow{PQ})$ is normal to the plane. We take

$$\mathbf{N}=\mathbf{i}\times\mathbf{V}(\overrightarrow{PQ})=\mathbf{i}\times(2\mathbf{i}-6\mathbf{j}+5\mathbf{k})=-5\mathbf{j}-6\mathbf{k}$$

Applying Theorem 10.4.2 with the normal vector $\mathbf{N}$ and the point P, we obtain

$$\begin{aligned}
0(x-1)-5(y-7)-6(z+3)&=0\\
5y+6z-17&=0
\end{aligned}$$

65. Let L be the plane through the points $P(-1,2,1)$, $Q(1,4,0)$, $R(1,-1,3)$. (a) $\mathbf{V}(\overrightarrow{PQ})=\langle 2,2,-1\rangle$ and $\mathbf{V}(\overrightarrow{PR})=\langle 2,-3,2\rangle$. A normal vector to L is $\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&-1\\2&-3&2\end{vmatrix}=\mathbf{i}-6\mathbf{j}-10\mathbf{k}$

L contains P; so an equation is $1(x+1)-6(y-2)-10(z-1)=0$; $x-6y-10z+23=0$.

(b) Because L contains P, the equation has the form $a(x+1)+b(y-2)+c(z-1)=0$.

Because Q and R lie on L, $\begin{cases}2a+2b+c=0\\2a-3b-2c=0\end{cases}$; so $a=-\frac{1}{10}c$, $b=\frac{3}{5}c$. Hence an equation is

$-\frac{1}{10}c(x+1)+\frac{3}{5}c(y-2)+c(z+1)=0$; $(x+1)-6(y-2)-10(z+1)=0$; $x-6y-10z+23=0$

66. $\mathbf{N}=\langle 3,2,1\rangle-\langle -3,0,4\rangle=\langle 6,2,-3\rangle$. $6(x-7)+2(y+2)-3(z+5)=0$; $6x+2y-3z-53=0$

67. $\mathbf{N}=\langle 5,-3,4\rangle$ is a normal vector to the plane and it contains the point $P(-6,3,-2)$. If d units is the distance from the origin O to the plane, then $d=\dfrac{\mathbf{N}\cdot\mathbf{V}(\overrightarrow{OP})}{\|\mathbf{N}\|}=\dfrac{\langle 5,-3,4\rangle\cdot\langle -6,3,-2\rangle}{\sqrt{25+9+16}}=\dfrac{-30-9-8}{5\sqrt{2}}=\frac{47}{10}\sqrt{2}$

68. Find two unit vectors orthogonal to $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and whose representations are parallel to the yz plane.

▹ If the representations of a vector are parallel to the yz plane, then the vector is orthogonal to the unit vector $\mathbf{i}$. Thus we want unit vectors that are orthogonal to both $\mathbf{i}$ and the given vector $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$. We find the cross product

$\mathbf{N}=\mathbf{i}\times(\mathbf{i}-3\mathbf{j}+4\mathbf{k})=-4\mathbf{j}-3\mathbf{k}$

Then the unit vectors are

$$\mathbf{U}=\pm\frac{\mathbf{N}}{\|\mathbf{N}\|}=\pm\frac{-4\mathbf{j}-3\mathbf{k}}{\sqrt{4^2+3^2}}=\pm(\tfrac{4}{5}\mathbf{i}+\tfrac{3}{5})$$

69. $A=(2,2,1)$, $B=(4,3,-1)$, $P=(4,6,-4)$. $\mathbf{V}(\overrightarrow{AP})=\langle 2,4,-5\rangle$, $\mathbf{V}(\overrightarrow{AB})=\langle 2,1,-2\rangle$, $\left\|\mathbf{V}(\overrightarrow{AB})\right\|=\sqrt{4+1+4}=3$.

If d units is the distance from P to line AB and θ is the angle between lines AB and AP, then

$$d=|\mathrm{AP}|\sin\theta=\frac{\left\|\mathbf{V}(\overrightarrow{AP})\times\mathbf{V}(\overrightarrow{AB})\right\|}{\left\|\mathbf{V}(\overrightarrow{AB})\right\|}=\frac{1}{3}\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&-5\\2&1&-2\end{vmatrix}\right\|=\frac{1}{3}\|-3\mathbf{i}-6\mathbf{j}-6\mathbf{k}\|=\|-\mathbf{i}-2\mathbf{j}-2\mathbf{k}\|=\sqrt{1+4+4}=3$$

70. Distance from $9x-2y+6z+44=0$ to $(-3,2,0)$ is $\dfrac{|9(-3)-2(2)+6(0)+44|}{\sqrt{81+4+36}}=\dfrac{13}{11}$

71. $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$. $\|\mathbf{A}\| = \sqrt{4+1+1} = \sqrt{6}$, $\|\mathbf{B}\| = \sqrt{16+9+5} = \sqrt{50} = 5\sqrt{2}$.

(a) If θ is the angle between **A** and **B**, $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{\langle 2,1,1\rangle \cdot \langle 4,-3,5\rangle}{\sqrt{6}\cdot 5\sqrt{2}} = \dfrac{8-3+5}{10\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

(b) $$\|\mathbf{A}\times\mathbf{B}\| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & -3 & 5 \end{vmatrix} \right\| = \|8\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}\| = \sqrt{64+36+100} = \sqrt{200} = 10\sqrt{2}$$

$\sin\theta = \dfrac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{10\sqrt{2}}{\sqrt{6}\cdot 5\sqrt{2}} = \dfrac{2}{\sqrt{6}}$; $\cos\theta = \pm\sqrt{1-\sin\theta} = \pm\sqrt{1-\frac{2}{3}} = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{3}\sqrt{3}$. The sign is unknown.

72. Prove that the lines

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-2}{2} \quad\text{and}\quad \frac{x-2}{2} = \frac{y-5}{3} = \frac{z-5}{1}$$

are skew lines and find the distance between them.

▹ Let L_1 be the first line and L_2 be the second line. Because the direction vector $\langle 1,2,2\rangle$ for L_1 is not a multiple of the direction vector $\langle 2,3,1\rangle$ for L_2, the lines are not parallel, and when the distance between then is shown to be positive, this proves the lines are skew. We have point $P_1(1,-2,2)$ in line L_1 and point $P_2(2,5,5)$ in line L_2. The distance between the lines is the absolute value of the scalar projection of vector $\mathbf{V}(\overrightarrow{P_1P_2})$ on a vector **N** whose representations are perpendicular to both line L_1 and line L_2. We have

$$\mathbf{V}(\overrightarrow{P_1P_2}) = \mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$$

We take for **N** the cross product of the direction vectors for lines L_1 and L_2. Thus, we take

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Then the distance between the lines is given by

$$\frac{\left|\mathbf{N}\cdot\mathbf{V}(\overrightarrow{P_1P_2})\right|}{\|\mathbf{N}\|} = \frac{\left|(-4\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+7\mathbf{j}+3\mathbf{k})\right|}{\sqrt{4^2+3^2+1^2}} = \frac{14}{\sqrt{26}}$$

73. The lines $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-2}{2}$ and $\frac{x-2}{2} = \frac{y-5}{3} = \frac{z-5}{1}$ have directions $\mathbf{A} = \langle 1,2,2\rangle$ and $\langle 2,3,1\rangle$.

The direction of a line L perpendicular to these is $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \langle -4,3,-1\rangle$.

Because L contains the origin, symmetric and parametric equations are
$\frac{x}{-4} = \frac{y}{3} = \frac{z}{-1}$ and $x = -4t$, $y = 3t$, $z = -t$

74. $P(-3,5,2)$, $Q(-1,-3,4)$. $\overrightarrow{PQ} = \langle 2,-8,2\rangle = 2\langle 1,-4,1\rangle$. $\frac{x+3}{1} = \frac{y-5}{-4} = \frac{z-2}{1}$; $x = -3+t$, $y = 5-4t$, $z = 2+t$

75. The given lines are L: $\frac{x-2}{3} = \frac{y+3}{-1} = \frac{z-5}{4}$ and M: $\frac{x+1}{-6} = \frac{y+2}{2} = \frac{z-1}{-8}$. Because the point $(2,-3,-5)$ of L lies on M and the point $(-1,-2,1)$ of M lies on L, the lines coincide.

76. Find an equation of the plane containing the line $\frac{1}{2}(x-3) = -(y+5) = -\frac{1}{3}(z+2)$ and the point $(5,0,-4)$.

▹ Because parametric equations of the given line are
$x = 3+2t \quad y = -5-t \quad z = -2-3t$
with $t = 0$ and $t = 1$ we get the points $(3,-5,-2)$ and $(5,-6,-5)$ of the plane. Using these with the given point $(5,0,-4)$, we get the equation

$$\begin{vmatrix} x & y & z & 1 \\ 3 & -5 & -2 & 1 \\ 5 & -6 & -5 & 1 \\ 5 & 0 & -4 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -5 & -2 & 1 \\ -6 & -5 & 1 \\ 0 & -4 & 1 \end{vmatrix} x - \begin{vmatrix} 3 & -2 & 1 \\ 5 & -5 & 1 \\ 5 & -4 & 1 \end{vmatrix} y + \begin{vmatrix} 3 & -5 & 1 \\ 5 & -6 & 1 \\ 5 & 0 & 1 \end{vmatrix} z - \begin{vmatrix} 3 & -5 & -2 \\ 5 & -6 & -5 \\ 5 & 0 & -4 \end{vmatrix} = 0$$

$17x - 2y + 12z - 37 = 0$

77. The cross section of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$ in the plane $z=4$ is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}+\frac{16}{25}=1$;
$\frac{x^2}{4}+\frac{y^2}{9}=\frac{9}{25}$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a^2=\frac{36}{25}$ and $b^2=\frac{81}{25}$. The measure of the area is $\pi ab=\pi\left(\frac{6}{5}\right)\left(\frac{9}{5}\right)=\frac{54}{25}\pi$.

78. Area $=\left|(2\mathbf{j}-3\mathbf{k})\times(5\mathbf{i}+4\mathbf{k})\right|=|8\mathbf{i}-15\mathbf{j}-10\mathbf{k}|=\sqrt{64+225+100}=\sqrt{389}$ sq units

79. Let $P=(1,3,0)$, $Q=(2,-1,3)$, $R=(-2,2,-1)$, $S=(-1,1,2)$. $\mathbf{V}(\overrightarrow{PQ})=\langle 1,-4,3\rangle$, $\mathbf{V}(\overrightarrow{PR})=\langle -3,-1,-1\rangle$, $\mathbf{V}(\overrightarrow{PS})=\langle -2,-2,2\rangle$. The measure of the volume of the parallepiped is

$$\left|\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})\cdot\mathbf{V}(\overrightarrow{PS})\right|=\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&3\\-3&-1&-1\end{vmatrix}\cdot\langle -2,-2,2\rangle\right|=\left|\langle 7,-8,-13\rangle\cdot\langle -2,-2,2\rangle\right|=|-14+16-26|=24$$

80. Prove by vector analysis that the diagonals of a parallelogram bisect each other.

▹ Let OABC be a parallelogram with $\mathbf{V}(\overrightarrow{OA})=\mathbf{V}(\overrightarrow{CB})=\mathbf{A}$ and $\mathbf{V}(\overrightarrow{OC})=\mathbf{V}(\overrightarrow{AB})=\mathbf{C}$. The position representations of the midpoints of diagonals OB and AC are

$$\tfrac{1}{2}\mathbf{V}(\overrightarrow{OB})=\tfrac{1}{2}[\mathbf{V}(\overrightarrow{OA})+\mathbf{V}(\overrightarrow{AB})]=\tfrac{1}{2}(\mathbf{A}+\mathbf{C})$$
$$\mathbf{V}(\overrightarrow{OA})+\tfrac{1}{2}\mathbf{V}(\overrightarrow{AC})=\mathbf{A}+\tfrac{1}{2}[\mathbf{V}(\overrightarrow{OC})-\mathbf{V}(\overrightarrow{AC})]=\mathbf{A}+\tfrac{1}{2}(\mathbf{C}-\mathbf{A})=\tfrac{1}{2}(\mathbf{A}+\mathbf{C})$$

Thus the two midpoint coincide; so the diagonals of a parallelogram bisect each other.

In Exercises 81 and 82, let $\mathbf{A}=\cos\alpha\mathbf{i}+\sin\alpha\mathbf{j}$ and $\mathbf{B}=\cos\beta\mathbf{i}+\sin\beta\mathbf{j}$. $\mathbf{A}$ and $\mathbf{B}$ are unit vectors.

81. $\cos(\alpha-\beta)=\cos(\mathbf{A},\mathbf{B})=\mathbf{A}\cdot\mathbf{B}=\langle\cos\alpha,\sin\alpha\rangle\cdot\langle\cos\beta,\sin\beta\rangle=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

82. $|\sin(\alpha-\beta)|=|\sin(\mathbf{A},\mathbf{B})|=\|\mathbf{A}\times\mathbf{B}\|=\|\langle\cos\alpha,\sin\alpha,0\rangle\times\langle\cos\beta,\sin\beta,0\rangle\|=\|\langle 0,0,\sin\alpha\cos\beta-\cos\alpha\sin\beta\rangle\|$
$=|\sin\alpha\cos\beta-\cos\alpha\sin\beta|$

83. If $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ then $\mathbf{A}\cdot\mathbf{i}=a_1$, $\mathbf{A}\cdot\mathbf{j}=a_2$, $\mathbf{A}_3\cdot\mathbf{k}=a_3$ and so $\mathbf{A}=(\mathbf{A}\cdot\mathbf{i})\mathbf{i}+(\mathbf{A}\cdot\mathbf{j})\mathbf{j}+(\mathbf{A}\cdot\mathbf{k})\mathbf{k}$

84. Let $\mathbf{A}$ and $\mathbf{B}$ be vectors in V_3, c_1, c_2, and c_3 be direction cosines of $\mathbf{A}$, and d_1, d_2, and d_3 be direction cosines of $\mathbf{B}$. If
$$\frac{c_1}{d_1}=\frac{c_2}{d_2}=\frac{c_3}{d_3}$$
prove that $\mathbf{A}$ and $\mathbf{B}$ are parallel.

▹ Let
$$\frac{c_1}{d_1}=\frac{c_2}{d_2}=\frac{c_3}{d_3}=t$$
Then
$$c_1=td_1\quad c_2=td_2\quad c_3=td_3$$
and so
$$\langle c_1,c_2,c_3\rangle=t\langle d_1,d_2,d_3\rangle \tag{1}$$
Because c_1, c_2, and c_3 are direction cosines of $\mathbf{A}$ and d_1, d_2, and d_3 are direction cosines of $\mathbf{B}$ then
$$\mathbf{A}=\|\mathbf{A}\|\langle c_1,c_2,c_3\rangle \text{ and } \mathbf{B}=\|\mathbf{B}\|\langle d_1,d_2,d_3\rangle \tag{2}$$
From (1) and (2) it follows that $\mathbf{A}$ is a scalar multiple of $\mathbf{B}$ and hence they are parallel.

85. Lagrange's Identity. We use Theorems 10.5.6 and 10.5.7.
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=\mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})]=\mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{D})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{D}]=(\mathbf{A}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})=\begin{vmatrix}\mathbf{A}\cdot\mathbf{C}&\mathbf{A}\cdot\mathbf{D}\\\mathbf{B}\cdot\mathbf{C}&\mathbf{B}\cdot\mathbf{D}\end{vmatrix}$$

86. Using position representations, the given implies
$\mathbf{d}-\mathbf{a}=\frac{1}{3}(\mathbf{b}-\mathbf{a})\quad\mathbf{e}-\mathbf{b}=\frac{1}{3}(\mathbf{c}-\mathbf{b})\quad\mathbf{f}-\mathbf{c}=\frac{1}{3}(\mathbf{a}-\mathbf{c})$
or, equivalently,
$\mathbf{d}=\frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}\quad\mathbf{e}=\frac{2}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}\quad\mathbf{f}=\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{c}$
Therefore,
$$\overrightarrow{AE}+\overrightarrow{BF}+\overrightarrow{CD}=(\mathbf{e}-\mathbf{a})+(\mathbf{f}-\mathbf{b})+(\mathbf{d}-\mathbf{c})=(-\mathbf{a}+\tfrac{2}{3}\mathbf{b}+\tfrac{1}{3}\mathbf{c})+(\tfrac{1}{3}\mathbf{a}-\mathbf{b}+\tfrac{2}{3}\mathbf{c})+(\tfrac{2}{3}\mathbf{a}+\tfrac{1}{3}\mathbf{b}-\mathbf{c})=\mathbf{0}$$

ELEVEN

VECTOR-VALUED FUNCTIONS

11.1 VECTOR-VALUED FUNCTIONS AND CURVES IN $\mathbb{R}^3$

11.1.1 Definition Let f, g, and h be real-valued functions of a real variable t. Then there is a *vector-valued function* $\mathbf{R}$, defined by
$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$
where t is any number in the domain common to f, g, and h. If $h \equiv 0$, we are in the plane. Operations on vector-valued functions are defined in the obvious way.

11.1.3 Definition Let $\mathbf{R}$ be a vector-valued function whose function values are given by
$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$
Then the *limit of* $\mathbf{R}(t)$ *as* t *approaches* a is defined by
$\lim\limits_{t\to a}\mathbf{R}(t) = [\lim\limits_{t\to a} f(t)]\mathbf{i} + [\lim\limits_{t\to a} g(t)]\mathbf{j} + [\lim\limits_{t\to a} h(t)]\mathbf{k}$
if $\lim\limits_{t\to a} f(t)$, $\lim\limits_{t\to a} g(t)$, $\lim\limits_{t\to a} h(t)$ all exist.

11.1.4 Definition The vector-valued function $\mathbf{R}$ is *continuous* at the number a if and only if the following three conditions are satisfied:
(i) $\mathbf{R}(a)$ exists;
(ii) $\lim\limits_{t\to a} \mathbf{R}(t)$ exists;
(iii) $\lim\limits_{t\to a} \mathbf{R}(t) = \mathbf{R}(a)$

Definition A *curve* is the graph of a continuous vector-valued function.

Exercises 11.1

In Exercises 1–8, find the domain of the vector-valued function.

1. $\text{Dom}[(1/t)\mathbf{i} + \sqrt{4-t}\mathbf{j}] = \text{Dom}(1/t) \cap \text{Dom}\sqrt{4-t} = \{t \neq 0\} \cap \{t \leq 4\} = (-\infty, 0) \cup (0, 4]$

2. $\text{Dom}[(t^2+3)\mathbf{i} + (t-1)^{-1}\mathbf{j}] = \text{Dom}(t^2+3) \cap \text{Dom}(t-1)^{-1} = \mathbb{R} \cap \{t \neq 1\} = \{t \neq 1\}$

3. $\text{Dom}[\sin^{-1} t\mathbf{i} + \ln(t+1)\mathbf{j}] = \text{Dom}(\sin^{-1} t) \cap \text{Dom}[\ln(t+1)] = [-1, 1] \cap \{t > -1\} = (-1, 1]$

4. $\mathbf{R}(t) = (\cos^{-1} t)\mathbf{i} + (\sec^{-1} t)\mathbf{j}$

▸ Because
$\text{Dom}(\cos^{-1} t) = [-1, 1]$
and
$\text{Dom}(\sec^{-1} t) = (-\infty, -1] \cup [1, +\infty)$
then $\text{Dom}(\mathbf{R})$ consist of the two numbers -1 and 1.

5. $\text{Dom}[\sqrt{t+2}\mathbf{i} + \sqrt{4-t}\mathbf{j} + \cot t\mathbf{k}] = [-2, +\infty) \cap (-\infty, 4] \cap \{t \neq k\pi\} = [-2, 0) \cup (0, \pi) \cup (\pi, 4]$

6. $\text{Dom}[\sqrt{t^2-9}\mathbf{i} + \ln|t-3|\mathbf{j} + (t^2+2t-8)\mathbf{k}] = \{(-\infty, -3] \cup [3, +\infty)\} \cap \{t \neq 3\} \cap \mathbb{R} = (-\infty, -3] \cup (3, +\infty)$

7. $\text{Dom}[\ln|\sin t|\mathbf{i} + \sqrt{16-t^2}\mathbf{j} + \ln|t+4|\mathbf{k}] = \{t \neq k\pi\} \cap [-4, 4] \cap \{t \neq -4\} = (-4, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 4]$

8. $\mathbf{R}(t) = \tan t\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + [1/(2+t)]\mathbf{k}$

▸ We have
$\text{Dom}(\tan t) = \{t \neq (k + \frac{1}{2})\pi\}$ where k is any integer
and
$\text{Dom}(\sqrt{4-t^2}) = [-2, 2]$
and
$\text{Dom}\left(\dfrac{1}{2+t}\right) = \{t \neq -2\}$
Because $\frac{1}{2}\pi \approx 1.57$, then
$\text{Dom}(\mathbf{R}) = (-2, -\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi, \frac{1}{2}\pi) \cup (\frac{1}{2}\pi, 2]$

In Exercises 9–12, find (a) $(\mathbf{F}+\mathbf{G})(t)$; (b) $(\mathbf{F}-\mathbf{G})(t)$; (c) $(\mathbf{F}\cdot\mathbf{G})(t)$; (d) $(\mathbf{F}\times\mathbf{G})(t)$

In Exercises 13–16, find: (a) $(f\mathbf{F})(t)$; (b) $(f\mathbf{G})(t)$; (c) $(\mathbf{F}\circ g)(t)$; (d) $(\mathbf{G}\circ g)(t)$

9. $\mathbf{F} = \langle t+1, t^2-1, t-1\rangle$, $\mathbf{G} = \langle t-1, 1, t+1\rangle$

▷ (a) $\mathbf{F}+\mathbf{G} = \langle 2t, t^2, 2t\rangle$ (b) $\mathbf{F}-\mathbf{G} = \langle 2, t^2-2, -2\rangle$

(c) $\mathbf{F}\cdot\mathbf{G} = (t^2-1)+(t^2-1)+(t^2-1) = 3t^2-3$

(d) $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t+1 & t^2-1 & t-1 \\ t-1 & 1 & t+1 \end{vmatrix}$

$= (t^3+t^2-2t)\mathbf{i} - 4t\mathbf{j} + (2t-t^3+t^2)\mathbf{k}$

10. $\mathbf{F} = \langle 4-t^2, 4, t^2-4\rangle$, $\mathbf{G} = \langle t^2, t^2-4, -4\rangle$

▷ (a) $\mathbf{F}+\mathbf{G} = \langle 4, t^2, t^2-8\rangle$ (b) $\mathbf{F}-\mathbf{G} = \langle 4-2t^2, 8-t^2, t^2\rangle$

(c) $\mathbf{F}\cdot\mathbf{G} = 4t^2-t^4+4t^2-16-4t^2+16 = 4t^2-t^4$

(d) $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4-t^2 & 4 & t^2-4 \\ t^2 & t^2-4 & -4 \end{vmatrix}$

$= \langle -t^4+8t^2-32, t^4-8t^2+16, -t^4+4t^2-16\rangle$

13. $f(t) = t-1$; $g(t) = t+1$

▷ (a) $f(t)\mathbf{F}(t) = \langle t^2-1, t^3-t^2-t+1, t^2-2t+1\rangle$

(b) $f(t)\mathbf{G}(t) = \langle t^2-2t+1, t-1, t^2-1\rangle$

(c) $\mathbf{F}(g(t)) = \langle t+2, t^2+2t, t\rangle$

(d) $\mathbf{G}(g(t)) = \langle t, 1, t+2\rangle$

14. $f(t) = 1/(2-t)$; $g(t) = 2-t$

▷ (a) $f(t)\mathbf{F}(t) = \langle 2+t, \frac{4}{2-t}, -(t+2)\rangle$

(b) $f(t)\mathbf{G}(t) = \langle \frac{t^2}{2-t}, -(t+2), \frac{4}{t-2}\rangle$

(c) $\mathbf{F}(g(t)) = \langle 4t-t^2, 4, t^2-4t\rangle$

(d) $\mathbf{G}(g(t)) = \langle t^2-4t+4, t^2-4t, -4\rangle$

11. $\mathbf{F} = \cos t\mathbf{i} - \sin t\mathbf{j} + t\mathbf{k}$, $\mathbf{G} = \sin t\mathbf{i} + \cos t\mathbf{j} - t\mathbf{k}$

▷ (a) $\mathbf{F}+\mathbf{G} = (\cos t+\sin t)\mathbf{i} + (\cos t-\sin t)\mathbf{j}$

(b) $\mathbf{F}-\mathbf{G} = (\cos t-\sin t)\mathbf{i} - (\cos t+\sin t)\mathbf{j} + 2t\mathbf{k}$

(c) $\mathbf{F}\cdot\mathbf{G} = \cos t\sin t - \sin t\cos t - t^2 = -t^2$

(d) $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sin t & t \\ \sin t & \cos t & -t \end{vmatrix}$

$= t(\sin t-\cos t)\mathbf{i} + t(\sin t+\cos t)\mathbf{j} + \mathbf{k}$

12. $\mathbf{F} = \sec t\mathbf{i} + \tan t\mathbf{j} - 2\mathbf{k}$; $\mathbf{G} = \sec t\mathbf{i} - \tan t\mathbf{j} + t\mathbf{k}$

▷ (a) $\mathbf{F}+\mathbf{G} = 2\sec t\mathbf{i} + (t-2)\mathbf{k}$

(b) $\mathbf{F}-\mathbf{G} = 2\tan t\mathbf{i} - (t+2)\mathbf{k}$

(c) $\mathbf{F}\cdot\mathbf{G} = \sec^2 t - \tan^2 t - 2t = -2t+1$

(d) $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sec t & \tan t & -2 \\ \sec t & -\tan t & t \end{vmatrix}$

$= (t-2)\tan t\mathbf{i} - (t+2)\sec t\mathbf{j} - 2\tan t\sec t\mathbf{k}$

15. $f(t) = \sin t$; $g(t) = \sin^{-1} t$

▷ (a) $f(t)\mathbf{F}(t) = \sin t\cos t\mathbf{i} - \sin^2 t\mathbf{j} + t\sin t\mathbf{k}$

(b) $f(t)\mathbf{G}(t) = \sin^2 t\mathbf{i} + \sin t\cos t\mathbf{j} - t\sin t\mathbf{k}$

(c) $\mathbf{F}(g(t)) = \sqrt{1-t^2}\mathbf{i} - t\mathbf{j} + \sin^{-1} t\mathbf{k}$

(d) $\mathbf{G}(g(t)) = t\mathbf{i} + \sqrt{1-t^2}\mathbf{j} - \sin^{-1} t\mathbf{k}$

16. $f(t) = \cos t$; $g(t) = \cos^{-1} t$

▷ (a) $f(t)\mathbf{F}(t) = \mathbf{i} + \sin t\mathbf{j} - 2\cos t\mathbf{k}$, $t \neq (k+\frac{1}{2})\pi$

(b) $f(t)\mathbf{G}(t) = \mathbf{i} - \sin t\mathbf{j} + t\cos t\mathbf{k}$, $t \neq (k+\frac{1}{2})\pi$

(c) $\mathbf{F}(g(t)) = \sec(\cos^{-1} t)\mathbf{i} + \tan(\cos^{-1} t)\mathbf{j} - 2\mathbf{k}$

$= (1/t)\mathbf{i} + (\sqrt{1-t^2}/t)\mathbf{j} - 2\mathbf{k}$

(d) $\mathbf{G}(g(t)) = (1/t)\mathbf{i} - (\sqrt{1-t^2}/t)\mathbf{j} + \cos^{-1} t\mathbf{k}$

In Exercises 17–24, find the indicated limit, if it exists.

17. $\lim_{t\to 2}\left[(t-2)\mathbf{i} + \frac{t^2-4}{t-2}\mathbf{j} + t\mathbf{k}\right] = 0\mathbf{i} + \lim_{t\to 2}(t+2)\mathbf{j} + 2t\mathbf{k} = 4\mathbf{j} + 2\mathbf{k}$

18. $\lim_{t\to -1}\left[\frac{t^2-1}{t+1}\mathbf{i} + \frac{t+1}{t-1}\mathbf{j} + |t+1|\mathbf{k}\right] = \lim_{t\to -1}(t-1)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = -2\mathbf{i}$

19. $\lim_{t\to 0}\left[\sin t\mathbf{i} + \cos t\mathbf{j} + \frac{\sin t}{t}\mathbf{k}\right] = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k} = \mathbf{j} + \mathbf{k}$

20. $\mathbf{R}(t) = \frac{1-\cos t}{t}\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$; $\lim_{x\to 0}\mathbf{R}(t)$

▷ Because

$\lim_{t\to 0}\frac{1-\cos t}{t} \overset{0/0}{=} \lim_{t\to 0}\frac{\sin t}{1} = 0$

$\lim_{t\to 0} e^t = e^0 = 1$

$\lim_{t\to 0} e^{-t} = e^0 = 1$

then

$\lim_{t\to 0}\mathbf{R}(t) = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k} = \mathbf{j} + \mathbf{k}$

21. $\lim_{t\to 1}\left[\frac{|t-2|}{t-2}\mathbf{i} + \frac{\sin \pi t}{t^2-1}\mathbf{j} + \frac{\tan \pi t}{t-1}\mathbf{k}\right] \overset{0/0}{=} -1\mathbf{i} + \lim_{t\to 1}\frac{\pi\cos \pi t}{2t}\mathbf{j} + \lim_{t\to 1}\frac{\pi\sec^2 \pi t}{1} = -\mathbf{i} - \frac{1}{2}\pi\mathbf{j} + \pi\mathbf{k}$

22. $\lim_{t\to 0}\left[\frac{1+\cos t}{1-\sin t}\mathbf{i} + \frac{1-\cos^2 t}{1-\cos t}\mathbf{j} + t\cdot\frac{t}{\sin t}\mathbf{k}\right] = 2\mathbf{i} + \lim_{t\to 0}(1+\cos t)\mathbf{j} + 0\cdot 1\mathbf{k} = 2\mathbf{i} + 2\mathbf{j}$

23. $\lim_{t\to 0}\left[e^{t+1}\mathbf{i} + e^{1-t}\mathbf{j} + (1+t)^{1/t}\mathbf{k}\right] = e(\mathbf{i} + \mathbf{j} + \mathbf{k})$

24. $\mathbf{R}(t) = \dfrac{\ln(t+1)}{t}\mathbf{i} + \sinh t\mathbf{j} + \cosh t\mathbf{k}$; $\lim\limits_{t\to 0}\mathbf{R}(t)$

▹ Because

$$\lim_{t\to 0}\frac{\ln(t+1)}{t} \overset{0/0}{=} \lim_{t\to 0}\frac{1/(t+1)}{1} = 1$$

then

$$\lim_{t\to 0}\mathbf{R}(t) = 1\mathbf{i} + \sinh 0\mathbf{j} + \cosh 0\mathbf{k} = \mathbf{i} + \mathbf{k}$$

In Exercises 25–30, determine the numbers at which the vector-valued function is continuous.

25. $t^2 + \ln(t-1)\mathbf{j} + \dfrac{1}{t-2}\mathbf{k}$

▹ $(1,2)\cup(2,+\infty)$

26. $(t-1)\mathbf{i} + \dfrac{1}{e^t-1}\mathbf{j} + \dfrac{|t-1|}{t-1}\mathbf{k}$

▹ $\{t \neq 0, 1\}$

27. $\cos t\mathbf{i} + \sec t\mathbf{j} + \tan t\mathbf{k}$

▹ $\{t \neq (k+\frac{1}{2})\pi$, k is any integer$\}$

28. $\mathbf{R}(t) = \sin \pi t\mathbf{i} - \tan \pi t\mathbf{j} + \cot \pi t\mathbf{k}$

▹ The tangent function is continuous except at the odd multiples of $\frac{1}{2}\pi$, the cotangent is continuous except at the even multiples of $\frac{1}{2}\pi$. Thus $\mathbf{R}$ is continuous everywhere except at $\frac{1}{2}k$, where k is any integer.

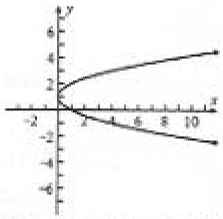
29. $\begin{cases} e^{1/t^2}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$ all real numbers

30. $\begin{cases} \dfrac{\sin t}{t}\mathbf{i} + \dfrac{1-\cos t}{t}\mathbf{j} + \dfrac{1-e^t}{t}\mathbf{k} & \text{if } t \neq 0 \\ \mathbf{i}-\mathbf{k} & \text{if } t = 0 \end{cases}$ all numbers

In Exercises 31–42, sketch the graph of the vector-valued function.

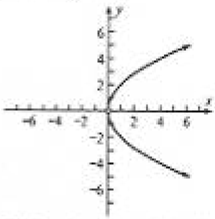
31. $\mathbf{R}(t) = t^2\mathbf{i} + (t+1)\mathbf{j}$

▹



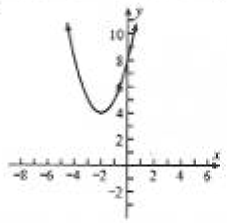
32. $\mathbf{R}(t) = \dfrac{4}{t^2}\mathbf{i} + \dfrac{4}{t}\mathbf{j}$

▹



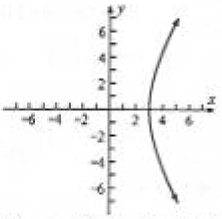
33. $\mathbf{R}(t) = (t-2)\mathbf{i} + (t^2+4)\mathbf{j}$

▹



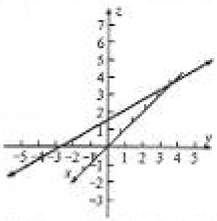
34. $\mathbf{R}(t) = 3\cosh t\mathbf{i} + 5\sinh t\mathbf{j}$

▹



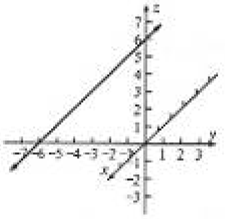
35. $\mathbf{R}(t) = t\mathbf{i} + (6-4t)\mathbf{j} + (5-2t)\mathbf{k}$

▹



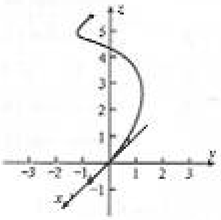
36. $\mathbf{R}(t) = (t+1)\mathbf{i} + (2t-3)\mathbf{j} + (2t+3)\mathbf{k}$

▹



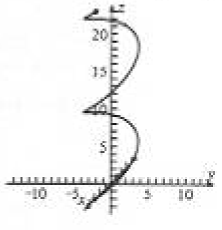
37. $\mathbf{R} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $[0, 2\pi]$

▹



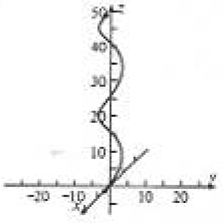
38. $\mathbf{R} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k}$, $[0, 4\pi]$

▹



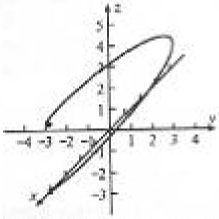
39. $\mathbf{R} = 2\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$, $[0, 4\pi]$

▹

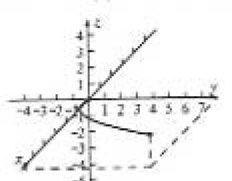


40. $\mathbf{R}(t) = 4\cos t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{2}t\mathbf{k},\ 0 \le t \le 2\pi$ **41.** $\mathbf{R}(t) = 3t\mathbf{i} + 2t^2\mathbf{j} + t\mathbf{k},\ [0,2]$ **42.** $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}t^3\mathbf{k},\ [0,2]$

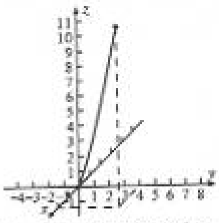
▷



▷



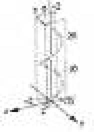
▷

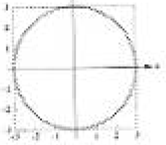


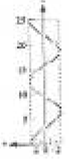
In Exercises 43–46, match each graph (a), (b), (c) of the same curve with one of the given viewing points.

43. Exercise 37 ▷ (a) $(0,8,0)$ (b) $(4,8,4)$ (c) $(0,0,8)$

44. Exercise 38







▷ (a) $(14,28,14)$ (b) $(0,0,28)$ (c) $(0,28,0)$

45. Exercise 41 ▷ (a) $(0,0,10)$ (b) $(-10,0,0)$ (c) $(10,0,0)$

46. Exercise 42 ▷ (a) $(-15,0,0)$ (b) $(0,0,15)$ (c) $(15,0,0)$

In Exercises 47–49, prove the limit theorem if $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are functions such that $\lim_{t\to a}\mathbf{U}(t)$ and $\lim_{t\to a}\mathbf{V}(t)$ exist.

▷ Let $\mathbf{U}(t) = \mathrm{U}_1(t)\mathbf{i} + \mathrm{U}_2(t)\mathbf{j} + \mathrm{U}_3(t)\mathbf{k}$ and $\mathbf{V}(t) = \mathrm{V}_1(t)\mathbf{i} + \mathrm{V}_2(t)\mathbf{j} + \mathrm{V}_3(t)\mathbf{k}$.

47.
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}+\mathbf{V}] &= \lim_{t\to a}[(\mathrm{U}_1+\mathrm{V}_1)\mathbf{i} + (\mathrm{U}_2+\mathrm{V}_2)\mathbf{j} + (\mathrm{U}_3+\mathrm{V}_3)\mathbf{k}] \\
&= [\lim_{t\to a}(\mathrm{U}_1+\mathrm{V}_1)]\mathbf{i} + [\lim_{t\to a}(\mathrm{U}_2+\mathrm{V}_2)]\mathbf{j} + [\lim_{t\to a}(\mathrm{U}_3+\mathrm{V}_3)]\mathbf{k} && \text{(Definition 11.1.3)} \\
&= [\lim_{t\to a}\mathrm{U}_1 + \lim_{t\to a}\mathrm{V}_1]\mathbf{i} + [\lim_{t\to a}\mathrm{U}_2 + \lim_{t\to a}\mathrm{V}_2] + [\lim_{t\to a}\mathrm{U}_3 + \lim_{t\to a}\mathrm{V}_3] && \text{(LT 4)} \\
&= [(\lim_{t\to a}\mathrm{U}_1)\mathbf{i} + (\lim_{t\to a}\mathrm{U}_2)\mathbf{j} + (\lim_{t\to a}\mathrm{U}_3)\mathbf{k}] + [(\lim_{t\to a}\mathrm{V}_1)\mathbf{i} + (\lim_{t\to a}\mathrm{V}_2)\mathbf{j} + (\lim_{t\to a}\mathrm{V}_3)\mathbf{k}] \\
&= \lim_{t\to a}\mathbf{U} + \lim_{t\to a}\mathbf{V} && \text{(Definition 11.1.3)}
\end{aligned}$$

48. $\lim_{t\to a}[\mathbf{U}(t)\cdot\mathbf{V}(t)] = \lim_{t\to a}\mathbf{U}(t)\cdot\lim_{t\to a}\mathbf{V}(t)$

▷
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}(t)\cdot\mathbf{V}(t)] &= \lim_{t\to a}[\mathrm{U}_1(t)\mathrm{V}_1(t) + \mathrm{U}_2(t)\mathrm{V}_2(t) + \mathrm{U}_3(t)\mathrm{V}_3(t)] \\
&= \lim_{t\to a}[\mathrm{U}_1(t)\mathrm{V}_1(t)] + \lim_{t\to a}[\mathrm{U}_2(t)\mathrm{V}_2(t)] + \lim_{t\to a}[\mathrm{U}_3(t)\mathrm{V}_3(t)] && \text{(LT 4)} \\
&= \lim_{t\to a}\mathrm{U}_1(t)\lim_{t\to a}\mathrm{V}_1(t) + \lim_{t\to a}\mathrm{U}_2(t)\lim_{t\to a}\mathrm{V}_2(t) + \lim_{t\to a}\mathrm{U}_3(t)\lim_{t\to a}\mathrm{V}_3(t) && \text{(LT 6)} \\
&= [\lim_{t\to a}\mathrm{U}_1(t)\mathbf{i} + \lim_{t\to a}\mathrm{U}_2(t)\mathbf{j} + \lim_{t\to a}\mathrm{U}_3(t)\mathbf{k}]\cdot[\lim_{t\to a}\mathrm{V}_1(t)\mathbf{i} + \lim_{t\to a}\mathrm{V}_2(t)\mathbf{j} + \lim_{t\to a}\mathrm{V}_3(t)\mathbf{k}] \\
&= \lim_{t\to a}\mathbf{U}(t)\cdot\lim_{t\to a}\mathbf{V}(t) && \text{(Definition 11.1.3)}
\end{aligned}$$

49.
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}\times\mathbf{V}] &= \lim_{t\to a}[(\mathrm{U}_2\mathrm{V}_3 - \mathrm{U}_3\mathrm{V}_2)\mathbf{i} + (\mathrm{U}_3\mathrm{V}_1 - \mathrm{U}_1\mathrm{V}_3)\mathbf{j} + (\mathrm{U}_1\mathrm{V}_2 - \mathrm{U}_2\mathrm{V}_1)\mathbf{k}] && \text{(Definition 10.5.1)} \\
&= [\lim_{t\to a}(\mathrm{U}_2\mathrm{V}_3 - \mathrm{U}_3\mathrm{V}_2)]\mathbf{i} + [\lim_{t\to a}(\mathrm{U}_3\mathrm{V}_1 - \mathrm{U}_1\mathrm{V}_3)]\mathbf{j} + [\lim_{t\to a}(\mathrm{U}_1\mathrm{V}_2 - \mathrm{U}_2\mathrm{V}_1)]\mathbf{k} && \text{(Definition 11.1.3)} \\
&= [\lim_{t\to a}\mathrm{U}_2\lim_{t\to a}\mathrm{V}_3 - \lim_{t\to a}\mathrm{U}_3\lim_{t\to a}\mathrm{V}_2]\mathbf{i} + [\lim_{t\to a}\mathrm{U}_3\lim_{t\to a}\mathrm{V}_1 - \lim_{t\to a}\mathrm{U}_1\lim_{t\to a}\mathrm{V}_3]\mathbf{j} + [\lim_{t\to a}\mathrm{U}_1\lim_{t\to a}\mathrm{V}_2 - \lim_{t\to a}\mathrm{U}_2\lim_{t\to a}\mathrm{V}_1]\mathbf{k} && \text{(LT4,6)} \\
&= [(\lim_{t\to a}\mathrm{U}_1)\mathbf{i} + (\lim_{t\to a}\mathrm{U}_2)\mathbf{j} + (\lim_{t\to a}\mathrm{U}_3)\mathbf{j}]\times[(\lim_{t\to a}\mathrm{V}_1)\mathbf{i} + (\lim_{t\to a}\mathrm{V}_2)\mathbf{j} + (\lim_{t\to a}\mathrm{V}_3)\mathbf{k}] && \text{(Definition 10.5.1)} \\
&= \lim_{t\to a}\mathbf{U}\times\lim_{t\to a}\mathbf{V} && \text{(Definition 11.1.3)}
\end{aligned}$$

50.
$$\begin{aligned}
&\lim_{t\to a}(f\mathbf{V}) = \lim_{t\to a}[(f\mathrm{V}_1)\mathbf{i} + (f\mathrm{V}_2)\mathbf{j} + (f\mathrm{V}_3)\mathbf{k}] \overset{11.1.3}{=} [\lim_{t\to a}(f\mathrm{V}_1)]\mathbf{i} + [\lim_{t\to a}(f\mathrm{V}_2)]\mathbf{j} + [\lim_{t\to a}(f\mathrm{V}_3)]\mathbf{k} \\
&\overset{\text{LT 6}}{=} (\lim_{t\to a}f)(\lim_{t\to a}\mathrm{V}_1)\mathbf{i} + (\lim_{t\to a}f)(\lim_{t\to a}\mathrm{V}_2)\mathbf{j} + (\lim_{t\to a}f)(\lim_{t\to a}\mathrm{V}_3)\mathbf{k} = \lim_{t\to a}f[(\lim_{t\to a}\mathrm{V}_1)\mathbf{i} + (\lim_{t\to a}\mathrm{V}_2)\mathbf{j} + (\lim_{t\to a}\mathrm{V}_3)\mathbf{k}] \\
&\overset{11.1.3}{=} [\lim_{t\to a}f][\lim_{t\to a}\mathbf{V}]
\end{aligned}$$

51. If $\mathbf{V}$ is continuous, then by Exercise 48, $\mathbf{V}\cdot\mathbf{V}$ is continuous, and by LT 10, so is $\|\mathbf{V}\| = \sqrt{\mathbf{V}\cdot\mathbf{V}}$.

11.2 CALCULUS OF VECTOR-VALUED FUNCTIONS

The definition of the derivative of a vector-valued function, differentiability of a vector-valued function, and the antiderivative of a vector-valued function are similar to the corresponding definitions for real-valued functions.

11.2.1 Definition If $\mathbf{R}$ is a vector-valued function, then the *derivative* of $\mathbf{R}$ is another vector-valued function, denoted by $\mathbf{R}'$ and defined by

$$\mathbf{R}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t}$$

if this limit exists.

The notation $D_t\mathbf{R}(t)$ is sometimes used in place of $\mathbf{R}'(t)$.

11.2.2 Theorem If $\mathbf{R}$ is a vector-valued function defined by

$$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{1}$$

then

$$\mathbf{R}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

if $f'(t)$, $g'(t)$, and $h'(t)$ exist. For each replacement of t, the direction of the vector $\mathbf{R}'(t)$ is along the tangent line to the curve whose vector equation is (1).

11.2.3 Definition A vector-valued function is said to be *differentiable* on a interval if $\mathbf{R}'(t)$ exists for all values of t in the interval.

While the mean-value theorem does not extend to vector-valued functions, the differentiation formulas for a vector-valued function are similar to the corresponding differentiation formulas for a real-valued function.

11.2.4 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R}+\mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t\mathbf{R}(t) + D_t\mathbf{Q}(t)$$

11.2.5 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R}\cdot\mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] = [D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) + \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)]$$

11.2.6 Theorem If $\mathbf{R}$ is a differentiable vector-valued function on an interval and f is a differentiable real-valued function on the interval, then

$$D_t[f(t)\mathbf{R}(t)] = [D_t f(t)]\mathbf{R}(t) + f(t)D_t\mathbf{R}(t)$$

11.2.7 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R}\times\mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t)\times\mathbf{Q}(t)] = [D_t\mathbf{R}(t)]\times\mathbf{Q}(t) + \mathbf{R}(t)\times[D_t\mathbf{Q}(t)]$$

11.2.8 Theorem Suppose that $\mathbf{F}$ is a vector-valued function and h is a real-valued function. If $\mathbf{F}'(h(t))$ exists and $h'(t)$ exists, then $D_t\mathbf{F}(h(t))$ exists and is given by

$$D_t\mathbf{F}(h(t)) = \mathbf{F}'(h(t))h'(t)$$

11.2.9 Theorem If $\mathbf{R}$ is a differentiable vector-valued function on an interval and $\|\mathbf{R}(t)\|$ is constant for all t in the interval, then the vectors $\mathbf{R}(t)$ and $D_t\mathbf{R}(t)$ are orthogonal.

11.2.10 Definition If $\mathbf{Q}$ is the vector-valued function given by

$$\mathbf{Q}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

then the *indefinite integral* and the *definite integral* from a to b of $\mathbf{Q}(t)$ are defined by

$$\int \mathbf{Q}(t)\,dt = \mathbf{i}\int f(t)\,dt + \mathbf{j}\int g(t)\,dt + \mathbf{k}\int h(t)\,dt \quad \text{and}$$

$$\int_a^b \mathbf{Q}(t)\,dt = \mathbf{i}\int_a^b f(t)\,dt + \mathbf{j}\int_a^b g(t)\,dt + \mathbf{k}\int_a^b h(t)\,dt$$

Further properties of integrals are given in Exercises 60–64.

11.2.11 Theorem Let the curve C have the vector equation $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f', g' and h' are continuous on the closed interval $[a, b]$. If L is the length of arc of C from the point $\mathbf{R}(a)$ to the point $\mathbf{R}(b)$ then

$$L = \int_a^b \|\mathbf{R}'(t)\|\,dt$$

Exercises 11.2

In Exercises 1–10, find $\mathbf{R}'(t)$ and $\mathbf{R}''(t)$.

1. $\mathbf{R}(t) = t\mathbf{i} + t^{-1}\mathbf{j}$. $\mathbf{R}'(t) = \mathbf{i} - t^{-2}\mathbf{j}$. $\mathbf{R}''(t) = 2t^{-3}\mathbf{j}$

2. $\mathbf{R}(t) = (t^2 - 3)\mathbf{i} + (2t + 1)\mathbf{j}$. $\mathbf{R}'(t) = 2t\mathbf{i} + 2\mathbf{j}$. $\mathbf{R}''(t) = 2\mathbf{i}$

3. $\mathbf{R}(t) = \frac{t-1}{t+1}\mathbf{i} + \frac{t-2}{t}\mathbf{j} = [1 - 2(t+1)^{-1}]\mathbf{i} + (1 - 2t^{-1})\mathbf{j}$. $\mathbf{R}'(t) = 2(t+1)^{-2}\mathbf{i} + 2t^{-2}\mathbf{j}$. $\mathbf{R}''(t) = -4(t+1)^{-3}\mathbf{i} - 4t^{-3}\mathbf{j}$

4. $\mathbf{R}(t) = (t^2 + 4)^{-1} + \sqrt{1 - 5t}\mathbf{j}$

▷ $\mathbf{R}'(t) = -2t(t^2+4)^{-2}\mathbf{i} - \frac{5}{2}(1-5t)^{-1/2}\mathbf{j}$

$$\mathbf{R}'' = [-2(t^2+4)^{-2} + (-2t)(-4t)(t^2+4)^{-3}]\mathbf{i} + \left[-\frac{5}{2}\left(-\frac{1}{2}\right)(1-5t)^{-3/2}(-5)\right]\mathbf{j}$$
$$= (t^2+4)^{-3}[-2(t^2+4) + 8t^2]\mathbf{i} - \frac{25}{4}(1-5t)^{-3/2}\mathbf{j} = (6t^2-8)(t^2+4)^{-3}\mathbf{i} - \frac{25}{4}(1-5t)^{-3/2}\mathbf{j}$$

5. $\mathbf{R}(t) = e^{2t}\mathbf{i} + \ln t\mathbf{j} + t^2\mathbf{k}$. $\mathbf{R}'(t) = 2e^{2t}\mathbf{i} + t^{-1}\mathbf{j} + 2t\mathbf{k}$, $t > 0$. $\mathbf{R}''(t) = 4e^{2t} - t^{-2}\mathbf{j} + 2\mathbf{k}$, $t > 0$.

6. $\mathbf{R}(t) = \cos 2t\mathbf{i} + \tan t\mathbf{j} + t\mathbf{k}$. $\mathbf{R}'(t) = -2\sin 2t\mathbf{i} + \sec^2 t\mathbf{j} + \mathbf{k}$. $\mathbf{R}''(t) = -2\cos 2t\mathbf{i} + 2\sec^2 t\tan t\mathbf{j}$

7. $\mathbf{R} = \tan^{-1} t\mathbf{i} + \sin^{-1} t\mathbf{j} + \cos^{-1} t\mathbf{k}$. $\mathbf{R}' = \frac{1}{1+t^2}\mathbf{i} + \frac{1}{\sqrt{1-t^2}}\mathbf{j} - \frac{1}{\sqrt{1-t^2}}\mathbf{k}$. $\mathbf{R}'' = -\frac{2t}{(1+t^2)^2}\mathbf{i} + \frac{t}{(1-t^2)^{3/2}}\mathbf{j} - \frac{t}{(1-t^2)^{3/2}}$

8. $\mathbf{R}(t) = (e^{3t} + 2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3 \cdot 2^t\mathbf{k}$

▷ $\mathbf{R}'(t) = 3e^{3t}\mathbf{i} + 6e^{3t}\mathbf{j} + 3(\ln 2)2^t\mathbf{k}$

$\mathbf{R}''(t) = 9e^{3t}\mathbf{i} + 18e^{3t}\mathbf{j} + 3(\ln 2)^2 2^t\mathbf{k}$

9. $\mathbf{R}(t) = 5\sin 2t\mathbf{i} - \sec 4t\mathbf{j} + 4\cos 2t\mathbf{k}$

▷ $\mathbf{R}''(t) = 10\cos 2t\mathbf{i} - 4\sec 4t\tan 4t\mathbf{j} - 8\sin 2t\mathbf{k}$. $\mathbf{R}''(t) = -20\sin 2t\mathbf{i} + (16\sec 4t - 32\sec^3 4t)\mathbf{j} - 16\cos 2t\mathbf{k}$

10. $\mathbf{R}(t) = \tan 3t\mathbf{i} + \ln\sin t\mathbf{j} - t^{-1}\mathbf{k}$. $\mathbf{R}''(t) = 3\sec^2 3t + \cot t\mathbf{j} + t^{-2}\mathbf{k}$. $\mathbf{R}''(t) = 18\sec^2 3t\tan 3t\mathbf{i} - \csc^2 t\mathbf{j} - 2t^{-3}\mathbf{k}$

In Exercises 11–14, find $D_t\|\mathbf{R}(t)\|$.

11. $\mathbf{R}(t) = (t-1)\mathbf{i} + (2-t)\mathbf{j}$

▷ $\|\mathbf{R}(t)\| = \sqrt{(t-1)^2 + (2-t)^2} = \sqrt{2t^2 - 6t + 5}$. $D_t\|\mathbf{R}(t)\| = \frac{4t-6}{2\sqrt{2t^2-6t+5}} = \frac{2t-3}{\sqrt{2t^2-6t+5}}$

12. $\mathbf{R}(t) = (e^t + 1)\mathbf{i} + (e^t - 1)\mathbf{j}$

▷ $\|\mathbf{R}(t)\| = \sqrt{(e^t+1)^2 + (e^t-1)^2} = \sqrt{e^{2t} + 2e^t + 1 + e^{2t} - 2e^t + 1} = \sqrt{2e^{2t} + 2}$

Thus,

$$D_t\|\mathbf{R}(t)\| = \tfrac{1}{2}(2e^{2t}t + 2)^{-1/2}(4e^{2t}) = \frac{2e^{2t}}{\sqrt{2e^{2t}+2}}$$

13. $\mathbf{R}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + 2e^{3t}\mathbf{k}$. $\|\mathbf{R}(t)\| = \sqrt{\sin^2 3t + \cos^2 3t + 4e^{6t}} = \sqrt{1 + 4e^{6t}}$. $D_t\|\mathbf{R}(t)\| = \frac{12e^{6t}}{\sqrt{1+4e^{6t}}}$

14. $\mathbf{R}(t) = \sqrt{t^2+1}\mathbf{i} + \sqrt{t^2-1}\mathbf{j} + t\mathbf{k}$. $\|\mathbf{R}(t)\| = \sqrt{t^2 + 1 + t^2 - 1 + t^2} = \sqrt{3}|t|$. $D_t\|\mathbf{R}(t)\| = \sqrt{3}\operatorname{sgn}(t)$

In Exercises 15 and 19, $\mathbf{R}(t) = (t^2 + e^t)\mathbf{i} + (t - e^{2t})\mathbf{j}$; $\mathbf{Q}(t) = (t^3 + 2e^t)\mathbf{i} - (3t + e^{2t})\mathbf{j}$

15. $D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t[(t^2 + t^3 + 3e^t)\mathbf{i} + (-2t - 2e^{2t})\mathbf{j}] = (2t + 3t^2 + 3e^t)\mathbf{i} + (-2 - 4e^{2t})\mathbf{j}$ and

$D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) = [(2t + e^t)\mathbf{i} + (1 - 2e^{2t})\mathbf{j}] + [(3t^2 + 2e^t)\mathbf{i} - (3 + 2e^{2t})\mathbf{j}] = (2t + 3t^2 + 3e^t)\mathbf{i} + (-2 - 4e^{2t})\mathbf{j}$

19. $D_t[\mathbf{R}(t) \cdot \mathbf{Q}(t)] = D_t[(t^2 + e^t)(t^3 + 2e^t) - (t - e^{2t})(3t + e^{2t})]$

$= (2t + e^t)(t^3 + 2e^t) + (t^2 + e^t)(3t^2 + 2e^t) - (1 - 2e^{2t})(3t + e^{2t}) - (t - e^{2t})(3 + 2e^{2t})$ and

$D_t\mathbf{R}(t) \cdot \mathbf{Q}(t) + \mathbf{R}(t) \cdot D_t\mathbf{Q}(t)$

$= [(2t + e^t)\mathbf{i} + (1 - 2e^{2t})\mathbf{j}] \cdot [(t^3 + 2e^t)\mathbf{i} - (3t + e^{2t})\mathbf{j}] + [(t^2 + e^t)\mathbf{i} + (t - e^{2t})\mathbf{j}] \cdot [(3t^2 + 2e^t)\mathbf{i} - (3 + 2e^{2t})\mathbf{j}]$

$= [(2t + e^t)(t^3 + 2e^t) - (1 - 2e^{2t})(3t + e^{2t})] + [(t^2 + e^t)(3t^2 + 2e^t) - (1 - e^{2t})(3 + 2e^{2t})]$

In Exercises 16 and 20, $\mathbf{R}(t) = \cos 2t\mathbf{i} - \sin 2t\mathbf{j}$ and $\mathbf{Q}(t) = \sin^2 t\mathbf{i} + \cos 2t\mathbf{j}$.

16. Verify Theorem 11.2.4 for the given **R** and **Q**.

▹
$$\begin{aligned} D_t[\mathbf{R}(t)+\mathbf{Q}(t)] &= D_t[(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) + (\sin^2 t\mathbf{i} + \cos 2t\mathbf{j})] \\ &= D_t[(\cos 2t + \sin^2 t)\mathbf{i} + (-\sin 2t + \cos 2t)\mathbf{j} \\ &= (-2\sin 2t + 2\sin t\cos t)\mathbf{i} + (-2\cos 2t - 2\sin 2t)\mathbf{j} \qquad (1) \end{aligned}$$

and

$$\begin{aligned} D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) &= D_t(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) + D_t(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= (-2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j}) + (2\sin t\cos t\mathbf{i} - 2\sin 2t)\mathbf{j} \\ &= (-2\sin 2t + 2\sin t\cos t)\mathbf{i} + (-2\cos 2t - 2\sin 2t)\mathbf{j} \qquad (2) \end{aligned}$$

and by comparing Eqs. (1) and (2) we verify Theorem 11.2.4. That is,

$$D_t[\mathbf{R}(t)+\mathbf{Q}(t)] = D_t\mathbf{R}(t) + D_t\mathbf{Q}(t)$$

20. Verify Theorem 11.2.5 for the vectors

▹
$$\begin{aligned} \mathbf{R}(t)\cdot\mathbf{Q}(t) &= (\cos 2t\mathbf{i} - \sin 2t\mathbf{j})\cdot(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= \sin^2 t\cos 2t - \sin 2t\cos 2t \end{aligned}$$

$$\begin{aligned} D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] &= \sin^2 t(-2\sin 2t) + \cos 2t(2\sin t\cos t) - [\sin 2t(-2\sin 2t) + \cos 2t(2\cos 2t)] \\ &= -2\sin^2 t\sin 2t + 2\sin t\cos t\cos 2t + 2\sin^2 2t - 2\cos^2 2t \qquad (1) \end{aligned}$$

Because

$$\begin{aligned} D_t\mathbf{R}(t) &= D_t(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) \\ &= -2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j} \end{aligned}$$

then

$$\begin{aligned} [D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) &= (-2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j})\cdot(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= -2\sin^2 t\sin 2t - 2\cos^2 2t \qquad (2) \end{aligned}$$

Furthermore,

$$\begin{aligned} D_t\mathbf{Q}(t) &= D_t(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= 2\sin t\cos t\mathbf{i} - 2\sin 2t\mathbf{j} \end{aligned}$$

and thus

$$\begin{aligned} \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)] &= (\cos 2t\mathbf{i} - \sin 2t\mathbf{j})\cdot(2\sin t\cos t\mathbf{i} - 2\sin 2t\mathbf{j}) \\ &= 2\sin t\cos t\cos 2t + 2\sin^2 2t \qquad (3) \end{aligned}$$

By adding the corresponding sides of Eqs. (2) and (3) we obtain

$$[D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) + \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)] = -2\sin^2 t\sin 2t - 2\cos^2 2t + 2\sin t\cos t\cos 2t + 2\sin^2 2t \qquad (4)$$

and by comparing Eqs. (1) and (4) we verify Theorem 14.4.47. That is,

$$D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] = [D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) + \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)]$$

In Exercises 17, 21 and 25, $\mathbf{R}(t) = 2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k}$ and $\mathbf{Q}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$

17.
$$\begin{aligned} D_t[\mathbf{R}(t)+\mathbf{Q}(t)] &= D_t[(2\sin t + \cos t)\mathbf{i} + (\cos t + 2\sin t)\mathbf{j} + (-\sin 2t + 1)\mathbf{k} \\ &= (2\cos t - \sin t)\mathbf{i} + (-\sin t + 2\cos t)\mathbf{j} + (-2\cos 2t)\mathbf{k} \qquad (1) \\ D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) &= (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k}) + (-\sin t\mathbf{i} + 2\cos t\mathbf{j}) = (1) \end{aligned}$$

21.
$$\begin{aligned} D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] &= D_t(2\sin t\cos t + 2\sin t\cos t - \sin 2t) = D_t\sin 2t = 2\cos 2t \\ \mathbf{R}'\cdot\mathbf{Q} + \mathbf{R}\cdot\mathbf{Q}' &= (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k})\cdot(\cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}) + \\ &\qquad (2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k})\cdot(-\sin t\mathbf{i} + 2\cos t\mathbf{j}) \\ &= (2\cos^2 t - 2\sin^2 t - 2\cos 2t) + (-2\sin^2 t + 2\cos^2 t) = 2\cos 2t \end{aligned}$$

25.
$$\begin{aligned} D_t[\mathbf{R}(t)\times\mathbf{Q}(t)] &= D_t\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sin t & \cos t & -\sin 2t \\ \cos t & 2\sin t & 1 \end{vmatrix} \\ &= D_t[(\cos t + 2\sin t\sin 2t)\mathbf{i} + (-2\sin t - \sin 2t\cos t)\mathbf{j} + (4\sin^2 t - \cos^2 t)\mathbf{k} \\ &= (-\sin t + 4\cos 2t\sin t + 2\sin 2t\cos t)\mathbf{i} + (-2\cos t - 2\cos 2t\cos t + \sin 2t\sin t)\mathbf{j} + 10\sin t\cos t\mathbf{k} \qquad (1) \end{aligned}$$

$$\begin{aligned} D_t\mathbf{R}\times\mathbf{Q} &= (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k})\times\mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos t & -\sin t & -2\cos 2t \\ \cos t & 2\sin t & 1 \end{vmatrix} \\ &= (-\sin t + 4\cos 2t\sin t)\mathbf{i} + (-2\cos 2t - 2\cos 2t\cos t)\mathbf{j} + 5\sin t\cos t\mathbf{k} \qquad (2) \end{aligned}$$

$$\mathbf{R}\times D_t\mathbf{Q} = (2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k})\times(-\sin t\mathbf{i} + 2\cos t\mathbf{j}) = 2\sin 2t\cos t\mathbf{i} + \sin 2t\sin t\mathbf{j} + 5\sin t\cos t \qquad (3)$$

and (2) + (3) = (1).

In Exercises 18, 22 and 26, $\mathbf{R}(t) = e^{3t}\mathbf{i} - 4e^{3t}\mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q}(t) = e^{t}\mathbf{i} - e^{t}\mathbf{j} - 2e^{4t}\mathbf{k}$

18. $\mathrm{D}_t(\mathbf{R}+\mathbf{Q}) = \mathrm{D}_t\mathbf{R} + \mathrm{D}_t\mathbf{Q} = (3e^{3t} + e^{t})\mathbf{i} + (-12e^{3t} - e^{t})\mathbf{j} + (-8e^{4t})$

22. $\mathrm{D}_t(\mathbf{R}\cdot\mathbf{Q}) = \mathbf{R}'\cdot\mathbf{Q} + \mathbf{R}\cdot\mathbf{Q}' = 36e^{4t}$

26. $\mathrm{D}_t(\mathbf{R}\times\mathbf{Q}) = \mathbf{R}'\times\mathbf{Q} + \mathbf{R}\times\mathbf{Q}' = (56e^{7t} - 2e^{t})\mathbf{i} + (14e^{7t} - 2e^{t})\mathbf{j} + 12e^{4t}\mathbf{k}$

23. $f(t) = \cos 2t$, $\mathbf{R}(t) = 5\sin 2t\mathbf{i} - \sec 4t\mathbf{j} + 4\cos 2t\mathbf{k}$

$\mathrm{D}_t(f\mathbf{R}) = \mathrm{D}_t[5\sin 2t\cos 2t\mathbf{i} - \cos 2t\sec 4t\mathbf{j} + 4\cos^2 2t\mathbf{k}]$

$= 10(\cos^2 2t - \sin^2 2t)\mathbf{i} + (2\sin 2t\sec 4t - 4\cos 2t\sec 4t\tan 4t)\mathbf{j} = f'\mathbf{R} + f\mathbf{R}'$

24. Verify Theorem 11.2.6 for the functions $f(t) = e^{t}$; $\mathbf{R}(t) = (e^{3t}+2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3\cdot 2^{t}\mathbf{k}$

▷ $\mathrm{D}_t[f(t)\mathbf{R}(t)] = \mathrm{D}_t[(e^{4t} + 2e^{t})\mathbf{i} + 2e^{4t}\mathbf{j} + 3(2e)^{t}] = (4e^{4t} + 2e^{t})\mathbf{i} + 8e^{4t}\mathbf{j} + 3(1+\ln 2)(2e)^{t}\mathbf{k}$ (1)

$f'(t)\mathbf{R}(t) + f(t)\mathbf{R}'(t) = e^{t}[(e^{3t}+2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3\cdot 2^{t}\mathbf{k}] + e^{t}[(4e^{4t} + 2e^{t})\mathbf{i} + 6e^{3t}\mathbf{j} + 3(\ln 2)2^{t}\mathbf{k}]$

$= (4e^{4t} + 2e^{t})\mathbf{i} + 8e^{4t}\mathbf{j} + 3(1+\ln 2)(2e)^{t}\mathbf{k}$ (2)

Comparing (1) and (2), we have verified Theorem 11.2.6.

In Exercises 27 and 28, verify Theorem 11.2.8 for the given functions.

27. $\mathbf{F}(\phi) = \phi\mathbf{i} + \phi^{2}\mathbf{j} + \ln\phi\mathbf{k}$ and $h(t) = e^{t}$. $\mathrm{D}_t(\mathbf{F}(h(t)) = \mathrm{D}_t(e^{t}\mathbf{i} + e^{2t}\mathbf{j} + t\mathbf{k}) = e^{t}\mathbf{i} + 2e^{2t}\mathbf{j} + \mathbf{k}$

$\mathbf{F}'(\phi) = \mathbf{i} + 2\phi\mathbf{j} + \phi^{-1}\mathbf{k}$. $\mathbf{F}'(h(t))h'(t) = (\mathbf{i} + 2e^{t}\mathbf{j} + e^{-t}\mathbf{k})e^{t} = e^{t}\mathbf{i} + 2e^{2t}\mathbf{j} + \mathbf{k}$

28. $\mathbf{F}(\phi) = \sin\phi\mathbf{i} + \cos\phi\mathbf{j} + \phi\mathbf{k}$ and $h(t) = \sin^{-1}t$

▷ $\mathrm{D}_t(\mathbf{F}(h(t)) = \mathrm{D}_t(t\mathbf{i} + \sqrt{1-t^2}\mathbf{j} + \sin^{-1}t\mathbf{k}) = \mathbf{i} - \dfrac{t}{\sqrt{1-t^2}}\mathbf{j} + \dfrac{1}{\sqrt{1-t^2}}\mathbf{k}$ (1)

We have

$$\mathbf{F}'(\phi) = \cos\phi\mathbf{i} - \sin\phi\mathbf{j} + \mathbf{k} \text{ and } h'(t) = \frac{1}{\sqrt{1-t^2}}$$

Thus,

$\mathbf{F}'(h(t))h'(t) = (\sqrt{1-t^2}\mathbf{i} - t\mathbf{j} + \mathbf{k})\dfrac{1}{\sqrt{1-t^2}} = \mathbf{i} - \dfrac{t}{\sqrt{1-t^2}}\mathbf{j} + \dfrac{1}{\sqrt{1-t^2}}\mathbf{k}$ (2)

Comparing (1) and (2), we have verified Theorem 11.2.8.

In Exercises 29–31, let $\mathbf{R}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}$ and $\mathbf{Q}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}$.

29. $\mathrm{D}_t[\mathbf{R}(t) + \mathbf{Q}(t)] = \mathrm{D}_t\{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]\}$

$= \mathrm{D}_t\{[f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j} + [h_1(t) + h_2(t)]\}$

$= [f_1'(t) + f_2'(t)]\mathbf{i} + [g_1'(t) + g_2'(t)]\mathbf{j} + [h_1'(t) + h_2'(t)]$

$= [f_1'(t)\mathbf{i} + g_1'(t)\mathbf{j} + h_1'(t)] + [f_2'(t)\mathbf{i} + g_2'(t)\mathbf{j} + h_2'(t)] = \mathrm{D}_t\mathbf{R}(t) + \mathrm{D}_t\mathbf{Q}(t)$

30. $\mathrm{D}_t[f\mathbf{R}] = \mathrm{D}_t[ff_1\mathbf{i} + fg_1\mathbf{j} + fh_1\mathbf{k}] = (f'f_1 + ff_1)\mathbf{i} + (f'g_1 + fg_1')\mathbf{j} + (f'h_1 + fh_1')\mathbf{k}$

$= f'(f_1\mathbf{i} + g_1\mathbf{j} + h_1\mathbf{k}) + f(f_1'\mathbf{i} + g_1'\mathbf{j} + h_1'\mathbf{k}) = f'\mathbf{R} + f\mathbf{R}'$

31. The derivative of $\mathbf{R}\times\mathbf{Q}$ is an immediate consequence of the following determinant theorem. When we use the rule for derivative of a product, we write the three terms vertically.

$$\mathrm{D}_t\begin{vmatrix} f & g & h \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix} = \mathrm{D}_t\left[f\begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix} - g\begin{vmatrix} f_1 & h_1 \\ f_2 & h_2 \end{vmatrix} + h\begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix}\right]$$

$= \mathrm{D}_t[fg_1h_2 - fg_2h_1 - f_1gh_2 + f_1gh_2 + f_2gh_1 + f_1g_2h - f_2g_1h]$

$= [f'g_1h_2 - f'g_2h_1 - f_1'gh_2 + f_1'gh_2 + f_2'gh_1 + f_1'g_2h - f_2'g_1h]$

$+ [fg_1'h_2 - fg_2'h_1 - f_1g'h_2 + f_1g'h_2 + f_2g'h_1 + f_1g_2'h - f_2g_1'h]$

$+ [fg_1h_2' - fg_2h_1' - f_1gh_2' + f_1gh_2' + f_2gh_1' + f_1g_2h' - f_2g_1h']$

$$= \begin{vmatrix} f' & g' & h' \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} f & g & h \\ f_1' & g_1' & h_1' \\ f_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} f & g & h \\ f_1 & g_1 & h_1 \\ f_2' & g_2' & h_2' \end{vmatrix}$$

If we replace f, g, h with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the first determinant is 0 because the the derivative of a constant is 0.

32. Prove Theorem 11.2.8

▷ We have $\phi = \mathrm{h}(t)$ and $\mathbf{F}(\mathrm{h}(t)) = f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}$

Applying the chain rule for real-valued function, we obtain

$$\begin{aligned}\mathrm{D}_t\mathbf{F}(h(t)) &= \mathrm{D}_t[f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}] = [\mathrm{D}_tf(\phi)]\mathbf{i} + [\mathrm{D}_tg(\phi)]\mathbf{j} + [\mathrm{D}_th(\phi)]\mathbf{k} \\ &= [\mathrm{D}_\phi f(\phi)\mathrm{D}_t\phi]\mathbf{i} + [\mathrm{D}_\phi g(\phi)\mathrm{D}_t\phi]\mathbf{j} + [\mathrm{D}_\phi h(\phi)\mathrm{D}_t\phi]\mathbf{k} = \{[\mathrm{D}_\phi f(\phi)]\mathbf{i} + [\mathrm{D}_\phi g(\phi)]\mathbf{j} + [\mathrm{D}_\phi h(\phi)]\mathbf{k}\}\mathrm{D}_t\phi \\ &= \mathrm{D}_\phi[f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}]\mathrm{D}_t\phi = \mathbf{F}'(h(t))\mathrm{D}_t\phi\end{aligned}$$

In Exercises 33–40, find the most general vector-valued function whose derivative has the given function value.

33. $\mathbf{i}\int \tan t\,dt - \mathbf{j}\int \frac{1}{t}\,dt = (\ln|\sec t| + C_1)\mathbf{i} - (\ln|t| + C_2)\mathbf{j} = \ln|\sec t|\mathbf{i} - \ln|t|\mathbf{j} + \mathbf{C}$

34. $\int [(t^2 - 9)\mathbf{i} + (2t - 5)\mathbf{j}]dt = (\frac{1}{3}t^3 - 9t)\mathbf{i} + (t^2 - 5t)\mathbf{j} + \mathbf{C}$

35. Let $\mathbf{R}'(t) = e^{3t}\mathbf{i} + \frac{1}{t-1}\mathbf{j}$. Then $\mathbf{R}(t) = \mathbf{i}\int e^{3t}dt + \mathbf{j}\int \frac{dt}{t-1} = \frac{1}{3}e^{3t}\mathbf{i} + \ln|t-1|\mathbf{j} + \mathbf{C}$.

36. $\frac{1}{4+t^2}\mathbf{i} - \frac{4}{1-t^2}\mathbf{j}$

▸ Let $\mathbf{R}'(t) = \ln t\mathbf{i} + t^2\mathbf{j}$. Then $\mathbf{R}(t) = \mathbf{i}\int \ln t\,dt + \mathbf{j}\int t^2dt = (t\ln t - t)\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + \mathbf{C}$.

37. $\int [e^{3t}\mathbf{i} + e^{-3t}\mathbf{j} - te^{3t}\mathbf{k}]dt = \frac{1}{3}e^{3t}\mathbf{i} - \frac{1}{3}e^{-3t}\mathbf{j} + (-\frac{1}{3}te^{3t} + \frac{1}{9}e^{3t})\mathbf{k} + \mathbf{C}$

38. $\int [3^t\mathbf{i} - 2^t\mathbf{j} + e^t\mathbf{k}]dt = \frac{3^t}{\ln 3}\mathbf{i} - \frac{2^t}{\ln 2}\mathbf{j} + e^t\mathbf{k} + \mathbf{C}$

39. $\int [\tan t\mathbf{i} + \sec t\mathbf{j} + \frac{1}{t}\mathbf{k}]dt = \ln|\sec t|\mathbf{i} + \ln|\sec t + \tan t|\mathbf{j} + \ln|t|\mathbf{k} + \mathbf{C}$

40. $t\sin t\mathbf{i} - t\cos t\mathbf{j} + t\mathbf{k}$

▸ We use integration by parts. Let

$$\begin{aligned} u &= t & dv &= \sin t\mathbf{i} + \cos t\mathbf{j} \\ du &= dt & v &= -\cos t\mathbf{i} + \sin t\mathbf{j} \end{aligned}$$

$$\begin{aligned} \int [t(\sin t\mathbf{i} + \cos t\mathbf{j}) + t\mathbf{k}]dt &= t(-\cos t\mathbf{i} + \sin t\mathbf{j}) - \int [-\cos t\mathbf{i} + \sin t\mathbf{j}]dt + \tfrac{1}{2}t^2\mathbf{k} \\ &= (-t\cos t + \sin t)\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \tfrac{1}{2}t^2\mathbf{k} + \mathbf{C} \end{aligned}$$

41. $\mathbf{R}(3) + \int_3^t \mathbf{R}'(t)dt = (2\mathbf{i} - 5\mathbf{j}) + \int_3^t (t^2\mathbf{i} + \frac{1}{t-2}\mathbf{j})dt = (2\mathbf{i} - 5\mathbf{j}) + \left[\frac{1}{3}t^3\mathbf{i} + \ln|t-2|\mathbf{j}\right]_3^t = (\frac{1}{3}t^3 - 7)\mathbf{i} + (\ln|t-2| - 5)\mathbf{j}$

42. $\mathbf{R}(t) = \int \mathbf{R}'(t)dt = \mathbf{i}\int \sin^2 t\,dt + \mathbf{j}\int 2\cos^2 t\,dt = \frac{1}{4}(2t - \sin 2t)\mathbf{i} + \frac{1}{2}(2t + \sin 2t)\mathbf{j} + \mathbf{C}$

$\mathbf{R}(\pi) = \mathbf{0} \Rightarrow \mathbf{0} = \frac{1}{2}\pi\mathbf{i} + \pi\mathbf{j} + \mathbf{C}$; $\mathbf{C} = -\frac{1}{2}\pi\mathbf{i} - \pi\mathbf{j}$. Thus $\mathbf{R}(t) = \frac{1}{2}(t - \frac{1}{2}\sin 2t - \pi)\mathbf{i} + (t + \frac{1}{2}\sin 2t - \pi)\mathbf{j}$

43. $\mathbf{R}(0) + \int_0^t \mathbf{R}'(t)dt = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \int_0^t (e^t\sin t\mathbf{j} + \cos t\mathbf{j} - e^t\mathbf{k})dt = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \left[\frac{1}{2}\sqrt{2}\sin(t - \frac{1}{4}\pi)\mathbf{i} + \sin t\mathbf{j} - e^t\mathbf{k}\right]_0^t$

$= [\frac{1}{2}e^t(\sin t - \cos t) + \frac{3}{2}]\mathbf{i} + (\sin t - 1)\mathbf{j} + (2 - e^t)\mathbf{k}$

44. If $\mathbf{R}'(t) = \frac{1}{t+1}\mathbf{i} - \tan t\mathbf{j} + \frac{t}{t^2-1}\mathbf{k}$ and $\mathbf{R}(0) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, find $\mathbf{R}(t)$.

▸ $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{R}'(t)dt = \mathbf{R}(0) + \int_0^t \left(\frac{1}{t+1}\mathbf{i} - \tan t\mathbf{j} + \frac{t}{t^2-1}\right)dt$

$= (4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + \left[\ln|t+1| + \ln|\cos t| + \frac{1}{2}\ln|t^2-1|\right]_0^t = (\ln|t+1| + 4)\mathbf{i} + (\ln|\cos t| - 3)\mathbf{j} + (\frac{1}{2}\ln|t^2-1| + 5)\mathbf{k}$

In Exercises 45 and 46, (a) find a cartesian equation of curve $\mathbf{R}'(t)$; (b) compute $\mathbf{R}(t)\cdot\mathbf{R}'(t)$. Interpret the result.

45. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. $\mathbf{R}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. Parametric equations for $\mathbf{R}'$ are $x = -\sin t$ and $y = \cos t$; a cartesian equation is $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, the unit circle. $\mathbf{R}(t)\cdot\mathbf{R}'(t) = -\cos t\sin t + \sin t\cos t = 0$ The representation of $\mathbf{R}'(t_1)$ having its initial point at $\mathbf{R}(t_1)$ is along the tangent line to the circle at $\mathbf{R}(t_1)$; the position representation of $\mathbf{R}(t_1)$ is the radius of the circle which is perpendicular to the tangent there.

46. $\mathbf{R}(t) = \cosh t\mathbf{i} - \sinh t\mathbf{j}$. $\mathbf{R}'(t) = \sinh t\mathbf{i} - \cosh t\mathbf{j}$. Parametric equations for $\mathbf{R}$ are $x = \cosh t$, $y = -\sinh t$; a cartesian equation is $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, $x \geq 1$, hyperbola. $\mathbf{R}(t)\cdot\mathbf{R}'(t) = 2\cosh t\sinh t = \sinh 2t$

In Exercises 47 and 48, if α is the radian measure of the angle between $\mathbf{R}(t)$ and $\mathbf{Q}(t)$, find $D_t\alpha(t)$.

47. $\mathbf{R}(t) = 3e^{2t}\mathbf{i} - 4e^{2t}\mathbf{j} = e^{2t}(3\mathbf{i} - 4\mathbf{j})$ and $\mathbf{Q}(t) = 6e^{3t}\mathbf{j}$. $\mathbf{R}(t)$ has the same direction as $3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{Q}(t)$ has the same direction as $\mathbf{j}$. Thus $\alpha(t)$ is a constant and $D_t\alpha(t) = 0$.

48. Given $\mathbf{R}(t) = 2t\mathbf{i} + (t^2 - 1)\mathbf{j}$ and $\mathbf{Q}(t) = 3t\mathbf{i}$.

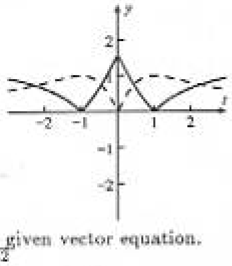
▹ The derivative will not exist at $t = 0$ because $\mathbf{Q}$ reverses its direction there, and at $t = \pm 1$ because the undirected angle switches abruptly between decreasing and increasing there. The figure shows $\cos\alpha(t)$ dashed and $\alpha(t)$ solid.

$$\cos\alpha(t) = \frac{\mathbf{R}(t)\cdot\mathbf{Q}(t)}{\|\mathbf{R}(t)\|\|\mathbf{Q}(t)\|} = \frac{6t^2}{\sqrt{(2t)^2 + (t^2-1)^2}\sqrt{(3t)^2}} = \frac{6t^2}{\sqrt{(t^2+1)^2}|3t|} = \frac{2|t|}{t^2+1}$$

Thus,

$$\alpha(t) = \cos^{-1}\frac{2|t|}{t^2+1}$$

Because $D_t|t| = t/|t|$, then



$$\alpha'(t) = \frac{-1}{\sqrt{1 - \left(\frac{2|t|}{t^2+1}\right)^2}}\cdot\frac{(t^2+1)\frac{2t}{|t|} - 2|t|(2t)}{(t^2+1)^2} = \frac{-(t^2+1)}{\sqrt{(t^2-1)^2}}\cdot\frac{-2t(t^2-1)}{|t|(t^2+1)^2}$$

$$= \frac{2t(t^2-1)}{|t||t^2-1|(t^2+1)} = \frac{2t}{t^2+1}\operatorname{sgn}[t(t^2-1)]$$

In Exercises 49–52, find the exact length L of arc from t_1 to t_2 of the curve having the given vector equation.

49. $\mathbf{R}(t) = (t+1)\mathbf{i} - t^2\mathbf{j} + (1-2t)\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|\mathbf{i} - 2t\mathbf{j} - 2\mathbf{k}\| = \sqrt{1 + 4t^2 + 4} = \sqrt{5 + 4t^2}$

$$L = \int_{-1}^{2}\|D_t\mathbf{R}(t)\|dt = \int_{-1}^{2}\sqrt{5+4t^2}\,dt = 2\int_{-1}^{2}\sqrt{t^2+\tfrac{5}{4}}\,dt = 2\left[\tfrac{t}{2}\sqrt{t^2+\tfrac{5}{4}} + \tfrac{5}{2}\ln\left|t + \sqrt{t^2+\tfrac{5}{4}}\right|\right]_{-1}^{2}$$

$$= 2\left(\sqrt{4+\tfrac{5}{4}} + \tfrac{5}{8}\ln\left|2+\sqrt{5+\tfrac{5}{4}}\right|\right) - 2\left(-\tfrac{1}{2}\sqrt{1+\tfrac{5}{4}} + \tfrac{5}{8}\ln\left|-1+\sqrt{1+\tfrac{5}{4}}\right|\right)$$

$$= \sqrt{21} + \tfrac{5}{4}\ln(2 + \tfrac{1}{2}\sqrt{21}) + \tfrac{3}{2} - \tfrac{5}{4}\ln\tfrac{1}{2} = \sqrt{21} + \tfrac{3}{2} + \tfrac{5}{4}\ln(4 + \sqrt{21})$$

50. $\mathbf{R}(t) = \sin 2t\mathbf{i} + \cos 2t\mathbf{j} + 2t^{3/2}\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j} + 3t^{1/2}\mathbf{k}\| = \sqrt{4\cos^2 2t + 4\sin^2 2t + 9t}$ $= \sqrt{4 + 9t}$. $L = \int_0^1 (4+9t)^{1/2}dt = \frac{1}{9}\cdot\frac{2}{3}(4+9t)^{3/2}\Big]_0^1 = \frac{2}{27}(13^{3/2} - 8)$

51. $\mathbf{R}(t) = 4t^{3/2}\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|6t^{1/2}\mathbf{i} - 3\cos t\mathbf{j} - 3\sin t\mathbf{k}\| = \sqrt{36t + 9\cos^2 t + 9\sin^2 t} = 3\sqrt{4t+1}$

$L = \int_0^2 3(4t+1)^{1/2}dt = 3\cdot\frac{1}{4}\cdot\frac{2}{3}(4t+1)^{3/2}\Big]_0^2 = \frac{1}{2}(27 - 1) = 13$

52. $\mathbf{R}(t) = t^2\mathbf{i} + (t + \frac{1}{3}t^3)\mathbf{j} + (t - \frac{1}{3}t^3)\mathbf{k}$; $t_1 = 0$; $t_2 = 1$

▹ $\mathbf{R}'(t) = 2t\mathbf{i} + (1 + t^2)\mathbf{j} + (1 - t^2)\mathbf{k}$

$$\|\mathbf{R}'(t)\| = \sqrt{4t^2 + (1 + 2t^2 + t^4) + (1 - 2t^2 + t^4)} = \sqrt{2 + 4t^2 + 2t^4} = \sqrt{2(1 + 2t^2 + t^4)} = \sqrt{2}(1 + t^2)$$

$$L = \int_0^1 \sqrt{2}(1+t^2)dt = \sqrt{2}(t + \tfrac{1}{3}t^3)\Big]_0^1 = \tfrac{4}{3}\sqrt{2}$$

In Exercises 53–56, use NINT to approximate to 4 digits the length L of arc from t_1 to t_2 of the given curve.

53. $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. $\mathbf{R}'(t) = \mathbf{j} + 2t\mathbf{j} + 3t^2\mathbf{k}$, $L = \int_0^2\|\mathbf{R}'(t)\|dt = \int_0^2\sqrt{1 + 4t^2 + 9t^4}\,dt = 9.57057 \approx 9.571$

54. $\mathbf{R}(t) = e^t\mathbf{i} + e^t\mathbf{j} + \ln t\mathbf{k}$. $\mathbf{R}'(t) = e^t\mathbf{i} + e^t\mathbf{j} + t^{-1}\mathbf{k}$. $L = \int_1^2\|\mathbf{R}'(t)\|dt = \int_1^2\sqrt{2e^{2t} + t^{-2}}\,dt = 6.6510 \approx 6.651$

55. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^3\mathbf{k}$. $\mathbf{R}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2\mathbf{k}$. $L = \int_{-1}^{1}\|\mathbf{R}'\|dt = \int_{-1}^{1}\sqrt{1 + 9t^4}\,dt = 3.0957 \approx 3.096$

56. $\mathbf{R}(t) = \sin 2t\mathbf{i} + \cos 2t\mathbf{j} + t^{1/2}\mathbf{k}$; $t_1 = 0$; $t_2 = 4$

▹ $\mathbf{R}'(t) = 2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j} + \frac{1}{2}t^{-1/2}\mathbf{k}$

$$\|\mathbf{R}'(t)\| = \sqrt{4\cos^2 2t + 2\sin^2 2t + \frac{1}{4t}} = \sqrt{4 + \frac{1}{4t}}$$

Because $\|\mathbf{R}'(t)\|$ is unbounded at 0, the arc length integral is improper. We let $t = u^2$, $dt = 2u\,du$ to get a proper integral, then use NINT.

$$L = \int_{t=0}^{4}\sqrt{4 + \frac{1}{4t}}\,dt = \int_{u=0}^{2}\sqrt{4 + \frac{1}{4u^2}}(2u\,du) = \int_0^2\sqrt{16u^2 + 1}\,du = 8.4093 \approx 8.409$$

The exact value of the integral is $\sqrt{65} + \frac{1}{16}\ln(129 + 16\sqrt{65})$.

57. $D_t[\mathbf{R}'(t)\cdot\mathbf{R}(t)] = \mathbf{R}''(t)\cdot\mathbf{R}(t) + \mathbf{R}'(t)\cdot\mathbf{R}'(t) = \mathbf{R}''(t)\cdot\mathbf{R}(t) + \|\mathbf{R}'(t)\|^2$

58. $\mathbf{R}(t)\cdot\mathbf{R}(t) = \|\mathbf{R}(t)\|^2 = h^2(t)$. Differentiating, $\mathbf{R}'(t)\cdot\mathbf{R}(t) + \mathbf{R}(t)\cdot\mathbf{R}'(t) = 2h(t)h'(t) \Rightarrow \mathbf{R}(t)\cdot\mathbf{R}'(t) = h(t)h'(t)$

59. From Theorem 11.2.6, $D_t\left[\dfrac{\mathbf{R}(t)}{f(t)}\right] = D_t[f(t)^{-1}\mathbf{R}(t)] = -f(t)^{-2}f'(t)\mathbf{R}(t) + f(t)^{-1}\mathbf{R}'(t) = \dfrac{f(t)\mathbf{R}'(t) - f'(t)\mathbf{R}(t)}{[f(t)]^2}$

60. Prove that if **A** and **B** are constant vectors and f and g are integrable functions, then

$$\int[\mathbf{A}f(t) + \mathbf{B}g(t)]dt = \mathbf{A}\int f(t)dt + \mathbf{B}\int g(t)dt$$

(*Hint:* Express **A** and **B** in terms of **i**, **j**, and **k**.)

▸ Because **A** and **B** are constant vectors, there are constants such that
$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
Therefore

$$\begin{aligned}
&\int[\mathbf{A}f(t) + \mathbf{B}g(t)]dt \\
&= \int[(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})f(t) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})g(t)]dt \\
&= \int[(a_1f(t) + b_1g(t))\mathbf{i} + (a_2f(t) + b_2g(t))\mathbf{j} + (a_3f(t) + b_3g(t))\mathbf{k}]dt \\
&= \mathbf{i}\int[a_1f(t) + b_1g(t)]dt + \mathbf{j}\int[a_2f(t) + b_2g(t)]dt + \mathbf{k}\int[a_3f(t) + b_3g(t)]dt \qquad \text{(Definition 11.2.10)} \\
&= \mathbf{i}\left[a_1\int f(t)dt + b_1\int g(t)dt\right] + \mathbf{j}\left[a_2\int f(t)dt + b_2\int g(t)dt\right] + \mathbf{k}\left[a_3\int f(t)dt + b_3\int g(t)dt\right] \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\int f(t)dt + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\int g(t)dt = \mathbf{A}\int f(t)dt + \mathbf{B}\int g(t)dt
\end{aligned}$$

61. Let $\mathbf{R}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}$ and $\mathbf{Q}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}$. If $\mathbf{R}'(t) \equiv \mathbf{Q}'(t)$ then $f_1'(t) \equiv f_2'(t)$, $g_1'(t) \equiv g_2'(t)$, $h_1'(t) \equiv h_2'(t)$. By Theorem 4.1.2, there are constants k, l, m such that $f_1(t) = f_2(t) + k$, $g_1(t) = g_2(t) + l$, $h_1(t) = h_2(t) + m$ and so $\mathbf{R}(t) = \mathbf{Q}(t) + \mathbf{K}$ where $\mathbf{K} = k\mathbf{i} + l\mathbf{j} + m\mathbf{k}$.

62. Let $\mathbf{G}(t)$ be any antiderivative of **R**. Then $\mathbf{G}'(t) = \mathbf{F}'(t)$. By Exercise 62, $\mathbf{G}(t) = \mathbf{F}(t) + \mathbf{C}$.

63. Let $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. Then $D_t\displaystyle\int_a^t \mathbf{R}(u)du = D_t\left[\int_a^t \{f(u)\mathbf{i} + g(u)\mathbf{j} + h(u)\mathbf{k}\}du\right]$

$$= \left(D_t\int_a^t f(u)du\right)\mathbf{i} + \left(D_t\int_a^t g(u)du\right)\mathbf{j} + \left(D_t\int_a^t h(u)du\right)\mathbf{k} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{R}(t)$$

64. Use the theorems of Exercises 62 and 63 to prove the following theorem that corresponds to the second fundamental theorem of the calculus (4.7.2): If the function **R** is continuous on the closed interval $[a, b]$ and if $\mathbf{F}(t)$ is any antiderivative of **R** on $[a, b]$, then $\displaystyle\int_a^b \mathbf{R}(u)du = \mathbf{F}(b) - \mathbf{F}(a)$.

▸ Because by hypothesis $\mathbf{F}'(t) = \mathbf{R}(t)$, and by Exercise 63, $D_t\int_a^t \mathbf{R}(u)du = \mathbf{R}(t)$, it follows from Exercise 62 that

$$\mathbf{F}(t) = \int_a^t \mathbf{R}(u)du + \mathbf{C}$$

where **C** is a constant vector. Because $\int_a^a \mathbf{R}(u)du = 0$, it follows that

$$\mathbf{F}(b) - \mathbf{F}(a) = \left(\int_a^b \mathbf{R}(u)du + \mathbf{C}\right) - \left(\int_a^a \mathbf{R}(u)du + \mathbf{C}\right) = \int_a^b \mathbf{R}(u)du$$

11.3 THE UNIT TANGENT AND UNIT NORMAL VECTORS AND ARC LENGTH AS A PARAMETER

We assume that a curve has a direction (or orientation) implied by increasing values of the parameter.

11.3.1 Definition If $\mathbf{R}(t)$ is the position vector of curve C at a point P on C, then the *unit tangent vector* of C at P, denoted by $\mathbf{T}(t)$, is the unit vector in the direction of $D_t\mathbf{R}(t)$ if $D_t\mathbf{R}(t) \neq 0$.

Thus, denoting t-derivatives by dots,

$$\mathbf{T}(t) = \frac{D_t\mathbf{R}(t)}{\|D_t\mathbf{R}(t)\|} = \frac{\dot{\mathbf{R}}}{\|\dot{\mathbf{R}}\|} \tag{1}$$

11.3.2 Definition If $\mathbf{T}(t)$ is the unit tangent vector of curve C at a point P on C, then the *unit normal vector*, denoted by $\mathbf{N}(t)$ is the unit vector in the direction of $D_t\mathbf{T}(t)$. Thus

$$\mathbf{N}(t) = \frac{D_t\mathbf{T}(t)}{\|D_t\mathbf{T}(t)\|} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} \tag{2}$$

Moving Trihedral The *unit binormal vector* to the curve C is given by $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ The planes at P having $\mathbf{B}$, $\mathbf{N}$, and $\mathbf{T}$ as respective normal vectors are the *osculating*, *rectifying* and *normal plane*.

If F, f, g, and h are real-valued functions and $\mathbf{R}$ is the vector-valued function defined by
$\mathbf{R}(t) = \mathrm{F}(t)[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
then the magnitude of $\mathbf{R}(t)$ is given by
$\|\mathbf{R}(t)\| = |\mathrm{F}(t)|\sqrt{[f(t)]^2 + [g(t)]^2 + [h(t)]^2}$

Finding T and N We may multiply $\dot{\mathbf{R}}$ and $\dot{\mathbf{T}}$ by any positive function before using (1) and (2). See Exercise 4. For $\mathbf{N}$, we may use the following version of the quotient rule.

$$\mathrm{D}_t\frac{f(t)}{\sqrt{g(t)}} = \mathrm{D}_t\{f(t)[g(t)]^{-1/2}\} = f'(t)[g(t)]^{-1/2} - \tfrac{1}{2}f(t)[g(t)]^{-3/2}g'(t) = \frac{f'(t)g(t) - \frac{1}{2}f(t)g'(t)}{[g(t)]^{3/2}}$$

11.3.3 Theorem If the vector equation of a curve C is $\mathbf{R}(s) = f(s)\mathbf{i} + g(s)\mathbf{j} + h(s)\mathbf{k}$, where s units is the length of arc measured from a particular point P_0 on C to the point P, then the unit tangent vector of C at P is given by $\mathbf{T}(s) = \mathrm{D}_s\mathbf{R}(s)$ if it exists. Furthermore

$$\left(\frac{df}{ds}\right)^2 + \left(\frac{dg}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2 = 1 \tag{8}$$

Helix A curve which lies on a cylinder or cone and cuts the elements under constant angle. If the cylinder is circular, we have a *circular helix* with equation $\mathbf{R} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$, which is a circle if $c = 0$. See Exercises 8 ff. A *conical helix* has equation $\mathbf{R} = ae^t\cos t\mathbf{i} + ae^t\sin t\mathbf{j} + ce^t\mathbf{k}$ which is a logarithmic spiral if $c = 0$. See Exercises 10 ff.

Exercises 11.3 (Starred numbers refer to Exercises 11.4.)

In Exercises 1–6, find $\mathbf{T}(t)$, $\mathbf{T}(t_1)$, $\mathbf{N}(t)$, $\mathbf{N}(t_1)$, sketch a portion of the curve and representations of $\mathbf{T}(t_1)$ and $\mathbf{N}(t_1)$. In Exercises 1*–6*, use the formula $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\|$ find the curvature K; find the radius of curvature ρ. Sketch the curve, unit tangent vector, and circle of curvature.

1. $\mathbf{R} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$; $t_1 = \frac{1}{2}\pi$. $\dot{\mathbf{R}} = -3\sin t\mathbf{i} + 3\cos t\mathbf{j}$. $\|\dot{\mathbf{R}}\| = 3\sqrt{\sin^2 t + \cos^2 t} = 3$. $\mathbf{T} = \dfrac{\dot{\mathbf{R}}}{\|\dot{\mathbf{R}}\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$.
$\mathbf{T}(\frac{1}{2}\pi) = -\mathbf{i}$. $\dot{\mathbf{T}} = -\cos t\mathbf{i} - \sin t\mathbf{j} = \mathbf{N}$ because $\|\dot{\mathbf{T}}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$

1*. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{3}$. $\rho = \dfrac{1}{\mathrm{K}} = 3$. The curve is a circle of radius 3; it is its own circle of curvature.

2. $\mathbf{R} = \cos 3t\mathbf{i} + \sin 3t\mathbf{j}$; $t = \frac{1}{3}\pi$. $\dot{\mathbf{R}} = -3\sin 3t\mathbf{i} + 3\cos 3t\mathbf{j}$. $\|\dot{\mathbf{R}}\| = 3\sqrt{\sin^2 3t + \cos^2 3t} = 3$.
$\mathbf{T} = \dot{\mathbf{R}}/\|\dot{\mathbf{R}}\| = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j}$. $\mathbf{T}(\frac{1}{3}\pi) = -\mathbf{i}$. $\dot{\mathbf{T}} = -3\cos 3t\mathbf{i} - 3\sin 3t\mathbf{j}$. $\|\dot{\mathbf{T}}\| = 3\sqrt{\cos^2 3t + \sin^2 3t} = 3$.
$\mathbf{N} = \dot{\mathbf{T}}/\|\dot{\mathbf{T}}\| = -\cos 3t\mathbf{i} - \sin 3t\mathbf{j}$. $\mathbf{N}(\frac{1}{3}\pi) = -\mathbf{j}$.

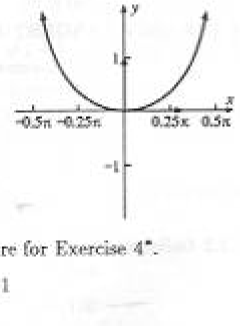
2*. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{3}{3} = 1$. $\rho = \frac{1}{\mathrm{K}} = 1$. The curve is a unit circle; it is its own circle of curvature.

3. $\mathbf{R} = \ln\sin t\mathbf{i} + t\mathbf{j}$; $0 < t < \pi$; $t_1 = \frac{1}{2}\pi$. $\dot{\mathbf{R}} = \frac{\cos t}{\sin t}\mathbf{i} + \mathbf{j}$. $(\sin t)\dot{\mathbf{R}} = \cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}$ because it is a unit vector.
$\mathbf{T}(\frac{1}{2}\pi) = \mathbf{j}$. $\dot{\mathbf{T}} = -\sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}$ because it is a unit vector. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{i}$

3*. $\dot{\mathbf{R}}(\frac{1}{2}\pi) = \mathbf{j}$. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{1} = 1$. $\rho = \frac{1}{\mathrm{K}} = 1$.

4. $\mathbf{R}(t) = t\mathbf{i} - \ln\cos t\mathbf{j}$; $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$; $t_1 = 0$

▹ $\mathrm{D}_t\mathbf{R}(t) = \mathbf{i} + \frac{\sin t}{\cos t}\mathbf{j}$
Because $\cos t > 0$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then
$(\cos t)\mathrm{D}_t\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}(t)$
because it has the same direction and it is a unit vector.
$\mathbf{T}(0) = \mathbf{i}$
$\mathrm{D}_t\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}(t)$
because it is a unit vector.
$\mathbf{N}(0) = \mathbf{j}$



The figure shows a portion of the curve, $\mathbf{T}(0)$ and $\mathbf{N}(0)$ and the circle of curvature for Exercise 4*.

4*. $\mathbf{R}(0) = \mathbf{0}$
$\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = 1/1 = 1$
$\mathbf{R}_c(0) = \mathbf{R}(0) + \rho\mathbf{N}(0) = \mathbf{j}$
$\|\dot{\mathbf{R}}(0)\| = \sqrt{1 + \tan^2 0} = 1$
$\rho = 1/\mathrm{K} = 1$

5. $\mathbf{R}(t) = (\frac{1}{3}t^3 - t)\mathbf{i} + t^2\mathbf{j};\ t_1 = 2.\ D_t\mathbf{R}(t) = (t^2-1)\mathbf{i} + 2t\mathbf{j}.$

$\|D_t\mathbf{R}(t)\| = \sqrt{(t^2-1)^2 + 4t^2} = \sqrt{(t^2+1)^2} = t^2+1.\ \mathbf{T}(t) = \dfrac{D_t\mathbf{R}(t)}{\|D_t\mathbf{R}(t)\|} = \dfrac{t^2-1}{t^2+1}\mathbf{i} + \dfrac{2t}{t^2+1}\mathbf{j}.\ \mathbf{T}(2) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

$$D_t\mathbf{T}(t) = \frac{2t(t^2+1) - 2t(t^2-1)}{(t^2+1)^2}\mathbf{i} + \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2}\mathbf{j} = \frac{4t}{(t^2+1)^2}\mathbf{i} + \frac{2(1-t^2)}{(t^2+1)^2}\mathbf{j}$$

$$\|D_t\mathbf{T}(t)\| = \sqrt{\frac{(4t)^2}{(t^2+1)^4} + \frac{4(1-t^2)^2}{(t^2+1)^4}} = \frac{2}{(t^2+1)^2}\sqrt{4t^2 + (1-2t^2+t^4)} = \frac{2}{(t^2+1)^2}\sqrt{(t^2+1)^2} = \frac{2}{t^2+1}$$

$\mathbf{N}(t) = \dfrac{D_t\mathbf{T}(t)}{\|D_t\mathbf{T}(t)\|} = \left[\dfrac{4t}{(t^2+1)^2}\mathbf{i} + \dfrac{2(1-t^2)}{(t^2+1)^2}\mathbf{j}\right]\dfrac{t^2+1}{2} = \dfrac{2t}{t^2+1}\mathbf{i} + \dfrac{1-t^2}{t^2+1}\mathbf{j}.\ \mathbf{N}(2) = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

5*. $K = \|\dot{\mathbf{T}}(2)\|/\|\dot{\mathbf{R}}(2)\| = \dfrac{2}{2^2+1}/(2^2+1) = \frac{2}{25}.\ \rho = \dfrac{1}{K} = \dfrac{25}{2}$

6. $\mathbf{R} = \frac{1}{2}t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j},\ t > 0;\ t_1 = 1.\ \dot{\mathbf{R}} = t\mathbf{i} + t^2\mathbf{j}.\ \mathbf{t} = \dot{\mathbf{R}}/t = \mathbf{i} + t\mathbf{j}.\ \|\mathbf{t}\| = \sqrt{1+t^2}.$

$\mathbf{T} = \dfrac{1}{\sqrt{1+t^2}}\mathbf{i} + \dfrac{t}{\sqrt{1+t^2}}\mathbf{j}.\ \mathbf{T}(1) = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}.\ \dot{\mathbf{T}} = -\dfrac{t}{(1+t^2)^{3/2}}\mathbf{i} + \dfrac{1(1+t^2) - \frac{1}{2}t(2t)}{(1+t^2)^{3/2}}\mathbf{j} = \dfrac{t\mathbf{i}+\mathbf{j}}{(1+t^2)^{3/2}}.\ \mathbf{n} = -t\mathbf{i} + \mathbf{j}$

$\|\mathbf{n}\| = \sqrt{t^2+1}.\ \mathbf{N} = -\dfrac{t}{\sqrt{t^2+1}}\mathbf{i} + \dfrac{1}{\sqrt{t^2+1}}\mathbf{j}.\ \mathbf{N}(1) = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

6*. $\|\dot{\mathbf{R}}(1)\| = \sqrt{1^2+1^2} = \sqrt{2}.\ \|\dot{\mathbf{T}}(1)\| = \dfrac{1}{2^{3/2}}\sqrt{1^2+1^2} = \frac{1}{2}.\ K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{4}.\ \rho = \dfrac{1}{K} = 4$

In Exercises 7–10, find $\mathbf{T}(t)$ and $\mathbf{N}(t)$. In Exercises 11–14, find the moving trihedral at $t = t_1$. In Exercises 15 and 16, find the moving trihedral at any point. In Exercises 17–22, find equations of the osculating, rectifying, and normal planes at $t = t_1$. In Exercises 31–36, find an equation of the having with arc length s as a parameter, where s increases with t from $t = 0$. Check by using Eq. (8). In Exercises 7*–10*, use the formula $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\|$ find the curvature K; in Exercises 11*–14*, use the formula $K(t) = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3$.
In Exercises 7, 11, 17, 31, 7*, and 11*, $\mathbf{R} = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + 2\mathbf{k}$.

7. $\dot{\mathbf{R}} = t\sin t\mathbf{i} + t\cos t\mathbf{j},\ t > 0.\ \mathbf{t} = \dot{\mathbf{R}}/t = \sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{T}.\ \dot{\mathbf{T}} = \cos t\mathbf{i} - \sin t\mathbf{j} = \mathbf{N}.$

11, 17. $t_1 = \frac{1}{2}\pi.\ \mathbf{R}(\frac{1}{2}\pi) = \mathbf{i} + \frac{1}{2}\pi\mathbf{j} + 2\mathbf{k}.\ \mathbf{T}(\frac{1}{2}\pi) = \mathbf{i}$, normal: $x = 1$. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$, rectifying: $y = \frac{1}{2}\pi$

$\mathbf{B}(\frac{1}{2}\pi) = \mathbf{i} \times (-\mathbf{j}) = -\mathbf{k}$, osculating: $z = 2$

31. $\dot{s} = \|\dot{\mathbf{R}}\| = t\sqrt{\sin^2 t + \cos^2 t} = t.\ s = \displaystyle\int_0^t u\,du = \tfrac{1}{2}u^2\Big]_0^t = \tfrac{1}{2}t^2;\ t = \sqrt{2s}$

$\mathbf{R} = (\sin\sqrt{2s} - \sqrt{2s}\cos\sqrt{2s})\mathbf{i} + (\cos\sqrt{2s} + \sqrt{2s}\sin\sqrt{2s})\mathbf{j} + 2\mathbf{k}$

7*. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = 1/t$

11*. $\ddot{\mathbf{R}} = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j}.\ \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\sin t & t\cos t & 0 \\ \sin t + t\cos t & \cos t - t\sin t & 0 \end{matrix} \right\| = |-t^2| = t^2$

$K = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = t^2/t^3 = 1/t$

In Exercises 8, 12, 20, 32, 8*, and 12*, the curve is the circular helix $\mathbf{R}(t) = \sin 3t\mathbf{i} - \cos 3t\mathbf{j} + 4t\mathbf{k}$

8. $\dot{\mathbf{R}}(t) = 3\cos 3t\mathbf{i} + 3\sin 3t\mathbf{j} + 4\mathbf{k}$

$\|\dot{\mathbf{R}}(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16} = \sqrt{9+16} = 5$

$\mathbf{T}(t) = \dfrac{\dot{\mathbf{R}}(t)}{\|\dot{\mathbf{R}}(t)\|} = \frac{3}{5}\cos 3t\mathbf{i} + \frac{3}{5}\sin 3t\mathbf{j} + \frac{4}{5}\mathbf{k}$

$\dot{\mathbf{T}}(t) = -\frac{9}{5}\sin 3t\mathbf{i} + \frac{9}{5}\cos 3t\mathbf{j}$

$\dot{\mathbf{T}}(t)/\frac{9}{5} = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j} = \mathbf{N}(t)$

12. $t_1 = \frac{1}{3}\pi.\ \mathbf{T}(\frac{1}{3}\pi) = \frac{3}{5}\cos\pi\mathbf{i} + \frac{3}{5}\sin\pi\mathbf{j} + \frac{4}{5}\mathbf{k} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

$\mathbf{N}(\frac{1}{3}\pi) = -\sin\pi\mathbf{i} + \cos\pi\mathbf{j} = -\mathbf{j}$

$\mathbf{B}(\frac{1}{3}\pi) = \mathbf{T}(\frac{1}{3}\pi) \times \mathbf{N}(\frac{1}{3}\pi) = (-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}) \times (-\mathbf{j}) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{k}$

20. $\mathbf{R}(\frac{1}{3}\pi) = \sin \pi\mathbf{i} - \cos \pi\mathbf{j} + \frac{4}{3}\pi\mathbf{k} = \mathbf{j} + \frac{4}{3}\mathbf{k}$

A normal vector to the osculating plane is $5\mathbf{B}(\frac{1}{3}\pi) = \langle 4, 0, 3\rangle$. An equation is $4x + 3(z - \frac{4}{3}) = 0$.

A normal vector to the rectifying plane is $-\mathbf{N}(\frac{1}{3}\pi) = \langle 0, 1, 0\rangle$. An equation is $y - 1 = 0$.

A normal vector to the normal plane is $5\mathbf{T}(\frac{1}{3}\pi) = \langle -3, 0, 4\rangle$. An equation is $-3x + 4(z - \frac{4}{3}) = 0$.

32. $\dot{s} = \|\dot{\mathbf{R}}(t)\| = 5$. $s = \int_0^t \|\dot{\mathbf{R}}(u)\| du = \int_0^t 5\, du = 5t$

Thus, $t = \frac{1}{5}s$. Hence,

$\mathbf{R}(s) = \sin \frac{3}{5}s\mathbf{i} - \cos \frac{3}{5}s\mathbf{j} + \frac{4}{5}s\mathbf{k}$

Because $f(s) = \sin \frac{3}{5}s$, $g(s) = -\cos \frac{3}{5}s$, and $h(s) = \frac{4}{5}s$,

$f'(s)^2 + g'(s)^2 + h'(s)^2 = (\frac{3}{5}\cos \frac{3}{5}s)^2 + (\frac{3}{5}\sin \frac{3}{5}s)^2 + (\frac{4}{5})^2 = \frac{9}{25}(\cos^2 \frac{3}{5}s + \sin^2 \frac{3}{5}s) + \frac{16}{25} = 1$

8*. $K = \|\dot{\mathbf{T}}\| / \|\dot{\mathbf{R}}\| = \frac{9}{5}/5 = \frac{9}{25}$

12* $\ddot{\mathbf{R}}(t) = -9 \sin 3t\mathbf{i} + 9 \cos 3t\mathbf{j}$.

$$\|\dot{\mathbf{R}}(t) \times \ddot{\mathbf{R}}(t)\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos 3t & 3 \sin 3t & 4 \\ -9 \sin 3t & 9 \cos 3t & 0 \end{matrix} \right\| = \|-36 \cos 3t\mathbf{i} - 36 \sin 3t\mathbf{j} + 27\mathbf{k}\| = 45$$

$K(t) = \|\dot{\mathbf{R}}(t) \times \ddot{\mathbf{R}}(t)\| / \|\dot{\mathbf{R}}(t)\|^3 = 45/5^3 = \frac{9}{25}$

In Exercises 9, 13, 19, 33, 9*, and 13*, $\mathbf{R} = \mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$, $t > 0$.

9. $\dot{\mathbf{R}} = t\mathbf{j} + t^2\mathbf{k}$. $\mathbf{t} = \dot{\mathbf{R}}/t = \mathbf{j} + t\mathbf{k}$. $\|\mathbf{t}\| = \sqrt{1 + t^2}$. $\mathbf{T} = \frac{\mathbf{t}}{\|\mathbf{t}\|} = \frac{1}{\sqrt{1+t^2}}\mathbf{j} + \frac{t}{\sqrt{1+t^2}}\mathbf{k}$.

$\dot{\mathbf{T}} = -\frac{t}{(1+t^2)^{3/2}}\mathbf{j} + \frac{1}{(1+t^2)^{3/2}}\mathbf{k}$. $\mathbf{n} = (1+t^2)^{3/2}\dot{\mathbf{T}} = -t\mathbf{j} + \mathbf{k}$. $\mathbf{N} = -\frac{t}{\sqrt{t^2+1}}\mathbf{j} + \frac{1}{\sqrt{t^2+1}}\mathbf{k}$

13. $t_1 = 1$. $\mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$. $\mathbf{N}(1) = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$. $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \mathbf{i}$

19. $\mathbf{R}(1) = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{3}\mathbf{k}$. osculating: $\mathbf{B}(1) = \mathbf{i}$, $x - 1 = 0$. rectifying: $\sqrt{2}\mathbf{N}(1) = -\mathbf{j} + \mathbf{k}$, $-(y - \frac{1}{2}) + (z - \frac{1}{3}) = 0$

normal: $\sqrt{2}\mathbf{T}(1) = \mathbf{j} + \mathbf{k}$, $(y - \frac{1}{2}) + (z - \frac{1}{3}) = 0$

33. $\dot{s} = \|\dot{\mathbf{R}}\| = t\sqrt{1+t^2}$. $s = \int_0^t u\sqrt{1+u^2}\,du = \frac{1}{3}(1+u^2)^{3/2}\Big]_0^t = \frac{1}{3}[(1+t^2)^{3/2} - 1]$, $t = \sqrt{(3s+1)^{2/3} - 1}$

$\mathbf{R}(s) = \mathbf{i} + \frac{1}{2}[(3s+1)^{2/3} - 1]\mathbf{j} + \frac{1}{3}[(3s+1)^{2/3} - 1]^{3/2}\mathbf{k}$

9*. $K = \|\dot{\mathbf{T}}\| / \|\dot{\mathbf{R}}\| = \frac{\sqrt{t^2+1}}{(1+t^2)^{3/2}} / t\sqrt{1+t^2} = \frac{1}{t(1+t^2)^{3/2}}$

13* $\ddot{\mathbf{R}} = \mathbf{j} + 2t\mathbf{k}$. $\|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & t & t^2 \\ 0 & 1 & 2t \end{matrix} \right\| = \|t^2\mathbf{i}\| = t^2$. $K = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| / \|\dot{\mathbf{R}}\|^3 = t^2/(t\sqrt{1+t^2})^3 = \frac{1}{t(1+t^2)^{3/2}}$

In Exercises 10, 14, 18, 34, 10*, and 14*, the curve is the conical helix $\mathbf{R} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$.

10. From §9.2, $\dot{\mathbf{R}} = \sqrt{2}e^t\cos(t + \frac{1}{4}\pi)\mathbf{i} + \sqrt{2}e^t\sin(t + \frac{1}{4}\pi)\mathbf{j} + e^t\mathbf{k}$. $\|\dot{\mathbf{R}}\| = e^t\sqrt{2\cos^2(t + \frac{1}{4}\pi) + 2\sin^2(t + \frac{1}{4}\pi) + 1} = \sqrt{3}e^t$

$\mathbf{T} = \sqrt{\frac{2}{3}}\cos(t+\frac{1}{4}\pi)\mathbf{i} + \sqrt{\frac{2}{3}}\sin(t+\frac{1}{4}\pi)\mathbf{j} + \sqrt{\frac{1}{3}}\mathbf{k}$. $\dot{\mathbf{T}} = -\sqrt{\frac{2}{3}}\sin(t+\frac{1}{4}\pi)\mathbf{i} + \sqrt{\frac{2}{3}}\cos(t+\frac{1}{4}\pi)\mathbf{j}$. $\mathbf{N} = -\sin(t+\frac{1}{4}\pi)\mathbf{i} + \cos(t+\frac{1}{4}\pi)\mathbf{j}$

14. $t_1 = 0$. $\mathbf{T}(0) = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\mathbf{i} + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\mathbf{j} + \sqrt{\frac{1}{3}}\mathbf{k} = \sqrt{\frac{1}{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. $\mathbf{N}(0) = \sqrt{\frac{1}{2}}(-\mathbf{i} + \mathbf{j})$. $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \sqrt{\frac{1}{6}}(-\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

18. $\mathbf{R}(0) = \mathbf{i} + \mathbf{k}$. osculating: $-\sqrt{6}\mathbf{B}(0) = \langle 1, 1, -2\rangle$, $(x-1) + y - 2(z-1) = 0$. rectifying: $\sqrt{2}\mathbf{N} = \langle -1, 1, 0\rangle$, $-(x-1) + y = 0$. normal: $\sqrt{3}\mathbf{T} = \langle 1, 1, 1\rangle$, $(x-1) + y + (z-1) = 0$

34. $\dot{s} = \|\dot{\mathbf{R}}\| = \sqrt{3}e^t$. $s = \int_0^t \sqrt{3}e^u du = \sqrt{3}e^u\Big]_0^t = \sqrt{3}(e^t - 1)$. $e^t = \frac{1}{3}\sqrt{3}s + 1$, $t = \ln(\frac{1}{3}\sqrt{3}s + 1)$

$\mathbf{R} = (\frac{1}{3}\sqrt{3}s + 1)[\cos \ln(\frac{1}{3}\sqrt{3}s + 1)\mathbf{i} + \sin \ln(\frac{1}{3}\sqrt{3}s + 1)\mathbf{j} + \mathbf{k}]$

10* $K = \|\dot{\mathbf{T}}\| / \|\dot{\mathbf{R}}\| = \sqrt{\frac{2}{3}} / \sqrt{3}e^t = \frac{1}{3}\sqrt{2}e^{-t}$

14. $\dot{\mathbf{R}} = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t\mathbf{k}$, $\ddot{\mathbf{R}} = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j} + e^t\mathbf{k}$

$$\|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\cos t + \sin t) & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{matrix} \right\|$$

$$= \|e^{2t}(\sin t - \cos t)\mathbf{i} - e^{2t}(\sin t + \cos t)\mathbf{j} + 2e^{2t}\mathbf{k}\| = \sqrt{6}e^{2t}.\ \mathrm{K} = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| / \|\dot{\mathbf{R}}\|^3 = \sqrt{6}e^{2t}/(\sqrt{3}e^t)^3 = \tfrac{1}{3}\sqrt{2}e^{-t}$$

In Exercises 15, 21 and 35, $\mathbf{R} = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j} + 2\mathbf{k}$, $0 \le t \le \frac{1}{2}\pi$

15. $\dot{\mathbf{R}} = -3 \cos^2 t \sin t\mathbf{i} + 3 \sin^2 t \cos t\mathbf{j}$. $\dot{\mathbf{R}}/3 \sin t \cos t = -\cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}$. $\dot{\mathbf{T}} = \sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}$. $\mathbf{B} = \mathbf{T} \times \mathbf{N} = -\mathbf{k}$

21. $t_1 = \frac{1}{4}\pi$. $\mathbf{R}(\frac{1}{4}\pi) = \frac{1}{4}\sqrt{2}\mathbf{i} + \frac{1}{4}\sqrt{2}\mathbf{j} + 2\mathbf{k}$. osculating: $\mathbf{B}(\frac{1}{4}\pi) = -\mathbf{k}$, $z = 2$. rectifying: $\sqrt{2}\mathbf{N} = \langle 1, 1, 0\rangle$,

$(x - \frac{1}{4}\sqrt{2}) + (y - \frac{1}{4}\sqrt{2}) = 0$. normal: $\sqrt{2}\mathbf{T} = \langle -1, 1, 0\rangle$, $-(x - \frac{1}{4}\sqrt{2}) + (y - \frac{1}{4}\sqrt{2}) = 0$

35. $\dot{s} = \|\dot{\mathbf{R}}\| = 3 \sin t \cos t$. $s = \displaystyle\int_0^t 3 \sin u \, d(\sin u) = \tfrac{3}{2} \sin^2 u \Big]_0^t = \tfrac{3}{2} \sin^2 t$, $\sin t = \sqrt{\tfrac{2}{3}s}$, $\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \tfrac{2}{3}s}$

$\mathbf{R} = (1 - \frac{2}{3}s)^{3/2}\mathbf{i} + (\frac{2}{3}s)^{3/2}\mathbf{j} + 2\mathbf{k}$

In Exercises 16, 22, and 36, $\mathbf{R}(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j} + t\mathbf{k}$.

16. $\dot{\mathbf{R}}(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$

$\|\dot{\mathbf{R}}(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2 \cosh^2 t} = \sqrt{2} \cosh t$

$\mathbf{T}(t) = \dot{\mathbf{R}}/\|\dot{\mathbf{R}}\| = \frac{1}{2}\sqrt{2}(\tanh t\mathbf{i} + \mathbf{j} + \operatorname{sech} t\mathbf{k})$

$\dot{\mathbf{T}}(t) = \frac{1}{2}\sqrt{2}(\operatorname{sech}^2 t\mathbf{i} - \operatorname{sech} t \tanh t\mathbf{k})$

$\sqrt{2}(\cosh t)\dot{\mathbf{T}}(t) = \operatorname{sech} t\mathbf{i} - \tanh t\mathbf{k} = \mathbf{N}(t)$

because

$\|\mathbf{N}(t)\| = \sqrt{\operatorname{sech}^2 t + \tanh^2 t} = 1$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \tanh t & 1 & \operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = -\tanh t\mathbf{i} + \mathbf{j} - \operatorname{sech} t\mathbf{k}$$

22. $t_1 = 0$. $\mathbf{R}(0) = \mathbf{i}$. osculate: $\mathbf{B}(0) = \mathbf{j} - \mathbf{k}$, $y - z = 0$. rectify: $\mathbf{N}(0) = \mathbf{i}$, $x - 1 = 0$. normal: $\mathbf{T}(0) = \mathbf{j} + \mathbf{k}$, $y + z = 0$

36. $\dot{s} = \|\dot{\mathbf{R}}(t)\| = \sqrt{2} \cosh t$

$s = \displaystyle\int_0^t \|\dot{\mathbf{R}}(u)\| du = \int_0^t \sqrt{2} \cosh u \, du = \sqrt{2} \sinh u \Big]_0^t = \sqrt{2} \sinh t$

$\sinh t = \frac{1}{2}\sqrt{2}s$

$\cosh t = \sqrt{\sinh^2 t + 1} = \sqrt{\frac{1}{2}s^2 + 1}$

$\mathbf{R}(s) = \sqrt{\frac{1}{2}s^2 + 1}\mathbf{i} + \frac{1}{2}\sqrt{2}s\mathbf{j} + \sinh^{-1}(\frac{1}{2}\sqrt{2}s)\mathbf{k}$

$f(s) = \sqrt{\frac{1}{2}s^2 + 1}$, $g(s) = \frac{1}{2}\sqrt{2}s$, $h(s) = \sinh^{-1}(\frac{1}{2}\sqrt{2}s)$

$$[f'(s)]^2 + [g'(s)]^2 + [h'(s)]^2 = \left[\frac{s}{2\sqrt{\frac{1}{2}s^2 + 1}}\right]^2 + [\tfrac{1}{2}\sqrt{2}]^2 + \left[\frac{\frac{1}{2}\sqrt{2}}{\sqrt{1 + \frac{1}{2}s^2}}\right]^2 = \frac{s^2}{2s^2 + 4} + \frac{1}{2} + \frac{1}{2 + s^2} = \frac{2s^2 + 4}{2s^2 + 4} = 1$$

23. $\mathbf{T}(\frac{1}{2}\pi) = \frac{1}{5}\sqrt{5}(-2\mathbf{i} + \mathbf{k})$; $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$; $\mathbf{B}(\frac{1}{2}\pi) = \frac{1}{5}\sqrt{5}(\mathbf{i} + 2\mathbf{k})$; osculating: $x + 2z - 4 = 0$; rectifying: $y = 1$;

normal: $2x - z + 2 = 0$

24. Find the cosine of the angle between the vectors $\mathbf{R}(2)$ and $\mathbf{T}(2)$ for the curve $\mathbf{R}(t) = 3t^2\mathbf{i} + (t^3 - 3t)\mathbf{j}$.

▹ $\mathrm{D}_t\mathbf{R}(t) = 6t\mathbf{i} + (3t^2 - 3)\mathbf{j}$

$\|\mathrm{D}_t\mathbf{R}(t)\| = \sqrt{36t^2 + 9t^4 - 18t^2 + 9} = \sqrt{9t^4 + 18t^2 + 9} = 3(t^2 + 1)$

$$\mathbf{T}(t) = \frac{\mathrm{D}_t\mathbf{R}(t)}{\|\mathrm{D}_t\mathbf{R}(t)\|} = \frac{2t}{t^2 + 1}\mathbf{i} + \frac{t^2 - 1}{t^2 + 1}\mathbf{j}.$$

$\mathbf{R}(2) = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{T}(2) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

The cosine of the angle between $\mathbf{R}(2)$ and $\mathbf{T}(2)$ is given by

$$\cos \theta = \frac{\mathbf{R}(2) \cdot \mathbf{T}(2)}{\|\mathbf{R}(2)\| \|\mathbf{T}(2)\|} = \frac{12 \cdot \frac{4}{5} + 2 \cdot \frac{3}{5}}{\sqrt{144 + 4}(1)} = \frac{\frac{48}{5} + \frac{6}{5}}{2\sqrt{37}} = \frac{27}{185}\sqrt{37}$$

25. $\mathbf{R} = 2\sin t\mathbf{i} + \sin 2t\mathbf{j} + \cos 3t\mathbf{k}$. $\dot{\mathbf{R}} = 2\cos t\mathbf{i} + 2\cos t\mathbf{j} - 3\sin 3t\mathbf{k}$. $\mathbf{R}(\frac{1}{6}\pi) = \langle 1, \frac{1}{2}\sqrt{3}, 0\rangle$. $\dot{\mathbf{R}}(\frac{1}{6}\pi) = \langle \sqrt{3}, 1, -3\rangle$

$$\cos\theta = \frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{\|\mathbf{R}\|\|\dot{\mathbf{R}}\|} = \frac{\sqrt{3}+\frac{1}{2}\sqrt{3}}{(\frac{1}{2}\sqrt{7})(\sqrt{13})} = \frac{3\sqrt{3}}{\sqrt{91}} = \frac{3}{91}\sqrt{273}$$

26. $\mathbf{R} = \cos 2t\mathbf{i} - 3t\mathbf{j} + 2\sin 2t\mathbf{k}$. $\dot{\mathbf{R}} = -2\sin 2t\mathbf{i} - 3\mathbf{j} + 4\cos 2t\mathbf{k}$. $\dot{\mathbf{R}}(\pi) = -3\mathbf{j} + 4\mathbf{k}$. $\cos\theta = \mathbf{j}\cdot\dot{\mathbf{R}}(\pi)/\|\dot{\mathbf{R}}(\pi)\| = -\frac{3}{5}$

27. $\mathbf{R} = (4-3t^2)\mathbf{i} + (t^3-3t)\mathbf{j}$. At $t = 1$, $\dot{\mathbf{R}} = -6t\mathbf{i} + (3t^2-3)\mathbf{j} = -6\mathbf{i}$, $\ddot{\mathbf{R}} = -6\mathbf{i} + 6t\mathbf{j} = -6\mathbf{i} + 6\mathbf{j}$.

$$\cos\theta = (-\mathbf{i})\cdot\tfrac{1}{2}\sqrt{2}(-\mathbf{i}+\mathbf{j}) = \tfrac{1}{2}\sqrt{2},\ \theta = \tfrac{1}{4}\pi$$

In Exercises 28 and 29, express the arc length s as a function of t, where s increases with t from $t = 0$.

28. The cycloid $\mathbf{R}(t) = 2(t - \sin t)\mathbf{i} + 2(1-\cos t)\mathbf{j}$

▷ $$\frac{ds}{dt} = \|\dot{\mathbf{R}}(t)\| = \|2(1-\cos t)\mathbf{i} + 2\sin t\mathbf{j}\| = 2\sqrt{(1-\cos t)^2 + \sin^2 t} = 2\sqrt{(1-\cos 2t)} = 2\sqrt{4\sin^2\tfrac{1}{2}t} = 4\sin\tfrac{1}{2}t$$

if $0 \le t \le 2\pi$. Then

$$s = \int_0^t \frac{ds}{du}du = \int_0^t 4\sin\tfrac{1}{2}u\,du = -8\cos\tfrac{1}{2}u\Big]_0^t = 8(1-\cos\tfrac{1}{2}t)$$

Because $s = 16$ when $t = 2\pi$, if $2\pi \le t \le 4\pi$, then

$$s = 16 + 8[1 - \cos\tfrac{1}{2}(t-2\pi)] = 16 + 8(1 + \cos\tfrac{1}{2}t)$$

In general,

$$s = 16[\![t/2\pi]\!] + 8[1 - (-1)^{[\![t/2\pi]\!]}\cos\tfrac{1}{2}t]$$

29. $\mathbf{R}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}$. $\dot{\mathbf{R}}(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$.

$$s = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t \sqrt{1+\tfrac{9}{4}u}\,du = \tfrac{1}{2}\int_0^t (4+9u)^{1/2}du = \tfrac{1}{27}(4+9u)^{3/2}\Big]_0^t = \tfrac{1}{27}(4+9t)^{3/2} - \tfrac{8}{27}$$

30. $f(s) = \frac{1}{27}[(27s+8)^{2/3} - 4]^{3/2}$, $g(s) = \frac{1}{2}[(27s+8)^{2/3} - 4]$, $h(s) = 4$

$$f'(s) = \tfrac{1}{18}[(27s+8)^{2/3} - 4]^{1/2}\left[\tfrac{2}{3}(27s+8)^{-1/3}(27)\right] = \frac{[(27s+8)^{2/3}-4]^{1/2}}{(27s+8)^{1/3}}$$

$$g'(s) = \tfrac{2}{27}(27s+8)^{-1/3}(27) = \frac{2}{(27s+8)^{1/3}},\ h'(s) = 0. \text{ Hence } (f')^2 + (g')^2 + (h')^2 = \frac{(27s+8)^{2/3} - 4 + 4}{(27s+8)^{2/3}} = 1.$$

37. $\mathbf{T}\cdot\mathbf{k} = \frac{1}{\sqrt{a^2+1}}(-a\sin t\mathbf{i} + b\cos t\mathbf{j} + \mathbf{k})\cdot\mathbf{k} = \frac{1}{\sqrt{a^2+1}}$, a constant

38. If a particle moves on a line, then $\mathbf{T}$ is constant and so $\dot{\mathbf{T}} \equiv \mathbf{0}$ and has no direction. Thus $\mathbf{N}$ is undefined.

11.4 CURVATURE

Curvature If $\mathbf{T}(t)$ is the unit tangent vector to a curve C at a point P, s is the arc length measured from an arbitrarily chosen point on C to P, and s increases as t increases, then for some scalar $\mathrm{K}(t)$, called the *curvature*,

$$D_s\mathbf{T}(t) = \mathrm{K}(t)\mathbf{N}(t) \tag{A}$$

Furthermore, for a scalar $\tau(t)$, called the *torsion*,

$$D_s\mathbf{B}(t) = -\tau(t)\mathbf{N}(t) \tag{B}$$

and

$$D_s\mathbf{N}(t) = -\mathrm{K}(t)\mathbf{T} + \tau(t)\mathbf{N} \tag{C}$$

Equations (A), (B), (C) together are known as Frenet's formulas.

Following are some formulas for the curvature.

$$\mathrm{K}(t) = \frac{\|\dot{\mathbf{T}}\|}{\|\dot{\mathbf{R}}\|} \tag{5}$$

$$\mathrm{K}(t) = \frac{\|\dot{\mathbf{R}}(t)\times\ddot{\mathbf{R}}(t)\|}{\|\dot{\mathbf{R}}(t)\|^3}$$ (Theorem 11.4.2. See Exercise 56)

In the Plane $$\mathrm{K}(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \tag{8}$$

and if $y = f(x)$ or $x = g(y)$

$$\mathrm{K}(x) = \frac{\left|d^2y/dx^2\right|}{\left[1+(dy/dx)^2\right]^{3/2}} \text{ or } \mathrm{K}(y) = \frac{\left|d^2x/dy^2\right|}{\left[1+(dx/dy)^2\right]^{3/2}} \tag{9}$$

Polar Coordinates Let a polar equation of a curve be $r = r(\theta)$. If β is the angle from the radius vector to the tangent line, then $\tan\beta = r/(dr/d\theta)$ and the curvature is

$$\mathrm{K}(\theta) = \frac{\left|r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\right|}{\left[r^2 + (dr/d\theta)^2\right]^{3/2}}$$

Circle of Curvature The circle on the concave side of C that is tangent to C at $\mathrm{P}(x, y)$ and whose radius is ρ is called the *circle of curvature* or the *osculating circle*. The center of this circle is called the *center of curvature* and is given by $\mathbf{R}_c = \mathbf{R} + \rho\mathbf{N} = (x_c, y_c)$ where (see Exercise 39)

$$x_c = x - \frac{dy}{dx}\cdot\frac{1+(dy/dx)^2}{d^2y/dx^2} \qquad y_c = y + \frac{1+(dy/dx)^2}{d^2y/dx^2}$$

or

$$x_c = x - \dot{y}\frac{(\dot{x})^2+(\dot{y})^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},\ y_c = y + \dot{x}\frac{(\dot{x})^2+(\dot{y})^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

The locus of the center of curvature of C is called the *evolute* of C.

Exercises 11.4

For Exercises 1–14, see Exercises 1*–14* of Exercises 11.3.

In Exercises 15 and 16, compute the curvature at the indicated point using $\mathrm{K} = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3$.

15. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. At $t = 0$, $\dot{\mathbf{R}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} = \mathbf{i}$, $\ddot{\mathbf{R}} = 2\mathbf{j} + 6t\mathbf{k} = 2\mathbf{j}$. $\mathrm{K} = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = \|2\mathbf{k}\|/\|\mathbf{i}\|^3 = 2$

16. $\mathbf{R}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k};\ t = 0$

▸ $\dot{\mathbf{R}}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}$ $\qquad$ $\dot{\mathbf{R}}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\ddot{\mathbf{R}}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$ $\qquad$ $\ddot{\mathbf{R}}(0) = \mathbf{i} + \mathbf{j}$

$$\left\|\dot{\mathbf{R}}(0)\times\ddot{\mathbf{R}}(0)\right\| = \left\|\begin{matrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 1 & 1 & 0\end{matrix}\right\| = \|-\mathbf{i}+\mathbf{j}+2\mathbf{k}\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$\left\|\dot{\mathbf{R}}(0)\right\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\mathrm{K}(0) = \left\|\dot{\mathbf{R}}(0)\times\ddot{\mathbf{R}}(0)\right\|/\left\|\dot{\mathbf{R}}(0)\right\|^3 = \sqrt{6}/2\sqrt{2} = \tfrac{1}{2}\sqrt{3}$$

In Exercises 17–26, find the curvature K and the radius of curvature ρ at the given point. Sketch portions of the curve and tangent line, and the circle of curvature at the given point.

17. $x = \frac{1}{1+t}$, $y = \frac{1}{1-t}$. $\frac{dx}{dt} = -\frac{1}{(1+t)^2}$, $\frac{dy}{dt} = \frac{1}{(1-t)^2}$; $\frac{d^2x}{dt^2} = \frac{2}{(1+t)^3}$, $\frac{d^2y}{dt^2} = \frac{2}{(1-t)^3}$.

$$\mathrm{K}(t) = \frac{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right|}{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}} = \frac{\left|\frac{-2}{(1+t)^2(1-t)^3} - \frac{2}{(1+t)^3(1-t)^2}\right|}{\left[\frac{1}{(1+t)^4}+\frac{1}{(1-t)^4}\right]^{3/2}}$$

$$= \frac{-2(1+t)^4(1-t)^3 - 2(1+t)^3(1-t)^4}{\left[(1-t)^4+(1+t)^4\right]^{3/2}} = \frac{-2(1+t)^3(1-t)^3[(1+t)+(1-t)]}{(1-4t+6t^2-4t^3+t^4+1+4t+6t^2+4t^3+t^4)^{3/2}}$$

$$= \frac{4|(1+t)(1-t)|^3}{(2+12t^2+2t^4)^{3/2}} = \frac{\sqrt{2}|1-t^2|^3}{(1+6t^2+4)^{3/2}}.$$ Therefore $\mathrm{K}(0) = \sqrt{2}$ and $\rho(0) = \frac{1}{2}\sqrt{2}$.

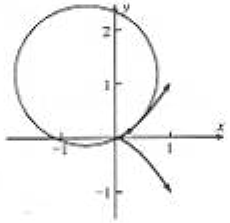
18. $x = e^t + e^{-t}$, $y = e^t - e^{-t}$. $\dot{x} = \ddot{y} = e^t - e^{-t}$, $\dot{y} = \ddot{x} = e^t + e^{-t}$. $\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$

$$= \frac{\left|(e^{2t}-2+e^{-2t})-(e^{2t}+2+e^{-2t})\right|}{\left[(e^{2t}-2+e^{-2t})+(e^{2t}+2+e^{-2t})\right]^{3/2}} = \frac{4}{\left[2(e^{2t}+e^{-2t})\right]^{3/2}} = \frac{\sqrt{2}}{(e^{2t}+e^{-2t})^{3/2}}.$$ $\mathrm{K}(0) = \frac{1}{2}$, $\rho(0) = 2$

19. $y = 2\sqrt{x}$. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$; $\dfrac{d^2y}{dx^2} = -\dfrac{1}{2x^{3/2}}$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{\frac{1}{2x^{3/2}}}{\left(1+\frac{1}{x}\right)^{3/2}} = \dfrac{1}{2(x+1)^{3/2}}$

At $(0,0)$, $\mathrm{K} = \frac{1}{2}$ and $\rho = 2$.

20. $y^2 = x^3$; $(\frac{1}{4}, \frac{1}{8})$



▹ We have the parametric equations

$x = t^2 \qquad\qquad y = t^3$

Calculating derivatives and evaluating at $t = \frac{1}{2}$, we have

$\dot{x} = 2t = 2(\frac{1}{2}) = 1 \qquad \dot{y} = 3t^2 = 3(\frac{1}{2})^2 = \frac{3}{4}$

$\ddot{x} = 2 \qquad \ddot{y} = 6t = 6(\frac{1}{2}) = 3$

The curvature is given by

$$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{\left|1\cdot 3 - \frac{3}{4}\cdot 2\right|}{[1^2 + (\frac{3}{4})^2]^{3/2}} = \frac{\frac{3}{2}}{(\frac{25}{16})^{3/2}} = \frac{\frac{3}{2}}{\frac{125}{64}} = \frac{96}{125}$$

The radius of curvature is the reciprocal of the curvature: $\rho = \frac{1}{\mathrm{K}} = \frac{125}{96}$

To find the center of curvature (x_c, y_c), let $c = \dfrac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \dfrac{\frac{25}{16}}{\frac{3}{2}} = \dfrac{25}{24}$. Then

$x_c = x - c\dot{y} = \frac{1}{4} - \frac{25}{24}\cdot\frac{3}{4} = \frac{7}{32} \qquad y_c = y + c\dot{x} = \frac{1}{8} + \frac{25}{24}\cdot 1 = \frac{7}{6}$

The figure shows portions of the curve and tangent line, and the circle of curvature at $(\frac{1}{4}, \frac{1}{8})$.

21. $y = e^x$. $\dfrac{dy}{dx} = e^x$; $\dfrac{d^2y}{dx^2} = e^x$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}}$

At $(0,1)$, $\mathrm{K} = \dfrac{1}{(1+1)^{3/2}} = \frac{1}{2} = \frac{1}{2}\sqrt{2}$ and $\rho = 2\sqrt{2}$.

22. $y = \ln x$. $\dfrac{dy}{dx} = x^{-1}$, $\dfrac{d^2y}{dx^2} = -x^{-2}$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{x^{-2}}{(1+x^{-2})^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}$.

At $(e,1)$, $\mathrm{K} = \dfrac{e}{(e^2+1)^{3/2}}$. $\rho = \dfrac{(e^2+1)^{3/2}}{e}$

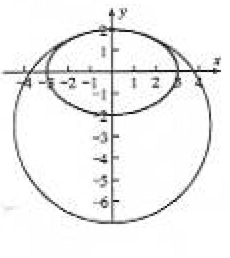
23. $x = \sin y$. $\dfrac{dx}{dy} = \cos y$; $\dfrac{d^2x}{dy^2} = -\sin y$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{|-\sin y|}{(1+\cos^2 y)^{3/2}} = \dfrac{|\sin y|}{(1+\cos^2 y)^{3/2}}$.

At $(\frac{1}{2}, \frac{1}{6}\pi)$, $\mathrm{K} = \dfrac{\sin\frac{1}{6}\pi}{(1+\cos^2\frac{1}{6}\pi)^{3/2}} = \dfrac{\frac{1}{2}}{(1+\frac{3}{4})^{3/2}} = \dfrac{4}{7^{3/2}} = \dfrac{4}{49}\sqrt{7}$ and $\rho = \frac{7}{4}\sqrt{7}$.

24. $4x^2 + 9y^2 = 36$; $(0,2)$

▹ We put the equation of the ellipse into standard form and write parametric equations.

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$



$x = 3\cos t \qquad\qquad y = 2\sin t$

We calculate derivatives and evaluate at $t = \frac{1}{2}\pi$.

$\dot{x} = -3\sin t = -3\sin\frac{1}{2}\pi = -3 \qquad \dot{y} = 2\cos t = 2\cos\frac{1}{2}\pi = 0$

$\ddot{x} = -3\cos t = -3\cos\frac{1}{2}\pi = 0 \qquad \ddot{y} = -2\sin t = -2\sin\frac{1}{2}\pi = -2$

The curvature is given by

$$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|(-3)(-2) - 0|}{[(-3)^3 + 0]^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

The radius of curvature is the reciprocal of the curvature: $\rho = \frac{1}{\mathrm{K}} = \frac{9}{2}$

The center of curvature is at $(0, -\frac{5}{2})$.

The figure shows portions of the curve and tangent line, and the circle of curvature at $(0,2)$.

25. $x = \sqrt{y-1}$. $\frac{dx}{dy} = \frac{1}{2\sqrt{y-1}}$; $\frac{d^2x}{dy^2} = \frac{-1}{4(y-1)^{3/2}}$. $\mathrm{K} = \frac{\left|d^2x/dy^2\right|}{[1+(dx/dy)^2]^{3/2}} = \frac{\frac{1}{4(y-1)^{3/2}}}{\left[1+\frac{1}{4(y-1)}\right]^{3/2}} = \frac{2}{(4y-3)^{3/2}}$

At (2,5), $\mathrm{K} = \frac{2}{17^{3/2}} = \frac{2}{289}\sqrt{17}$ and $\rho = \frac{17}{2}\sqrt{17}$.

26. $x = \tan y$. At $y = \frac{1}{4}\pi$, $\frac{dx}{dy} = \sec^2 y = 2$; $\frac{d^2x}{dy^2} = 2\sec^2 y \tan y = 4$. $\mathrm{K} = \frac{\left|d^2x/dy^2\right|}{[1+(dx/dy)^2]^{3/2}} = \frac{4}{(1+2^2)^{3/2}} = \frac{4}{25}\sqrt{5}$

$\rho = \frac{5}{4}\sqrt{5}$

In Exercises 27–34, find the radius of curvature at any point.

27. $y = \sin^{-1}x$. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$; $\frac{d^2y}{dx^2} = \frac{x}{(1-x^2)^{3/2}}$.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+\frac{1}{1-x^2}\right)^{3/2}}{\frac{|x|}{(1-x^2)^{3/2}}} = \frac{(1-x^2+1)^{3/2}}{|x|} = \frac{(2-x^2)^{3/2}}{|x|}$$

28. $y = \ln \sec x$

▷ $$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x, \ \sec x > 0$$
$$\frac{d^2y}{dx^2} = \sec^2 x$$

Hence,

$$\rho(x) = \frac{1}{\mathrm{K}(x)} = \frac{[1+(dy/dx)^2]^{3/2}}{\left|d^2y/dx^2\right|} = \frac{(1+\tan^2 x)^{3/2}}{\sec^2 x} = \frac{(\sec^2 x)^{3/2}}{\sec^2 x} = \frac{\left|\sec^3 x\right|}{\sec^2 x} = \sec x$$

29. The hyperbola $4x^2 - 9y^2 = 16$. $8x - 18\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{4x}{9y}$. $\frac{d^2y}{dx^2} = \frac{4}{9}\cdot\frac{y - x\frac{dy}{dx}}{y^2} = \frac{4}{9}\cdot\frac{y - x\frac{4x^2}{9y}}{y^2} = \frac{4(9y^2-4x^2)}{81y^3}$

$$= \frac{4(-16)}{81y^3} = -\frac{64}{81y^3}.\ \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+\frac{16x^2}{81y^2}\right)^{3/2}}{\frac{64}{81|y^3|}} = \frac{(16x^2+81y^2)^{3/2}}{576}$$

30. $x = \tan^{-1}y$, $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$, $\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$. $\rho = \frac{[1+(\mathrm{D}_x y)^2]^{3/2}}{\left|\mathrm{D}_x{}^2 y\right|} = \frac{(1+\sec^4 x)^{3/2}}{2\sec^2 x|\tan x|}$

31. The parabola $x^{1/2} + y^{1/2} = a^{1/2}$. $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\mathrm{D}_x y = 0$; $\mathrm{D}_x y = -\frac{y^{1/2}}{2} = \frac{a^{1/2} - x^{1/2}}{2} = 1 - a^{1/2}x^{-1/2}$.

$\mathrm{D}_x{}^2 y = \frac{1}{2}a^{1/2}x^{-3/2}$. $\rho = \frac{[1+(\mathrm{D}_x y)^2]^{3/2}}{\left|\mathrm{D}_x{}^2 y\right|} = \frac{[1+(-y^{1/2}x^{-1/2})^2]^{3/2}}{\left|\frac{1}{2}a^{1/2}x^{-3/2}\right|} = \frac{2(x+y)^{3/2}}{a^{1/2}}$.

32. The logarithmic spiral $\mathbf{R}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$

▷ $x = e^t \sin t$ $\quad y = e^t \cos t$
$\dot{x} = e^t(\cos t + \sin t)$ $\quad \dot{y} = e^t(\cos t - \sin t)$
$\ddot{x} = 2e^t \cos t$ $\quad \ddot{y} = -2e^t \sin t$

$$\mathrm{K}(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{\left|-2e^{2t}\sin t(\cos t + \sin t) - 2e^{2t}\cos t(\cos t - \sin t)\right|}{[e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2]^{3/2}}$$

$$= \frac{2e^{2t}\left|\sin t\cos t + \sin^2 t + \cos^2 t - \sin t \cos t\right|}{e^{3t}[\cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t]^{3/2}} = \frac{2}{e^t(2)^{3/2}} = \frac{1}{\sqrt{2}e^t}$$

Thus,

$$\rho(t) = \frac{1}{\mathrm{K}(t)} = \sqrt{2}e^t$$

33. The cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$. $x' = a(1 - \cos t)$, $x'' = a \sin t$; $y' = a \sin t$, $y'' = a \cos t$.

$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = \frac{[a^2(1 - \cos t)^2 + a^2\sin^2 t]^{3/2}}{|a(1 - \cos t)(a \cos t) - (a \sin t)(a \sin t)|}$$

$$= \frac{(a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t)^{3/2}}{|a^2\cos t - a^2\cos^2 t - a^2\sin^2 t|} = \frac{[2a^2(1 - \cos t)]^{3/2}}{|a^2(\cos t - 1)|} = \frac{2^{3/2}|a^3|(1 - \cos t)^{3/2}}{a^2(1 - \cos t)}$$

$$= 2^{3/2}|a|(1 - \cos t)^{1/2} = 4a\left(\frac{1 - \cos t}{2}\right)^{1/2} = 4\left|a \sin \tfrac{1}{2}t\right|$$

34. The tractrix. Let $t = au$. $x = t - a \tanh(t/a) = au - a \tanh u$, $y = a \operatorname{sech}(t/a) = a \operatorname{sech} u$. We reduce $\operatorname{sech}^2 u = 1 - \tanh^2 u$. $\dot{x} = a - a \operatorname{sech}^2 u = a \tanh^2 u$, $\dot{y} = -a \operatorname{sech} u \tanh u$.
$\ddot{x} = 2a \operatorname{sech}^2 u \tanh u = 2a(\tanh u - \tanh^3 u)$, $\ddot{y} = a(\operatorname{sech} u \tanh^2 u - \operatorname{sech}^3 u) = a \operatorname{sech} u(2 \tanh^2 u - 1)$
$[\dot{x}^2 + \dot{y}^2]^{3/2} = [a^2(\tanh^4 u + \operatorname{sech}^2 u \tanh^2 u)]^{3/2} = [a^2\tanh^2 u]^{3/2} = |a^3\tanh^3 u|$

$$|\dot{x}\ddot{y} - \dot{y}\ddot{x}| = a^2\left|\tanh^2 u \operatorname{sech} u(2 \tanh^2 u - 1) + 2 \operatorname{sech} u \tanh u(\tanh u - \tanh^3 u)\right| = a^2\tanh^2 u|\operatorname{sech} u|$$

$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = |a \sinh u| = |a \sinh(t/a)|$$

In Exercises 35–38, find a point on the given curve at which the curvature is an absolute maximum.

35. $y = e^x$. $y' = e^x$, $y'' = e^x$. $\mathrm{K} = \dfrac{e^x}{(1 + e^{2x})^{3/2}}$. $\mathrm{K}' = e^x(-\tfrac{3}{2})(1 + e^{2x})^{-5/2}(2e^{2x}) + (1 + e^{2x})^{-3/2}e^x = \dfrac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$

Because $\mathrm{K}'(x) > 0$ if $e^{2x} < \frac{1}{2}$ and $\mathrm{K}'(x) < 0$ if $e^{2x} > \frac{1}{2}$, then K has an absolute maximum value when $x = -\frac{1}{2} \ln 2$ and $y = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$.

36. $y = x^2 - 2x + 3$

▷ $\dfrac{dy}{dx} = 2x - 2$ and $\dfrac{d^2y}{dx^2} = 2$

$$\mathrm{K}(x) = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \frac{2}{[1 + (2x - 2)^2]^{3/2}} = \frac{2}{[1 + 4(x - 1)^2]^{3/2}}$$

Because $1 + 4(x - 1)^2$ has an absolute minimum at $x = 1$, then K has an absolute maximum at $x = 1$. If $x = 1$, then $y = 2$. Thus, the curvature is an absolute maximum at the point $(1, 2)$. Note that $y = (x - 1)^2 + 2$ is a parabola and that $(1, 2)$ is the vertex of the parabola.

37. $\mathbf{R}(t) = (2t - 3)\mathbf{i} + (t^2 - 1)\mathbf{j}$. $x = 2t - 3$, $y = t^2 - 1$. $\dot{x} = 2$, $x'' = 0$, $\dot{y} = 2t$, $y'' = 2$.

$$\mathrm{K} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[(\dot{x})^2 + (\dot{y})^2]^{3/2}} = \frac{4}{(4 + 4t^2)^{3/2}} = \frac{1}{2(1 + t^2)^{3/2}}$$

$1 + t^2$ has an absolute minimum when $t = 0$ at $(-3, -1)$, so K has an absolute maximum there.

38. $y = \sin x$. $\dfrac{dy}{dx} = \cos x$, $\dfrac{d^2y}{dx^2} = -\sin x$. $\mathrm{K} = \dfrac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{|\sin x|}{[1 + \cos^2 x]^{3/2}} = \dfrac{|\sin x|}{(2 - \sin^2 x)^{3/2}}$

When $x = (n + \frac{1}{2})\pi$, for any integer n, the numerator has an absolute maximum and the denominator an absolute minimum. Thus K has an absolute maximum.

39. The circle of curvature is the circle $(x - x_c)^2 + (y - y_c)^2 = r^2$ such that y, y', y'' have the same values as on the curve. Differentiating twice with respect to x we get
$2(x - x_c) + 2(y - y_c)y' = 0$; $(x_c - x) + (y_c - y)y' = 0$
$-1 - (y')^2 + (y_c - y)y'' = 0$; $(y_c - y)y' = 1 + (y')^2$
Solving for $x_c - x$ and $y_c - y$ we get

$$x_c = x - \frac{y'[1 + (y')^2]}{y''} \text{ and } y_c = y + \frac{1 + (y')^2}{y''}$$

In Exercises 40–42, find the curvature K, the radius of curvature ρ, and the center of curvature at the given point. Sketch the curve and the circle of curvature.

40. $y = \ln x$; $(1, 0)$

▸ We have

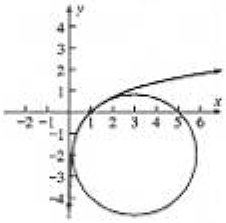
$\dfrac{dy}{dx} = \dfrac{1}{x}$ and $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$

When $x = 1$, we get

$\dfrac{dy}{dx} = 1$ and $\dfrac{dy^2}{dx^2} = -1$

Thus,

$$K = \frac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \frac{1}{2^{2/3}} \text{ and } \rho = \frac{1}{K} = 2\sqrt{2}$$



The coordinates of the center of curvature at $(1, 0)$ are given by

$$x_c = x - \frac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2} = 1 - \frac{1[1 + 1^2]}{-1} = 3 \text{ and } y_c = y + \frac{1 + (dy/dx)^2}{d^2y/dx^2} = 0 + \frac{1^2 + 1}{-1} = -2$$

The center of curvature is $(3, -2)$. The figure shows the curve and the circle of curvature at $(1, 0)$.

41. $y = x^4 - x^2$; $y' = 4x^3 - 2x$; $y'' = 12x^2 - 2$. At $(0,0)$, $y = 0$, $y' = 0$, $y'' = 2$, and we have

$$K = \frac{y''}{[1 + (y')^2]^{3/2}} = \frac{|-2|}{[1 + 0]^{3/2}} = 2,\ \rho = \tfrac{1}{2},\ x_c = x - \frac{y'[1 + (y')^2]}{y''} = 0 - \frac{[1 + 0^2]}{-2} = 0,$$

$$y_c = y + \frac{1 + (y')^2}{y''} = 0 + \frac{1 + 0}{-2} = -\frac{1}{2}.$$ The center of curvature is at $(0, -\frac{1}{2})$.

42. $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$. At $(\frac{1}{3}\pi, \frac{1}{2})$, $y = \frac{1}{2}$, $y' = \frac{1}{2}\sqrt{3}$, $y'' = -\frac{1}{2}$, and we have

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{2}}{[1 + \frac{3}{4}]^{3/2}} = \frac{4}{7\sqrt{7}},\ \rho = \tfrac{7}{4}\sqrt{7},\ x_c = x - \frac{y'[1 + (y')^2]}{y''} = \tfrac{1}{3}\pi - \frac{\frac{1}{2}\sqrt{3}(1 + \frac{3}{4})}{\frac{1}{2}} = \tfrac{1}{3}\pi - \tfrac{7}{4}\sqrt{3}$$

$$y_c = y + \frac{1 + (y')^2}{y''} = \tfrac{1}{2} + \frac{1 + \frac{3}{4}}{-\frac{1}{2}} = -3.$$ The center of curvature is at $(\frac{1}{3}\pi - \frac{7}{4}\sqrt{3}, -3)$.

In Exercises 43–46, find the coordinates of the center of curvature at any point.

43. $y^2 = 4px$, $y = 2p^{1/2}x^{1/2}$, $y' = p^{1/2}x^{-1/2}$, $y'' = -\frac{1}{2}p^{1/2}x^{-3/2}$.

$$x_c = x - \frac{y'[1 + (y')^2]}{y''} = x - \frac{p^{1/2}x^{-1/2}[1 + px^{-1}]}{-\frac{1}{2}p^{1/2}x^{-3/2}} = x + 2x + 2p = 3x + 2p$$

$$y_c = y + \frac{1 + (y')^2}{y''} = 2p^{1/2}x^{1/2} + \frac{1 + px^{-1}}{-\frac{1}{2}p^{1/2}x^{-3/2}} = 2p^{1/2}x^{1/2} - 2p^{-1/2}x^{3/2} - 2p^{1/2}x^{1/2} = -2p^{-1/2}x^{3/2}$$

Therefore, the center of curvature is $(3x + 2p, \mp 2p^{-1/2}x^{3/2}) = (3x + 2p, -\frac{1}{4}p^{-2}y^3)$.

44. $y^3 = a^2x$.

▸ Solving for y and differentiating, we have

$y = a^{2/3}x^{1/3}$ $\quad \dfrac{dy}{dx} = \frac{1}{3}a^{2/3}x^{-2/3}$ $\quad \dfrac{d^2y}{dx^2} = -\frac{2}{9}a^{2/3}x^{-5/3}$

The coordinates of the center of curvature are given by

$$x_c = x - \frac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2} = x - \frac{\frac{1}{3}a^{2/3}x^{-2/3}[1 + (\frac{1}{3}a^{2/3}x^{-2/3})^2]}{-\frac{2}{9}a^{2/3}x^{-5/3}}$$

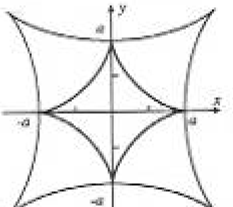
$$= x + \tfrac{3}{2}x(1 + \tfrac{1}{9}a^{4/3}x^{-4/3}) = x + \tfrac{3}{2}x + \tfrac{1}{6}a^{4/3}x^{-1/3} = \tfrac{5}{2}x + \tfrac{1}{6}a^{4/3}x^{-1/3}$$

$$y_c = y + \frac{1 + (dy/dx)^2}{d^2y/dx^2} = y + \frac{(\frac{1}{3}a^{2/3}x^{-2/3})^2 + 1}{-\frac{2}{9}a^{2/3}x^{-5/3}} = a^{2/3}x^{1/3} + \frac{\frac{1}{9}a^{4/3}x^{-4/3} + 1}{-\frac{2}{9}a^{2/3}x^{-5/3}}$$

$$= a^{2/3}x^{1/3} - \tfrac{1}{2}x^{2/3}x^{1/3} - \tfrac{9}{2}a^{-2/3}x^{5/3} = \tfrac{1}{2}a^{2/3}x^{1/3} - \tfrac{9}{2}a^{-2/3}x^{5/3}$$

45. $\mathbf{R}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j}$. Parametric equations of the ellipse are $x = a\cos t$ and $y = b\sin t$. Following the same steps as in the solution of Exercise 39, but differentiating with respect to t, we get the system
$$\begin{cases} \dot{x}(x_c - x) + \dot{y}(y_c - y) = 0 \\ \ddot{x}(x_c - x) + \ddot{y}(y_c - y) = (\dot{x})^2 + (\dot{y})^2 \end{cases}, \quad x_c = x - \frac{\dot{y}[(\dot{x})^2 + (\dot{y})^2]}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}, \quad y_c = y + \frac{\dot{x}[(\dot{x})^2 + (\dot{y})^2]}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$
$\dot{x} = -a\sin t$, $\ddot{x} = -a\cos t$; $\dot{y} = b\cos t$, $\ddot{y} = -b\sin t$
$\dot{x}y'' - \dot{y}x'' = (-a\sin t)(-b\sin t) - (b\cos t)(-a\cos t) = ab$
$$x_c = a\cos t - \frac{b\cos t(a^2\sin^2 t + b^2\cos^2 t)}{ab} = \frac{a^2\cos t(1 - \sin^2 t) - b^2\cos^3 t}{a} = \frac{a^2 - b^2}{a}\cos^3 t$$
$$y_c = b\sin t + \frac{(-a\sin t)(a^2\sin^2 t + b^2\cos^2 t)}{ab} = \frac{b^2\sin t(1 - \cos^2 t) - a^2\sin^3 t}{b} = \frac{b^2 - a^2}{b}\sin^3 t$$
The evolute is a squashed astroid.

46. Astroid $x = a\cos^3 t$, $y = a\sin^3 t$, $a > 0$.



▷ $\dot{x} = -3a\cos^2 t\sin t$, $\dot{y} = 3a\sin^2 t\cos t$. In $(0, \frac{1}{2}\pi)$,
$\dot{s} = 3a[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t]^{1/2} = 3a[\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)]^{1/2}$
$= 3a\sin t\cos t > 0$. $\ddot{x} = 3a(2\cos t\sin^2 t - \cos^3 t)$, $\ddot{y} = 3a(2\sin t\cos^2 t - \sin^3 t)$,
$\dot{x}\ddot{y} - \dot{y}\ddot{x} = 9a^2[(-\cos^2 t\sin t)(2\sin t\cos^2 t - \sin^3 t) - (\sin^2 t\cos t)(2\cos t\sin^2 t - \cos^3 t)]$
$= -9a^2(\sin^2 t\cos^4 t + \sin^4 t\cos^2 t) = -9a^2\sin^2 t\cos^2 t$.
$\mathrm{K} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|/s'^3 = 9a^2\sin^2 t\cos^2 t/27a^3\sin^3 t\cos^3 t = 1/(3a\sin t\cos t)$
$= 2/(3a\sin 2t)$. Remark. $s'^2/\mathrm{D} = 9a^2\sin^2 t\cos^2 t/(-9a^2\sin^2 t\cos^2 t) = -1$.
$\mathbf{C} = [a\cos^3 t - (3a\sin^2 t\cos t)(-1)]\mathbf{i} + [a\sin^3 t + (-3a\cos^2 t\sin t)(-1)]\mathbf{j}$.
To show the evolute is an astroid twice as large we rotate axes 45° to u and v. Then
$u = 2^{-1/2}(y + x) = 2^{-1/2}a(\sin^3 t + 3\sin^2 t\cos t + 3\sin t\cos^2 t + \cos^3 t) = 2^{-1/2}a(\sin t + \cos t)^3 = 2a\sin^3(t + \frac{1}{4}\pi)$
$v = 2^{-1/2}(y - x) = 2^{-1/2}a[\sin^3 t - 3\sin^2 t\cos t + 3\sin t\cos^2 t - \cos^3 t) = 2^{-1/2}a(\sin t - \cos t)^3 = 2a\cos^3(t + \frac{1}{4}\pi)$.

47. $y = \frac{1}{81}x^4$. At $x = 3$, $y = 1$, $y' = \frac{4}{81}x^3 = \frac{4}{3}$, $y'' = \frac{4}{27}x^2 = \frac{4}{3}$. $\mathrm{K} = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{\frac{4}{3}}{(1 + \frac{16}{9})^{3/2}} = \dfrac{\frac{4}{3}}{\frac{125}{9}} = \dfrac{36}{125}$, $\rho = \dfrac{125}{36}$
$c = \dfrac{1 + (y')^2}{y''} = \dfrac{\frac{25}{9}}{\frac{4}{3}} = \dfrac{25}{12}$. $x_c = x - cy' = 3 - \frac{25}{12}\cdot\frac{4}{3} = \frac{2}{9}$, $y_c = y + c = 1 + \frac{25}{12} = \frac{37}{12}$

48. Show that the curvature of a line is zero at every point.

▷ An equation of a straight line is $y = mx + \mathrm{b}$. $y' = \mathrm{m}$, $y'' = 0$. At every point (x,y), we have
$$\mathrm{K} = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{0}{(1 + \mathrm{m}^2)^{3/2}} = 0$$

49. The circle of curvature is the circle for which the values of y, y', y'' at $x = 0$ equal those of the curve $y = e^x$. Let an equation of the circle be
$(x - h)^2 + (y - k)^2 = r^2$. Because $y = e^0 = 1$, we have $h^2 + (1 - k)^2 = r^2$.
$2(x - h) + 2(y - k)y' = 0$; $(x - h) + (y - k)y' = 0$. $y' = e^x = e^0 = 1$, so we have $-h + (1 - k) = 0$.
$1 + (y')^2 + (y - k)y'' = 0$. Because $y'' = e^x = e^0 = 1$, we have $1 + 1 + (1 - k) = 0$.
Therefore $k = 3$, $h = -2$, $r^2 = 8$ and the equation is $(x + 2)^2 + (y - 3)^2 = 8$.

50. Prove the formula for curvature in polar coordinates.

▷ $\tan(\beta + \theta) = \tan\varphi = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(r\sin\theta)'}{(r\cos\theta)'} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \dfrac{\tan\theta + r/r'}{1 - (r/r')\tan\theta}$. Thus $\tan\varphi = r/r'$,
$\varphi = \theta + \beta = \theta + \tan^{-1}(r/r')$, $\mathrm{d}\varphi/d\theta = 1 + [(r'^2 - rr'']/r'^2[1 + (r/r')^2] = (r^2 + 2r'^2 - rr'')/(r^2 + r'^2)$.
$\mathrm{K}(\theta) = (\mathrm{d}\varphi/d\theta)/(ds/d\theta) = (r^2 + 2r'^2 - rr'')/(r^2 + r'^2)^{3/2}$

In Exercises 51–54, find the curvature and the radius of curvature at the indicated point of the polar curve.

51. $r = 4\cos 2\theta$. $\dfrac{\mathrm{d}r}{d\theta} = -8\sin 2\theta$, $\dfrac{\mathrm{d}^2 r}{d\theta^2} = -16\cos 2\theta$.

At $\theta = \frac{\pi}{12}$, $r = 4\cos\frac{\pi}{6} = 2\sqrt{3}$, $\dfrac{\mathrm{d}r}{d\theta} = -8\sin\frac{\pi}{6} = -4$, $\dfrac{\mathrm{d}^2 r}{d\theta^2} = -16\cos\frac{\pi}{6} = -8\sqrt{3}$.
$$\mathrm{K} = \frac{(2\sqrt{3})^2 + 2(-4)^2 - (2\sqrt{3})(-8\sqrt{3})}{[(2\sqrt{3})^2 + (-4)^2]^{3/2}} = \frac{12 + 32 + 48}{(12 + 16)^{3/2}} = \frac{92}{28^{3/2}} = \frac{23}{98}\sqrt{7}, \ \rho = \frac{14}{23}\sqrt{7}$$

52. $r = 1 - \sin\theta;\ \theta = 0$

▹ We have

$$\frac{dr}{d\theta} = -\cos\theta \text{ and } \frac{d^2r}{d\theta^2} = \sin\theta$$

When $\theta = 0$, we get

$$r = 1 \quad \frac{dr}{d\theta} = -1 \quad \frac{d^2r}{d\theta^2} = 0$$

Thus,

$$K = \frac{\left|r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\right|}{\left[r^2 + (dr/d\theta)^2\right]^{3/2}} = \frac{\left|1^2 + 2(-1)^2 - 1(0)\right|}{\left[1^2 + (-1)^2\right]^{3/2}} = \frac{3}{2^{3/2}}$$

Hence,

$$\rho = \frac{1}{K} = \tfrac{2}{3}\sqrt{2}$$

53. $r = a\sec^2\tfrac{1}{2}\theta$. $\frac{dr}{d\theta} = a\sec^2\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta$, $\frac{d^2r}{d\theta^2} = a\sec^2\tfrac{1}{2}\theta\tan^2\tfrac{1}{2}\theta + \frac{1}{2}a\sec^4\tfrac{1}{2}\theta$. At $\theta = \frac{2}{3}\pi$,

$r = a\sec^2\frac{\pi}{3} = 4a$; $\frac{dr}{d\theta} = a\sec^2\frac{\pi}{3}\tan\frac{\pi}{3} = 4\sqrt{3}a$, $\frac{d^2r}{d\theta^2} = a\sec^2\frac{\pi}{3}\tan^2\frac{\pi}{3} + \frac{1}{2}a\sec^4\frac{\pi}{3} = 12a + 8a = 20a$,

$$K = \frac{(4a)^2 - 2(4\sqrt{3}a)^2 - (4a)(20a)}{\left[(4a)^2 + (4\sqrt{3}a)^2\right]^{3/2}} = \frac{16a^2 + 98a^2 - 80a^2}{(16a^2 + 48a^2)^{3/2}} = \frac{32a^2}{8^3|a|^3} = \frac{1}{16|a|},\ \rho = 16|a|$$

54. At $\theta = 1$, $r = a\theta = a$, $r' = a$, $r'' = 0$. $K = \frac{\left|a^2 + 2a^2 - 0\right|}{(a^2 + a^2)^{3/2}} = \frac{3a^2}{2^{3/2}|a|^3} = \frac{3\sqrt{2}}{4|a|}$, $\rho = \frac{2}{3}\sqrt{2}|a|$

55. $[K(t)]^2 = \|D_s\mathbf{T}(t)\|^2 = D_s\mathbf{T}(t)\cdot D_s\mathbf{T}(t)$ and by Theorem 11.3.3, $\mathbf{T}(t) = D_s\mathbf{R}(t)$. Because $\mathbf{T}(t)$ is a unit vector,
$0 = \mathbf{T}(t)\cdot D_s\mathbf{T}(t)$. Therefore
$0 = D_s[\mathbf{T}(t)\cdot D_s\mathbf{T}(t)] = D_s\mathbf{T}(t)\cdot D_s\mathbf{T}(t) + \mathbf{T}(t)\cdot D_s{}^2\mathbf{T}(t) = [K(t)]^2 + D_s\mathbf{R}(t)\cdot D_s{}^3\mathbf{R}(t)$
Hence $D_s\mathbf{R}(t)\cdot D_s{}^3\mathbf{R}(t) = -[K(t)]^4$.

56. Prove Theorem 11.4.2.

▹ We have by the chain rule

$$\dot{\mathbf{R}} = \frac{d\mathbf{R}}{ds}\cdot\frac{ds}{dt} = \dot{s}\mathbf{T}$$

$$\ddot{\mathbf{R}} = \ddot{s}\mathbf{T} + \dot{s}\dot{\mathbf{T}} = \ddot{s}\mathbf{T} + \dot{s}^2K\mathbf{N}$$

Because $\mathbf{T}\times\mathbf{T} = \mathbf{0}$, we have

$$\dot{\mathbf{R}}\times\ddot{\mathbf{R}} = \dot{s}^3K(\mathbf{T}\times\mathbf{N}) = \dot{s}^3K\mathbf{B}$$

Because $\mathbf{T}$ and $\mathbf{B}$ are unit vectors, then

$$\|\dot{\mathbf{R}}\| = |\dot{s}| \text{ and } \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\| = |\dot{s}|^3K = \|\dot{\mathbf{R}}\|^3K$$

Solving for K, we find the formula of Theorem 11.4.2

$$K = \frac{\|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|}{\|\dot{\mathbf{R}}\|^3}$$

57. $y = a\cosh\frac{x}{a}$. $\frac{dy}{dx} = \sinh\frac{x}{a}$; $\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\frac{x}{a}$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|\frac{1}{a}\cosh\frac{x}{a}\right|}{\left[1 + \sinh^2\frac{x}{a}\right]^{3/2}} = \frac{\frac{1}{a}\cosh\frac{x}{a}}{\cosh^3\frac{x}{a}} = \frac{a}{a^2\cosh^2\frac{x}{a}} = \frac{a}{y^2}$$

Because $y = a\cosh\frac{x}{a}$ has an absolute minimum at (0,a), K has an absolute maximum there.

58. $\mathbf{R} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$, $a > 0$. $\dot{\mathbf{R}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b$, $\ddot{\mathbf{R}} = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$$\|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{matrix}\right\| = \|ab\sin t\mathbf{i} - ab\cos t\mathbf{j} - a^2\mathbf{k}\| = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2 + b^2}$$

$$K = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = a\sqrt{a^2 + b^2}/(\sqrt{a^2 + b^2})^3 = a/(a^2 + b^2)$$

59. $K = \frac{1}{a + b^2/a} = \frac{1}{(\sqrt{a} - b/\sqrt{a})^2 + 2b}$ has an absolute maximum value of $\frac{1}{2b}$ when $\sqrt{a} = b/\sqrt{a}$, $b = a$.

11.5 CURVILINEAR MOTION

11.5.1 Definition Let C be the curve having vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

If a particle is moving along C so that its position at any time t units is is the point $P(f(t), g(t), h(t))$, and then the velocity vector $\mathbf{V}(t)$ and the acceleration vector $\mathbf{A}(t)$ at the point P are defined by

$\mathbf{V}(t) = \mathbf{R}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

$\mathbf{A}(t) = \mathbf{V}'(t) = \mathbf{R}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$

Initial Value Reversing the above equations, we have

$\mathbf{V}(t) = \mathbf{V}(a) + \int_a^t \mathbf{A}(u)du$ and $\mathbf{R}(t) = \mathbf{R}(a) + \int_a^t \mathbf{V}(u)du$

If a representation of the velocity vector $\mathbf{V}(t)$ is chosen so that its initial point is on the curve C, then the representation is tangent to C at all t. The measure of the *speed* of the particle at t is a scalar and is the magnitude of the velocity vector. Because the number of units in the speed is ds/dt, we have

$v(t) = \frac{ds}{dt} = \|\mathbf{V}(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

The acceleration vector may be expressed as a linear combination of the unit tangent vector and the unit normal vector as follows.

$\mathbf{A}(t) = A_T(t)\mathbf{T}(t) + A_N(t)\mathbf{N}(t)$

where

$A_T(t) = \frac{d^2s}{dt^2}$ is the *tangential component* of acceleration

and

$A_N(t) = K(t)\left(\frac{ds}{dt}\right)^2 = \frac{v^2}{\rho}$ is the *normal component* of acceleration.

It follows that

$A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2}$ and $K(t) = A_N(t)/v(t)^2$

Planar Motion If

$\mathbf{T} = a\mathbf{i} + b\mathbf{j}$

then

$\mathbf{N} = \begin{cases} -b\mathbf{i} + b\mathbf{j} & \text{if } \mathbf{A}\cdot\mathbf{N} \geq 0 \\ b\mathbf{i} - a\mathbf{j} & \text{if otherwise} \end{cases}$

If a representation of the acceleration vector $\mathbf{A}(t)$ is chosen so that its initial point in on the curve C, then the representation is directed toward the "inside" or concave side of the curve.

If a projectile is shot from a gun having angle of elevation of measure α with a muzzle velocity of v_0 ft/sec, then the position vector of the projectile at t sec is $\mathbf{R}(t)$, given by

$\mathbf{R}(t) = tv_0\cos\alpha\mathbf{i} + (tv_0\sin\alpha - \frac{1}{2}gt^2)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \quad \mathbf{V}(t) = v_0\cos\alpha\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}$

y is maximum when $t = T = v_0g^{-1}\sin\alpha$. The range is $x(2T) = v_0{}^2g^{-1}\sin 2\alpha$.

where g ft/sec^2 is the acceleration due to the force of gravity ($g = 32$).

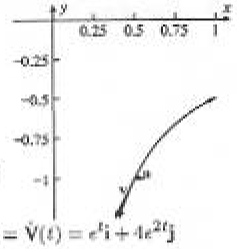
Exercises 11.5

In Exercises 1–10, the position $\mathbf{R}$ of a particle moving in the xy plane at time t units is given. (a) Find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $v(t) = \|\mathbf{V}(t)\|$, $a(t) = \|\mathbf{A}(t)\|$. (b) Find $\mathbf{V}(t_1)$ and $\mathbf{A}(t_1)$ and (c) sketch their representations on the particle's path.

1. $\mathbf{R}(t) = (t^2 + 4)\mathbf{i} + (t - 2)\mathbf{j}$; $t_1 = 3$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2t\mathbf{i} + \mathbf{j}$; $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{i}$;
$v(t) = \sqrt{4t^2 + 1}$; $a(t) = 2$ (b) $\mathbf{V}(3) = 6\mathbf{i} + \mathbf{j}$ $\mathbf{A}(3) = 2\mathbf{i}$

2. $\mathbf{R}(t) = (1 + t)\mathbf{i} + (t^2 - 1)\mathbf{j}$; $t_1 = 1$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + 2t\mathbf{j}$; $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{j}$;
$v(t) = \sqrt{1 + 4t^2}$; $a(t) = 2$ (b) $\mathbf{V}(1) = \mathbf{i} + 2\mathbf{j}$ $\mathbf{A}(1) = 2\mathbf{j}$

3. $\mathbf{R}(t) = 5\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$; $t_1 = \frac{1}{4}\pi$ (a) $\mathbf{V} = \dot{\mathbf{R}} = -10\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j}$; $\mathbf{A} = \dot{\mathbf{V}} = -20\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j}$;
$v = 2\sqrt{25\sin^2 2t + 9\cos^2 2t}$; $a = 4\sqrt{25\cos^2 2t + 9\sin^2 2t}$ (b) $\mathbf{V}(\frac{1}{4}\pi) = -10\mathbf{i}$; $\mathbf{A}(\frac{1}{4}\pi) = -12\mathbf{j}$



4. $\mathbf{R}(t) = \frac{2}{t}\mathbf{i} - \frac{1}{4}t\mathbf{j}$; $t_1 = 4$

▸ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = -\frac{2}{t^2}\mathbf{i} - \frac{1}{4}\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = \frac{4}{t^3}\mathbf{i}$

$\|\mathbf{V}(t)\| = \sqrt{4/t^4 + \frac{1}{16}}$ $\quad \|\mathbf{A}(t)\| = \frac{4}{|t|^3}$

(b) $\mathbf{V}(4) = -\frac{1}{8}\mathbf{i} - \frac{1}{4}\mathbf{j}$ $\quad \mathbf{A}(4) = \frac{1}{16}\mathbf{i}$

(c) $\mathbf{R}(4) = \frac{1}{2}\mathbf{i} - \mathbf{j}$. The figure shows the path (a hyperbola) and representations of the velocity and acceleration vectors at $(\frac{1}{2}, -1)$.

5. $\mathbf{R}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}$, $t_1 = \ln 2$ $\quad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = e^t\mathbf{i} + 2e^{2t}\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = e^t\mathbf{i} + 4e^{2t}\mathbf{j}$

$v(t) = e^t\sqrt{1 + 4e^{2t}}$; $a(t) = e^t\sqrt{1 + 16e^{2t}}$ (b) $\mathbf{V}(\ln 2) = 2\mathbf{i} + 8\mathbf{j}$ $\quad \mathbf{A}(\ln 2) = 2\mathbf{i} + 16\mathbf{j}$

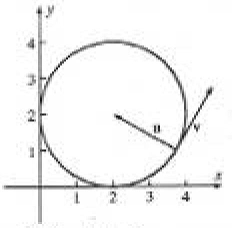
6. $\mathbf{R}(t) = e^{2t}\mathbf{i} + e^{3t}\mathbf{j}$, $t_1 = 0$ $\quad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2e^{2t}\mathbf{i} + 3e^{3t}\mathbf{j}$ $\quad \mathbf{A}(t) = 4e^{2t}\mathbf{i} + 9e^{3t}\mathbf{j}$

$v(t) = e^{2t}\sqrt{4 + 9e^{2t}}$; $a(t) = e^{2t}\sqrt{16 + 81e^{2t}}$ (b) $\mathbf{V}(0) = 2\mathbf{i} + 3\mathbf{j}$ $\quad \mathbf{A}(0) = 4\mathbf{i} + 9\mathbf{j}$

7. $\mathbf{R}(t) = t\mathbf{i} + (\ln \sec t)\mathbf{j}$; $t_1 = \frac{1}{4}\pi$ $\quad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + \tan t\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = \sec^2 t\mathbf{j}$

$v(t) = \sqrt{1 + \tan^2 t} = |\sec t|$; $a(t) = \sec^2 t$ (b) $\mathbf{V}(\frac{1}{4}\pi) = \mathbf{i} + \mathbf{j}$ $\quad \mathbf{A}(\frac{1}{4}\pi) = 2\mathbf{j}$

8. $\mathbf{R}(t) = 2(1 - \cos t)\mathbf{i} + 2(1 - \sin t)\mathbf{j}$; $t_1 = \frac{5}{6}\pi$



▸ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2 \sin t\mathbf{i} - 2 \cos t\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

$\|\mathbf{V}(t)\| = 2\sqrt{\sin^2 t + \cos^2 t} = 2$ $\quad \|\mathbf{A}(t)\| = 2\sqrt{\cos^2 t + \sin^2 t} = 2$

(b) $\mathbf{V}(\frac{5}{6}\pi) = 2(\frac{1}{2})\mathbf{i} - 2(-\frac{1}{2}\sqrt{3})\mathbf{j} = \mathbf{i} + \sqrt{3}\mathbf{j}$; $\mathbf{A}(\frac{5}{6}\pi) = 2(-\frac{1}{2}\sqrt{3})\mathbf{i} + 2(\frac{1}{2})\mathbf{j} = -\sqrt{3}\mathbf{i} + \mathbf{j}$

(c) $\mathbf{R}(\frac{5}{6}\pi) = 2(1 + \frac{1}{2}\sqrt{3})\mathbf{i} + 2(1 - \frac{1}{2})\mathbf{j} = (2 + \sqrt{3})\mathbf{i} + \mathbf{j}$

The figure shows the circular path and representations of the velocity and acceleration vectors at $(1 + \sqrt{3}, 1)$.

9. $\mathbf{R}(t) = (t^2 + 3t)\mathbf{i} + (1 - 3t^2)\mathbf{j}$; $t_1 = \frac{1}{2}$ $\quad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = (2t + 3)\mathbf{i} - 6t\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{i} - 6\mathbf{j}$

$v = \sqrt{(2t + 3)^2 + 36t^2} = \sqrt{40t^2 + 12t + 9}$; $a = 2\sqrt{1^2 + 3^2} = 2\sqrt{10}$ (b) $\mathbf{V}(\frac{1}{2}) = 4\mathbf{i} - 3\mathbf{j}$; $\mathbf{A}(\frac{1}{2}) = 2\mathbf{i} - 6\mathbf{j}$

10. $\mathbf{R}(t) = \ln(t + 2)\mathbf{i} + \frac{1}{3}t^2\mathbf{j}$; $t_1 = 1$ $\quad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \frac{1}{t + 2}\mathbf{i} + \frac{2}{3}t\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = -\frac{1}{(t + 2)^2}\mathbf{i} + \frac{2}{3}\mathbf{j}$

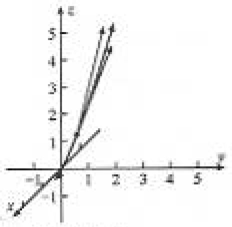
$v = \sqrt{(t + 2)^{-2} + \frac{4}{9}t^2}$; $a = \sqrt{(t + 2)^{-4} + \frac{4}{9}}$ (b) $\mathbf{V}(1) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$ $\quad \mathbf{A}(1) = -\frac{1}{9}\mathbf{i} + \frac{2}{3}\mathbf{j}$

In Exercises 11–16, the position $\mathbf{R}$ of a particle moving in space at time t units is given. Find $\mathbf{V}(t_1)$, $\mathbf{A}(t_1)$ and $v(t_1)$ and sketch representations of the vectors on the particle's path.

11. $\mathbf{R} = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}$; $t_1 = \frac{1}{2}\pi$ $\quad \mathbf{V} = \dot{\mathbf{R}} = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k} = -2\mathbf{i} + \mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = -2 \cos t\mathbf{i} - 2 \sin t\mathbf{j} = -2\mathbf{j}$ $\quad v(\frac{1}{2}\pi) = \sqrt{2^2 + 1^2} = \sqrt{5}$

12. $\mathbf{R}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$; $t_1 = 2$



▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = \mathbf{j} + 2t\mathbf{k}$

$\mathbf{V}(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ $\quad \mathbf{A}(2) = \mathbf{j} + 4\mathbf{k}$

$v(2) = \|\mathbf{V}(2)\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$

$\mathbf{R}(2) = 2\mathbf{i} + 2\mathbf{j} + \frac{8}{3}\mathbf{j}$

The figure shows the path and representations of the velocity and acceleration vectors at $(2, 2, \frac{8}{3})$

13. $\mathbf{R} = t\mathbf{i} + (t^2 - 2t)\mathbf{j} + 2(t - 1)\mathbf{k}$; $t_1 = 2$ $\quad \mathbf{V} = \dot{\mathbf{R}} = \mathbf{i} + (2t - 2)\mathbf{j} + 2\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = 2\mathbf{j}$ $\quad v(2) = \sqrt{1^2 + 2^2 + 2^2} = 3$

14. $\mathbf{R} = 3 \cos t\mathbf{i} + 4 \sin t\mathbf{j} + 2t\mathbf{k}$; $t_1 = \frac{1}{2}\pi$ $\quad \mathbf{V} = \dot{\mathbf{R}} = -3 \sin t\mathbf{i} + 4 \cos t\mathbf{j} + 2\mathbf{k} = -3\mathbf{i} + 2\mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = -3 \cos t\mathbf{i} - 4 \sin t\mathbf{j} = -4\mathbf{j}$ $\quad v(\frac{1}{2}\pi) = \sqrt{3^2 + 2^2} = \sqrt{13}$

15. $\mathbf{R} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k};\ t_1 = 0$ $\quad \mathbf{V} = \dot{\mathbf{R}} = \sqrt{2}e^t\cos(t + \frac{1}{4}\pi)\mathbf{i} + \sqrt{2}e^t\sin(t + \frac{1}{4}\pi)\mathbf{j} + e^t\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = 2e^t\cos(t + \frac{1}{2}\pi)\mathbf{i} + 2e^t\sin(t + \frac{1}{2}\pi)\mathbf{j} + e^t\mathbf{k} = 2\mathbf{j} + \mathbf{k}$ $\quad v(0) = \sqrt{3}$

16. $\mathbf{R}(t) = \frac{1}{2}(t^2+1)^{-1}\mathbf{i} + \ln(1+t^2)\mathbf{j} + \tan^{-1}t\mathbf{k};\ t_1 = 1$

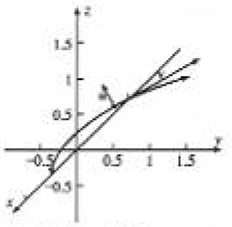
▷ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = t(t^2+1)^{-2}\mathbf{i} + 2t(1+t^2)^{-1}\mathbf{j} + (1+t^2)^{-1}\mathbf{k}$

$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = (3t^2-1)(t^2+1)^{-3}\mathbf{i} + 2(1-t^2)(1+t^2)^{-2}\mathbf{j} - 2t(1+t^2)^{-2}\mathbf{k}$

$\mathbf{V}(1) = -\frac{1}{4}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$ $\quad \mathbf{A}(1) = \frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{k}$

$v(1) = \|\mathbf{V}(1)\| = \sqrt{\frac{1}{16} + 1 + \frac{1}{4}} = \frac{1}{4}\sqrt{21}$ $\quad \mathbf{R}(1) = \frac{1}{4}\mathbf{i} + \ln 2\mathbf{j} + \frac{1}{4}\pi\mathbf{k}$

The figure shows the path and representations of the velocity and acceleration vectors at (0.25, 0.69, 0.79).



In Exercises 17–24, find a vector equation of the path satisfying the given equation and initial conditions.

17. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 3\mathbf{i} + 2\mathbf{j} + \int_0^t [(u+1)^{-2}\mathbf{i} + (u+1)\mathbf{j}]du$

$= 3\mathbf{i} + 2\mathbf{j} + \Big[-(u+1)^{-1}\mathbf{i} + (\frac{1}{2}u^2 + u)\mathbf{j}\Big]_0^t = \Big(4 - \frac{1}{t+1}\Big)\mathbf{i} + (\frac{1}{2}t^2 + t + 2)\mathbf{j}$

18. $\mathbf{R}(t) = \mathbf{R}(1) + \int_1^t \mathbf{V}(u)du = 4\mathbf{i} - 3\mathbf{j} + \int_1^t [(2u-1)\mathbf{i} + 3u^{-2}\mathbf{j}]du$

$= 4\mathbf{i} - 3\mathbf{j} + [(u^2 - u)\mathbf{i} - 3u^{-1}]_1^t = (t^2 - t + 4)\mathbf{i} - 3t^{-1}\mathbf{j}$

19. $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 2\mathbf{i} + \mathbf{j} + \int_0^t (e^{-u}\mathbf{i} + 2e^{3u}\mathbf{j})du = 2\mathbf{i} + \mathbf{j} + \Big[-e^{-u}\mathbf{i} + \frac{2}{3}e^{3u}\mathbf{j}\Big]_0^t = (3 - e^{-t})\mathbf{i} + (\frac{2}{3}e^{3t} + \frac{1}{3})\mathbf{j}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 3\mathbf{j} + \int_0^t [(3 - e^{-u})\mathbf{i} + (\frac{2}{3}e^{3u} + \frac{1}{3})\mathbf{j}]du = 3\mathbf{j} + \Big[(3u + e^{-u})\mathbf{i} + (\frac{2}{9}e^{3u} + \frac{1}{3}u)\Big]_0^t$

$= (3t + e^{-t} - 1)\mathbf{i} + (\frac{2}{9}e^{3t} + \frac{1}{3}t + \frac{25}{9})\mathbf{j}$

20. $\mathbf{A}(t) = 2\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j}$, $\mathbf{V}(0) = \mathbf{i} + \mathbf{j}$, and $\mathbf{R}(0) = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

▷ $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = \mathbf{i} + \mathbf{j} + \int_0^t (2\cos 2u\mathbf{i} + 2\sin 2u\mathbf{j})du = \mathbf{i} + \mathbf{j} + [\sin 2u\mathbf{i} - \cos 2u\mathbf{j}]_0^t$

$= (\sin 2t + 1)\mathbf{i} + (2 - \cos 2t)\mathbf{j}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \int_0^t (\sin 2u + 1)\mathbf{i} + (2 - \cos 2t)\mathbf{j}$

$= \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \Big[(-\frac{1}{2}\cos 2u + u)\mathbf{i} + (2u - \frac{1}{2}\sin 2u)\mathbf{j}\Big]_0^t = (-\frac{1}{2}\cos 2t + t + 1)\mathbf{i} + (2t - \frac{1}{2}\sin 2t - \frac{1}{2})\mathbf{j}$

21. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = \mathbf{i} + 2\mathbf{j} + \int_0^t (\mathbf{i} + \mathbf{j} - 32u\mathbf{k})du = \mathbf{i} + 2\mathbf{j} + [u\mathbf{i} + 2u\mathbf{j} - 16u^2\mathbf{k}]_0^t = (t+1)\mathbf{i} + (2t+2)\mathbf{j} - 16t^2\mathbf{k}$

22. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 2\mathbf{j} + \mathbf{k} + \int_0^t [(u^2 + 2u)\mathbf{i} + 2u\mathbf{j} + 3u^2\mathbf{k}]du = 2\mathbf{j} + \mathbf{k} + \Big[(\frac{1}{3}u^3 + u^2)\mathbf{i} + u^2\mathbf{j} + u^3\mathbf{k}\Big]_0^t$

$= (\frac{1}{3}t^3 + t^2)\mathbf{i} + (t^2 + 2)\mathbf{j} + (t^3 + 1)\mathbf{k}$

23. $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 2\mathbf{i} + 3\mathbf{j} + \int_0^t (6u\mathbf{i} + 12u^2\mathbf{j} + \mathbf{k})du = 2\mathbf{i} + 3\mathbf{j} + [3u^2\mathbf{i} + 4u^3\mathbf{j} + u\mathbf{k}]_0^t$

$= (3t^2 + 2)\mathbf{i} + (4t^3 + 3)\mathbf{j} + t\mathbf{k}$. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 4\mathbf{k} + \int_0^t [(3u^2 + 2)\mathbf{i} + (4u^3 + 3)\mathbf{j} + u\mathbf{k}]du$

$= 4\mathbf{k} + \Big[(u^3 + 2u)\mathbf{i} + (u^4 + 3u)\mathbf{j} + \frac{1}{2}u^2\mathbf{k}\Big]_0^t = (t^3 + 2t)\mathbf{i} + (t^4 + 3t)\mathbf{j} + (\frac{1}{2}t^2 + 4)\mathbf{k}$

24. $\mathbf{A}(t) = -32\mathbf{k}$, $\mathbf{V}(0) = 4\mathbf{i} + 4\mathbf{j}$, $\mathbf{R}(0) = 60\mathbf{k}$

▷ $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 4\mathbf{i} + 4\mathbf{j} + \int_0^t (-32\mathbf{k})du = 4\mathbf{i} + 4\mathbf{j} - 32t\mathbf{k}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 60\mathbf{k} + \int_0^t (4\mathbf{i} + 4\mathbf{j} - 32u\mathbf{k})du = 60\mathbf{k} + [4u\mathbf{i} + 4u\mathbf{j} - 16u^2\mathbf{k}]_0^t = 4t\mathbf{i} + 4t\mathbf{j} + (60 - 16t^2)\mathbf{k}$

In Exercises 25–30, find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\|\mathbf{V}(t)\|$, $\mathrm{A}_T(t)$, $\mathrm{A}_N(t)$ and $\mathrm{K}(t)$; and their values when $t = t_1$. Sketch the curve and representations of $\mathbf{V}(t_1)$, $\mathbf{A}(t_1)$ and its tangential and normal components.

25. $\mathbf{R}(t) = (2t+3)\mathbf{i} + (t^2-1)\mathbf{j}$. $\mathbf{V}(t) = 2\mathbf{i} + 2t\mathbf{j}$. $\mathbf{A}(t) = 2\mathbf{j}$. $\mathrm{D}_t s = \|\mathbf{V}(t)\| = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$.

$\mathrm{A}_T(t) = \mathrm{D}_t{}^2 s = \dfrac{2t}{\sqrt{1+t^2}}$. $\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{4 - \dfrac{4t^2}{1+t^2}} = \sqrt{\dfrac{4+4t^2-4t^2}{1+t^2}} = \dfrac{2}{\sqrt{1+t^2}}$

$\mathrm{K}(t) = \dfrac{\mathrm{A}_N(t)}{(\mathrm{D}_t s)^2} = \dfrac{2}{\sqrt{1+t^2}} \cdot \dfrac{1}{4(1+t^2)} = \dfrac{1}{2(1+t^2)^{3/2}}$. $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{1}{\sqrt{1+t^2}}\mathbf{i} + \dfrac{t}{\sqrt{1+t^2}}\mathbf{j}$.

$\mathbf{N}(t) = \dfrac{-t}{\sqrt{1+t^2}}\mathbf{i} + \dfrac{1}{\sqrt{1+t^2}}\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 0$.

$\mathbf{V}(2) = 2\mathbf{i} + 4\mathbf{j}$. $\mathbf{A}(2) = 2\mathbf{j}$. $\mathbf{T}(2) = \dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$. $\quad \mathbf{N}(2) = \dfrac{-2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}$. $\|\mathbf{V}(2)\| = \sqrt{4+16} = 2\sqrt{5}$.

$\mathrm{A}_T(2) = \dfrac{4}{\sqrt{5}}$. $\mathrm{A}_N(2) = \dfrac{2}{\sqrt{5}}$. $\mathrm{K}(2) = \dfrac{1}{2\sqrt{125}} = \dfrac{1}{10\sqrt{5}}$

26. $\mathbf{R}(t) = (t-1)\mathbf{i} + t^2\mathbf{j}$. $\mathbf{V}(t) = \mathbf{i} + 2t\mathbf{j}$. $\mathbf{A}(t) = 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}(t)\| = \sqrt{1+4t^2}$. $\mathrm{A}_T(t) = \ddot{s} = \dfrac{4t}{\sqrt{1+4t^2}}$

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{4 - \dfrac{16t^2}{1+4t^2}} = \sqrt{\dfrac{4}{1+4t^2}} = \dfrac{2}{\sqrt{1+4t^2}}$

$\mathrm{K}(t) = \dfrac{\mathrm{A}_N(t)}{\dot{s}^2} = \dfrac{2}{\sqrt{1+4t^2}} \cdot \dfrac{1}{1+4t^2} = \dfrac{2}{(1+4t^2)^{3/2}}$ $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{\mathbf{i}+2\mathbf{j}}{\sqrt{1+4t^2}}$. $\mathbf{N}(t) = \dfrac{-2\mathbf{i}+\mathbf{j}}{\sqrt{1+4t^2}}$

$\mathbf{V}(1) = \mathbf{i} + 2\mathbf{j}$. $\mathbf{A}(1) = 2\mathbf{j}$. $\mathbf{T}(1) = \frac{1}{5}\sqrt{5}(\mathbf{i}+2\mathbf{j})$. $\quad \mathbf{N}(1) = \frac{1}{5}\sqrt{5}(-2\mathbf{i}+\mathbf{j})$. $\|\mathbf{V}(1)\| = \sqrt{5}$.

$\mathrm{A}_T(1) = \frac{4}{5}\sqrt{5}$. $\mathrm{A}_N(1) = \frac{2}{5}\sqrt{5}$. $\mathrm{K}(1) = \frac{2}{25}\sqrt{5}$

27. $\mathbf{R}(t) = 5\cos 3t\mathbf{i} + 5\sin 3t\mathbf{j}$. $\mathbf{V}(t) = -15\sin 3t\mathbf{i} + 15\cos 3t\mathbf{j}$. $\mathbf{A}(t) = -45\cos 3t\mathbf{i} - 45\sin 3t\mathbf{j}$.

$\mathrm{D}_t s = \|\mathbf{V}(t)\| = 15\sqrt{(-\sin 3t)^2 + \cos^2 3t} = 15$. $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j}$.

$\mathbf{N}(t) = -\cos 3t\mathbf{i} - \sin 3t\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 0$. $\quad \mathrm{A}_T(t) = \mathrm{D}_t{}^2 s = \mathrm{D}_t(15) = 0$.

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{\|\mathbf{A}(t)\|^2} = \|\mathbf{A}(t)\| = 45\sqrt{(-\cos 3t)^2 + \sin^2 3t} = 45$

$\mathrm{K}(t) = \dfrac{\mathrm{A}_N(t)}{(\mathrm{D}_t s)^2} = \dfrac{45}{225} = \dfrac{1}{5}$. $\mathbf{V}(\frac{1}{3}\pi) = -15\sin\pi\mathbf{i} + 15\cos\pi\mathbf{j} = -15\mathbf{j}$. $\mathbf{A}(\frac{1}{3}\pi) = -45\cos\pi\mathbf{i} - 45\sin\pi\mathbf{j} = 45\mathbf{i}$.

$\mathbf{T}(\frac{1}{3}\pi) = -\sin\pi\mathbf{i} + \cos\pi\mathbf{j} = -\mathbf{j}$. $\mathbf{N}(\frac{1}{3}\pi) = -\cos\pi\mathbf{i} - \sin\pi\mathbf{j} = \mathbf{i}$. $\|\mathbf{V}(\frac{1}{3}\pi)\| = 15$.

28. $\mathbf{R}(t) = 3t^2\mathbf{i} + 2t^3\mathbf{j}$; $t_1 = 1$

▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 6t\mathbf{i} + 6t^2\mathbf{j}$ $\quad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = 6\mathbf{i} + 12t\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = 6\sqrt{t^2+t^4} = 6|t|\sqrt{1+t^2} = 6t\sqrt{1+t^2}$

We have simplified by omitting the factor $\operatorname{sgn}(t)$ from the last expression and every subsequent expression.

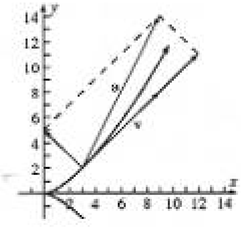
$\mathrm{A}_T = \ddot{s} = 6\sqrt{1+t^2} + 6t\dfrac{t}{\sqrt{1+t^2}} = \dfrac{6(1+2t^2)}{\sqrt{1+t^2}}$

$\mathrm{A}_N = \sqrt{\|\mathbf{A}(t)\|^2 - \mathrm{A}_T{}^2} = 6\sqrt{(1+4t^2) - \dfrac{1+4t2+4t4}{1+t2}} = \dfrac{6t}{\sqrt{1+t^2}}$

$\mathrm{K}(t) = \dfrac{\mathrm{A}_N}{\dot{s}^2} = \dfrac{6t}{\sqrt{1+t^2}} \cdot \dfrac{1}{36t^2(1+t^2)} = \dfrac{1}{6t(1+t^2)^{3/2}}$

$\mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{6t\mathbf{i}+6t^2\mathbf{j}}{6t\sqrt{1+t^2}} = \dfrac{\mathbf{i}+t\mathbf{j}}{\sqrt{1+t^2}}$

$\mathbf{N}(t) = \dfrac{-t\mathbf{i}+\mathbf{j}}{\sqrt{1+t^2}}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) = \dfrac{6t}{\sqrt{1+t^2}} \geq 0$



Substituting $t = 1$ in each formula, we obtain

$\mathbf{V}(1) = 6\mathbf{i} + 6\mathbf{j}$ $\quad \mathbf{A}(1) = 6\mathbf{i} + 12\mathbf{j}$ $\quad \mathbf{T}(1) = \frac{1}{2}\sqrt{2}(\mathbf{i}+\mathbf{j})$ $\quad \mathbf{N}(1) = \frac{1}{2}\sqrt{2}(-\mathbf{i}+\mathbf{j})$

$\|\mathbf{V}(1)\| = 6\sqrt{2}$ $\mathrm{A}_T(1) = 9\sqrt{2}$ $\mathrm{A}_N(1) = 3\sqrt{2}$ $\mathrm{K}(1) = \frac{1}{24}\sqrt{2}$

$\mathbf{R}(1) = 3\mathbf{i} + 2\mathbf{j}$. The figure shows the curve and the vectors with terminal point $(3,2)$.

29. $\mathbf{R}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$. $\mathbf{V}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$. $\mathbf{A}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$. $\mathrm{D}_t s = \|\mathbf{V}(t)\| = \sqrt{e^{2t} + e^{-2t}}$.

$$\mathbf{T}(t) = \frac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \frac{e^t}{\sqrt{e^{2t}+e^{-2t}}}\mathbf{i} - \frac{e^{-t}}{\sqrt{e^{2t}+e^{-2t}}}\mathbf{j}$$

$\mathbf{N}(t) = \dfrac{e^{-t}}{\sqrt{e^{2t}+e^{-2t}}}\mathbf{i} + \dfrac{e^t}{\sqrt{e^{2t}+e^{-2t}}}\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 0$. $\mathrm{A}_T(t) = \mathrm{D}_t{}^2 s = \dfrac{e^{2t}-e^{-2t}}{\sqrt{e^{2t}+e^{-2t}}}$.

$$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{e^{2t}+e^{-2t} - \frac{(e^{2t}-e^{-2t})^2}{e^{2t}+e^{-2t}}} = \sqrt{\frac{e^{4t}+2+e^{-4t}-(e^{4t}-2+e^{-4t})}{e^{2t}+e^{-2t}}}$$

$$= \frac{2}{\sqrt{e^{2t}+e^{-2t}}}. \ \mathrm{K}(t) = \frac{\mathrm{A}_N(t)}{(\mathrm{D}_t s)^2} = \frac{2}{\sqrt{e^{2t}+e^{-2t}}}\cdot\frac{1}{e^{2t}+e^{-2t}} = \frac{2}{(e^{2t}+e^{-2t})^{3/2}}.$$

$\mathbf{V}(0) = \mathbf{i} - \mathbf{j}$. $\mathbf{A}(0) = \mathbf{i} + \mathbf{j}$. $\mathbf{T}(0) = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{N}(0) = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$.

$\|\mathbf{V}(0)\| = \sqrt{2}$. $\mathrm{A}_T(0) = 0$. $\mathrm{A}_N(0) = \sqrt{2}$. $\mathrm{K}(0) = \dfrac{1}{\sqrt{2}}$

30. $\mathbf{R}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j}$. $\mathbf{V}(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j}$. $\mathbf{A}(t) = (-2\sin t^2 - 4t^2\cos t^2)\mathbf{i} + (2\cos t^2 - 4t^2\sin t^2)\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = 2t\,\mathrm{sgn}(t)$. $\mathrm{A}_T = \ddot{s} = 2\,\mathrm{sgn}(t)$. $\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - \mathrm{A}_T(t)^2} = 4t^2$. $\mathrm{K}(t) = \dfrac{\mathrm{A}_N}{\dot{s}^2} = 1$.

$\mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \mathrm{sgn}(t)(-\sin t^2\mathbf{i} + \cos t^2\mathbf{j})$. $\mathbf{N}(t) = -\cos t^2\mathbf{i} - \sin t^2\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 2 > 0$

$t_1 = \frac{1}{2}\sqrt{\pi}$. $\mathbf{V}(t_1) = \sqrt{\frac{1}{2}\pi}(-\mathbf{i}+\mathbf{j})$. $\mathbf{A}(t_1) \approx -3.6\mathbf{i} - 0.8\mathbf{j}$. $\mathbf{T}(t_1) = \frac{1}{2}\sqrt{2}(-\mathbf{i}+\mathbf{j})$. $\mathbf{N}(t_1) = -\frac{1}{2}\sqrt{2}(\mathbf{i}+\mathbf{j})$

$\|\mathbf{V}(t_1)\| = \sqrt{\pi}$. $\mathrm{A}_T(t_1) = 2$. $\mathrm{A}_N(t_1) = \pi$. $\mathrm{K}(t_1) = 1$

In Exercises 31–36, write $\mathbf{A}(t) = \mathrm{A}_T(t)\mathbf{T}(t) + \mathrm{A}_N(t)\mathbf{N}(t)$ without computing $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

31. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$. $\mathbf{V} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$. $\mathbf{A} = 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}\| = \sqrt{1+4t^2+1} = \sqrt{4t^2+2}$. $\mathrm{A}_T = \ddot{s} = \dfrac{4t}{\sqrt{4t^2+2}}$

$\mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{4 - \dfrac{16t^2}{4t^2+2}} = \sqrt{\dfrac{8}{4t^2+2}} = \dfrac{2\sqrt{2}}{\sqrt{4t^2+2}}$. $\mathbf{A} = \dfrac{4t\mathbf{T} + 2\sqrt{2}\mathbf{N}}{\sqrt{4t^2+2}}$

32. $\mathbf{R}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j} + \sqrt{2}t\mathbf{k}$

▷ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = -e^{-t}\mathbf{i} + e^t\mathbf{j} + \sqrt{2}\mathbf{k}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = \sqrt{e^{-2t}+e^{2t}+2} = e^{-t} + e^t$ $\qquad$ $\mathrm{A}_T = \ddot{s} = -e^{-t} + e^t$

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - \mathrm{A}_T(t)^2} = \sqrt{4} = 2$ $\qquad$ $\mathbf{A}(t) = (-e^{-t} + e^t)\mathbf{T}(t) + 2\mathbf{N}(t)$

33. $\mathbf{R}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 2\mathbf{k}$, $t \geq 0$.
$\mathbf{V}(t) = \mathrm{D}_t\mathbf{R}(t) = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} = t\cos t\mathbf{i} + t\sin t\mathbf{j}$

$\mathbf{A}(t) = \mathrm{D}_t\mathbf{V}(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$. $\|\mathbf{A}(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$

$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} = \sqrt{(\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{1+t^2}$. $\mathrm{D}_t s = \|\mathbf{V}(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t$. $\mathrm{A}_T = \mathrm{D}_t{}^2 s = 1$.

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{(1+t^2)-1} = \sqrt{t^2} = |t| = t$. $\mathbf{A}(t) = \mathbf{T}(t) + t\mathbf{N}(t)$

34. $\mathbf{R} = 2t^2\mathbf{i} + t^2\mathbf{j} + 4t\mathbf{k}$. $\mathbf{V} = 4t\mathbf{i} + 2t\mathbf{j} + 4\mathbf{k}$. $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}\| = 2\sqrt{4t^2+t^2+4} = 2\sqrt{5t^2+4}$.

$\mathrm{A}_T = \ddot{s} = \dfrac{10}{\sqrt{5t^2+4}}$ $\mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{20 - \dfrac{100}{5t^2+4}} = \sqrt{\dfrac{100t^2-20}{5t^2+4}} = 2\sqrt{\dfrac{25t^2-5}{5t^2+4}}$

$\mathbf{A} = \dfrac{10}{\sqrt{5t^2+4}}\mathbf{T} + 2\sqrt{\dfrac{25t^2-5}{5t^2+4}}\mathbf{N}$

35. $\mathbf{R} = t^2\mathbf{i} + (\frac{1}{3}t^3 + t)\mathbf{j} + (\frac{1}{3}t^3 - t)\mathbf{k}$. $\mathbf{V} = 2t\mathbf{i} + (t^2+1)\mathbf{j} + (t^2-1)\mathbf{k}$. $\mathbf{A} = 2\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{j}$.
$\dot{s} = \|\mathbf{V}\| = \sqrt{4t^2 + (t^4 + 2t^2 + 1) + (t^4 - 2t^2 + 1)} = \sqrt{2t^4 + 4t^2 + 2} = \sqrt{2}(t^2+1)$. $A_T = \ddot{s} = 2\sqrt{2}t$
$A_N = \sqrt{\|\mathbf{A}\|^2 - {A_T}^2} = \sqrt{4 + 4t^2 + 4t^2 - 8t^2} = 2$. $\mathbf{A} = 2\sqrt{2}t\mathbf{T} + 2\mathbf{N}$

36. $\mathbf{R}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$
▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}$
$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j}$
$\dot{s} = \|\mathbf{V}(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} = \sqrt{2+t^2}$
$A_T(t) = \ddot{s} = \dfrac{t}{\sqrt{2+t^2}}$ $\qquad A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - {A_T}^2}$

$$= \sqrt{(-2\sin t - t\cos t)^2 + (2\cos t - t\sin t)^2 - \frac{t^2}{2+t^2}} = \sqrt{4 + t^2 - \frac{t^2}{2+t^2}} = \sqrt{\frac{4t^4 + 5t^2 + 8}{2+t^2}}$$

$$\mathbf{A}(t) = \frac{t}{\sqrt{2+t^2}}\mathbf{T} + \sqrt{\frac{4t^4+5t^2+8}{2+t^2}}\mathbf{N}$$

37. This is Theorem 11.2.9.

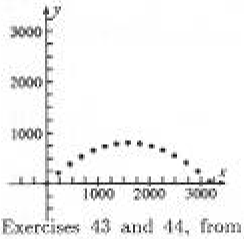
38. $\mathbf{R} = \tan t\mathbf{i} + \sinh 2t\mathbf{j} + \operatorname{sech} t\mathbf{k}$. $\mathbf{V} = \sec^2 t\mathbf{i} + 2\cosh 2t\mathbf{j} - \operatorname{sech} t\tanh t\mathbf{k}$.
$\mathbf{A} = 2\sec^2 t\tan t + 4\sinh 2t\mathbf{j} + \operatorname{sech} t(2\tanh^2 t - 1)$. $\mathbf{V}(0)\cdot\mathbf{A}(0) = (i + 2\mathbf{j})\cdot(-\mathbf{k}) = 0$

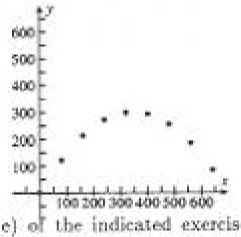
39. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. $\dot{\mathbf{R}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. $\mathbf{T} = \dfrac{1}{\sqrt{1+4t^2+9t^4}}(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$.
$\mathbf{n} = (-4t - 18t^2)\mathbf{i} + (2 - 18t^4)\mathbf{j} + (6t + 12t^3)\mathbf{k}$. $\mathbf{R}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. $\mathbf{T}(1) = \dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.
$\mathbf{n}(1) = -22\mathbf{i} - 16\mathbf{j} + 18\mathbf{k}$. $\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = 14(3\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Osculating: $3(x-1) - 3(y-1) + (z-1) = 0$

40. Prove that for the twisted cubic $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, if $t \neq 0$, no two of the vectors $\mathbf{R}(t)$, $\mathbf{V}(t)$, and $\mathbf{A}(t)$ are orthogonal.
▸ $\mathbf{V}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$ $\qquad \mathbf{A}(t) = 2\mathbf{j} + 6t\mathbf{k}$
Two vectors are orthogonal if and only if their dot product is zero.
$\mathbf{R}(t)\cdot\mathbf{V}(t) = (t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k})\cdot(\mathbf{i} + 2t\mathbf{j} + 3t^3\mathbf{k}) = t + 2t^3 + 3t^5 = t(1 + 2t^2 + 3t^4)$
Because $t \neq 0$ and $3t^4 + 2t^2 + 1 = 2t^4 + (t^2+1)^2 > 0$, then $\mathbf{R}(t)\cdot\mathbf{V}(t) \neq 0$
$\mathbf{R}(t)\cdot\mathbf{A}(t) = (t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k})\cdot(2\mathbf{j} + 6t\mathbf{k}) = 2t^2 + 6t^4 = 2t^2(1 + 3t^2)$
Because $t^2 \neq 0$ and $3t^2 + 1 > 0$, then $\mathbf{R}(t)\cdot\mathbf{A}(t) \neq 0$.
$\mathbf{V}(t)\cdot\mathbf{A}(t) = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\cdot(2\mathbf{j} + 6t\mathbf{k}) = 4t + 18t^3 = t(4 + 18t^2)$
Because $t \neq 0$ and $4 + 18t^2 > 0$, then $\mathbf{V}(t)\cdot\mathbf{A}(t) \neq 0$. Thus no two of the vectors are orthogonal.

41. (a) $\mathbf{V}(0) = 320\cos\frac{1}{4}\pi\mathbf{i} + 320\sin\frac{1}{4}\pi\mathbf{j} = 160\sqrt{2}\mathbf{i} + 160\sqrt{2}\mathbf{j}$
(b) $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = 160\sqrt{2}t\mathbf{i} + (160\sqrt{2}t - 16t^2)\mathbf{j}$; (c) $2T = 2\dfrac{v_0\sin\alpha}{g} = 2\dfrac{320(\frac{1}{2}\sqrt{2})}{32} = 10\sqrt{2}$ sec
(d) range $= \dfrac{{v_0}^2}{g}\sin 2\alpha = \dfrac{(320)^2}{32}\cdot 1 =$ 3200 ft (e) $y_{max} = \dfrac{{v_0}^2}{2g}\sin^2\alpha = \dfrac{(320)^2}{2(32)}\cdot\frac{1}{2} = 800$ ft
(f) $\mathbf{V}(2T) = 160\sqrt{2}\mathbf{i} - 160\sqrt{2}\mathbf{j}$; $v(2T) = v_0 = 320$ ft/sec (g) $\mathbf{R}(6) = 960\sqrt{2}\mathbf{i} + (960\sqrt{2} - 576)\mathbf{j}$;
$\mathbf{V}(6) = 160\sqrt{2}\mathbf{i} + (160\sqrt{2} - 192)\mathbf{j}$; $v(6) = 64\sqrt{34 - 15\sqrt{2}} \approx 229$ ft/sec; (h) $y = x - \dfrac{x^2}{3200}$

42. (a) $\mathbf{V}(0) = 160\cos\frac{1}{3}\pi\mathbf{i} + 160\sin\frac{1}{3}\pi\mathbf{j} = 80\mathbf{i} + 80\sqrt{3}\mathbf{j}$ (b) $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = 80t\mathbf{i} + (80\sqrt{3}t - 16t^2)\mathbf{j}$
(c) $2T = 2\dfrac{v_0\sin\alpha}{g} = 2\dfrac{160(\frac{1}{2}\sqrt{3})}{32} = 5\sqrt{3}$ sec (d) range $= \dfrac{{v_0}^2}{g}\sin 2\alpha = \dfrac{(160)^2}{32}(\frac{1}{2}\sqrt{3}) = 400\sqrt{3}$ ft ≈ 692.8 ft
(e) $y_{max} = \dfrac{{v_0}^2}{2g}\sin^2\alpha = \dfrac{(160)^2}{32}\cdot\dfrac{3}{4} = 600$ ft (f) $\mathbf{V}(2T) = 80\mathbf{i} - 80\sqrt{3}\mathbf{j}$; $v(2T) = v_0 = 160$ ft/sec
(g) $\mathbf{R}(4) = 320\mathbf{i} + (320\sqrt{3} - 256)\mathbf{j}$; $\mathbf{V}(4) = 80\mathbf{i} + (80\sqrt{3} - 128)\mathbf{j}$; $v(4) = \sqrt{80^2 + (80\sqrt{3} - 128)^2}$
$= 32\sqrt{41 - 20\sqrt{3}} \approx 80.7$ ft/sec (h) $t = \frac{x}{80}$. $y = 80\sqrt{3}\cdot\frac{x}{80} - 16\left(\frac{x}{80}\right)^2 = \sqrt{3}x - \frac{x^2}{400}$





In Exercises 43 and 44, from the answers in parts (b)–(e) of the indicated exercises, (a) plot the path of the projectile and simulate its motion; (b) plot the path of the projectile in dot mode and (c) sketch what you see. (d) What do the dots tell you about the speed of the projectile?

43. Exercise 41. See the figure, above left.

44. Exercise 42

▷ In Exercise 42, we found the parametric equations of the path of the projectile to be
$x = 80t$ and $y = 80\sqrt{3}t - 16t^2$
A sketch of the plot in dot mode is shown above right. The dots are closer at the vertex, indicating that the projectile's speed is least there.

45. $y_0 = 96$, $v_0 = 1600$, $\alpha = 30°$. At t sec, $R(t) = (t \cdot 1600 \cos 30°)i + (96 + t \cdot 1600 \sin 30° - 16t^2)$
$= 800\sqrt{3}ti + (96 + 800t - 16t^2)j$ and $x = 800\sqrt{3}t$, $y = 96 + 800t - 16t^2$.
Set $y = 0$: $16t^2 - 800t - 96 = 0$. Because $t > 0$, $t = 25 + \sqrt{631} \approx 50.1$ is the time of flight.
The number of feet traveled is $x(25 + \sqrt{631}) = 20{,}000\sqrt{3} + 800\sqrt{1893} \approx 69{,}450$.

46. range $= 400 = \frac{v_0^2}{g}\sin 2\alpha$, $\sin 2\alpha = \frac{400 \cdot 32}{160^2} = \frac{1}{2}$. $2\alpha = 30°$, $\alpha = 15°$ or $2\alpha = 150°$, $\alpha = 75°$.

47. At t sec, $R(t) = v_0\cos\alpha ti + (v_0\sin\alpha t - 16t^2)j$ so $x = v_0\cos\alpha t$ and $y = \sin\alpha t - 16t^2$.
Because the range is 2000 ft, then $x = 2000$ when $y = 0$. When $y = 0$, $t = \frac{v_0\sin\alpha}{16}$ and then
$2000 = \frac{v_0^2\sin\alpha\cos\alpha}{16}$; $v_0^2\sin\alpha\cos\alpha = 32{,}000$ (1)
Because the maximum height is 1000 ft, then $y = 1000$ when $D_t y = 0$; $v_0\sin\alpha - 32t = 0$;
$t = \frac{v_0\sin\alpha}{32}$. Then $\frac{v_0^2\sin^2\alpha}{32} - \frac{v_0\sin^2\alpha}{64} = 1000$; $v_0^2 = 64{,}000$ (2)
From (1) and (2) we obtain $\tan\alpha = 2$. Therefore $\sin\alpha = \frac{1}{\sqrt{5}}$ and so from (2) we get
$v_0^2 = 80{,}000$; thus $v_0 = 200\sqrt{2} \approx 283$. Hence the muzzle speed of the gun is 283 ft/sec.

48. A ball is thrown horizontally from the top of a cliff 256 ft high with an initial speed of 50 ft/sec. Find the time of flight of the ball and the distance from the base of the cliff to the point where the ball lands.

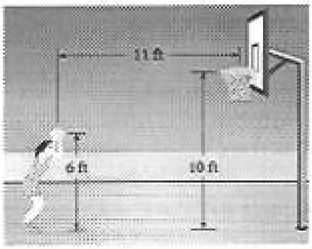
▷ We have
$256 = \frac{1}{2}gt^2 = 16t^2$; $\quad t^2 = 16$; $\quad t = 4$
The time of flight is 4 seconds and the horizontal distance is $4(50) = 200$ feet.

49. $y(t) = 15 \cdot \frac{1}{2}\sqrt{2}t - 16t^2 + 60 = 0$ if $t = \frac{1}{64}(15\sqrt{2} + \sqrt{15810}) \approx 2.30$ sec. $x(2.30) = 15 \cdot \frac{1}{2}\sqrt{2}(2.30) \approx 24.4 < 28$

50. $v_0 = 15$, $\alpha = 0°$, $y_0 = 60$. $R(t) = 15t\cos 0° + (60 + 15t\sin 0° - 16t^2)j = 15ti + (60 - 16t^2)j$ at t sec and $x = 15t$,
$y = 60 - 16t^2$. Set $y = 0$: $60 - 16t = 0$; $t^2 = \frac{15}{4}$; $t = \frac{1}{2}\sqrt{15}$ sec at impact. $x(\frac{1}{2}\sqrt{15}) = 15(\frac{1}{2}\sqrt{15}) \approx 29.0 > 28$

51. $v_0 = 60$, $\alpha = 60°$. $R(t) = t \cdot 60\cos 60°i + (5 + t \cdot 60\sin 60° - 16t^2)j = 30ti + (5 + 30\sqrt{3}t - 16t^2)$ at t sec and $x = 30t$, $y = 5 + 30\sqrt{3}t - 16t^2$. The ball hits the building when $x = 25$, that is, when $t = \frac{5}{6}$. When $t = \frac{5}{6}$, $y = 25\sqrt{3} - \frac{55}{9} \approx 37$. Because $y > 0$, the ball does hit the building.
$V(t) = D_t R(t) = 30i + (30\sqrt{3} - 32t)j$; $V(\frac{5}{6}) = 30i + (30\sqrt{3} - \frac{80}{3})j$
If θ degrees is the direction of the ball when it hits, then $\tan\theta = \frac{30\sqrt{3} - \frac{80}{3}}{30} = \frac{9\sqrt{3} - 8}{9} \approx 0.8431$; $\theta = 40.1°$

52. A basketball player is at a horizontal distance of 11 feet from the center of a basket 10 feet from the ground. Determine the angle α at which she should throw the ball at an initial speed of 25 ft/sec if her hands are 6 ft above the floor.



▹ The basket is 4 feet above her head The equations of motion are
$x(t) = 25(\cos\alpha)t$ and $y(t) = 25(\sin\alpha)t - 16t^2$

The ball reaches the basket when $x = 11$ and $y = 4$ Then

$$11 = 25(\cos\alpha)t;\ t = \frac{11}{25\cos\alpha}$$

and

$$4 = 25(\sin\alpha)\frac{11}{25\cos\alpha} - 16\left(\frac{11}{25\cos\alpha}\right)^2 = 11\tan\alpha - 3.0976\sec^2\alpha$$
$$= 11\tan\alpha - 3.0976\tan^2\alpha - 3.0976$$

We have the quadratic equation

$$0 = 3.0976\tan^2\alpha - 11\tan\alpha + 7.0976$$
$$\tan\alpha = \frac{11 \pm \sqrt{11^2 - 4(3.0976)(7.0976)}}{2(3.0976)} = 2.7036 \text{ or } 0.8475$$
$$\alpha = 69.7^\circ \text{ or } 40.3^\circ$$

She should throw the ball at an angle of 69.7° or 40.3°

53. $x(t) = 40\sqrt{2}t = 100$ if $t = \frac{100}{40\sqrt{2}} = \frac{5}{4}\sqrt{2}$. $y(t) = 40\sqrt{2}t - 16t^2$. $y(\frac{5}{4}\sqrt{2}) = 40\sqrt{2}(\frac{5}{4}\sqrt{2}) - 16(\frac{5}{4}\sqrt{2})^2 = 50 > 45$
$y(t) = 0$ when $t = \frac{40\sqrt{2}}{16} = \frac{5}{2}\sqrt{2}$. $x(\frac{5}{2}\sqrt{2}) = 40\sqrt{2}(\frac{5}{2}\sqrt{2}) = 200$, 25 feet from the pin.

54. Because range $= \frac{v_0{}^2}{g}\sin 2\alpha$, range is maximum when $\sin 2\alpha = 1$, $2\alpha = 90^\circ$, $\alpha = 45^\circ$.

55. Uniform circular motion. $\mathbf{R}(t) = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}$ (a) $\mathbf{V}(t) = -r\omega\sin\omega t\mathbf{i} + r\omega\cos\omega t\mathbf{j}$. $v = \|\mathbf{V}\| = r\omega\sqrt{\sin^2\omega t + \cos^2\omega t} = r\omega$ (b) $\mathbf{A}(t) = -r\omega^2\cos\omega t\mathbf{i} - r\omega^2\sin\omega t\mathbf{j} = -\omega^2\mathbf{R}(t) \Rightarrow$ the direction is opposite.
(c) $\mathbf{T} = \mathbf{V}/v = -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}$, $\mathbf{N} = \dot{\mathbf{T}}/\omega = -\cos\omega t\mathbf{i} - \sin\omega t\mathbf{i}$. $A_T = \dot{v} = 0$, $A_N(t) = \|\mathbf{A}\| = r\omega^2$
(d) If ω is doubled, A_N is quadrupled.

Miscellaneous Exercises for Chapter 11

In Exercises 1–4, find the domain of the vector-valued function.

1. $\mathrm{Dom}\left(\frac{1}{t-3}\mathbf{i} + \sqrt{t}\mathbf{j}\right) = \mathrm{Dom}\left(\frac{1}{t-3}\right) \cap \mathrm{Dom}(\sqrt{t}) = \{t \neq 3\} \cap \{t \geq 0\} = [0,3) \cup (3,+\infty)$

2. $\mathrm{Dom}[\ln(t+1)\mathbf{i} + e^{1/t}\mathbf{j}] = \mathrm{Dom}[\ln(t+1)] \cap \mathrm{Dom}(e^{1/t}) = \{t > -1\} \cap \{t \neq 0\} = (-1,0) \cup (0,+\infty)$

3. $\mathrm{Dom}(\ln|\cos t|\mathbf{i} + \sqrt{4t^2-1}\mathbf{j} + \sqrt{4-t^2}\mathbf{k}) = \{t \neq (k+\frac{1}{2})\pi\} \cap \{|t| \geq \frac{1}{2}\} \cap \{|t| \leq 2\} = [-2,-\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi,-\frac{1}{2}] \cup [\frac{1}{2},\frac{1}{2}\pi) \cup (\frac{1}{2}\pi,2]$

4. $\mathbf{R}(t) = \sqrt{9-t^2}\mathbf{i} + \tan t\mathbf{j} + \frac{t}{t-1}\mathbf{k}$

▹ $\mathrm{Dom}(\mathbf{R}) = \mathrm{Dom}(\sqrt{9-t^2}) \cap \mathrm{Dom}(\tan t) \cap \mathrm{Dom}\left(\frac{t}{t-1}\right) = \{-3 \leq t \leq 3\} \cap \{t \neq (k+\frac{1}{2}\pi)\} \cap \{t \neq 1\}$
$= [-3,-\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi,1) \cup (1,\frac{1}{2}\pi) \cup (\frac{1}{2}\pi,3]$

In Exercises 5–8, find the indicated limit, if it exists.

5. $\lim_{t\to 1}\left(\frac{1}{t+1}\mathbf{i} + \frac{\sqrt{t}-1}{t-1}\mathbf{j}\right) = \lim_{t\to 1}\left(\frac{1}{t+1}\mathbf{i} + \frac{1}{\sqrt{t}+1}\mathbf{j}\right) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

6. $\lim_{t\to 0}\left(\cos t\mathbf{i} + \frac{\sin t}{t}\mathbf{j}\right) = \mathbf{i} + \mathbf{j}$

7. $\lim_{t\to 0^+}\left(e^{-1/t}\mathbf{i} + \frac{1-\cos t}{t}\mathbf{j} + \frac{\sin^{-1}t}{t}\mathbf{k}\right) = \mathbf{k}$

8. $\lim_{t\to 4}\left(\frac{t^2-16}{t-4}\mathbf{i} + \frac{\tan^{-1}(t-4)}{t-4}\mathbf{j} + |t-4|\mathbf{k}\right)$

▹ We apply L'Hôpital's rule to the middle term. The limit is
$\lim_{t\to 4}\left((t+4)\mathbf{i} + \frac{1}{1+(t-4)^2}\mathbf{j} + |t-4|\mathbf{k}\right) = 8\mathbf{i} + \mathbf{j}$

In Exercises 9 – 12, determine the numbers at which the vector-valued function is continuous.

9. $e^{-t}\mathbf{i} + \ln t\mathbf{j} + \frac{1}{\sqrt{4-t}}\mathbf{k}$. $\{t > 0\} \cap \{t < 4\} = (0,4)$

10. $\ln|\cos t|\mathbf{i}+\frac{1}{|t|-1}\mathbf{j}+\sqrt{1-t^2}\mathbf{k}$. $\{t \neq (k+\frac{1}{2}\pi)\} \cap \{t \neq \pm 1\} \cap \{-1 \le t \le 1\} = (-1,1)$

11. $\begin{cases} \frac{\sin(t-1)}{t-1}\mathbf{i}+\frac{t^2-1}{2(t-1)}\mathbf{j}+(t-1)\ln|t-1|\mathbf{k} & \text{if } t \neq 1 \\ \mathbf{i}+\mathbf{j} & \text{if } t = 1 \end{cases}$. $(-\infty,+\infty)$ because $\lim\limits_{t\to 1}\frac{\sin(t-1)}{t-1}=1$,

$\lim\limits_{t\to 1}\frac{t+1}{2}=1$ and $\lim\limits_{t\to 1}(t-1)\ln|t-1|=0$.

12. $\mathbf{R}(t)=\begin{cases} (1+t)^{1/t}\mathbf{i}+e^{1+t}\mathbf{j}+\frac{e^{1-t}}{1-t}\mathbf{k} & \text{if } t \neq 0 \\ e(\mathbf{i}+\mathbf{j}+\mathbf{k}) & \text{if } t \neq 0 \end{cases}$

▹ $\lim\limits_{t\to 0}(1+t)^{1/t}=e^1=e$

$\lim\limits_{t\to 0}e^{1+t}=e^1=e$

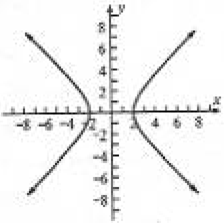
$\lim\limits_{t\to 0}\frac{e^{1-t}}{1-t}=\frac{e^1}{1}=e$

but $\frac{e^{1-t}}{1-t}$ is not defined at $t=1$. Thus, $\mathbf{R}(t)$ is continuous on $(-\infty,1)\cup(1,+\infty)$.

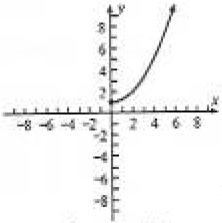
In Exercises 13–16, sketch the graph of the vector-valued function.

13. $\mathbf{R}(t)=2\sec t\mathbf{i}+2\tan t\mathbf{j}$

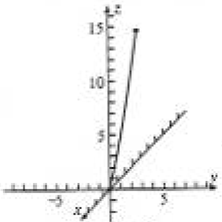
14. $\mathbf{R}(t)=2\sqrt{t}\mathbf{i}+(t+1)\mathbf{j}$



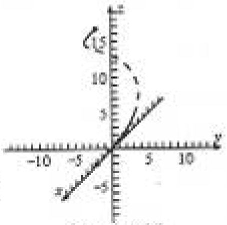
Exercise 13



Exercise 14



Exercise 15



Exercise 16

15. $\mathbf{R}(t)=t\mathbf{i}+t^2\mathbf{j}+2t^3\mathbf{k},\ 0 \le t \le 2$

16. $\mathbf{R}(t)=3\cos t\mathbf{i}+3\sin t\mathbf{j}+3t\mathbf{k},\ 0 \le t \le 2\pi$

In Exercises 17 and 18, find $\mathbf{R}'(t)$ and $\mathbf{R}''(t)$.

17. $\mathbf{R}(t)=\frac{1}{t^2+9}\mathbf{i}+\frac{1}{t}\mathbf{j}-\ln t\mathbf{k}$. $\mathbf{R}'(t)=-\frac{2t}{(t^2+9)^2}\mathbf{i}-\frac{1}{t^2}\mathbf{j}-\frac{1}{t}\mathbf{k}$. $\mathbf{R}''(t)=6\frac{t^2-3}{(t^2+9)^2}\mathbf{i}+\frac{2}{t^3}\mathbf{j}+\frac{1}{t^2}\mathbf{k},\ t>0$

18. $\mathbf{R}(t)=\frac{t}{t-1}\mathbf{i}-\frac{t+1}{t}\mathbf{j}+\frac{t+1}{t-1}\mathbf{k}=\left(1+\frac{1}{t-1}\right)\mathbf{i}-\left(1+\frac{1}{t}\right)\mathbf{j}+\left(1+\frac{2}{t-1}\right)\mathbf{k}$

$\mathbf{R}'(t)=-\frac{1}{(t-1)^2}\mathbf{i}+\frac{1}{t^2}\mathbf{j}-\frac{2}{(t-1)^2}\mathbf{k}$. $\mathbf{R}''(t)=\frac{2}{(t-1)^3}\mathbf{i}-\frac{2}{t^3}\mathbf{j}+\frac{4}{(t-1)^3}\mathbf{k}$

In Exercises 19 and 20, find $D_t\|\mathbf{R}(t)\|$ and $\|D_t\mathbf{R}(t)\|$.

19. $\mathbf{R}(t)=2(e^t-1)\mathbf{i}+2(e^t+1)\mathbf{j}+e^t\mathbf{k}$. $D_t\sqrt{4(e^t-1)^2+4(e^t+1)^2+e^{2t}}=D_t\sqrt{9e^{2t}+8}=\frac{9e^{2t}}{\sqrt{9e^{2t}+8}}$

$\|2e^t\mathbf{i}+2e^t\mathbf{j}+e^t\mathbf{k}\|=\sqrt{4e^{2t}+4e^{2t}+e^{2t}}=3e^t$

20. $\mathbf{R}(t)=\cos 2t\mathbf{i}+\sin 2t\mathbf{j}+2t^2\mathbf{k}$

▹ $D_t\|\mathbf{R}(t)\|=D_t\sqrt{\cos^2 2t+\sin^2 2t+4t^4}=D_t\sqrt{1+4t^4}=\frac{8t^3}{\sqrt{1+4t^4}}$

$\|D_t\mathbf{R}(t)\|=\|-2\sin 2t\mathbf{i}+2\cos 2t\mathbf{j}+4t\mathbf{k}\|=\sqrt{4\sin^2 2t+4\cos^2 2t+16t^2}=2\sqrt{1+4t^2}$

21. $\mathbf{R}(t)=\mathbf{R}(0)+\int_0^t\left(\frac{1}{u+2}\mathbf{i}+\frac{1}{u^2+1}\mathbf{j}\right)du=\mathbf{j}+\left[\ln(u+2)\mathbf{i}+\tan^{-1}u\mathbf{j}\right]_0^t=[\ln(t+2)-\ln 2]\mathbf{i}+(\tan^{-1}t+1)\mathbf{j}$

22. $\mathbf{R}(t)=\mathbf{R}(1)+\int_1^t\left(\frac{1}{u}\mathbf{i}+\frac{2\ln u}{u}\mathbf{j}\right)du=3\mathbf{i}-2\mathbf{j}+\left[\ln u\,\mathbf{i}+(\ln u)^2\mathbf{j}\right]_1^t=(\ln t+3)\mathbf{i}+[(\ln t)^2-2]\mathbf{j},\ t>0$

23. $\mathbf{R}(t)=\mathbf{R}(2)+\int_2^t(2e^{u/2}\mathbf{i}-2e^{-u/2}\mathbf{j}+2\cosh\frac{1}{2}u\mathbf{k})du=2e\mathbf{i}-2e^{-1}\mathbf{j}+\mathbf{k}+\left[4e^{u/2}\mathbf{i}+4e^{-u/2}\mathbf{j}+4\sinh\frac{1}{2}u\mathbf{k}\right]_2^t$
$=(4e^{t/2}-2e)\mathbf{i}+(4e^{-t/2}-6e^{-1})\mathbf{j}+(4\sinh\frac{1}{2}t-4\sinh 1+1)\mathbf{k}$

24. If $\mathbf{R}'(t) = \cos^2 t\mathbf{i} + \cos 2t\mathbf{j} - 2\sin 2t\mathbf{k}$ and $\mathbf{R}(0) = \mathbf{i}$, find $\mathbf{R}(t)$.

▸ $$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{R}'(u)du = \mathbf{i} + \int_0^t [\tfrac{1}{2}(1+\cos 2u)\mathbf{i} + \cos 2u\mathbf{j} - 2\sin 2u\mathbf{k}]du$$
$$= \mathbf{i} + \Big[(\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u)\mathbf{i} + \tfrac{1}{2}\sin 2u\mathbf{j} + \cos 2u\mathbf{k}\Big]_0^t = (\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t + 1)\mathbf{i} + \tfrac{1}{2}\sin 2t\mathbf{j} + (\cos 2t - 1)\mathbf{k}$$

In Exercises 25 and 26, find the exact length of arc L from t_1 to t_2.

25. $\|\mathbf{R}'\| = \|(\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}\| = \sqrt{1 + t^2 + 1}$. $\mathrm{L} = \int_0^{\pi/2}\sqrt{t^2+2}\,dt = \frac{1}{8}\pi\sqrt{8+\pi^2} + \sinh^{-1}\frac{1}{4}\sqrt{2}\pi$

26. $\|\mathbf{R}'(t)\| = \|-3\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}\| = \sqrt{25+4t^2}$. $\mathrm{L} = \int_0^{5/2}\sqrt{25+4t^2}\,dt = \frac{3}{4}\sqrt{34} + \frac{25}{4}\sinh^{-1}\frac{3}{5}$

In Exercises 27 and 28, use NINT find to four significant digits the length of arc L from t_1 to t_2.

27. $\|\mathbf{R}\| = \left\|\frac{1}{t}\mathbf{i} - \frac{1}{t^2}\mathbf{j} + \frac{1}{t^2}\right\| = \sqrt{\frac{1}{t^2} + \frac{2}{t^4}} = \frac{\sqrt{t^2+2}}{t^2}$. $\mathrm{L} = \int_1^2 \sqrt{t^2+2}\frac{dt}{t^2} = \left[\sinh^{-1}\frac{t}{\sqrt{2}} - \frac{\sqrt{t^2+2}}{t}\right]_1^2$
$= \sqrt{3} - \frac{1}{2}\sqrt{6} + \sinh^{-1}\sqrt{2} - \sinh^{-1}\frac{1}{\sqrt{2}} = 0.99504 \approx 0.9950$

28. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + e^{2t}\mathbf{k}$; $t_1 = 0$; $t_2 = \frac{1}{2}\pi$

▸ $\|\mathbf{R}'(t)\| = \|-\sin t\mathbf{j} + \cos t\mathbf{j} + 2e^{2t}\| = \sqrt{\sin^2 t + \cos^2 t + 4e^{4t}} = \sqrt{1+4e^{4t}}$
$$\mathrm{L} = \int_0^{\pi/2}\sqrt{1+4e^{4t}}\,dt = 22.258 \approx 22.26$$
using NINT. To get the exact value, let $u^2 = 1 + 4e^{4t}$. Then
$$\frac{2u\,du}{u^2-1} = \frac{16e^{4t}dt}{4e^{4t}} = 4\,dt$$
and so
$$\mathrm{L} = \int_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}} u\cdot\frac{u\,du}{2(u^2-1)} = \frac{1}{2}\int_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}}\Big(1 + \frac{1}{u^2-1}\Big)du = \frac{1}{2}u + \frac{1}{4}\ln\left|\frac{u-1}{u+1}\right|\Big]_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}}$$
$$= \frac{1}{2}\sqrt{1+4e^{2\pi}} - \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln\frac{\sqrt{1+4e^{2\pi}}-1}{\sqrt{1+4e^{2\pi}}+1} - \frac{1}{4}\ln\frac{\sqrt{5}-1}{\sqrt{5}+1}$$

In Exercises 29 and 30, find $\mathbf{T}(t)$ and $\mathbf{N}(t)$; also $\mathbf{T}(t_1)$ and $\mathbf{N}(t_1)$ and sketch their representations at $t = t_1$. In Exercises 31–34, find $\mathbf{T}(t)$, $\mathbf{N}(t)$. In Exercise 35–38, find the moving trihedral and equations of its planes at $t = t_1$. In Exercises 39 and 40, express the arc length s as a function of t measured from $t = 0$. In Exercises 41–44, find an equation of the curve having s as parameter. In Exercises 45 and 46, find the curvature K, the radius of curvature ρ and the center of curvature at $t = t_1$; in Exercises 47–50, just K.

29 and 45. $\mathbf{R}(t) = \frac{1}{2}e^{2t}\mathbf{i} + t\mathbf{j}$. $\dot{\mathbf{R}}(t) = e^{2t}\mathbf{i} + \mathbf{j}$. $\mathbf{T}(t) = \frac{1}{\sqrt{e^{4t}+1}}(e^{2t}\mathbf{i} + \mathbf{j})$. $\ddot{\mathbf{R}}(t) = 2e^{2t}\mathbf{i}$. $\mathbf{N}(t) = \frac{1}{\sqrt{e^{4t}+1}}(\mathbf{i} - e^{2t}\mathbf{j})$
because $\mathbf{N}\cdot\ddot{\mathbf{R}} = e^{2t} > 0$. At $t = \ln 2$, $e^{2t} = 4$, $\mathbf{T} = \frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$, $\mathbf{N} = \frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$, $x = 2$, $y = \ln 2$, $\dot{x} = 4$,
$\dot{y} = 1$, $\ddot{x} = 8$, $\ddot{y} = 0$. $\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|4(0) - 1(8)|}{(4^2+1^2)^{3/2}} = \frac{8}{17^{3/2}}$, $\rho = \frac{17^{3/2}}{8}$, $c = \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{4^2+1^2}{4(0)-1(8)} = -\frac{17}{8}$.
$x_c = x - \dot{y}c = 2 - 1(-\frac{17}{8}) = \frac{33}{8}$, $y_c = y + \dot{x}c = \ln 2 + 4(-\frac{17}{8}) = \ln 2 - \frac{17}{2}$

30, 42, and 46. $\mathbf{R}(t) = t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j}$, $t \geq 0$. $\dot{\mathbf{R}}(t) = 2t\mathbf{i} + t^2\mathbf{j}$. $\|\dot{\mathbf{R}}(t)\| = t\sqrt{4+t^2}$. $\mathbf{T}(t) = \frac{1}{\sqrt{4+t^2}}(2\mathbf{i} + t\mathbf{j})$. $\ddot{\mathbf{R}} = 2\mathbf{i} + 2t\mathbf{j}$.
$\mathbf{N}(t) = \frac{1}{\sqrt{4+t^2}}(-t\mathbf{i} + 2\mathbf{j})$ because $\mathbf{N}\cdot\ddot{\mathbf{R}} = \frac{2t}{\sqrt{4+t^2}} \geq 0$. $\mathbf{T}(1) = \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$. $\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-\mathbf{i} + 2\mathbf{j})$
$s = \int_0^t u\sqrt{4+u^2}\,du = \frac{1}{3}(4+u^2)^{3/2}\Big]_0^t = \frac{1}{3}[(4+t^2)^{3/2} - 8]$. $3s + 8 = (4+t^2)^{3/2}$; $t = [(3s+8)^{2/3} - 4]^{1/2}$
$\mathbf{R}(s) = [(3s+8)^{2/3} - 4]\mathbf{i} + \frac{1}{3}[(3s+8)^{2/3} - 4]^{3/2}\mathbf{j}$. At $t = 1$, $x = 1$, $y = \frac{1}{3}$, $\dot{x} = 2$, $\dot{y} = 1$, $\ddot{x} = 2$, $\ddot{y} = 2$.
$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|2(2) - 1(2)|}{(2^2+1^2)^{3/2}} = \frac{2}{5^{3/2}}$, $\rho = \frac{5^{3/2}}{2}$, $c = \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{5}{2}$, $x_c = x - \dot{y}c = 1 - 1(\frac{5}{2}) = -\frac{3}{2}$,
$y_c = y + \dot{x}c = \frac{1}{3} + 2(\frac{5}{2}) = \frac{16}{3}$

31, 35 39, and 47. $\mathbf{R}(t) = \frac{1}{3}t^3\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$. $\dot{\mathbf{R}}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$. $\|\dot{\mathbf{R}}(t)\| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$.

$\mathbf{T}(t) = \frac{t^2}{t^2+2}\mathbf{i} + \frac{2t}{t^2+2}\mathbf{j} + \frac{2}{t^2+2}\mathbf{j}$. $\dot{\mathbf{T}}(t) = \frac{2}{(t^2+2)^2}[2t\mathbf{i} - (t^2-2)\mathbf{j} - 2t\mathbf{k}]$. $\|\dot{\mathbf{T}}(t)\| = \frac{2}{t^2+2}$.

$$\mathbf{N}(t) = \frac{1}{t^2+2}[2t\mathbf{i} - (t^2-2)\mathbf{j} - 2t\mathbf{k}].\ \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{(t^2+2)^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & 2t & 2 \\ 2t & 2-t^2 & -2t \end{vmatrix}$$

$= -\frac{2}{t^2+2}\mathbf{i} + \frac{2t}{t^2+2}\mathbf{j} - \frac{t^2}{t^2+2}$. $\mathbf{R}(1) = \frac{1}{3}\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. $\mathbf{B}(1) = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \Rightarrow$ osculating:
$-2(x - \frac{1}{3}) + 2(y-1) - (z-2) = 0$; $\mathbf{N}(1) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow$ rectifying: $2(x-\frac{1}{3}) + (y-1) - 2(z-2) = 0$;
$\mathbf{T}(1) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow$ normal: $(x-\frac{1}{3}) + 2(y-1) + 2(z-2) = 0$. $s(t) = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t (u^2+2)du = \frac{1}{3}t^3 + 2t$.
$\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{2}{(t^2+2)^2}$

32, 36, 40, and 48. $\mathbf{R}(t) = e^t\mathbf{i} + 2e^{-t}\mathbf{j} + 2t\mathbf{k}$

▹ $\dot{\mathbf{R}}(t) = e^t\mathbf{i} - 2e^{-t}\mathbf{j} + 2\mathbf{k}$

$\|\dot{\mathbf{R}}(t)\| = \sqrt{e^{2t} + 4e^{-2t} + 4} = e^t + 2e^{-t}$

$$\mathbf{T}(t) = \frac{e^t}{e^t+2e^{-t}}\mathbf{i} - \frac{2e^{-t}}{e^t+2e^{-t}}\mathbf{j} + \frac{2}{e^t+2e^{-t}}\mathbf{k}$$

$$\dot{\mathbf{T}}(t) = \frac{e^t(e^t+2e^{-t}) - e^t(e^t-2e^{-t})}{(e^t+2e^{-t})^2}\mathbf{i} - 2\frac{-e^{-t}(e^t+2e^{-t}) - e^{-t}(e^t-2e^{-t})}{(e^t+2e^{-t})^2}\mathbf{j} - 2\frac{e^t-2e^{-t}}{(e^t+2e^{-t})^2}\mathbf{k}$$

$$= \frac{2}{(e^t+2e^{-t})^2}[2\mathbf{i} + 2\mathbf{j} - (e^t - 2e^{-t})\mathbf{k}]$$

$$\|\dot{\mathbf{T}}(t)\| = \frac{2}{(e^t+2e^{-t})^2}\sqrt{4 + 4 + (e^t-2e^{-t})^2} = \frac{2}{e^t+2e^{-t}}$$

$$\mathbf{N}(t) = \frac{1}{e^t+2e^{-t}}[2\mathbf{i} + 2\mathbf{j} - (e^t-2e^{-t})\mathbf{k}]$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{(e^t+2e^{-t})^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -2e^{-t} & 2 \\ 2 & 2 & 2e^{-t} - e^t \end{vmatrix}$$

$$= \frac{1}{(e^t+2e^{-t})^2}[-2(1+2e^{-2t})\mathbf{i} + (e^{2t}+2)\mathbf{j} + (2e^t + 4e^{-t})\mathbf{k}] = \frac{1}{e^t+2e^{-t}}(-2e^{-t}\mathbf{i} + e^t\mathbf{j} + 2\mathbf{k})$$

$\mathbf{R}(0) = \mathbf{i} + 2\mathbf{j}$

$\mathbf{B}(0) = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ An equation of the osculating plane is $-2(x-1) + (y-2) + 2z = 0$

$\mathbf{N}(0) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ An equation of the rectifying plane is $2(x-1) + 2(y-2) + z = 0$

$\mathbf{T}(0) = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ An equation of the normal plane is $(x-1) - 2(y-2) + 2z = 0$

$$s(t) = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t (e^u + 2e^{-u})du = e^u - 2e^{-u}\Big]_0^t = e^t - 2e^{-t} + 1$$

$$\mathrm{K} = \|\dot{\mathbf{T}}(t)\|/\|\dot{\mathbf{R}}(t)\| = \frac{2}{(e^t+2e^{-t})^2}$$

33, 37, 43, and 49. $\mathbf{R}(t) = 3\sin 2t\mathbf{i} + 4t\mathbf{j} + 3\cos 2t\mathbf{k}$. $\dot{\mathbf{R}}(t) = 6\cos 2t\mathbf{i} + 4\mathbf{j} - 6\sin 2t\mathbf{k}$. $\|\dot{\mathbf{R}}(t)\| = \sqrt{36+16} = 2\sqrt{13}$

$\mathbf{T}(t) = \frac{1}{\sqrt{13}}(3\cos 2t\mathbf{i} + 2\mathbf{j} - 3\sin 2t\mathbf{k})$. $\dot{\mathbf{T}}(t) = \frac{6}{\sqrt{13}}(-\sin 2t\mathbf{i} - \cos 2t\mathbf{k})$. $\|\dot{\mathbf{T}}(t)\| = \frac{6}{\sqrt{13}}$. $\mathbf{N}(t) = -\sin 2t\mathbf{i} - \cos 2t\mathbf{k}$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{13}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos 2t & 4t & -3\sin 2t \\ -\sin 2t & 0 & -\cos 2t \end{vmatrix} = \frac{1}{\sqrt{13}}(-2\cos 2t\mathbf{i} + 3\mathbf{j} + 2\sin 2t\mathbf{k}).\ \mathbf{R}(0) = 3\mathbf{k}.$$

$\mathbf{B}(0) = \frac{1}{\sqrt{13}}(-2\mathbf{i} + 3\mathbf{j}) \Rightarrow$ osculating: $-2x + 3y = 0$; $\mathbf{N}(0) = -\mathbf{k} \Rightarrow$ rectifying: $z = 3$; $\mathbf{T}(0) = \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j}) \Rightarrow$
normal: $3x + 2y = 0$. $s = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t 2\sqrt{13}du = 2\sqrt{13}t$, $t = \frac{1}{26}\sqrt{13}s$.

$\mathbf{R}(s) = 3\sin\frac{1}{13}\sqrt{13}s\mathbf{i} + \frac{2}{13}\sqrt{13}s\mathbf{j} + 3\cos\frac{1}{13}\sqrt{13}s\mathbf{k}$. $\mathrm{K} = \|\dot{\mathbf{T}}(t)\|/\|\dot{\mathbf{R}}(t)\| = \frac{3}{13}$

34 and 50. $\mathbf{R}(t)=3(\cos t+t\sin t)\mathbf{i}+3(\sin t-t\cos t)\mathbf{j}+3\mathbf{k}$, $t\geq 0$. $\dot{\mathbf{R}}(t)=3t\cos t\mathbf{i}+3t\sin t\mathbf{j}$. $\left\|\dot{\mathbf{R}}(t)\right\|=3t$
$\mathbf{T}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}$. $\dot{\mathbf{T}}(t)=-\sin t\mathbf{i}+\cos t\mathbf{j}=\mathbf{N}(t)$. $\mathrm{K}=\left\|\dot{\mathbf{T}}(t)\right\|/\left\|\dot{\mathbf{R}}(t)\right\|=1/(3t)$

41. $\mathbf{R}(t)=4t\mathbf{i}+\frac{1}{3}(2t+1)^{3/2}\mathbf{j}$. $\dot{\mathbf{R}}(t)=4\mathbf{i}+(2t+1)^{1/2}\mathbf{j}$. $\left\|\dot{\mathbf{R}}(t)\right\|=\sqrt{16+2t+1}=\sqrt{2t+17}$.
$s=\int_0^t\sqrt{2u+17}\,du=\frac{1}{3}(2t+17)^{3/2}-\frac{1}{3}(17)^{3/2}$; $3s+17\sqrt{17}=(2t+17)^{3/2}$; $2t=(3s+17\sqrt{17})^{2/3}-17$
$\mathbf{R}(s)=[2(3s+17\sqrt{17})^{2/3}-34]\mathbf{i}+\frac{1}{3}[(3s+17\sqrt{17})^{2/3}-16]^{3/2}\mathbf{j}$

44. $\mathbf{R}(t)=3(\cos t+t\sin t)\mathbf{i}+3(\sin t-t\cos t)\mathbf{j}+3\mathbf{k}$, $t\geq 0$
▸ $\dot{\mathbf{R}}(t)=3t\cos t\mathbf{i}+3t\sin t\mathbf{j}$ $\qquad\left\|\dot{\mathbf{R}}(t)\right\|=3t\sqrt{\cos^2 t+\sin^2 t}=3t$
$s=\int_0^t\left\|\dot{\mathbf{R}}(u)\right\|du=\int_0^t 3u\,du=\frac{3}{2}u^2\Big]_0^t=\frac{3}{2}t^2$ $\qquad t=\sqrt{\frac{2}{3}s}$
$\mathbf{R}(s)=3(\cos\sqrt{\frac{2}{3}s}+\sqrt{\frac{2}{3}s}\sin\sqrt{\frac{2}{3}s})\mathbf{i}+3(\sin\sqrt{\frac{2}{3}s}-\sqrt{\frac{2}{3}s}\cos\sqrt{\frac{2}{3}s})\mathbf{j}+3\mathbf{k}$

51. $y=\ln x$, where $x>0$. $y'=x^{-1}$. $y''=-x^{-2}$. $\mathrm{K}=\frac{y''}{[1+(y')^2]^{3/2}}=\frac{|-x^{-2}|}{(1+x^{-2})^{3/2}}=\frac{x}{(x^2+1)^{3/2}}$.
$\mathrm{D}_x\mathrm{K}=\frac{(x^2+1)^{3/2}-3x^2(x^2+1)^{1/2}}{(x^2+1)^3}=\frac{x^2+1-3x^2}{(x^2+1)^{5/2}}=\frac{1-2x^2}{(x^2+1)^{3/2}}$
Set $\mathrm{D}_x\mathrm{K}=0$: $x=\frac{1}{2}\sqrt{2}$. Because $\mathrm{D}_x>0$ if $0<x<\frac{1}{2}\sqrt{2}$ and $\mathrm{D}_x<0$ if $x>\frac{1}{2}\sqrt{2}$, then K has an absolute maximum value when $x=\frac{1}{2}\sqrt{2}$. Then $y=\ln\frac{1}{\sqrt{2}}=-\frac{1}{2}\ln 2$ and $\mathrm{K}=\frac{1}{\sqrt{2}}/\left(\frac{1}{2}+1\right)^{3/2}=\frac{1}{\sqrt{2}}/\frac{3}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}}=\frac{2}{3\sqrt{3}}=\frac{2}{9}\sqrt{3}$.

52. Find the curvature at any point of the branch of the parametric hyperbola $x=a\cosh t$ and $y=b\sinh t$.
▸ $\dot{x}=a\sinh t$ $\qquad\dot{y}=b\cosh t$
$\ddot{x}=a\cosh t$ $\qquad\ddot{y}=b\sinh t$

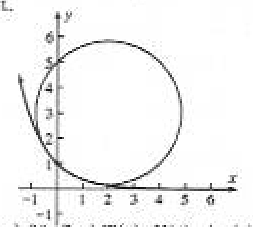
$$\mathrm{K}(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}=\frac{|(a\sinh t)(b\sinh t)-(b\cosh t)(a\cosh t)|}{(a^2\sinh^2 t+b^2\cosh^2 t)^{3/2}}=\frac{|ab|}{(a^2\sinh^2 t+b^2\cosh^2 t)^{3/2}}$$

53 and 55. $x=3t^2$, $y=t^3-3t$. $\dot{x}=6t$, $\dot{y}=3t^2-3$. $\ddot{x}=6$, $\ddot{y}=6t$.

$$\mathrm{K}(2)=\frac{\dot{x}(2)\ddot{y}(2)-\dot{y}(2)\ddot{x}(2)}{[\dot{x}(2)^2+\dot{y}(2)^2]^{3/2}}=\frac{12\cdot 12-9\cdot 6}{[12^2+9^2]^{3/2}}=\frac{90}{225^{3/2}}=\frac{2}{75};\ \rho(2)=\frac{75}{2}.\ c=\frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{225}{90}=\frac{5}{2}$$

$x_c=x-c\dot{y}=12-\frac{5}{2}(9)=-\frac{21}{2}$. $y_c=y+c\dot{x}=2+\frac{5}{2}(12)=32$

54 and 56. Find the curvature, radius of curvature and center of curvature (x_c,y_c) of the curve $y=e^{-x}$ at the point $(0,1)$. Sketch a portion of the curve and the circle of curvature at that point.
▸ $y'=-e^{-x}$ $\qquad y'(0)=-1$
$y''=e^{-x}$ $\qquad y''(0)=1$
$\mathrm{K}=\frac{|y''|}{(1+y'^2)^{3/2}}=\frac{1}{2^{3/2}}$ $\qquad\rho=\frac{1}{\mathrm{K}}=2^{3/2}$
$c=\frac{1+y'^2}{y''}=\frac{2}{1}=2$. $x_c=x-cy'=0-2(-1)=2$. $y_c=y+c=1+2=3$
The figure shows the curve and circle of curvature at $(0,1)$.



In Exercises 57 and 58, find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $\|\mathbf{V}(t)\|$, $\|\mathbf{A}(t)\|$, $\mathbf{V}(1)$, $\mathbf{A}(1)$. In Exercises 67 and 68, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathrm{A}_T(t)$, $\mathrm{A}_N(t)$, $\mathrm{K}(t)$. Find the particular values when $t=1$. Sketch representations of $\mathbf{V}(1)$, $\mathbf{A}(1)$, $\mathrm{A}_T(1)\mathbf{T}(1)$, and $\mathrm{A}_N(1)\mathbf{N}(1)$ on the path.

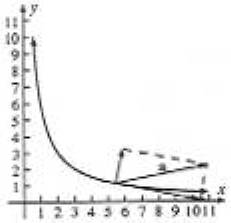
57 and 67. $\mathbf{R}(t)=3t\mathbf{i}+(4t-t^2)\mathbf{j}$. $\mathbf{V}(t)=3\mathbf{i}+(4-2t)\mathbf{j}$. $\mathbf{A}(t)=-2\mathbf{j}$. $\|\mathbf{V}(t)\|=\sqrt{3^2+(4-2t)^2}=\sqrt{4t^2-16t+25}$.
$\mathbf{V}(1)=3\mathbf{i}+2\mathbf{j}$. $\mathbf{A}(1)=-2\mathbf{j}$. $\mathbf{T}(t)=\frac{1}{\sqrt{4t^2-16t+25}}[3\mathbf{i}+(4-2t)\mathbf{j}]$. $\mathbf{N}(t)=\frac{1}{\sqrt{4t^2-16t+25}}[(4-2t)\mathbf{i}-3\mathbf{j}]$.

$$\mathrm{A}_T(t)=\dot{\mathrm{V}}(t)=\frac{4t-8}{\sqrt{4t^2-16t+25}}.\ \mathrm{A}_N(t)=\sqrt{\|\mathbf{A}\|^2-\mathrm{A}_T{}^2}=\sqrt{4-\frac{16t^2-64t+64}{4t^2-16t+25}}=\frac{6}{\sqrt{4t^2-16t+25}}.$$

$K(t) = \dfrac{A_N(t)}{\|\mathbf{V}(t)\|^2} = \dfrac{6}{(4t^2 - 16t + 25)^{3/2}}$. $\mathbf{T}(1) = \dfrac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$. $\mathbf{N}(1) = \dfrac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$. $\|\mathbf{V}(1)\| = \sqrt{13}$. $A_T(1) = \dfrac{-4}{\sqrt{13}}$

$A_N(1) = \dfrac{6}{\sqrt{13}}$. $K(1) = \dfrac{6}{13^{3/2}}$

58 and 68. $\mathbf{R}(t) = 2e^t\mathbf{i} + 3e^{-t}\mathbf{j}$



▷ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2e^t\mathbf{i} - 3e^{-t}\mathbf{j}$ $\qquad$ $\mathbf{V}(1) = 2e\mathbf{i} - 3e^{-1}\mathbf{j} \approx 5.44\mathbf{i} - 1.10\mathbf{j}$

$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2e^t\mathbf{i} + 3e^{-t}\mathbf{j}$ $\qquad$ $\mathbf{A}(1) = 2e\mathbf{i} + 3e^{-1}\mathbf{j} \approx 5.44\mathbf{i} + 1.10\mathbf{j}$

$v(t) = \|\mathbf{V}(t)\| = \sqrt{4e^{2t} + 9e^{-2t}}$ $\qquad$ $\|\mathbf{V}(1)\| = \sqrt{4e^2 + 9e^{-2}}$

$\mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{2e^t\mathbf{i} - 3e^{-t}\mathbf{j}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $\qquad$ $\mathbf{T}(1) = \dfrac{2e\mathbf{i} - 3e^{-1}\mathbf{j}}{\sqrt{4e^2 + 9e^{-2}}}$

$\mathbf{N}(t) = \dfrac{3e^{-t}\mathbf{i} + 2e^t\mathbf{j}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $(\mathbf{N} \cdot \mathbf{A} = 6 > 0)$ $\qquad$ $\mathbf{N}(1) = \dfrac{3e^{-1}\mathbf{i} + 2e\mathbf{j}}{\sqrt{4e^2 + 9e^{-2}}}$

$A_T(t) = \dot{v}(t) = \dfrac{4e^{2t} - 9e^{-2t}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $\qquad$ $A_T(1) = \dfrac{4e^2 - 9e^{-2}}{\sqrt{4e^2 + 9e^{-2}}}$

$A_N(t) = \sqrt{\|\mathbf{A}\|^2 - {A_T}^2} = \sqrt{(4e^{2t} + 9e^{-2t}) - \dfrac{(4e^{2t} - 9e^{-2t})^2}{4e^{2t} + 9e^{-2t}}} = \dfrac{12}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $\qquad$ $A_N(1) = \dfrac{12}{\sqrt{4e^2 + 9e^{-2}}}$

$K(t) = \dfrac{A_N(t)}{v(t)^2} = \dfrac{12}{(4e^{2t} + 9e^{-2t})^{3/2}}$ $\qquad$ $K(1) = \dfrac{12}{(4e^2 + 9e^{-2})^{3/2}}$

The figure shows the hyperbola and $\mathbf{V}(1)$, $\mathbf{A}(1)$, $A_T(1)\mathbf{T}(1) \approx 5.01\mathbf{i} - 1.02\mathbf{j}$ and $A_N(1)\mathbf{N}(1) \approx 0.43\mathbf{i} + 2.12\mathbf{j}$.

In Exercises 69 and 70, find $A_T(t)$ and $A_N(t)$ and write $\mathbf{A}(t) = A_T(t)\mathbf{T}(t) + A_N(t)\mathbf{N}(t)$. Don't find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

69. $\mathbf{R} = 2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$. $\mathbf{V} = 2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$. $\mathbf{A} = e^t\mathbf{j} + e^{-t}\mathbf{k}$. $v = \|\mathbf{V}\| = \sqrt{4 + e^{2t} + e^{-2t}}$. $A_T = \dot{v} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{4 + e^{2t} + e^{-2t}}}$

$A_N = \sqrt{\|\mathbf{A}\|^2 - {A_T}^2} = \sqrt{e^{2t} + e^{-2t} - \dfrac{e^{4t} - 2 + e^{-4t}}{4 + e^{2t} + e^{-2t}}} = \sqrt{\dfrac{4 + 4e^{2t} + 4e^{-2t}}{4 + e^{2t} + e^{-2t}}}$

$\mathbf{A}(t) = \dfrac{e^{2t} - e^{-2t}}{\sqrt{4 + e^{2t} + e^{-2t}}}\mathbf{T}(t) + \sqrt{\dfrac{4 + 4e^{2t} + 4e^{-2t}}{4 + e^{2t} + e^{-2t}}}\mathbf{N}(t)$

70. $\mathbf{R} = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$. $\mathbf{V} = 2t\mathbf{i} + 2\mathbf{j}$. $\mathbf{A} = 2\mathbf{i}$. $v = \|\mathbf{V}\| = 2\sqrt{t^2 + 1}$. $A_T = \dot{v} = \dfrac{2t}{\sqrt{t^2 + 1}}$.

$A_N = \sqrt{\|\mathbf{A}\|^2 - {A_T}^2} = \sqrt{4 - \dfrac{4t}{t^2 + 1}} = 2\sqrt{\dfrac{t^2 - t + 1}{t^2 + 1}}$. $\mathbf{A} = \dfrac{2t}{\sqrt{t^2 + 1}}\mathbf{T} + 2\sqrt{\dfrac{t^2 - t + 1}{t^2 + 1}}\mathbf{N}$

In Exercises 71 and 72, find (a) velocity, acceleration, (b) speed, (c) tangential, normal components of acceleration.

71. $\mathbf{R}(t) = \cosh 2t\mathbf{i} + \sinh 2t\mathbf{j}$. $\mathbf{V}(t) = 2\sinh 2t\mathbf{i} + 2\cosh 2t\mathbf{j}$. $\mathbf{A}(t) = 4\cosh 2t\mathbf{i} + 4\sinh 2t\mathbf{j}$.

$\|\mathbf{V}(t)\| = \sqrt{4\sinh^2 2t + 4\cosh^2 2t} = 2\sqrt{\sinh^2 2t + \cosh^2 2t} = 2\sqrt{\cosh 4t} = D_t s$

$A_T(t) = {D_t}^2 s = \dfrac{4\sinh 4t}{\sqrt{\cosh 4t}}$ $\|\mathbf{A}(t)\| = \sqrt{16\cosh^2 2t + 16\sinh^2 2t} = 4\sqrt{\cosh 4t}$

$A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2} = \sqrt{16\cosh 4t - \dfrac{16\sinh^2 4t}{\cosh 4t}} = 4\sqrt{\dfrac{\cosh^2 4t - \sin^2 4t}{\cosh 4t}} = 4\sqrt{\dfrac{1}{\cosh 4t}} = \dfrac{4}{\sqrt{\cosh 4t}}$

72. $\mathbf{R}(t) = (2\tan^{-1}t - t)\mathbf{i} + \ln(1 + t^2)\mathbf{j} + 2\mathbf{k}$

(a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \dfrac{1 - t^2}{1 + t^2}\mathbf{i} + \dfrac{2t}{1 + t^2}\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = \dfrac{-4t}{(1 + t^2)^2}\mathbf{i} + \dfrac{2(1 - t^2)}{(1 + t^2)^2}\mathbf{j}$

(b) $v = \|\mathbf{V}(t)\| = \dfrac{1}{1 + t^2}\sqrt{(1 - t^2)^2 + (2t)^2} = 1$

(c) $A_T = \dot{v} = 0$ and so $A_N = \|\mathbf{A}\| = \dfrac{2}{(1 + t^2)^2}\sqrt{(-2t)^2 + (1 - t^2)^2} = \dfrac{2}{1 + t^2}$

73 and 74. $\mathbf{V}_0 = 150\cos 30^\circ\mathbf{i} + 150\sin 30\mathbf{j} = 75\sqrt{3}\mathbf{i} + 75\mathbf{j}$ (a) $\mathbf{R} = 75\sqrt{3}t\mathbf{i} + (75t - 16t^2)\mathbf{j}$ (b) $y = 0$ when $t = \frac{75}{16}$, range $= x(\frac{75}{16}) = 75\sqrt{3} \cdot \frac{75}{16} = \frac{5625}{16}\sqrt{3} \approx 609$ ft (c) $y(\frac{75}{32}) = \frac{5625}{64} \approx 87.9$ ft (d) speed of impact is 150 ft/sec. The dots are closer at the vertex of the parabolic path indicating the speed is least there.

75. Range $= \frac{{v_0}^2}{g}\sin 2\theta = \frac{{v_0}^2}{32}\sin 80^\circ \geq 300$, $v_0 \geq \sqrt{\frac{9600}{\sin 80^\circ}} = 98.73$ ft/sec

76. Find the maximum height reached by a projectile of muzzle speed v_0 feet per second and angle of elevation α.

▹ We complete the square on the vertical component.

$$y(t) = v_0 \sin \alpha t - \frac{1}{2}gt^2 = -\frac{g}{2}\left(t^2 - \frac{2v_0}{g}\sin \alpha t + \frac{{v_0}^2 \sin^2\alpha}{g^2}\right) + \frac{{v_0}^2\sin^2\alpha}{2g} = \frac{{v_0}^2\sin^2\alpha}{2g} - \frac{g}{2}\left(t - \frac{v_0 \sin \alpha}{g}\right)^2$$

Hence the maximum height is $\frac{{v_0}^2\sin^2\alpha}{2g}$ feet.

77. (a) $y = 20 - 24\sin 25^\circ t - 16t^2 = 0$ when $t = \frac{1}{32}(-24\sin 25^\circ + \sqrt{24^2\sin^2 25^\circ + 1820}) \approx 0.845$ sec
(b) $x = 20\cos 25^\circ(0.845) = 18.38$ ft

78. (a) $y = 20 - 16t^2 = 0$ when $t = \frac{1}{2}\sqrt{5} \approx 1.12$ sec (b) $x = 20(\frac{1}{2}\sqrt{5}) = 10\sqrt{5} = 22.36$ ft

79. $\mathbf{R} \cdot \dot{\mathbf{R}} = 0 \Rightarrow D_t\|\mathbf{R}\|^2 = 0 \Rightarrow \|\mathbf{R}\|$ is constant and so the particle moves in a circle.

80. If a particle is moving along a curve, under what condition will the acceleration vector and the unit tangent vector have the same or opposite directions?

▹ For some scalar function $a(t)$ we have
$\dot{\mathbf{R}}(t) = a(t)\mathbf{T}(t)$
Differentiating with respect to t,
$\mathbf{A}(t) = \ddot{\mathbf{R}}(t) = \dot{a}(t)\mathbf{T}(t) + a(t)\dot{\mathbf{T}}(a)$
This will have the same or opposite direction as $\mathbf{T}(t)$ if and only if $\dot{\mathbf{T}}(t) = 0$. Thus $\mathbf{T}(t)$ is a constant vector and the curve is a straight line.

81. From Exercise 11.4.56, **T** and **B** have the direction of **V** and $\mathbf{V} \times \mathbf{A}$. Thus **N** has the direction of $(\mathbf{V} \times \mathbf{A}) \times \mathbf{V} = (\mathbf{V} \cdot \mathbf{V})\mathbf{A} - (\mathbf{V} \cdot \mathbf{A})\mathbf{V}$. Dividing by the magnitude gives a unit vector.

82. Because the scalar triple product is a determinant, the result is immediate from Exercise 11.2.31

TWELVE

DIFFERENTIAL CALCULUS OF FUNCTIONS OF MORE THAN ONE VARIABLE

12.1 FUNCTIONS OF MORE THAN ONE VARIABLE

Let P be an ordered pair of real numbers (x, y), and let z be a real number. A *function of two variables* is a set of ordered pairs of the form (P, z) in which no two distinct pairs have the same first element. The set of all possible replacements of P is called the *domain* of the function, and the set of all possible values of z is called the *range* of the function. The variables x and y are the *independent variables*, and z is the *dependent variable*. If f is a function of two variables, then we denote the function value of f at P by either $f(\mathrm{P})$ or $f(x, y)$. A *polynomial function* f of two variables x and y is the sum of terms of the form $cx^n y^m$ where c is a real number and n and m are nonnegative integers. The *degree* of a term is the sum of the exponents; the largest degree is the degree of f. If all terms have the same degree, f is *homogeneous*.

12.1.1 Definition The set of all ordered n-tuples of real numbers is called the *n-dimensional number space* and is denoted by R^n. Each ordered n-tuple $(x_1, x_2, \ldots, x_n)$ is called a *point* in the n-dimensional number space.

12.1.2 Definition A *function of n variables* is a set of ordered pairs of the form (P, w) in which no two distinct ordered pairs have the same first element, where P is a point in n-dimensional number space and w is a real number. The set of all possible values of P is called the *domain* of the function, and the set of all possible values of w is called the *range* of the function.

12.1.3 Definition If f is a function of a single variable and g is a function of two variables, then the *composite function* $f \circ g$ is the function of two variables defined by $(f \circ g)(x, y) = f(g(x, y))$ and the domain of $f \circ g$ is the set of all points (x, y) in the domain of g such that $g(x, y)$ is in the domain of f.

12.1.4 Definition If f is a function of a single variable and g is a function of n variables, then the *composite function* $f \circ g$ is the function of n variables defined by

$(f \circ g)(x_1, x_2, \ldots, x_n) = f(g(x_1, x_2, \ldots, x_n))$

and the domain of $f \circ g$ is the set of all points $(x_1, x_2, \ldots, x_n)$ in the domain of g such that $g(x_1, x_2, \ldots, x_n)$ is in the domain of f.

12.1.5 Definition If f is a function of two variables, then the graph of f is the set of all points (x, y, z) in R^3 for which (x, y) is a point in the domain of f and $z = f(x, y)$.

Thus, the graph of a function of two variables is a surface in R^3. Because there is only one value of z for each replacement of (x, y) from the domain of the function, a line perpendicular to the xy plane intersects the surface in no more than one point.

12.1.6 Definition If f is a function of n variables, then the *graph* of f is the set of all points $(x_1, x_2, \ldots, x_n, w)$ in R^{n+1} for which $(x_1, x_2, \ldots, x_n)$ is a point in the domain of f and $w = f(x_1, x_2, \ldots, x_n)$.

Let C be the intersection of the surface $z = f(x, y)$ and the plane $z = k$. The projection of C on the xy plane, that is, the graph of $f(x, y) = k$, is called the *level curve* of the function f at k. Level curves are called *equipotential curves* if f gives electric potential. The set of all level curves of f at k if $k = k_1, k_2, \ldots, k_n$ is called a *contour map* of the function f. If the numbers k_1, k_2, …, k_n are equally spaced, then the resulting contour map indicates the steepness of the surface that is the graph of f. Closely spaced contour curves indicate that the surface is steep; widely space contour curves indicate that the surface is not steep; and a contour "curve" that is a region indicates that the graph is flat. Similarly, functions of three variables have *level surfaces*. The *inside* of conic section or quadric surface Q is the set of points through which there are no lines tangent to Q.

Exercises 12.1

1. $f(x,y) = \dfrac{x+y}{x-y}$

▸ (a) $f(-3,4) = \dfrac{-3+4}{-3-4} = -\frac{1}{7}$ (c) $f(x+1,y-1) = \dfrac{(x+1)+(y-1)}{(x+1)-(y-1)} = \dfrac{x+y}{x-y+2}$

(b) $f(\frac{1}{2},\frac{1}{3}) = \dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} = \dfrac{\frac{5}{6}}{\frac{1}{6}} = 5$ (d) $f(-x,y) - f(x,-y) = \dfrac{-x+y}{-x-y} - \dfrac{x-y}{x+y} = \dfrac{x-y}{x+y} - \dfrac{x-y}{x+y} = 0$

2. $g(x,y) = \sqrt{x^2 - y}$

▸ (a) $g(3,5) = \sqrt{3^2 - 5} = \sqrt{4} = 2$ (c) $g(x+2, 4x+4) = \sqrt{(x+2)^2 - (4x+4)} = \sqrt{x^2} = |x|$

(b) $g(-4,-9) = \sqrt{(-4)^2 - (-9)} = \sqrt{25} = 5$ (d) $g\left(\frac{1}{x}, \frac{-3}{x^2}\right) = \sqrt{\left(\frac{1}{x}\right)^2 - \left(\frac{-3}{x^2}\right)} = \sqrt{\frac{4}{x^2}} = \frac{2}{|x|}$

3. $g(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$

(a) $g(1,-1,-1) = \sqrt{4-1-1-1} = 1$ (b) $g(-1,\frac{1}{2},\frac{3}{2}) = \sqrt{4 - 1 - \frac{1}{4} - \frac{9}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$

(c) $g(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z) = \sqrt{4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 - \frac{1}{4}z^2} = \frac{1}{2}\sqrt{16 - x^2 - y^2 - z^2}$

(d) $$\begin{aligned}[g(x,y,z)]^2 - [g(x+2,y+2,z)]^2 &= (4 - x^2 - y^2 - z^2) - [4 - (x+2)^2 - (y+2)^2 - z^2] \\ &= -x^2 - y^2 + x^2 + 4x + 4 + y^2 + 4y + 4 = 4x + 4y + 8\end{aligned}$$

4. Let the function f of three variables x, y, and z be the set of all ordered pairs of the form (P, w) such that

$w = \dfrac{4}{x^2 + y^2 + z^2 - 9} \Leftrightarrow f(x,y,z) = \dfrac{4}{x^2 + y^2 + z^2 - 9}$

Find: (a) $f(1,2,3)$; (b) $f(2, -\frac{1}{2}, \frac{3}{2})$; (c) $f\left(-\frac{2}{x}, \frac{2}{x}, -\frac{1}{x}\right)$; (d) $f(x+2), 1, x-2)$

▸ (a) We let $x = 1$, $y = 2$, and $z = 3$. Thus,

$$f(1,2,3) = \frac{4}{1^2 + 2^2 + 3^2 - 9} = \frac{4}{5}$$

(b) $f(2, -\frac{1}{2}, \frac{3}{2}) = \dfrac{4}{2^2 + (-\frac{1}{2})^2 + (\frac{3}{2})^2 - 9} = \dfrac{4}{4 + \frac{1}{4} + \frac{9}{4} - 9} = -\frac{8}{5}$

(c) $f\left(-\frac{2}{x}, \frac{2}{x}, -\frac{1}{x}\right) = \dfrac{4}{\left(-\frac{2}{x}\right)^2 + \left(\frac{2}{x}\right)^2 + \left(-\frac{1}{x}\right)^2 - 9} = \dfrac{4}{\frac{9}{x^2} - 9} = \dfrac{4x^2}{9(1-x^2)}, \; x \neq 0$

In Exercises 5–20, determine the domain of f and sketch the domain as a region in R^2. Use dashed curves to indicate any part of the boundary not in the domain and solid curves to indicate parts of the boundary in the domain.

5. $f(x,y) = \dfrac{1}{x^2 + y^2 - 1}$

▸ $\mathrm{Dom}(f) = \{(x,y) \mid x^2 + y^2 \neq 1\}$, the set of all points (x,y) in R^2 except those on the circle $x^2 + y^2 = 1$.

6. $f(x,y) = \dfrac{4}{4 - x^2 - y^2}$

▸ $\mathrm{Dom}(f) = \{(x,y) \mid x^2 + y^2 \neq 4\}$, the set of all points (x,y) in R^2 except those on the circle $x^2 + y^2 = 4$.

7. $f(x,y) = \sqrt{1 - x^2 - y^2}$

▸ $\mathrm{Dom}(f) = \{(x,y) \mid 1 - x^2 - y^2 \geq 0\}$, the set of all points (x,y) in R^2 inside or on the circle $x^2 + y^2 = 1$.

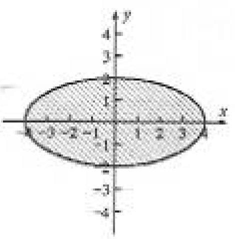
8. $f(x,y) = \sqrt{16 - x^2 - 4y^2}$

▸ Because the square root of a negative number is not real, the value of the expression under the radical sign must be greater than or equal to zero. That is

$$16 - x^2 - 4y^2 \geq 0$$

$$x^2 + 4y^2 \leq 16 \qquad (1)$$

The domain of the function f is the set of all points (x,y) that satisfy (1). The graph of this set is the region in the xy plane that is inside or on the ellipse $x^2/16 + y^2/4 = 1$, as shown in the figure.



9. $f(x,y) = \sqrt{x^2 - y^2 - 1}$

▹ $\text{Dom}(f) = \{(x,y) \mid x^2 - y^2 \geq 1\}$, the set of all points (x,y) in R^2 on and inside the hyperbola $x^2 - y^2 = 1$.

10. $f(x,y) = \sqrt{x^2 - 4y^2 + 16}$

▹ $\text{Dom}(f) = \{(x,y) \mid 4y^2 - x^2 \leq 16\}$, the set of all points (x,y) in R^2 on and outside the hyperbola $4y^2 - x^2 = 16$.

11. $f(x,y) = \sqrt{x^2 + y^2 - 1}$

▹ $\text{Dom}(f) = \{(x,y) \mid x^2 + y^2 \geq 1\}$, the set of all points (x,y) in R^2 on and outside the circle $x^2 + y^2 = 1$.

12. $f(x,y) = \sqrt{x^2 + 4y^2 - 16}$

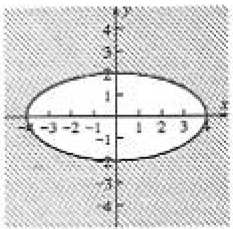
▹ If

$x^2 + 4y^2 - 16 \geq 0$

then

$$x^2 + 4y^2 \geq 16 \qquad (1)$$

The domain of f is the set of all points that satisfy (1). This is the region in the xy plane that is on or outside the ellipse $x^2/16 + y^2/4 = 1$, as shown in the figure.



13. $f(x,y) = \dfrac{1}{\sqrt{1 - x^2 - y^2}}$

▹ The domain of f is $\{(x,y) \mid x^2 + y^2 < 1\}$, the set of all points (x,y) in R^2 inside the circle $x^2 + y^2 = 1$.

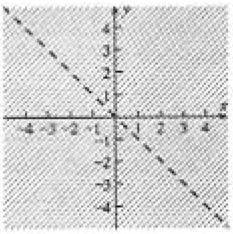
14. $f(x,y) = \dfrac{1}{\sqrt{16 - x^2 - 4y^2}}$

▹ The domain of f is $\{(x,y) \mid x^2 + 4y^2 < 16\}$, the set of all points (x,y) in R^2 inside the ellipse $x^2 + 4y^2 = 16$.

15. $f(x,y) = \dfrac{x^4 - y^4}{x^2 - y^2}$. $\text{Dom}(f) = \{(x,y) \mid y \neq \pm x\}$, the set of all points (x,y) in R^2 not on the lines $y = \pm x$.

16. $f(x,y) = \dfrac{x - y}{x + y}$

▹ The domain of f consists of all points in the xy plane except those on the line $x + y = 0$, as shown in the figure.



17. $f(x,y) = \cos^{-1}(x - y)$. The domain of f is $\{(x,y) \mid -1 \leq x - y \leq 1\}$, the set of all points (x,y) in R^2 on and between the lines $y = x - 1$ and $y = x + 1$.

18. $f(x,y) = \ln(x^2 + y)$. The domain of f is $\{(x,y) \mid y > -x^2\}$, the set of all points (x,y) in R^2 outside the parabola $y = -x^2$.

19. $f(x,y) = \ln(xy - 1)$. The domain of f is $\{(x,y) \mid xy > 1\}$, the set of all points (x,y) in R^2 inside the hyperbola $xy = 1$.

20. $f(x,y) = \sin^{-1}(x + y)$

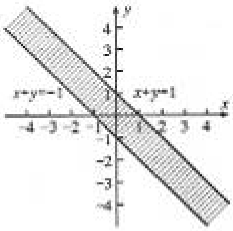
▹ Because the domain of the inverse sine function is $[-1,1]$, the domain of f is

$\{(x,y) \mid |x + y| \leq 1\}$

Thus the domain is the set of all points on or between the parallel lines

$x + y = 1$ and $x + y = -1$

See the figure.



In Exercises 21-28, determine the domain of f and describe as a region in R^3 the set of points in the domain.

21. $f(x,y,z) = \dfrac{x + y + z}{x - y - z}$

▹ The domain of f is $\{(x,y,z) \mid x - y - z \neq 0\}$, the set of all points (x,y,z) in R^3 not on the plane $x - y - z = 0$.

22. $f(x,y,z) = \dfrac{z}{x^2 - y}$

▹ The domain of f is $\{(x,y,z) \mid y \neq x^2\}$, the set of all points (x,y,z) in R^3 not on the parabolic cylinder $y = x^2$.

23. $f(x,y,z) = \sqrt{16 - x^2 - 4y^2 - z^2}$. The domain of f is $\{(x,y,z) \mid x^2 + 4y^2 + z^2 \leq 16\}$, the set of all points (x,y,z) in R^3 on and inside the ellipsoid $x^2/16 + y^2/4 + z^2/16 = 1$.

24. $f(x,y,z)=\sqrt{9-x^2-y^2-z^2}$

▹ The domain consists of the points of R^3 for which

$$9-x^2-y^2-z^2\geq 0$$

or, equivalently,

$$x^2+y^2+z^2\leq 9$$

The domain is the set of points inside or on a sphere of radius 3 centered at the origin.

25. $f(x,y,z)=\sin^{-1}x+\sin^{-1}y+\sin^{-1}z$. The domain of f is $\{(x,y,z)\,|\,|x|\leq 1,\ |y|\leq 1,\ |z|\leq 1\}$, the set of all points (x,y,z) in R^3 on and inside the cube $|x|\leq 1, |y|\leq 1, |z|\leq 1$.

26. $f(x,y,z)=\ln x+\ln y+\ln z$. $\mathrm{Dom}(f)=\{(x,y,z)\,|\,x>0,\ y>0,\ z>0\}$, the first octant in R^3.

27. $f(x,y,z)=\ln(4-x^2-y^2)+|z|$

▹ $\mathrm{Dom}(f)=\{(x,y,z)\,|\,x^2+y^2<4\}$, the set of all points (x,y,z) in R^3 inside the circular cylinder $x^2+y^2=4$.

28. $f(x,y,z)=xz\cos^{-1}(y^2-1)$

▹ Because the domain of the inverse cosine function is $[-1,1]$, we must have

$$-1\leq y^2-1\leq 1$$

$$0\leq y^2\leq 2$$

$$-\sqrt{2}\leq y\leq\sqrt{2}$$

The domain is the set of points of R^3 on and between the parallel planes $y=-\sqrt{2}$ and $y=\sqrt{2}$.

In Exercises 29–36, determine the domain of f and sketch the graph of f.

In Exercises 37-46, sketch a contour map of f showing the level curves at the given numbers.

29 and 37. $f(x,y)=\sqrt{16-x^2-y^2}$. The domain of f is the set of points (x,y) of R^2 on and inside the circle $x^2+y^2=16$. The graph of $z=\sqrt{16-x^2-y^2}$ is the top half of the sphere $x^2+y^2+z^2=16$. If $z=k$ we have $k^2=16-x^2-y^2$; $x^2+y^2=16-k^2$. The level curves are circles centered at the origin of radius $\sqrt{16-k^2}$, $k=0,1,2,3,4$ (a point).

30 and 38. $f(x,y)=6-2x+2y$. $\mathrm{Dom}(f)=\mathrm{R}^2$. The graph of $z=6-2x+2y$ is a plane. If $z=k$ we have $k=6-2x+2y$; $y=x+\frac{1}{2}k-3$. The level curves are the lines for $k=10$, 6, 2, 0, -2, -6, and -10.

31 and 39. $f(x,y)=16-x^2-y^2$. The domain of f is the set of all points (x,y) of R^2. The graph of $z=16-x^2-y^2$ is a hyperbolic paraboloid. If $z=k$ we have $k=16-x^2-y^2$; $x^2+y^2=16-k$. The level curves are circles of radius $\sqrt{16-k}$, $k=16$ (a point), 12, 7, 0, -20.

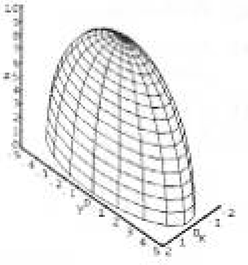
32. $f(x,y)=\sqrt{100-25x^2-4y^2}$

▹ Let $z=f(x,y)$. Then

$$z^2=100-25x^2-4y^2$$

$$25x^2+4y^2+z^2=100$$

$$\frac{x^2}{4}+\frac{y^2}{25}+\frac{z^2}{100}=1$$



Because $z\geq 0$, the graph of the function f is that part of an ellipsoid of semiaxes 2, 5, and 10, that is not below the xy plane, as shown in the figure. The domain of f is $\{(x,y)\,|\,100-25x^2-4y^2\geq 0\}=\{(x,y)\,|\,25x^2+4y^2\leq 100\}$, the set of points in the xy plane inside or on the ellipse $x^2/4+y^2/25=1$. Note that the domain of f is the projection on the xy plane of the surface that is the graph of f.

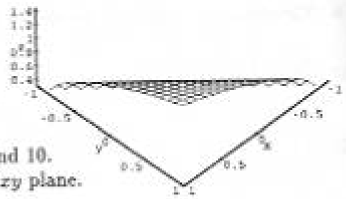
33 and 41. $f(x,y)=x^2-y^2$. The domain of f is the set of all points (x,y) of R^2. The graph of $z=x^2-y^2$ is a hyperbolic paraboloid. If $z=k$, the level curves are the hyperbolas $x^2-y^2=k$, 16, 9, 4, 0, -4, -9, -16.

34. $f(x,y)=144-9x^2-16y^2$. The domain of f is the set of all points (x,y) of R^2. The graph of $z=144-9x^2-16y^2$ is an elliptical paraboloid.

35. $f(x,y)=4x^2+9y^2$. The domain of f is the set of all points (x,y) of R^2. The graph of $z=4x^2+9y^2$ is an elliptical paraboloid.

36. $f(x,y) = \sqrt{x+y}$

▹ The domain is $\{(x,y) \mid x+y \geq 0\}$. The graph of f is the graph of the equation $z = \sqrt{x+y}$. It is the upper half of a parabolic cylinder with rulings parallel to its trace in the xy plane, the line $x+y=0$. Its trace in the plane $z=k$ is the line $x+y=k^2$. Its trace in the perpendicular plane $x-y=0$ is $z=\sqrt{2x}$ or $z^2=4x$, $z \geq 0$. See the figure.

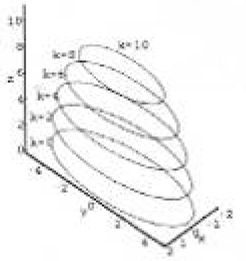


40. The function f for which $f(x,y) = \sqrt{100-25x^2-4y^2}$ at 0, 2, 4, 6, 8, and 10.

▹ The level curves of f at the number k is the graph of $f(x,y)=k$ in the xy plane. Then

$$\begin{aligned}\sqrt{100-25x^2-4y^2} &= k\\ 100-25x^2-4y^2 &= k^2\\ 25x^2+4y^2 &= 100-k^2\\ \frac{x^2}{\frac{100-k^2}{25}}+\frac{y^2}{\frac{100-k^2}{4}} &= 1\end{aligned}$$

This is an ellipse centered at the origin of semiaxes $\frac{1}{5}\sqrt{100-k^2}$ and $\frac{1}{2}\sqrt{100-k^2}$. The semiaxes for the given values of k are shown in the table. See the sketch



k	$\sqrt{100-k^2}$	$\frac{1}{5}\sqrt{100-k^2}$	$\frac{1}{2}\sqrt{100-k^2}$
0	10	2	5
2	$\sqrt{96} \approx 9.8$	1.96	4.9
4	$\sqrt{84} \approx 9.2$	1.83	4.6
6	8	1.6	4
8	6	1.2	3
10	0	0	0

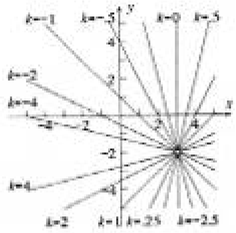
42. The level curves are the lines $x+y=k^2$, $k=10, 8, 6, 5$ and 0

43. $f(x,y)=\frac{1}{2}(x^2+y^2)$. If $f=k$ we have $k=\frac{1}{2}(x^2+y^2)$; $x^2+y^2=2k$. The level curves are circles centered at the origin of radius $\sqrt{2k}$, $k=8, 6, 4, 2, 0$.

44. $f(x,y)=(x-3)/(y+2)$ at 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, 0, $-\frac{1}{4}$, $-\frac{1}{2}$, -1, -2, and -4.

▹ If $f(x,y)=k$, then

$$\begin{aligned}\frac{x-3}{y+2} &= k\\ x-3 &= ky+2k,\ y \neq -2\\ x &= ky+2k+3,\ y \neq 2\end{aligned}$$



Thus each graph is a line with the point $(3,-2)$ deleted. The figure shows the contour map obtained by letting k take on each of the given values.

45. $f(x,y)=e^{xy}$. If $f=k$ we have $k=e^{xy}$; $xy=\ln k$. The level curves are hyperbolas whose asymptotes are the axes for $k=1, 2, e, 4, \frac{1}{2}, e^{-1}, \frac{1}{4}$; $\ln k = 0$ (the axes), ln 2, 1, ln 4, −ln 2, −1, −ln 4.

46. $f(x,y)=\ln xy$. If $f=k$ we have $k=\ln xy$; $xy=e^k$. The level curves are hyperbolas in the first and third quadrants whose asymptotes are the axes, for $k=0, 1, 2, 4, -1, -1, -4$.

47. $f(x,y)=x-y$, $g(t)=\sqrt{t}$, and $h(s)=s^2$.

(a) $(\mathrm{g}\circ \mathrm{f})(5,1)=g(f(5,1))=g(5-1)=g(4)=\sqrt{4}=2$

(b) $f(h(3),g(9))=f(9,3)=9-3=6$

(c) $f(g(x),h(y))=f(\sqrt{x},y^2)=\sqrt{x}-y^2$

(d) $g((h\circ f)(x,y))=g(h(f(x,y)))=g(h(x-y))=g((x-y)^2)=\sqrt{(x-y)^2}=|x-y|$

(e) $(\mathrm{g}\circ h)(f(x,y))=(g\circ h)(x-y)=g(h(x-y))=|x-y|$

48. Given $f(x,y)=x/y^2$, $g(x)=x^2$, $h(x)=\sqrt{x}$. Find: (a) $(h\circ f)(2,1)$; (b) $f(g(2),h(4))$; (c) $f(g(\sqrt{x}),h(x^2))$; (d) $h((g\circ f)(x,y))$; (e) $(h\circ g)(f(x,y))$.

▸ (a) $(h\circ f)(2,1)=h(f(2,1))=h\left(\frac{2}{1^2}\right)=h(2)=\sqrt{2}$

(b) $f(g(2),h(4))=f(2^2,\sqrt{4})=f(4,2)=\frac{4}{2^2}=1$

(c) $f(g(\sqrt{x}),h(x^2))=f((\sqrt{x})^2,\sqrt{x^2})=f(x,x)=\frac{x}{x^2}=\frac{1}{x},\ x>0$

(d) $h((g\circ f)(x,y))=h(g(f(x,y)))=h\left(g\left(\frac{x}{y^2}\right)\right)=h\left(\left(\frac{x}{y^2}\right)^2\right)=h\left(\frac{x^2}{y^4}\right)=\sqrt{\frac{x^2}{y^4}}=\frac{|x|}{y^2}$

(e) $(h\circ g)(f(x,y))=h(g(f(x,y)))=\frac{|x|}{y^2}$

In Exercises 49 and 50, find $h=f\circ g$ and its domain.

49. $f(t)=\sin^{-1}t$; $g(x,y)=\sqrt{1-x^2-y^2}$; $h(x,y)=(f\circ g)(x,y)=f(\sqrt{1-x^2-y^2})=\sin^{-1}\sqrt{1-x^2-y^2}$.
The domain of h is $\{(x,y)\mid x^2+y^2\le 1\}$. Each of these points (x,y) lies in the domain of g and $g(x,y)$ is in $[-1,1]$ which is the domain of f.

50. $f(t)=e^t$; $g(x,y)=y\ln x$. $h(x,y)=(f\circ g)(x,y)=e^{y\ln x}=xe^y,\ x>0$.

In Exercises 51–54, match the function with one of the surfaces (a)–(d) and with one of the contour maps (i)–(iv).

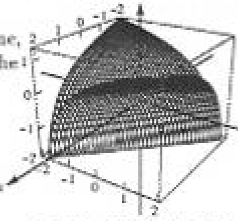
51. $f(x,y)=\frac{x^2}{x^2+y^2}$. On the x axis, $y=0$ and $z=1$ while on the y axis $x=0$ and $z=1$, creating a rift at the origin. Thus figure (b). If $f=k$ we have $k=\frac{x^2}{x^2+y^2}$, $kx^2+ky^2=x^2$; $y^2=\frac{1-k}{k}x^2$. The level curves are lines intersecting at the origin, with the origin deleted, figure (iii)

52. $f(x,y)=\ln|x+y|$

▸ As we approach the line $x+y=0$ in the xy plane, the graph of $z=\ln|x+y|$ approaches $-\infty$ and the graph is symmetric with respect to the plane $x+y=0$. The graph is figure (d). If

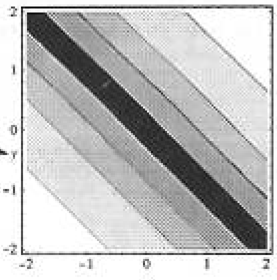
$$\begin{aligned} f(x,y)&=k\\ \ln|x+y|&=k\\ |x+y|&=e^k\\ x+y&=\pm e^k \end{aligned}$$

Thus, the level curve are lines parallel to $x+y=0$, as shown in figure (i).



[-2, 2] by [-2, 2] by [-2, 1.5]

(d)



(i)

53. $f(x,y)=e^y\cos x$. We note the periodic behavior of the cosine curve decaying exponentially as y decreases; figure (a). If $f=k$ we have $k=e^y\cos x$; $e^y=k\sec x$; $y=\ln|k\sec x|$. This is figure (iv).

54. $f(x,y)=\sin x+\cos y$. We note the periodic behavior in the direction of both axes in figure (c). If $f=0$, $\sin x=-\cos y$; $\cos(\frac{1}{2}\pi+x)=\cos(y+n\pi)$; $\frac{1}{2}\pi-x=y-n\pi$, $x+y=(n+\frac{1}{2}\pi)$ or $\frac{1}{2}\pi-x=n\pi-y$, $y-x=(n-\frac{1}{2})\pi$. These diagonal lines form a grid separating the groups of curves in figure (ii).

55. Let $C(x,y)$ dollars be the cost when the width is x ft and the height is y ft. Because the volume is 16 ft^3, the depth must be $16/xy$ ft. $C(x,y)=2(\text{cost of top})+2(\text{cost of front})+2(\text{cost of other side})$
$=2\left(.18x\frac{16}{xy}+.16xy+.12y\frac{16}{xy}\right)=.32\left(\frac{18}{y}+xy+\frac{12}{x}\right)$, $x>0$, $y>0$. $C(2,4)=.32(4.5+8+6)=5.92$

56. A rectangular box without a top is to made at a cost of \$10 for the material. The material for the bottom costs \$0.15 per square foot and the material for the sides costs \$0.30 per square foot. (a) Find a mathematical model for for the volume of the box as a function of the dimensions of the bottom. State the domain of the function. (b) What is the volume of the box if the bottom is a square of side 3 ft?

▸ Let $V(x,y)$ ft^3 be the volume when the length and width are x ft and y ft, and the height is h ft.

(a) Because the cost is \$10, we have

$$\begin{aligned} 10&=2(\text{cost of front})+2(\text{cost of side})+(\text{cost of bottom})\\ 10&=2(0.30)xh+2(0.30)yh+0.15xy\\ 10&=0.6h(x+y)+0.15xy\\ h&=\frac{10-0.15xy}{0.6(x+y)} \end{aligned}$$

Then

$$V(x,y) = xyh = \frac{xy(10 - 0.15xy)}{0.6(x+y)},\ x > 0,\ y > 0$$

(b) $V(3,3) = \dfrac{9(10 - 0.15 \cdot 9)}{0.6(6)} = 21.625$

The volume is 21.625 ft^3.

57 and 58. $z = \frac{1}{2}(6 - x - 3y)$. $V = xyz = \frac{1}{2}xy(6 - x - 3y)$. $V(1.25, 1.25) = \frac{1}{2}(1.25)^2(1) = 0.78125\ \text{ft}^3$.
$S = 2(xy + yz + zx) = 2[xy + \frac{1}{2}(x+y)(6 - x - 3y)]$. $S(1.25, 1.25) = 2(1.25^2 + 1.25) = 5.625\ \text{ft}^2$.
The domain of each function is the triangle bounded by the coordinate axes and the line $x + 3y = 6$.

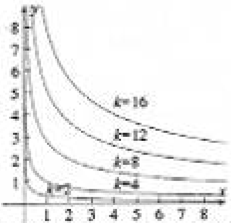
59. $V(x,y) = \dfrac{4}{\sqrt{9 - x^2 - y^2}}$. If $V(x,y) = V$ we have $9 - x^2 - y^2 = \dfrac{16}{V^2}$; $x^2 + y^2 = 9 - \dfrac{16}{V^2}$. The equipotential curves are circles centered at the origin of radius $\sqrt{9 - 16/V^2}$, $V = 16, 12, 8, 4$.

60. A production function f for a commodity has function values $f(x,y) = 4x^{1/3}y^{2/3}$, where x and y give the amounts of two inputs. Draw a contour map of f showing the constant product curves at 16, 12, 8, 4, and 2.

▸ If $f = k$, then

$$k = 4x^{1/3}y^{2/3}$$
$$k^3 = 64xy^2$$
$$x = \frac{k^3}{64y^2}$$

The figure shows the constant product curves for $k = 16, 12, 8, 4$, and 2.



61. $z = 6xy$. The constant product curves for $z = 30, 24, 18, 12$, and 6 are the first quadrant branches of the hyperbolas $xy = 5$, $xy = 4$, $xy = 3$, $xy = 2$, and $xy = 1$. The curve is restricted to the first quadrant because $x > 0$ and $y > 0$.

62. If $(x,y) = k$, then $4x^2 + 2y^2 = k$; $\dfrac{x^2}{k/4} + \dfrac{y^2}{k/2} = 1$, an ellipse centered at the origin of semiaxes $\sqrt{k/4}$ and $\sqrt{k/2}$, $k = 12, 18, 4, 1$, and 0 (point).

63. $k = 4e^{-(x^2+y^2+z^2)}$; $x^2 + y^2 + z^2 = \ln(4/k)$. The level surfaces are spheres of radius $\sqrt{\ln(4/k)}$, $k = 4, 2, 1, \frac{1}{2}$.

64. The electric potential at a point (x, y, z) in three-dimensional space is $V(x,y,z)$ volts where $V(x,y,z) = \dfrac{8}{\sqrt{16x^2 + 4y^2 + z^2}}$. The level surfaces of V are *equipotential surfaces*. Describe these surfaces at 4, 2, 1, and $\frac{1}{2}$.

▸ If $V(x, yz) = k$, then

$$k = \frac{8}{\sqrt{16x^2 + 4y^2 + z^2}}$$
$$\sqrt{16x^2 + 4y^2 + z^2} = \frac{8}{k}$$
$$16x^2 + 4y^2 + z^2 = \frac{64}{k^2}$$
$$\frac{x^2}{4/k^2} + \frac{y^2}{16/k^2} + \frac{z^2}{64/k^2} = 1$$

Thus the equipotential surfaces are ellipsoids of semiaxes $2/k$, $4/k$ and $8/k$ for $k = 4, 2, 1, \frac{1}{2}$.

12.2 LIMITS AND CONTINUITY OF FUNCTIONS OF MORE THAN ONE VARIABLE

We first consider functions of two variables. The *open disk* $B((x_0, y_0); r)$ is the set of all points in the xy plane that are inside the circle with center at the point (x_0, y_0) and radius r. Thus a point (x, y) is in $B((x_0, y_0); r)$ if and only if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.

12.2.5 Definition Let f be a function of two variables that is defined on some open disk $B((x_0, y_0); r)$, except possibly at the point (x_0, y_0) itself. Then the *limit of* $f(x, y)$ *as* (x, y) *approaches* (x_0, y_0) *is* L, written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = 1$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

if $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ then $|f(x,y) - L| < \epsilon$

Let S be a set of points in the xy plane. A point (x_0, y_0) is said to be an *accumulation* point of S if and only if every open disk $B((x_0, y_0); r)$ contains infinitely many points of S.

We now extend to functions of two variables the concept of one-sided limits for functions of two variables.

12.2.8 Definition Let f be a function defined on a set of points in R^2 and let (x_0, y_0) be an accumulation point of S. Then the *limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) in S is L*, is written as

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (P \text{ in } S)}} f(x, y) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that
if $0 < \|(x, y) - (x_0, y_0)\| < \delta$ and $(x, y) \in S$ then $|f(x, y) - L| < \epsilon$

If S is the domain of f, this is the definition of limit in advanced calculus. This distinction is illustrated in Exercise 24.

12.2.9 Theorem Suppose that the function f is defined for all points on an open disk having its center at (x_0, y_0), except possibly at (x_0, y_0) itself,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

and S is any set of points in R^2 having (x_0, y_0) as an accumulation point. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ (P \text{ in } S)}} f(x, y) = L$$

The contrapositive of this theorem may be used to prove a limit does not exist.

12.2.10 Theorem If the function f has different limits as (x, y) approaches (x_0, y_0) through distinct sets of points having (x_0, y_0) as an accumulation point, then

$\lim_{(x,y)\to(x_0,y_0)}$ does not exist.

The limit theorems for functions of one variable, with minor modifications, apply to functions of two variables. We also have the following theorem regarding the limit of a composite function of two variables.

12.2.6 Theorem If g is a function of two variables and $\lim_{(x,y)\to(x_0,y_0)} = b$, and f is a function of a single variable continuous at b, then

$$\lim_{(x,y)\to(x_0,y_0)} (f \circ g)(x, y) = f(b) \Leftrightarrow$$

$$\lim_{(x,y)\to(x_0,y_0)} f(g(x, y)) = f\Big(\lim_{(x,y)\to(x_0,y_0)} g(x, y)\Big)$$

We may be able to find the limit of a function of two variables by applying the limit theorem for a function of one variable together with Theorem 12.2.6. If these theorems fail, as they do, for example, when both the numerator and denominator of a fraction have limit zero, then the following steps may sometimes be used to determine if $\lim_{(x,y)\to(x_0,y_0)}$ exists.

1. Let S_1 be a set of points on a curve containing (x_0, y_0) and find L_1, it it exists, where
   $L_1 = \lim_{\substack{(x,y)\to(x_0,y_0)\\ (P \text{ in } S_1)}} f(x, y)$
2. If L_1 does not exist, then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.
3. If L_1 does exist, then let S_2 be a set of points on a different curve containing (x_0, y_0), and find L_2, it it exists, where
   $L_2 = \lim_{\substack{(x,y)\to(x_0,y_0)\\ (P \text{ in } S_2)}} f(x, y)$
4. If L_2 does not exist, or if $L_1 \neq L_2$, then $\lim_{(x,y)\to(x_0,y_0)} (x, y)$ does not exist.
5. If $L_1 = L_2$, try to use Definition 12.2.5 with $L = L_1 = L_2$. (It is still possible that the limit does not exist.)

In most of Exercises 11–24, the numerator is homogeneous of degree m and the denominator is homogeneous of degree n. Then the limit at $(0, 0)$ exists and is zero if $m > n$ and does not exist if $m \leq n$.

12.2.12 Definition The function f of two variables x and y is said to be *continuous at the point* (x_0, y_0) if and only if the following three conditions are satisfied:

(i) $f(x_0, y_0)$ exists;

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists;

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

If a function f of two variables is discontinuous at the point (x_0, y_0) but $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then f is said to have a *removable discontinuity* at (x_0, y_0) because if f is redefined at (x_0, y_0) so that $f(x_0, y_0) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ then the new function is continuous at (x_0, y_0). If the discontinuity is not removable, it is an *essential discontinuity*.

12.2.13 Theorem If f and g are two functions continuous at the point (x_0, y_0), then

(i) $f + g$ is continuous at (x_0, y_0);

(ii) $f - g$ is continuous at (x_0, y_0);

(iii) fg is continuous at (x_0, y_0);

(iv) f/g is continuous at (x_0, y_0), provided that $g(x_0, y_0) \neq 0$.

12.2.14 Theorem A polynomial function of two variables is continuous at every point in R^2.

12.2.15 Theorem A rational function of two variables is continuous at every point in its domain.

12.2.17 Theorem Suppose that f is a function of a single variable and g is a function of two variables such that g is continuous at (x_0, y_0) and f is continuous at $g(x_0, y_0)$. Then the composite function $f \circ g$ is continuous at (x_0, y_0).

The following extend the definitions and theorems to functions of more than two variables.

12.2.1 Definition If $P(x_1, x_2, \ldots, x_n)$ and $A(a_1, a_2, \ldots, a_n)$ are two points in R^n, then the distance between P and A is given by

$$\|P - A\| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}$$

12.2.2 Definition If A is a point in R^n and r is a positive number, then the *open ball* $B(A; r)$ is the set of all points P in R^n such that $\|P - A\| < r$.

12.2.4 Definition Let f be a function of n variables that is defined on some open ball $B(A; r)$, except possibly at the point A itself. Then the *limit of $f(P)$ as P approaches A is L*, written as $\lim\limits_{P\to A} f(P) = L$ if for any $\epsilon > 0$, however small, the exists a $\delta > 0$ such that

if $\|P - A\| < \delta$ then $|f(P) - L| < \epsilon$

12.2.7 Definition A point P is said to be an *accumulation point* of a set S of points in R^n if every open ball $B(P; r)$ contains infinitely many points of S.

12.2.11 Definition Suppose that f is a function of n variables and A is a point in R^n. Then f is said to be *continuous* at the point A if and only if the following three conditions are satisfied:

(i) $f(A)$ exists;

(ii) $\lim\limits_{P\to A} f(P)$ exists;

(iii) $\lim\limits_{P\to A} f(P) = f(A)$.

12.2.16 Definition A function of n variables is said to be *continuous on an open ball* if it is continuous at every point of the open ball.

Exercises 12.2

In Exercises 1–6, evaluate the limit by use of limit theorems.

1. $\lim\limits_{(x,y)\to(2,3)} (3x^2 + xy - 2y^2) = \lim\limits_{(x,y)\to(2,3)} 3x^2 + \lim\limits_{(x,y)\to(2,3)} xy + \lim\limits_{(x,y)\to(2,3)} (-2y^2) = 12 + 6 - 18 = 0$

2. $\lim\limits_{(x,y)\to(-1,4)} (5x^2 - 2xy + y^2) = \lim\limits_{(x,y)\to(-1,4)} 5x^2 - \lim\limits_{(x,y)\to(-1,4)} 2xy + \lim\limits_{(x,y)\to(-1,4)} y^2 = 5 - (-8) + 16 = 29$

3. $\lim\limits_{(x,y)\to(2,-1)} = \dfrac{\lim\limits_{(x,y)\to(2,-1)}(3x-2y)}{\lim\limits_{(x,y)\to(2,-1)}(x+4y)} = \dfrac{6+2}{2-4} = \dfrac{8}{-2} = -4$

4. $\lim\limits_{(x,y)\to(-2,4)} y\sqrt[3]{x^3+2y}$

▸ $\lim\limits_{(x,y)\to(-2,4)} y\sqrt[3]{x^3+2y} = \left(\lim\limits_{y\to 4} y\right)\cdot\sqrt[3]{\lim\limits_{(x,y)\to(-2,4)}(x^3+2y)} = 4\sqrt[3]{0} = 0$

5. $\lim\limits_{(x,y)\to(0,1)} \dfrac{x^4-(y-1)^4}{x^2-(y-1)^2} = \lim\limits_{(x,y)\to(0,1)} \dfrac{[x^2-(y-1)^2][x^2+(y-1)^2]}{x^2-(y-1)^2} = \lim\limits_{(x,y)\to(0,1)} [x^2+(y-1)^2] = 0$

6. $\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)^{4/3}-(y-1)^{4/3}}{(x-1)^{2/3}+(y-1)^{2/3}} = \lim\limits_{(x,y)\to(1,1)} \dfrac{[(x-1)^{2/3}-(y-1)^{2/3}][(x-1)^{2/3}+(y-1)^{2/3}]}{(x-1)^{2/3}+(y+1)^{2/3}}$

$= \lim\limits_{(x,y)\to(1,1)} [(x-1)^{2/3}-(y-1)^{2/3}] = 0-0 = 0$

In Exercises 7–10, establish the limit by finding a $\delta > 0$ such that Definition 12.2.5 holds.

7. Because $3x-4y$ is defined at every point (x,y), any open disk centered at $(3,2)$ will satisfy the first requirement of Definition 12.2.5. To prove that $\lim\limits_{(x,y)\to(3,2)}(3x-4y) = 1$, we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$|3x-4y-1| < \epsilon$ whenever $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$ (1)

$|3x-4y-1| = |3(x-3)-4(y-2)| \le 3|x-3|+4|y-2| \le 3\sqrt{(x-3)^2+(y-2)^2}+4\sqrt{(x-3)^2+(y-2)^2}$

Thus $|3x-4y-1| \le 3\delta+4\delta = 7\delta$ whenever $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$. Hence if $\delta = \frac{1}{7}\epsilon$, statement (1) holds.

8. $\lim\limits_{(x,y)\to(-2,1)}(5x+4y) = -6$

▸ Because $5x+4y$ is defined everywhere, it is defined on an open disk having $(-2,1)$ as its center. We use the Cauchy-Schwarz inequality:

$$|a_1b_1+a_2b_2| \le \sqrt{a_1{}^2+a_2{}^2}\sqrt{b_1{}^2+b_2{}^2}$$

Thus, if

$0 < \|(x,y)-(2,-1)\| = \sqrt{(x+2)^2+(y-1)^2} < \delta$

then

$$|(5x+4y)-(-6)| = |5(x+2)+4(y-1)| \le \sqrt{5^2+4^2}\sqrt{(x+2)^2+(y-1)^2} < \sqrt{41}\,\delta$$

Therefore, if $\delta = \epsilon/\sqrt{41}$ then $|(5x+4y)-(-6)| < \epsilon$.

9. Because $3x-2y$ is defined at every point (x,y), any open disk centered at $(-1,3)$ will satisfy the first requirement of Definition 12.2.5. To prove that $\lim\limits_{(x,y)\to(0,0)}(3x-2y) = -9$, we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$|3x-2y+9| < \epsilon$ whenever $0 < \sqrt{(x+1)^2+(y-3)^2} < \delta$ (1)

$|3x-2y+9| = |3(x+1)-2(y-3)| \le 3|x+1|+2|y-3| \le 3\sqrt{(x+1)^2+(y-3)^2}+2\sqrt{(x+1)^2+(y-3)^2}$

Thus $|3x-2y+9| \le 3\delta+2\delta = 5\delta$ whenever $0 < \sqrt{(x+2)^2+(y-3)^2} < \delta$. Thus, if $\delta = \frac{1}{5}\epsilon$, statement (1) holds.

10. Because $5x-3y$ is defined at every point (x,y), any open disk centered at $(2,4)$ will satisfy the first requirement of Definition 12.2.5. If $0 < \sqrt{(x-2)^2+(y-4)^2} < \delta$, then by the Cauchy-Schwarz inequality

$|(5x-3y)-(-2)| = |5(x-2)-3(y-4)| \le \sqrt{5^2+3^2}\sqrt{(x+2)^2+(y-4)^2} < \sqrt{34}\delta$

Therefore, if $\delta = \epsilon/\sqrt{34}$ then $|(5x-3y)-(-2)| < \epsilon$.

In Exercises 11–16, prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

11. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = \lim\limits_{x\to 0} 1 = 1$; $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = \lim\limits_{y\to 0}(-1) = -1$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

12. $f(x,y) = \dfrac{x^2}{x^2+y^2}$

▸ We use Theorem 12.2.10 with $S_1 = \{(x,y) \mid y = 0\}$ and $S_2 = \{(x,y) \mid x = 0\}$.

$$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^2}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x^2}{x^2} = \lim\limits_{x\to 0} 1 = 1$$

$$\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2}{x^2+y^2} = \lim\limits_{y\to 0} \frac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0$$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

13. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{x^4y^4}{(x^2+y^4)^3} = \lim\limits_{x\to 0} \dfrac{0}{x^6} = \lim 0 = 0$; $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=y^2)}} \dfrac{x^4y^4}{(x^2+y^4)^3} = \lim\limits_{y\to 0} \dfrac{y^8y^4}{(y^4+y^4)^3} = \lim\limits_{y\to 0} \dfrac{1}{8} = \dfrac{1}{8}$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ does not exist.

14. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} = \lim\limits_{x\to 0} \dfrac{x^4}{x^4} = \lim\limits_{x\to 0} 1 = 1$; $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \dfrac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} = \lim\limits_{y\to 0} 0 = 0$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$ does not exist.

15. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{x^9y}{(x^6+y^2)} = \lim\limits_{x\to 0} \dfrac{0}{x^{12}} = 0$; $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x^3)}} \dfrac{x^9y}{(x^6+x^6)^2} = \lim\limits_{x\to 0} \dfrac{x^{12}}{4x^{12}} = \lim\limits_{x\to 0} \dfrac{1}{4} = \dfrac{1}{4}$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^9y}{(x^6+y^2)^2}$ does not exist.

16. $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$

▸ We use Theorem 12.2.10 with $S_1 = \{(x,y) \mid y = 0\}$ and $S_2 = \{(x,y) \mid y = x\}$.

$$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^2y^2}{x^4+y^4} = \lim\limits_{x\to 0} \frac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$$

$$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} \frac{x^2y^2}{x^4+y^4} = \lim\limits_{x\to 0} \frac{x^4}{2x^4} = \lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Because these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+y^4}$ does not exist.

In Exercises 17–20, prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

17. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y+xy^2}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^2y+xy^2}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$\left|\dfrac{x^2y+xy^2}{x^2+y^2}\right| \le \dfrac{x^2|y|+|x|y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2}$. If $\delta = \epsilon$, then statement (1) holds.

18. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \tag{1}$$

$\left|\dfrac{x^3+y^3}{x^2+y^2}\right| \le \dfrac{x^2|x|+|y|y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2}$. If $\delta = \epsilon$, then statement (1) holds.

19. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \tag{1}$$

$\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| = \dfrac{|x||y|}{\sqrt{x^2+y^2}} \le \dfrac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$. Hence, if $\delta = \epsilon$, then statement (1) holds.

20. $f(x,y) = \dfrac{x^2+2xy}{\sqrt{x^2+y^2}}$

▸ $$\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2+2xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{0}{|y|} = 0$$

Therefore, if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists, then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. We use Definition 12.2.5 to prove that the limit is 0. For any $\epsilon > 0$, we must find a $\delta > 0$ such that

$$\left|\frac{x^2+2xy}{\sqrt{x^2+y^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \tag{1}$$

Because $|x| \le \sqrt{x^2+y^2}$ and $|y| \le \sqrt{x^2+y^2}$ and by the triangle inequality $|x+2y| \le |x| + |2y|$, then

$$\left|\frac{x^2+2xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||x+2y|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}\cdot 3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 3\sqrt{x^2+y^2}$$

Hence, if $\delta = \frac{1}{3}\epsilon$, then statement (1) holds.

In Exercises 21–24, determine if the limit exists.

21. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^2y^2}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \tag{1}$$

$\left|\dfrac{x^2y^2}{x^2+y^2}\right| = \dfrac{x^2y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)(x^2+y^2)}{x^2+y^2} = x^2+y^2$. Therefore if $\delta = \sqrt{\epsilon}$, statement (1) holds.

22. Because $y^4 \le x^4 + y^4$, then $0 \le \dfrac{x^2y^4}{x^4+y^4} \le x^2$. Because $\lim\limits_{(x,y)\to(0,0)} x^2 = 0$, then by the squeeze theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^4}{x^4+y^4} = 0$.

23. $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \dfrac{x^2+y}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{y}{y^2} = \lim\limits_{y\to 0} \dfrac{1}{y}$ does not exist. Therefore $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y}{x^2+y^2}$ does not exist.

24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$

▹ Let $f(x,y) = \dfrac{x^2y^2}{x^3+y^3}$. Because $x^3+y^3 = (x+y)(x^2-xy+x^2)$, the domain of consists of all points of R^2 except those on the line $x+y=0$. Therefore f is not defined on any open disk centered at the origin and so the limit does not exist according to Definition 12.2.5. To show the limit does not exist using the advanced definition, we use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{x^2y^2}{x^3+y^3} = \frac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r^3\cos^3\theta + r^3\sin^3\theta} = r\frac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$$

Because for any fixed value of r, no matter how small, the value of this expression is arbitrarily large when θ is near $\frac{3}{4}\pi$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

In Exercises 25–28, show the application of Theorem 12.2.6 to find the limit.

25. Because $\tan^{-1}t$ is continuous at $t=1$ it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(2,2)} \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\lim_{(x,y)\to(2,2)}\frac{y}{x}\right) = \tan^{-1}\frac{2}{2} = \tan^{-1}1 = \tfrac{1}{4}\pi$$

26. Because $\displaystyle\lim_{(x,y)\to(\ln 3,\ln 2)}(x-y) = \ln 3 - \ln 2 = \ln\tfrac{3}{2}$ and e^t is continuous at $t = \ln\frac{3}{2}$, it follows by Theorem 12.2.6 that $\displaystyle\lim_{(x,y)\to(\ln 3,\ln 2)} e^{x-y} = \exp\left(\lim_{(x,y)\to(\ln 3,\ln 2)}(x-y)\right) = \exp(\ln\tfrac{3}{2}) = \tfrac{3}{2}$

27. Because $\sqrt{t}$ is continuous at $t=\frac{1}{4}$ it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(4,2)}\sqrt{\frac{1}{3x-4y}} = \sqrt{\lim_{(x,y)\to(4,2)}\frac{1}{3x-4y}} = \sqrt{\frac{1}{3\cdot 4-4\cdot 2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

28. $\displaystyle\lim_{(x,y)\to(-2,3)}\llbracket 5x+\tfrac{1}{2}y^2\rrbracket$

▹ Because

$$\lim_{(x,y)\to(-2,3)}(5x+\tfrac{1}{2}y^2) = -10+\tfrac{9}{2} = -\tfrac{11}{2}$$

and $\llbracket t\rrbracket$ is continuous at any noninteger, and hence at $-\frac{11}{2}$, it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(-2,3)}\llbracket 5x+\tfrac{1}{2}y^2\rrbracket = \llbracket\lim_{(x,y)\to(-2,3)}(5x+\tfrac{1}{2}y^2)\rrbracket = \llbracket -\tfrac{11}{2}\rrbracket = -6$$

In Exercises 29–52, determine all points at which the function is continuous.

29. $f(x,y) = \dfrac{x^2}{y-1}$. Because f is a rational function, then by Theorem 12.2.15, f is continuous at every point in its domain, which is $\{(x,y) \mid y \neq 1\}$. Thus f is continuous at every point (x,y) in R^2 that is not on line $y=1$.

30. $F(x,y) = \dfrac{1}{x-y}$ is continuous at every point (x,y) in R^2 that is not on the line $y=x$.

31. $h(x,y) = \sin\frac{y}{x}$. Let $g(x,y) = \frac{y}{x}$. Because g is a rational function, then by Theorem 12.2.15 it is continuous at every point in its domain which is $\{(x,y) \mid x \neq 0\}$. Let $f(t) = \sin t$. Because the sine function is continuous everywhere, then f is continuous at all t. The function h is the composite function $f \circ g$, which by Theorem 12.2.17 is continuous at every point (x,y) in R^2 that is not on the y axis.

32. $f(x,y) = \ln xy^2$

▹ Because $xy^2 > 0$ if $x > 0$ and $y \neq 0$, and the natural logarithmic function is continuous at every positive number, f is continuous at every point in the first and fourth quadrant.

33. $f(x,y) = \dfrac{4x^2y+3y^2}{2x-y}$. Because f is a rational function, then by Theorem 12.2.15, f is continuous at every point in its domain, which is $\{(x,y) \mid y \neq 2x\}$. Thus, f is continuous at every point (x,y) in R^2 not on line $y=2x$.

34. $g(x,y) = \dfrac{5xy^2+2y}{16-x^2-4y^2}$ is continuous except if $16-x^2-4y^2=0$, that is, except on the ellipse $4x^2+y^2=16$.

35. $g(x,y) = \ln(25 - x^2 - y^2)$. Let $\text{h}(x,y) = 25 - x^2 - y^2$. Because h is a polynomial function, then by Theorem 12.2.14 it is continuous at every point in R^2. Let $f(t) = \ln t$. Because the natural logarithmic function is continuous at all positive numbers, then f is continuous at all $t > 0$. The function g is the composite function $f \circ g$, which by Theorem 12.2.17 is continuous at every point (x,y) in R^2 inside the circle $x^2 + y^2 = 25$.

36. $f(x,y) = \cos^{-1}(x+y)$

▸ The inverse cosine function is continuous at every point in the open interval $(-1,1)$. Thus, f is continuous at every point in the strip $\{(x,y) \mid -1 < x + y < 1\}$.

37. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ f is continuous at (x,y). From Exercise 19 we have

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$. Thus, f is continuous at $(0,0)$. Hence, f is continuous at every point in R^2.

38. $h(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▸ From Example 7, $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does not exist. Hence f is continuous at every point of R^2 except $(0,0)$.

39. $f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ f is continuous at (x,y). Test if f is continuous at $(0,0)$.

$\lim\limits_{\substack{(x,y)\to(0,0) \\ (y=x)}} f(x,y) = \lim\limits_{\substack{(x,y)\to(0,0) \\ (y=x)}} \dfrac{x+y}{x^2+y^2} = \lim\limits_{x\to0} \dfrac{2x}{2x^2} = \lim\limits_{x\to0} \dfrac{1}{x}$. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist because this limit does not exist. Hence f is not continuous at $(0,0)$. Thus f is continuous at every point $(x,y) \neq (0,0)$ in R^2.

40. $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▸ In Exercise 18 it was shown that

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

Because $f(0,0) = 0$, then the three conditions of Definition 12.2.12 are satisfied and hence f is continuous at $(0,0)$. If $(x_0,y_0) \neq (0,0)$, then by Theorem 12.2.15, f is continuous at (x_0,y_0). Therefore, f is continuous everywhere.

41. $g(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ G is continuous at (x,y). To show that G is continuous at $(0,0)$ we must show that $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ exists and is equal to $g(0,0) = 0$ which is done by showing that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy}{|x|+|y|}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \tag{1}$$

$$\left|\frac{xy}{|x|+|y|}\right| \leq \frac{|x||y|}{|x|} = |y| \leq \sqrt{x^2+y^2}$$

Therefore, if $\delta = \epsilon$ statement (1) holds. Hence G is continuous at every point in R^2.

42. $F(x,y) = \begin{cases} \dfrac{x^2y^2}{|x^3|+|y^3|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Because $x^2 = |x^3|^{2/3} \le (|x^3|+|y^3|)^{2/3}$ and $y^2 \le (|x^3|+|y^3|)^{2/3}$, then

$$0 \le \frac{x^2y^2}{|x^3|+|y^3|} \le \frac{(|x^3|+|y^3|)^{4/3}}{|x^3|+|y^3|} = (|x^3|+|y^3|)^{1/3} \to 0 \text{ as } (x,y) \to (0,0)$$

By the squeeze theorem, F is continuous at $(0,0)$ and hence at every point in R^2. Contrast with Exercise 24.

43. $f(x,y) = \dfrac{xy}{\sqrt{16-x^2-y^2}}$. By Thms 12.2.15 and 12.2.17 the region of continuity of f is the set of all points (x,y) in R^2 for which $16 - x^2 - y^2 > 0$. This is the set of all points in R^2 that are inside the circle $x^2 + y^2 = 16$.

44. $f(x,y) = \dfrac{y}{\sqrt{x^2-y^2-4}}$

▸ Because $g(x,y) = y$ is continuous at all points in R^2 and $h(x,y) = \sqrt{x^2-y^2-4}$ is continuous at all (x,y) for which

$$x^2 - y^2 - 4 > 0$$
$$x^2 - y^2 > 4$$

then, by Theorem 12.2.13(iv), f is continuous at all points in the interior of the hyperbola $x^2 - y^2 = 4$.

45. $f(x,y) = \dfrac{x}{\sqrt{4x^2+9y^2-36}}$. By Thms 12.2.15 and 12.2.17 the region of continuity of f is the set of all points (x,y) in R^2 for which $4x^2 + 9y^2 - 36 > 0$, the set of all points in R^2 that are outside the ellipse $4x^2 + 9y^2 = 36$

46. $f(x,y) = \dfrac{x^2+y^2}{\sqrt{9-x^2-y^2}}$ is continuous if $9 - x^2 - y^2 > 0$, that is, inside the circle $x^2 + y^2 = 9$.

47. $f(x,y) = \sec^{-1}(xy)$. If $g(t) = \sec^{-1}t$, the domain of g is $\{t \mid |t| \ge 1\}$, and g is continuous on its domain. Thus the region of continuity of f is the set of all points (x,y) in R^2 for which $|xy| \ge 1$.

48. $f(x,y) = \ln(x^2+y^2-9) - \ln(1-x^2-y^2)$

▸ Because the natural logarithmic function is defined for positive numbers only, $f(x,y)$ is defined only when both
$x^2 + y^2 - 9 > 0$ and $1 - x^2 - y^2 > 0$
or, equivalently, only when both
$x^2 + y^2 > 9$ and $x^2 + y^2 < 1$
which is impossible. Thus, f is continuous at no point of R^2.
Note that the function $\ln[(x^2+y^2-9)/(1-x^2-y^2)]$ is continuous in the annulus $\{(x,y) \mid 1 < x^2 + y^2 < 9\}$.

49. $f(x,y) = \sin^{-1}(x+y) + \ln(xy)$

▸ If $g(t) = \sin^{-1}t$, the domain of g is $\{t \mid |t| \le 1\}$ and g is continuous on its domain. If $h(t) = \ln t$, the domain of h is $\{t \mid t > 0\}$, and h is continuous on its domain. Therefore f is continuous at all points (x,y) in R^2 for which $|x+y| \le 1$ and $xy > 0$. These are all the points in the first and third quadrants for which $|x+y| \le 1$.

50. $f(x,y) = \sin^{-1}(xy)$ is continuous if $|xy| < 1$.

51. $f(x,y) = \begin{cases} \dfrac{\sin(x+y)}{x+y} & \text{if } x+y \neq 0 \\ 1 & \text{if } x+y = 0 \end{cases}$. Let $h(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$ and let $g(x,y) = x + y$. Then $f(x,y) = (h \circ g)(x,y)$. Observe that $\lim\limits_{t \to 0} h(t) = \lim\limits_{t \to 0} \dfrac{\sin t}{t} = 1 = h(0)$. Thus h is continuous at 0, and continuous everywhere else. The domain of g is R^2 and g is continuous on its domain. Therefore by Theorem 12.2.17 f is continuous at all points in R^2.

52. $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x - y} & \text{if } x \neq y \\ x - y & \text{if } x = y \end{cases}$

▷ By Theorem 12.2.15, f is continuous at all points not on the line $y = x$.
Let (x_0, y_0) be a point such that $x_0 = y_0$. Let S_1 be the line $y = x$ and S_2 the rest of R^2.

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (P \text{ in } S_1)}} f(x,y) = \lim_{x\to 0}(x - x) = 0$$

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (P \text{ in } S_2)}} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(x_0,y_0)} (x + y) = x_0 + x_0 = 2x_0$$

Thus, f is continuous if $x_0 = 0$, that is at the origin, and f is discontinuous at any other point of line $y = x$.

In Exercises 53–59, f is discontinuous at the origin because $f(0,0)$ does not exist. Determine if the discontinuity is removable or essential. If the discontinuity is removable, redefine $f(0,0)$ so that the new function is continuous.

53. $f(x,y) = \dfrac{xy}{x^2 + xy + y^2}$

▷ $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=x)}} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}$; $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=-x)}} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{-x^2}{x^2} = \lim_{x\to 0} -1 = -1$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$ does not exist. Therefore, the discontinuity is essential.

54. $f(x,y) = \dfrac{x}{x^2 + y^2}$. $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=0)}} \frac{x}{x^2 + y^2} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ does not exist. Thus the discontinuity is essential.

55. $f(x,y) = (x + y)\sin \dfrac{x}{x^2 + y^2}$. Because $|\sin t| \leq 1$, then $0 \leq |f(x,y)| \leq |x + y|$. Because $\displaystyle\lim_{(x,y)\to(0,0)} (x + y) = 0$, it follows from the squeeze theorem that $\lim f(x,y) = 0$; if we define $f(0,0) = 0$ the discontinuity is removed.

56. $f(x,y) = \dfrac{x^2y^2}{x^2 + y^2}$

▷ In Exercise 21 it was shown that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Hence the discontinuity at the origin may be removed by defining $f(0,0) = 0$.

57. $f(x,y) = \dfrac{x^3y^2}{x^6 + y^4}$. $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=0)}} \frac{x^3y^2}{x^6 + y^4} = \lim_{x\to 0} \frac{0}{x^6} = \lim_{x\to 0} 0 = 0$; $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=x^{3/2})}} \frac{x^3y^2}{x^6 + y^4} = \lim_{x\to 0} \frac{x^6}{x^6 + x^6} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

Because these two limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6 + y^4}$ does not exist. Thus, the discontinuity is essential.

58. $f(x,y) = \dfrac{2y^2 - 3xy}{\sqrt{x^2 + y^2}}$. $0 \leq |f(x,y)| = \dfrac{|2y^2 - 3xy|}{\sqrt{x^2 + y^2}} = |y|\dfrac{2|y| + 3|x|}{\sqrt{x^2 + y^2}} \leq |y|\dfrac{5\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = |y| \to 0$.

If we define $f(0,0) = 0$ the discontinuity is removed.

59. $f(x,y) = \dfrac{x^3 - 4xy^2}{x^2 + y^2}$. $0 \leq |f(x,y)| = \left|\dfrac{x^3 - 4xy^2}{x^2 + y^2}\right| \leq |x|\dfrac{x^2 + 4y^2}{x^2 + y^2} \leq |x|\dfrac{4x^2 + 4y^2}{x^2 + y^2} = 4|x|$

From the squeeze theorem we get $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. If we define $f(0,0) = 0$, the discontinuity is removed.

60. (a) Give a definition, similar to Definition 12.2.5, of the limit of a function of three variables as a point (x,y,z) approaches a point (x_0, y_0, z_0).
(b) Give a definition, similar to Definition 12.2.8, of the limit of a function of three variables as a point (x,y,z) approaches a point (x_0, y_0, z_0) in a specific set of points S in R^3.

▷ (a) Let f be a function of three variables that is defined on some open disk $B((x_0, y_0, z_0); r)$, except possibly at the point (x_0, y_0, z_0) itself. Then

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that if $0 < \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < \delta$ then $|f(x,y,z)) - \mathrm{L}| < \epsilon$

(b) Let f be a function defined on a set of points in R^3 and let (x_0, y_0, z_0) be an accumulation point of S. Then

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\(P\ in\ S)}} f(x,y,z) = \mathrm{L}$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that
if $0 < \|(x,y,z)-(x_0,y_0,z_0)\| < \delta$ and $(x,y,z) \in \mathrm{S}$ then $|f(x,y,z) - \mathrm{L}| < \epsilon$

61. (a) A theorem similar to Theorem 12.2.9 for a function of three variables is:
Suppose that the function f is defined for all points on an open ball having its center at (x_0,y_0,z_0) except possibly at (x_0,y_0,z_0) itself, and $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \mathrm{L}$.
If S is any set of points in R^3 having (x_0,y_0,z_0) as an accumulation point, then
$\lim\limits_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\(P\ in\ S)}} f(x,y,z)$ exists and always has the value L.

(b) A theorem similar to Theorem 12.2.10 for a function of three variables is:
If the function f has different limits as (x,y,z) approaches (x_0,y_0,z_0) through two distinct sets of points having (x_0,y_0,z_0) as an accumulation point, then $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ does not exist.

In Exercises 62–65, use definitions and theorems of Ex. 60–61 to prove that $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

62. $f(x,y,z) = \dfrac{x^3+yz^2}{x^4+y^2+z^4}$

▹ $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\y=z=0}} f(x,y,z) = \lim\limits_{x\to0}\dfrac{x^3}{x^4} = \lim\limits_{x\to0}\dfrac{1}{x}$ does not exist. Then $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

63. $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\ z=0)}} \dfrac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim\limits_{x\to0}\dfrac{x^2}{x^2} = \lim\limits_{x\to0} 1 = 1$; $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(x=0,\ y=0)}} \dfrac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim\limits_{z\to0}\dfrac{-z^2}{z^2} = \lim\limits_{z\to0}(-1) = -1$

Because these two limits are not equal, $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2+y^2-z^2}{x^2+y^2+z^2}$ does not exist.

64. $f(x,y,z) = \dfrac{x^4+yx^3+z^2x^2}{x^4+y^4+z^4}$

▹ $(0,0,0)$ is an accumulation point of the sets $\{(x,y,z) \mid x = 0\}$ and $\{(x,y,z) \mid x = y = z\}$. Furthermore,

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\x=0}} f(x,y,z) = \lim_{(y,z)\to(0,0)} \frac{0}{y^4+z^4} = 0$$

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\x=y=z}} f(x,y,z) = \lim_{x\to0}\frac{3x^4}{3x^4} = 1$$

Because these limits are not equal, then $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

65. $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\ z=0)}} \dfrac{x^2y^2z^2}{x^6+y^6+z^6} = \lim\limits_{x\to0}\dfrac{0}{x^6} = \lim\limits_{x\to0} 0 = 0$; $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(x=y=z)}} \dfrac{x^2y^2z^2}{x^6+y^6+z^6} = \lim\limits_{x\to0}\dfrac{x^6}{3x^6} = \lim\limits_{x\to0}\dfrac{1}{3} = \dfrac{1}{3}$

Because these two limits are not equal, $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^6+y^6+z^6}$ does not exist.

In Exercises 66 and 67, use the definition in Exercise 60(a) to prove that $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ exists.

66. To prove $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{y^3+xz^2}{x^2+y^2+z^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{y^3+xz^2}{x^2+y^2+z^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2+z^2} < \delta \tag{1}$$

$$\left|\frac{y^3+xz^2}{x^2+y^2+z^2}\right| \le \frac{|y^2|^{3/2}+|x|z^2}{x^2+y^2+z^2} \le \frac{2(x^2+y^2+z^2)^{3/2}}{x^2+y^2+z^2} = 2\sqrt{x^2+y^2+z^2} < 2\delta$$

Hence if $\delta = \frac{1}{2}\epsilon$, then statement (1) holds.

67. To prove $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2+z^2} < \delta \qquad (1)$$

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right| \le \frac{|x||y|+|x||y|+|y||z|}{\sqrt{x^2+y^2+z^2}} \le \frac{3\sqrt{(x^2+y^2+z^2)^2}}{\sqrt{x^2+y^2+z^2}} = 3\sqrt{x^2+y^2+z^2}$$

Hence if $\delta = \frac{1}{3}\epsilon$, then statement (1) holds.

68. (a) The function f of three variables x, y, z is said to be continuous at the point (x_0, y_0, z_0) if and only if the following three conditions are satisfied:
(i) $f(x_0, y_0, z_0)$ exists; (ii) $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists; (iii) $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$
(b) If f and g are two functions continuous at the point (x_0, y_0, z_0), then
(i) $f+g$ is continuous at (x_0,y_0,z_0); (ii) $f-g$ is continuous at (x_0,y_0,z_0); (iii) fg is continuous at (x_0,y_0,z_0); (iv) f/g is continuous at (x_0,y_0,z_0), provided that $g(x_0,y_0,z_0) \neq 0$.

(c) Suppose that f is a function of a single variable and g is a function of three variables such that g is continuous at (x_0,y_0,z_0) and f is continuous at $g(x_0,y_0,z_0)$. Then the composite function $f \circ g$ is continuous at (x_0,y_0,z_0).

In Exercises 69–72, use the definitions and theorems of Exercise 68 to determine where the function is continuous.

69. $f(x,y,z) = \dfrac{xz}{\sqrt{x^2+y^2+z^2-1}}$. By Ex. 68(b)(iv) and (c), the region of continuity of f is the set of all points (x,y,z) in R^3 for which $x^2+y^2+z^2-1 > 0$, the set of all points in R^3 outside the sphere $x^2+y^2+z^2 = 1$.

70. $f(x,y,z) = \ln(36-4x^2-y^2-9z^2)$. Applying Ex. 68(c) to the natural logarithmic function, we see that f is continuous when $36-4x^2-y^2-9z^2 < 0$, that is, for points inside the ellipsoid $4x^2-y^2-9z^2 = 36$.

71. $f(x,y,z) = \begin{cases} \dfrac{3xyz}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$
The function f is continuous at every point (x,y,z) in R^3 for which $(x,y,z) \neq (0,0,0)$ because it is a rational function. We show that f is also continuous at $(0,0,0)$ by proving that $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$.

$$0 \le \left|\frac{3xyz}{x^2+y^2+z^2}\right| = \frac{3|x||y||z|}{x^2+y^2+z^2} \le \frac{3(\sqrt{x^2+y^2+z^2})^3}{x^2+y^2+z^2} = 3\sqrt{x^2+y^2+z^2}$$

Because $\lim\limits_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2) = 0$, it follows by the squeeze theorem that $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$.
So, f is continuous at all points in R^3.

72. $f(x,y,z) = \begin{cases} \dfrac{xz-y^2}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$
The function f is continuous at every point (x,y,z) in R^3 for which $(x,y,z) \neq (0,0,0)$ because it is a rational function. We show that f is not continuous at $(0,0,0)$ by considering the sets $\{(x,y,z) \mid y=0,\ z=0\}$ and $\{(x,y,z) \mid y=0,\ z=x\}$, each of which has $(0,0,0)$ as an accumulation point.

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\,z=0)}} f(x,y,z) = \lim_{x\to 0} \frac{0}{x^2} = 0 \qquad \lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\,z=x)}} f(x,y,z) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Because these limits are not equal, f is continuous everywhere except $(0,0,0)$.

73. $g(x,y) = \begin{cases} x^2+4y^2 & \text{if } x^2+4y^2 \le 5 \\ 3 & \text{if } x^2+4y^2 > 5 \end{cases}$

If ${x_0}^2 + 4{y_0}^2 < 5$ then $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} (x^2+4y^2) = {x_0}^2 + 4{y_0}^2 = g(x_0,y_0)$.

If ${x_0}^2 + 4{y_0}^2 > 5$ then $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} 3 = g(x_0,y_0)$.

Thus, G is continuous at all points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 \ne 5$.

Consider the points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 = 5$.

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2=5)}} g(x,y) = \lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2=5)}} (x^2+4y^2) = \lim_{(x,y)\to(x_0,y_0)} 5 = 5; \quad \lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2>5)}} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} 3 = 3$$

Because these two limits are not equal we conclude that $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$ does not exist. Thus G is discontinuous at all points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 = 5$. Hence G is continuous at all points of R^2 except those on the ellipse $x^2+4y^2=5$.

74. $F(x,y) = \begin{cases} x^2-3y^2 & \text{if } x^2-3y^2 \le 1 \\ 2 & \text{if } x^2-3y^2 > 1 \end{cases}$. If (x,y) is inside the hyperbola x^2-3y^2, then F is continuous because it is a polynomial; if (x,y) is outside, it is continuous because it is a constant. For (x_0,y_0) on the hyperbola, the limit from points inside is 1 while the limit from points outside is 2; thus the limit does not exist.

75. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{f(x,y)}{g(x,y)} = \lim\limits_{x\to 0} \dfrac{f(x,0)}{g(x,0)} = \lim\limits_{x\to 0} \dfrac{x^n f(1,0)}{x^n g(1,0)} = \dfrac{f(1,0)}{g(1,0)}$ which exists by (ii)

$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} \dfrac{f(x,y)}{g(x,y)} = \lim\limits_{x\to 0} \dfrac{f(x,x)}{g(x,x)} = \lim\limits_{x\to 0} \dfrac{x^n f(1,1)}{x^n g(1,1)} = \dfrac{f(1,1)}{g(1,1)}$ which also exists. But by (iii) the limits are unequal.

12.3 PARTIAL DERIVATIVES

12.3.1 Definition Let f be a function of two variables, x and y. The *partial derivative of f with respect to x* is that function, denoted by $D_1 f$, such that its function value at any point (x,y) in the domain of f is given by

$$D_1 f(x,y) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{u\to x} \frac{f(u,y) - f(x,y)}{u - x}$$

if this limit exists. Similarly, the *partial derivative of f with respect to y* is that function, denoted by $D_2 f$ such that its function value at any point (x,y) in the domain of f is given by

$$D_2 f(x,y) = \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{v\to y} \frac{f(x,v) - f(x,y)}{v - y}$$

if this limit exists.

There are various notations for partial derivative function values. If $z = f(x,y)$, then

$$D_1 f(x,y) = f_1(x,y) = f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \quad \text{and} \quad D_2 f(x,y) = f_2(x,y) = f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

We do not usually have to apply Definition 12.3.1 to find the partial derivatives of a function. To find $D_1 f(x,y)$, differentiate with respect to x, with y regarded as a constant, and use the differentiation formulas for ordinary derivatives. To find $D_2 f(x,y)$, differentiate with respect to y, with x regarded as a constant, and use the differentiation formulas for ordinary derivatives.

The line tangent to the curve that is the intersection of the surface $z = f(x,y)$ and the plane $x = x_0$ at the point $(x_0, y_0, f(x_0,y_0))$ has $[0, 1, D_2 f(x_0,y_0)]$ for a set of direction numbers. Thus, equations of the line are

$x = x_0 \quad z = z_0 + D_2 f(x_0,y_0)(y - y_0)$

The line tangent to the curve that is the intersection of the surface $z = f(x,y)$ and the plane $y = y_0$ at the point $(x_0, y_0, f(x_0,y_0))$ has $[1, 0, D_1 f(x_0,y_0)]$ for a set of direction numbers. Thus, equations of the line are

$y = y_0 \quad z = z_0 + D_1 f(x_0,y_0)(x - x_0)$

Partial derivatives for functions of more variables are defined as follows.

12.3.2 Definition Let $P(x_1, x_2, \ldots, x_3)$ be a point in R^n, and let f be a function of the n variables $x_1, x_2, \ldots, x_n$. Then the partial derivative with respect to x_k is that function, denoted by $D_k f$, such that its function value at any point P in the domain of f is given by

$$D_k f(x_1, x_2, \ldots, x_n) = \lim_{\Delta x_k \to 0} \frac{f(x_1, x_2, \ldots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_k}$$

if this limit exists.

We may find $D_k f(x_1, x_2, \ldots, x_n)$ by using the formulas for ordinary differentiation if we regard all variables except x_k as constants and differentiate with respect to x_k.

Higher Order If f is a function of x and y, then the second partial derivatives of f obtained by first partial-differentiating f with respect to x and then partial-differentiating the result with respect to y is represented by each of the following notations.

$$D_2(D_1 f) \quad D_{12} f \quad f_{12} \quad f_{xy} \quad \frac{\partial^2 f}{\partial y \partial x}$$

Note the order of the symbols in the last form. The second partial derivative of f obtained by partial-differentiating twice with respect to x is represented by each of the following notations.

$$D_1(D_1 f) \quad D_{11} f \quad f_{11} \quad f_{xx} \quad \frac{\partial^2 f}{\partial x^2}$$

The second partial derivative of f obtained by partial-differentiating twice with respect to y is represented by each of the following notations.

$$D_2(D_2 f) \quad D_{22} f \quad f_{22} \quad f_{yy} \quad \frac{\partial^2 f}{\partial y^2}$$

There is another possible mixed partial derivative, represented by f_{21}, f_{yx}, etc., but as the following theorem states, it is often true (but not always) that $f_{21} = f_{12}$, $f_{yx} = f_{xy}$, etc.

12.3.3 Theorem (Equality of mixed partials) Suppose that f is a function of two variables x and y defined on an open disk $B((x_0, y_0); r)$ and f_x, f_y, and f_{xy} are also defined on B. Furthermore, suppose that f_{xy} is continuous at (x_0, y_0). Then $f_{yx}(x_0, y_0)$ exists and $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

A *partial differential equation* has partial derivatives with respect to more than one variable.

Laplace's Equation in R^2 is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$; in R^3 it is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$. A solution is *harmonic*. See Exercises 49–53.

Exercises 12.3

In Exercises 1–6, apply definition 12.3.1 to find the partial derivative.

1. $f(x, y) = 6x + 3y - 7$

▸ $D_1 f(x, y) = \displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(6x + 6\Delta x + 3y - 7) - (6x + 3y - 7)}{\Delta x} = \lim_{\Delta x \to 0} \frac{6\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 6 = 6$

2. $f(x, y) = 4x^2 - 3xy$. $D_1 f(x, y) = \displaystyle\lim_{u \to x} \frac{f(u, y) - f(x, y)}{u - x} = \lim_{u \to x} \frac{(4u^2 - 3uy) - (4x^2 - 3xy)}{u - x}$

$$= \lim_{u \to x}\left[4\frac{u^2 - x^2}{u - x} - 3y\frac{u - x}{u - x}\right] = \lim_{u \to x}[4(u + x) - 3y] = 8x - 3y$$

3. $f(x, y) = 3xy + 6x - y^2$.

$$D_2 f(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[3x(y + \Delta y) + 6x - (y + \Delta y)^2] - (3xy + 6x - y^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3xy + 3x\Delta y + 6x - y^2 - 2y\Delta y - (\Delta y)^2 - 3xy - 6x + y^2}{\Delta y} = \lim_{\Delta y \to 0} \frac{(3x - 2y - \Delta y)\Delta y}{\Delta y}$$

$= \displaystyle\lim_{\Delta y \to 0} (3x - 2y - \Delta y) = 3x - 2y$. Alternatively, by formula (4),

$$D_2 f(x, y) = \lim_{v \to y} \frac{f(x, v) - f(x, y)}{v - y} = \lim_{v \to y} \frac{(3xv + 6x - v^2) - (3xy + 6x - y^2)}{v - y} = \lim_{v \to y} \frac{3x(v - y) - (v^2 - y^2)}{v - y}$$

$$= \lim_{v \to y}[3x - (v + y)] = 3x - 2y$$

4. $f(x,y) = xy^2 - 5y + 6$; $D_2 f(x,y)$

▸ $$D_2 f(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x(y+\Delta y)^2 - 5(y+\Delta y) + 6] - [xy^2 - 5y + 6]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{xy^2 + 2xy\Delta y + x(\Delta y)^2 - 5y - 5\Delta y + 6 - xy^2 + 5y - 6}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y(2xy + x\Delta y - 5)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (2xy + x\Delta y - 5) = 2xy - 5$$

5. $f(x,y) = \sqrt{x^2+y^2}$. $D_1 f(x,y) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{(x+\Delta x)^2 + y^2} - \sqrt{x^2+y^2}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(\sqrt{(x+\Delta x)^2 + y^2} - \sqrt{x^2+y^2})(\sqrt{(x+\Delta x)^2+y^2} + \sqrt{x^2+y^2})}{\Delta x(\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2})}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + y^2 - x^2 - y^2}{\Delta x[\sqrt{(x+\Delta x)^2+y^2} + \sqrt{x^2+y^2}]} = \lim_{\Delta x \to 0} \frac{2x + \Delta x}{\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2}} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

6. $\dfrac{\partial}{\partial y} \dfrac{x+2y}{x^2-y} = \lim\limits_{v \to y} \dfrac{1}{v-y}\left(\dfrac{x+2v}{x^2-v} - \dfrac{x+2y}{x^2-y}\right) = \lim\limits_{v \to y} \dfrac{(x+2v)(x^2-y) - (x+2y)(x^2-v)}{(v-y)(x^2-v)(x^2-y)} = \lim\limits_{v \to y} \dfrac{(2x^2+x)(v-y)}{(v-y)(x^2-v)(x^2-y)}$

$$= \lim_{v \to y} \frac{2x^2+x}{(x^2-v)(x^2-y)} = \frac{2x^2+x}{(x^2-y)^2}$$

In Exercises 7–10, apply Definition 12.3.2 to find the partial derivative.

7. $f(x,y,z) = x^2y - 3xy^2 + 2yz$

$$D_2 f(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2(y+\Delta y) - 3x(y+\Delta y)^2 + 2(y+\Delta y)z - (x^2y - 3xy^2 + 2yz)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y - 3xy^2 - 6xy\Delta y - 3x(\Delta y)^2 + 2yz + 2z\Delta y - x^2y + 3xy^2 - 2yz}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(x^2 - 6xy - 3x\Delta y + 2z)\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (x^2 - 6xy - 3x\Delta y + 2z) = x^2 - 6xy + 2z$$

Alternatively, $D_2 f(x,y,z) = \lim\limits_{v \to y} \dfrac{f(x,v,z) - f(x,y,z)}{v-y} = \lim\limits_{v \to y} \dfrac{x^2v - 3xv^2 + 2vz - (x^2y - 3xy^2 + 2yz)}{v-y}$

$$= \lim_{v \to y} \frac{x^2(v-y) - 3x(v^2-y^2) + 2z(v-y)}{v-y} = \lim_{v \to y} [x^2 - 3x(v+y) + 2z] = x^2 - 6xy + 2z$$

8. $f(x,y,z) = x^2 + 4y^2 + 9z^2$; $D_1 f(x,y,z)$

▸ $$D_1 f(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 4y^2 + 9z^2] - [x^2 + 4y^2 + 9z^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

9. $f(x,y,z,r,t) = xyr + yzt + yrt + zrt$

$$D_4 f(x,y,z,r,t) = \lim_{\Delta r \to 0} \frac{f(x,y,z,r+\Delta r,t) - f(x,y,z,r,t)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{xy(r+\Delta r) + yzt + y(r+\Delta r)t + z(r+\Delta r)t - (xyr + yzt + yrt + zrt)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{xyr + xy\Delta r + yzt + yrt + yt\Delta r + zrt + zt\Delta r - xyr - yzt - yrt - zrt}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{(xy + yt + zt)\Delta r}{\Delta r} = \lim_{\Delta r \to 0} (xy + yt + zt) = xy + yt + zt$$

10. $\frac{\partial}{\partial v}(3r^2st + st^2v - 2tuv^2 - tvw + 3uw^2)$

$= \lim\limits_{z \to v} \frac{(3r^2st + st^2z - 2tuz^2 - tzw + 3uw^2) - (3r^2st + st^2v - 2tuv^2 - tvw + 3uw^2)}{z - v}$

$= \lim\limits_{z \to v}\left[st^2\frac{z-v}{z-v} - 2tu\frac{z^2-v^2}{z-v} - tw\frac{z-v}{z-v}\right] = \lim\limits_{z \to v}[st^2 - 2tu(z+v) - tw] = st^2 - 4tuv - tw$

In Exercises 11 and 12, $f(x,y) = x^2 - 9y^2$.

11. (a) $D_1 f(2,1) = \lim\limits_{\Delta x \to 0} \frac{f(2+\Delta x,1) - f(2,1)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{(2+\Delta x)^2 - 9 - (-5)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{4 + 4\Delta x + (\Delta x)^2 - 4}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \frac{\Delta x(4+\Delta x)}{\Delta x} = \lim\, (4 + \Delta x) = 4$

(b) $D_1 f(2,1) = \lim\limits_{x \to 2} \frac{f(x,1) - f(2,1)}{x-2} = \lim\limits_{x \to 2} \frac{x^2 - 9 - (2^2 - 9)}{x-2} = \lim\limits_{x \to 2} \frac{x^2 - 2^2}{x-2} = \lim\limits_{x \to 2}(x+2) = 4$

(c) $D_1 f(x,y) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta y,y) - f(x,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 9y^2 - (x^2 - 9y^2)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim\limits_{\Delta x \to 0}(2x + \Delta x) = 2x$. Thus at (2,1), $D_1 f(x,y) = 4$.

12. Find $D_2 f(2,1)$ by: (a) applying formula (2), (b) applying formula (4), and (c) applying definition 12.3.1 and then replacing x and y by 2 and 1, respectively.

(a) $D_2 f(2,1) = \lim\limits_{\Delta y \to 0} \frac{f(2,1+\Delta y) - f(2,1)}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{[4 - 9(1+\Delta y)^2] - [4-9]}{\Delta y}$

$= \lim\limits_{\Delta y \to 0} \frac{4 - 9 - 18\Delta y - 9(\Delta y)^2 + 5}{\Delta y} = \lim\limits_{\Delta y \to 0}(-18 - 9\Delta y) = -18$

(b) $D_2 f(2,1) = \lim\limits_{y \to 1} \frac{f(2,y) - f(2,1)}{y-1} = \lim\limits_{y \to 1} \frac{(4 - 9y^2) - (-5)}{y-1} = \lim\limits_{y \to 1} \frac{-9(y+1)(y-1)}{y-1} = \lim\limits_{y \to 1}[-9(y+1)] = -18$

(c) $D_2 f(x,y) = \lim\limits_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{[x^2 - 9(y+\Delta y)^2 - [x^2 - 9y^2]}{\Delta y}$

$= \lim\limits_{\Delta y \to 0} \frac{x^2 - 9y^2 - 18y\Delta y - 9(\Delta y)^2 - x^2 + 9y^2}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{\Delta y(-18y - 9\Delta y)}{\Delta y} = \lim\limits_{\Delta y \to 0}(-18y - 9\Delta y) = -18y$

Thus, $D_2 f(2,1) = -18(1) = -18$.

In Exercises 13–24, find the partial derivative by holding all but one of the variables constant and applying theorems for ordinary differentiation.

13. $D_1 f(x,y) = \frac{\partial}{\partial x}[4y^3 + \sqrt{x^2+y^2}] = \frac{\partial}{\partial x}[4y^3 + (x^2+y^2)^{1/2}] = 0 + \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$

14. $\frac{\partial}{\partial y} \frac{x+y}{\sqrt{y^2-x^2}} = \frac{1(y^2-x^2) - \frac{1}{2}(x+y)(2y)}{(y^2-x^2)^{3/2}} = \frac{-x^2 - xy}{(y^2-x^2)^{3/2}}$

15. $D_2 f(\theta,\phi) = \frac{\partial}{\partial \phi}[\sin 3\theta \cos 2\phi] = \sin 3\theta[-\sin 2\phi(2)] = -2 \sin 3\theta \sin 2\phi$

16. $f(r,\theta) = r^2 \cos\theta - 2r\tan\theta$; $f_\theta(r,\theta)$

▸ We differentiate with respect to θ with r regarded as a constant. Thus,
$f_\theta(r,\theta) = -r^2 \sin\theta - 2r\sec^2\theta$

17. $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[e^{y/x} \ln\frac{x^2}{y}\right] = \left[e^{y/x}\left(\frac{1}{x}\right)\right]\ln\frac{x^2}{y} + e^{y/x}\left[\frac{y}{x^2}\left(-\frac{x^2}{y^2}\right)\right] = e^{y/x}\left(\frac{1}{x}\ln\frac{x^2}{y} - \frac{1}{y}\right) = \frac{e^{y/x}}{xy}\left(y \ln\frac{x^2}{y} - x\right)$

18. $\frac{\partial}{\partial \theta} e^{-\theta}\cos(\theta + \phi) = -e^{-\theta}\cos(\theta+\phi) - e^{-\theta}\sin(\theta+\phi)$

19. $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = \frac{-z}{(x^2+y^2+z^2)}$

20. $u = \tan^{-1}(xyzw)$; $\dfrac{\partial u}{\partial w}$

▸ We differentiate with respect to w with x, y and z held constant. Thus,

$$\frac{\partial u}{\partial w} = \frac{1}{1+(xyzw)^2}\frac{\partial}{\partial w}(wxyz) = \frac{xyz}{1+x^2y^2z^2w^2}$$

21. $f_3(x,y,z) = \dfrac{\partial}{\partial z}[4xyz + \ln(2xyz)] = 4xy + \dfrac{1}{2xyz}(2xy) = 4xy + \dfrac{1}{z}$

22. $\dfrac{\partial}{\partial z}(e^{xy}\sinh 2z - e^{xy}\cosh 2z) = 2e^{xy}(\cosh 2z - \sinh 2z)$

23. $f_3(x,y,z) = \dfrac{\partial}{\partial z}\left[e^{xyz} + \tan^{-1}\left(\dfrac{3xy}{z^2}\right)\right] = xze^{xyz} + \dfrac{1}{1+\left(\dfrac{3xy}{z^2}\right)^2}\left(\dfrac{3x}{z^2}\right) = xze^{xyz} + \dfrac{3xz^2}{9x^2y^2+z^4}$

24. $f(r,\theta,\phi) = 4r^2\sin\theta + 5e^r\cos\theta\sin\phi - 2\cos\phi$; $f_2(r,\theta,\phi)$

▸ We differentiate with respect to θ with r and ϕ held constant. Thus,
$f_2(r,\theta,\phi) = 4r^2\cos\theta - 5e^r\sin\theta\sin\phi$

25. $f(r,\theta) = r\tan\theta - r^2\sin\theta$

▸ (a) $f_1(r,\theta) = \tan\theta - 2r\sin\theta$; $f_1(\sqrt{2}, \frac{1}{4}\pi) = \tan\frac{1}{4}\pi - 2\sqrt{2}\sin\frac{1}{4}\pi = 1 - 2\sqrt{2}(\frac{1}{2}\sqrt{2}) = -1$

(b) $f_2(r,\theta) = r\sec^2\theta - r^2\cos\theta$; $f_2(3,\pi) = 3\sec^2\pi - 9\cos\pi = 3(-1)^2 - 9(-1) = 12$

26. $f(x,y,z) = e^{xy^2} + \ln(y+z)$ (a) $f_1(x,y,z) = y^2e^{xy^2}$; $f_1(3,0,17) = 0$

(b) $f_2(x,y,z) = 2xye^{xy^2} + \dfrac{1}{y+z}$; $f_2(1,0,2) = \frac{1}{2}$ (c) $f_3(x,y,z) = \dfrac{1}{y+z}$; $f_3(0,0,1) = 1$

In Exercises 27 and 28, find $f_x(x,y)$ and $f_y(x,y)$.

27. $f(x,y) = \displaystyle\int_0^y \ln\sin t\,dt$; so $f_y(x,y) = \ln\sin y$. $f(x,y) = -\displaystyle\int_0^x \ln\sin t\,dt$; so $f_x(x,y) = -\ln\sin x$.

28. $f(x,y) = \displaystyle\int_x^y e^{\cos t}dt$

▸ The first fundamental theorem of the calculus gives the derivative with respect to the upper limit. If we hold x constant and differentiate with respect to y, we get
$f_y(x,y) = e^{\cos y}$
If we hold y constant and differentiate with respect to x, we must first interchange the limits of integration. Thus

$$f_x(x,y) = \frac{\partial}{\partial x}\int_y^x -e^{\cos t}dt = -e^{\cos x}$$

In Exercises 29–38 (a) find $D_{11}f(x,y)$, (b) find $D_{22}f(x,y)$, and (c) show that $D_{12}f(x,y) = D_{21}f(x,y)$.

29. $f(x,y) = \dfrac{x^2}{y} - \dfrac{y}{x^2}$

(a) $D_1f(x,y) = \dfrac{2x}{y} + \dfrac{2y}{x^3}$; $D_{11}f(x,y) = \dfrac{2}{y} - \dfrac{6y}{x^4}$

(b) $D_2f(x,y) = -\dfrac{x^2}{y^2} - \dfrac{1}{x^2}$; $D_{22}f(x,y) = \dfrac{2x^2}{y^3}$

(c) $D_{12}f(x,y) = -\dfrac{2x}{y^2} + \dfrac{2}{x^3}$; $D_{21}f(x,y) = -\dfrac{2x}{y^2} + \dfrac{2}{x^3}$

30. $f(x,y) = 2x^3 - 3x^2y + xy^2$

(a) $D_1f(x,y) = 6x^2 - 6xy + y^2$; $D_{11}f(x,y) = 12x - 6y$

(b) $D_2f(x,y) = -3x^2 + 2xy$; $D_{22}f(x,y) = 2x$

(c) $D_{12}f(x,y) = -6x + 2y$; $D_{21}f(x,y) = -6x + 2y$

31. $f(x,y) = e^{2x}\sin y$

(a) $D_1f(x,y) = 2e^{2x}\sin y$; $D_{11}f(x,y) = 4e^{2x}\sin y$

(b) $D_2f(x,y) = e^{2x}\cos y$; $D_{22}f(x,y) = -e^{2x}\sin y$

(c) $D_{12}f(x,y) = 2e^{2x}\cos y$; $D_{21}f(x,y) = 2e^{2x}\cos y$

32. $f(x,y) = e^{-x/y} + \ln\dfrac{y}{x}$

▸ Simplifying $f(x,y)$, we have, in quadrants 1 and 3,

$f(x,y) = e^{-x/y} + \ln|y| - \ln|x|$
Hence,

$$D_1f(x,y) = -\frac{1}{y}e^{-x/y} - \frac{1}{x} \tag{1}$$

and

$$D_2f(x,y) = \frac{x}{y^2}e^{-x/y} + \frac{1}{y} \tag{2}$$

(a) We partial-differentiate on both sides of (1) with respect to x.

$\mathrm{D}_{11}f(x,y) = \frac{1}{y^2}e^{-x/y} + \frac{1}{x^2}$

(b) We partial-differentiate on both sides of (2) with respect to y.

$\mathrm{D}_{22}f(x,y) = \frac{x}{y^2}\cdot\frac{x}{y^2}e^{-x/y} + e^{-x/y}\cdot\frac{-2x}{y^3} - \frac{1}{y^2} = \left(\frac{x^2}{y^4} - \frac{2x}{y^3}\right)e^{-x/y} - \frac{1}{y^2}$

(c) We partial-differentiate on both sides of (1) with respect to y.

$\mathrm{D}_{12}f(x,y) = \left(-\frac{1}{y}\right)\left(\frac{x}{y^2}\right)e^{-x/y} + e^{-x/y}\left(\frac{1}{y^2}\right) = \left(\frac{1}{y^2} - \frac{x}{y^3}\right)e^{-x/y}$ (3)

We partial-differentiate on both sides of (2) with respect to x.

$\mathrm{D}_{21}f(x,y) = \left(\frac{x}{y^2}\right)\left(-\frac{1}{y}\right)e^{-x/y} + e^{-x/y}\left(\frac{1}{y^2}\right) = \left(\frac{1}{y^2} - \frac{x}{y^3}\right)e^{-x/y}$ (4)

From (3) and (4) we show that $\mathrm{D}_{12}f(x,y) = \mathrm{D}_{21}f(x,y)$.

33. $f(x,y) = (x^2+y^2)\tan^{-1}\frac{y}{x}$

▸ (a) $\mathrm{D}_1f(x,y) = 2x\tan^{-1}\frac{y}{x} + (x^2+y^2)\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) = 2x\tan^{-1}\frac{y}{x} - y$

$\mathrm{D}_{11}f(x,y) = 2\tan^{-1}\frac{y}{x} + 2x\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) = 2\tan^{-1}\frac{y}{x} - \frac{2xy}{x^2+y^2}$

(b) $\mathrm{D}_2f(x,y) = 2y\tan^{-1}\frac{y}{x} + (x^2+y^2)\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = 2y\tan^{-1}\frac{y}{x} + x$

$\mathrm{D}_{22}f(x,y) = 2\tan^{-1}\frac{y}{x} + 2y\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = 2\tan^{-1}\frac{y}{x} + \frac{2xy}{x^2+y^2}$

(c) $\mathrm{D}_{12}f(x,y) = \frac{2x^3}{x^2+y^2}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - \frac{x^2+y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

$\mathrm{D}_{21}f(x,y) = \frac{2x^2y}{x^2+y^2}\left(-\frac{y}{x^2}\right) + 1 = \frac{-2y^2}{x^2+y^2} + \frac{x^2+y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

34. $f(x,y) = \sin^{-1}\frac{3y}{x^2}$. (a) $\mathrm{D}_1f(x,y) = \frac{1}{\sqrt{1-(3y/x^2)^2}}\left(-\frac{6y}{x^3}\right) = \frac{-6y}{x\sqrt{x^4-9y^2}}$;

$\mathrm{D}_{11}f(x,y) = \frac{6y}{x^2\sqrt{x^4-9y^2}} + \frac{12x^2y}{(x^4-9y^2)^{3/2}} = \frac{18y(x^4-3y^2)}{x^2(x^4-9y^2)^{3/2}}$

(b) $\mathrm{D}_2f(x,y) = \frac{1}{\sqrt{1-(3y/x^2)^2}}\cdot\frac{3}{x^2} = \frac{3}{\sqrt{x^4-9y^2}}$; $\mathrm{D}_{22}f(x,y) = \frac{27y}{(x^4-9y^2)^{3/2}}$

(c) $\mathrm{D}_{12}f(x,y) = \frac{-6}{x}\cdot\frac{1(x^4-9y^2)-\frac{1}{2}y(-18y)}{(x^4-9y^2)^{3/2}} = \frac{-6x^3}{(x^4-9y^2)^{3/2}} = \mathrm{D}_{21}f(x,y)$

35. $f(x,y) = 4x\sinh y + 3y\cosh x$

(a) $\mathrm{D}_1f(x,y) = 4\sinh y + 3y\sinh x$; $\mathrm{D}_{11}f(x,y) = 3y\cosh x$

(b) $\mathrm{D}_2f(x,y) = 4x\cosh y + 3\cosh x$; $\mathrm{D}_{22}f(x,y) = 4x\sinh y$

(c) $\mathrm{D}_{12}f(x,y) = 4\cosh y + 3\sinh x$; $\mathrm{D}_{21}f(x,y) = 4\cosh y + 3\sinh x$

36. $f(x,y) = x\cos y - ye^x$

▸ $\mathrm{D}_1f(x,y) = \cos y - ye^x$ (1)
$\mathrm{D}_2f(x,y) = -x\sin y - e^x$ (2)
(a) From (1) we obtain $\mathrm{D}_{11}f(x,y) = -ye^x$

(b) From (2), we obtain $\mathrm{D}_{22}f(x,y) = -x\cos y$

(c) From (1), we obtain $\mathrm{D}_{12}f(x,y) = -\sin y - e^x$

From (2), we obtain $\mathrm{D}_{21}f(x,y) = -\sin y - e^x$
Thus $\mathrm{D}_{12}f(x,y) = \mathrm{D}_{21}f(x,y)$.

37. $f(x,y) = e^x\cos y + \tan^{-1}x(\ln y)$

▸ (a) $D_1f(x,y) = e^x\cos y + \dfrac{1}{1+x^2}(\ln y)$; $D_{11}f(x,y) = e^x\cos y - \dfrac{2x\ln y}{(1+x^2)^2}$

(b) $D_2f(x,y) = -e^x\sin y + \tan^{-1}x\left(\dfrac{1}{y}\right)$; $D_{22}f(x,y) = -e^x\cos y - \dfrac{\tan^{-1}x}{y^2}$

(c) $D_{12}f(x,y) = -e^x\sin y + \dfrac{1}{y(1+x^2)}$; $D_{21}f(x,y) = -e^x\sin y + \dfrac{1}{y(1+x^2)}$

38. $f(x,y) = 3x\cosh y - y\sin^{-1}e^x$

▸ (a) $D_1f(x,y) = 3\cosh y - \dfrac{ye^x}{\sqrt{1-e^{2x}}}$; $D_{11}f(x,y) = -y\dfrac{e^x(1-e^{2x}) - \frac{1}{2}e^x(-2e^{2x})}{(1-e^{2x})^{3/2}} = \dfrac{-ye^x}{(1-e^{2x})^{3/2}}$

(b) $D_2f(x,y) = 3x\sinh y - \sin^{-1}e^x$; $D_{22}f(x,y) = 3x\cosh y$ (c) $D_{12}f(x,y) = 3\sinh y - \dfrac{e^x}{\sqrt{1-e^{2x}}} = D_{21}f(x,y)$

In Exercises 39–46, find the indicated partial derivatives.

39. $f(x,y) = 2x^3y + 5x^2y^2 - 3xy^2$

▸ (a) $f_1(x,y) = 6x^2y + 10xy^2 - 3y^2$; $f_{12}(x,y) = 6x^2 + 20xy - 6y$; $f_{121}(x,y) = 12x + 20y$

(b) $f_2(x,y) = 2x^3 + 10x^2y - 6xy$; $f_{21}(x,y) = 6x^2 + 20xy - 6y$; $f_{211}(x,y) = 12x + 20y$

40. $G(x,y) = 3x^3y^2 + 5x^2y^3 + 2x$; (a) $G_{yyx}(x,y)$; (b) $G_{yxy}(x,y)$

▸ (a) $G_y(x,y) = (G(x,y))_y = 6x^3y + 15x^2y^2$

$G_{yy}(x,y) = (G_y(x,y))_y = 6x^3 + 30x^2y$

$G_{yyx}(x,y) = (G_{yy}(x,y))_x = 18x^2 + 60xy$

(b) $G_{yx}(x,y) = (G_y(x,y))_x = 18x^2y + 30xy^2$

$G_{yxy}(x,y) = (G_{yx}(x,y))_y = 18x^2 + 60xy$

41. $f(x,y,z) = ye^x + xe^y + e^z$ (a) $f_x(x,y,z) = ye^x$; $f_{xz}(x,y,z) = 0$ (b) $f_y(x,y,z) = e^x + xe^y$; $f_{yx}(x,y,z) = e^y$

42. $g(x,y,z) = \sin xyz$ (a) $g_2(x,y,z) = xz\cos xyz$; $g_{23}(x,y,z) = x\cos xyx - x^2yz\sin xyz$

(b) $g_1(x,y,z) = yz\cos xyz$; $g_{12}(x,y,z) = z\cos xyz - xyz^2\sin xyz$

43. $f(w,z) = w^2\cos e^z$ (a) $f_1(w,z) = 2w\cos e^z$; $f_{12}(w,z) = -2we^z\sin e^z$; $f_{121}(w,z) = -2e^z\sin e^z$

(b) $f_2(w,z) = -w^2e^z\sin e^z$; $f_{21}(w,z) = -2we^z\sin e^z$; $f_{212}(w,z) = -2we^z\sin e^z - 2we^{2z}\cos e^z$

44. $f(u,v) = \ln\cos(u-v)$; (a) $f_{uuv}(u,v)$; (b) $f_{vuv}(u,v)$

▸ (a) $f_u(u,v) = -\dfrac{\sin(u-v)}{\cos(u-v)} = -\tan(u-v)$

$f_{uu}(u,v) = (f_u(u,v))_u = -\sec^2(u-v)$

$f_{uuv}(u,v) = (f_{uu}(u,v))_v = 2\sec^2(u-v)\tan(u-v)$

(b) $f_v(u,v) = \dfrac{\sin(u-v)}{\cos(u-v)} = \tan(u-v)$

$f_{vu}(u,v) = (f_v(u,v))_u = \sec^2(u-v)$

$f_{vuv}(u,v) = (f_{vu}(u,v))_v = -2\sec^2(u-v)\tan(u-v)$

45. $g(r,s,t) = \ln(r^2 + 4s^2 - 5t^2)$

▸ (a) $g_1(r,s,t) = \dfrac{2r}{r^2+4s^2-5t^2}$; $g_{13}(r,s,t) = \dfrac{20rt}{(r^2+4s^2-5t^2)^2}$; $g_{132}(r,s,t) = \dfrac{-2(20rt)(8s)}{(r^2+4s^2-5t^2)^3} = -\dfrac{320rst}{(r^2+4s^2-5t^2)^3}$

(b) $g_{12}(r,s,t) = \dfrac{-16rs}{(r^2+4s^2-5t^2)^2}$. Using the product rule,

$$g_{122}(r,s,t) = \frac{-16r}{(r^2+4s^2-5t^2)^2} + \frac{(-16rs)(-2)(8s)}{(r^2+4s^2-5t^2)^3} = \frac{-16r(r^2+4s^2-5t^2)+256rs^2}{(r^2+4s^2-5t^2)^3} = \frac{192rs^2-16r^3+80rt^2}{(r^2+4s^2-5t^2)^3}$$

46. $f(x,y,z) = \tan^{-1} 3xyz$. $f_1 = \frac{3yz}{1+9x^2y^2z^2}$ (a) $f_{11} = \frac{-54xy^3z^3}{(1+9x^2y^2z^2)^2}$; $f_{113}(x,y,z) = \frac{162y^3z^2(1+9x^2y^2z^2)+12z^2}{(1+9x^2y^2z^2)^3}$

(b) $f_{12}(x,y,z) = 3z\frac{1(1+9x^2y^2z^2)-y(18x^2yz^2)}{(1+9x^2y^2z^2)^2} = \frac{3z(1-9x^2y^2z^2)}{(1+9x^2y^2z^2)^2}$; $f_{123}(x,y,z) = 3\frac{1-54x^2y^2z^2+81x^4y^4z^4}{(1+9x^2y^2z^2)^3}$

47. $u = \sin\frac{r}{t} + \ln\frac{t}{r}$. $t\frac{\partial u}{\partial t} + r\frac{\partial u}{\partial r} = t\left[\cos\frac{r}{t}\left(-\frac{r}{t^2}\right)+\frac{1}{t}\right] + r\left[\cos\frac{r}{t}\left(\frac{1}{t}\right)-\frac{1}{r}\right] = -\frac{r}{t}\cos\frac{r}{t} + 1 + \frac{r}{t}\cos\frac{r}{t} - 1 = 0.$

48. Given $w = x^2y + y^2z + z^2x$. Verify $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = (x+y+z)^2$.

▷ $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = (2xy + z^2) + (x^2 + 2yz) + (y^2 + 2zx) = (x+y+z)^2$

In Exercises 49–53, show that u satisfies Laplace's equation in R^2 or R^3.

49. $u(x,y) = \ln(x^2+y^2)$

▷ $\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$; $\frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$; $\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2} = -\frac{\partial^2 u}{\partial x^2}$. Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

50. $u(x,y) = \tan^{-1}\frac{2xy}{x^2-y^2} = 2\tan^{-1}\frac{y}{x}$. $\frac{\partial u}{\partial x} = \frac{2\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}} = \frac{-2y}{x^2+y^2}$; $\frac{\partial^2 u}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2}$. $\frac{\partial u}{\partial y} = \frac{\frac{2}{x}}{1+\frac{y^2}{x^2}} = \frac{2x}{x^2+y^2}$;

$\frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2} = -\frac{\partial^2 u}{\partial^2 y}$. Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

51. $u(x,y) = \tan^{-1}\frac{y}{x} + \frac{x}{x^2+y^2}$

▷ $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) + \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2-x^2y-y^3}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x^2} = \frac{-2x-2xy}{(x^2+y^2)^2} - \frac{2(y^2-x^2-x^2y-y^3)(2x)}{(x^2+y^2)^3} = \frac{2x^3-6xy^2+2x^3y+2xy^3}{(x^2+y^2)^3}$

$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) - \frac{2xy}{(x^2+y^2)^2} = \frac{x^3+xy^2-2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{2xy-2x}{(x^2+y^2)^2} - \frac{2(x^3+xy^2-2xy)(2y)}{(x^2+y^2)^3} = \frac{-2x^3+6xy^2-2x^3y-2xy^3}{(x^2+y^2)^3} = -\frac{\partial^2 u}{\partial x^2}$. Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

52. $u(x,y) = e^x\sin y + e^y\cos x$

▷ $u_x = e^x\sin y - e^y\sin x$

$u_{xx} = e^x\sin y - e^y\cos x \qquad (1)$

and

$u_y = e^x\cos y + e^y\cos x$

$u_{yy} = -e^x\sin y + e^y\cos x \qquad (2)$

Adding the members of (1) and (2), we obtain

$u_{xx} + u_{yy} = 0$

53. $u(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}$. $\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\cdot\frac{(-x)(2x)}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\frac{\partial^2 u}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$. Therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

54. $36x^2 - 9y^2 + 4z^2 + 36 = 0$. We hold x constant and differentiate implicitly with respect to y.

$-18y + 8z\frac{\partial z}{\partial y} = 0$; $\frac{\partial z}{\partial y} = \frac{9y}{4z}$. At $(1, \sqrt{12}, -3)$, $\frac{\partial z}{\partial y} = \frac{9(\sqrt{12})}{4(-3)} = -\frac{3}{2}\sqrt{3}$

55. $z = x^2 + y^2$. Therefore $\frac{\partial z}{\partial x} = 2x$. At $(2,1,5)$, $\frac{\partial z}{\partial x} = 4$.

56. Find equations of the tangent line to the curve of intersection of the surface $x^2 + y^2 + z^2 = 9$ with the plane $y = 2$ at the point $(1,2,2)$.

▶ We hold y constant and differentiate implicitly with respect to x on both sides of the equation. Thus

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

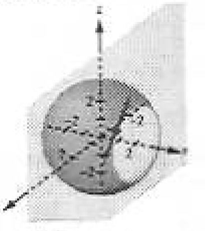
$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

At the point $(1,2,2)$, we have

$$\frac{\partial z}{\partial x} = -\frac{1}{2}$$

Thus, equations of the tangent line are

$z = 2 - \frac{1}{2}(x - 1)$ and $y = 2$



57. $T = 54 - \frac{2}{3}x^2 - 4y^2$. $\frac{\partial T}{\partial x} = -\frac{4}{3}x$ and $\frac{\partial T}{\partial y} = -8y$. At the point $(3,1)$, $\frac{\partial T}{\partial x} = -4$ and $\frac{\partial T}{\partial y} = -8$. Hence, the rate of change of the temperature with respect to the distance moved in the direction of the positive x axis is -4 deg/cm and in the direction of the positive y axis is -8 deg/cm.

58. Gas law: $PV = kT$. $\frac{\partial}{\partial T}V(P,T) \cdot \frac{\partial}{\partial P}T(P,V) \cdot \frac{\partial}{\partial V}P(T,V) = \frac{\partial}{\partial T}\left(\frac{kT}{P}\right) \cdot \frac{\partial}{\partial P}\left(\frac{PV}{k}\right) \cdot \frac{\partial}{\partial V}\left(\frac{kT}{V}\right) = \frac{k}{P} \cdot \frac{V}{k} \cdot \frac{-kT}{V^2} = \frac{-kT}{PV} = -1$

59. $V = 100\left[\frac{1 - (1+i)^{-t}}{i}\right]$. (a) $\frac{\partial V}{\partial i} = 100\left[\frac{it(1+i)^{-t-1} - 1 + (1+i)^{-t}}{i^2}\right] = \frac{100}{i^2}\left[\frac{ti}{(1+i)^{t+1}} + \frac{1}{(1+i)^t} - 1\right]$

When $t = 8$, $\frac{\partial V}{\partial i} = \frac{100}{i^2}\left[\frac{8i}{(1+i)^9} + \frac{1}{(1+i)^8} - 1\right] = \frac{100}{i^2}\left[\frac{9i+1}{(1+i)^9} - 1\right]$ (b) If $i = 0.06$, $\Delta i = 0.01$ and $t = 8$, then $\Delta V \approx \frac{\partial V}{\partial i}\Delta i = \frac{1}{0.0036}\left[\frac{1.54}{1.06^9} - 1\right] \approx -24.4$. Hence, the present value decreases approximately \$24.40. (c) $\frac{\partial V}{\partial t} = \frac{100}{i}(1+i)^{-t}\ln(1+i)$. When $i = 0.06$, $\frac{\partial V}{\partial t} = \frac{100}{0.06 \cdot 1.06^t}\ln(1.06) = \frac{5000\ln(1.06)}{3 \cdot 1.06^t}$. (d) If $t = 8$, $\Delta t = -1$ and $i = 0.06$, $\Delta V \approx \frac{\partial V}{\partial t}\Delta t = \frac{5000\ln(1.06)}{3 \cdot 1.06^8}(-1) \approx -60.9$. Thus, the present value is decreased by about \$60.90.

60. Suppose that $10{,}000x$ dollars is the inventory carried in a store employing y clerks, P dollars is the weekly profit of the store, and

$P = 3000 + 240y + 20y(x - 2y) - 10(x - 12)^2$

where $15 \le x \le 25$ and $5 \le y \le 12$. At present the inventory is \$180,000 and there are 8 clerks. (a) Find the instantaneous rate of change of P per unit change in x if y remains constant. (b) Use the result of part (a) to find the approximate change in weekly profit in the inventory changes from \$180,000 to \$200,000 and the number of clerks remains fixed at 8. (c) Find the instantaneous rate of change of P per unit change in y if x remains fixed at 18. (d) Use the result of part (c) to find the approximate change in the weekly profit if the number of clerks is increased from 8 to 10 and the inventory remains fixed at \$180,000.

▶ (a) We hold y constant and differentiate with respect to x.

$P_x(18,8) = 20y - 20(x - 12)\Big]_{\substack{x=18\\y=8}} = 20(8) - 20(18 - 12) = 40$

The instantaneous rate of change of P is \$40 for each \$10,000 increase in inventory. (b) The approximate change in profit when the inventory is increased by $200{,}000 - 180{,}000 = 20{,}000$ is $2(\$40) = \$80$.

(c) We hold x constant and differentiate with respect to y.

$P_y(18,8) = 240 + 20x - 80y\Big]_{\substack{x=18\\y=8}} = 240 + 20(18) - 80(8) = -40$

The instantaneous rate of change is a decrease in profit of \$40 for each additional clerk. (d) Two additional clerks will cause a decrease of about $2(\$40) = \$80$.

61. $f(x,y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. (a) $f_1(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x - 0} = \lim_{x \to 0} 1 = 1$

(b) $f_2(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y \to 0} \frac{y - 0}{y - 0} = \lim_{y \to 0} 1 = 1$

In Exercises 62 and 63, $f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

62. (a) If $(x,y) \neq (0,0)$, $f_1(x,y) = \dfrac{(2x-y)(x+y) - (x^2 - xy)1}{(x+y)^2} = \dfrac{x^2 + 2xy - y^2}{(x+y)^2}$. If $y \neq 0$, $f_1(0,y) = \dfrac{-y^2}{y^2} = -1$

(b) $f_1(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x \to 0} \frac{x}{x} = 1$

63. (a) If $(x,y) \neq (0,0)$, $f_2(x,y) = \dfrac{-x(x+y) - 1(x^2 - xy)}{(x+y)^2} = \dfrac{-2x^2}{(x+y)^2}$. If $x \neq 0$, $f_2(x,0) = \dfrac{-2x^2}{x^2} = -2$

(b) $f_2(0,0) = \lim\limits_{y \to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y \to 0} \dfrac{0 - 0}{y} = \lim\limits_{y \to 0} 0 = 0$

64. For the function of Example 11, show that f_{12} is discontinuous at $(0,0)$ and hence the hypothesis of Theorem 12.3.3 is not satisfied if $(x_0, y_0) = (0,0)$.

▸ The function of Example 4 is defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0) \end{cases}$$

We show that $\lim\limits_{(x,y) \to (0,0)} f_{12}(x,y)$ does not exist. If $(x,y) \neq (0,0)$, then

$$f(x,y) = \frac{x^3 y - xy^3}{x^2 + y^2}$$

$$f_1(x,y) = \frac{(x^2 + y^2)(3x^2 y - y^3) - (x^3 y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

$$f_{12}(x,y) = \frac{(x^2 + y^2)^2(x^4 + 12x^2 y^2 - 5y^4) - (x^4 y + 4x^2 y^3 - y^5)(4y)(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}$$

Along the x axis we have

$$\lim_{\substack{(x,y) \to (0,0) \\ (y=0)}} f_{12}(x,y) = \lim_{x \to 0} \frac{x^6}{x^6} = 1$$

while along the y axis we have

$$\lim_{\substack{(x,y) \to (0,0) \\ (x=0)}} f_{12}(x,y) = \lim_{y \to 0} \frac{-y^6}{y^6} = -1$$

Because these two limits are not equal, then $\lim\limits_{(x,y) \to (0,0)} f_{12}(x,y)$ does not exist. Hence f_{12} is discontinuous at $(0,0)$.

In Exercises 65–67, find $f_{12}(0,0)$ and $f_{21}(0,0)$ if they exist.

65. $f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. $f_1(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - (0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0 - 0}{\Delta x} = 0$

If $y \neq 0$, $f_1(0,y) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x} \cdot \dfrac{2(\Delta x) y}{(\Delta x)^2 + y^2} = \dfrac{2}{y}$.

$f_{12}(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f_1(0, \Delta y) - f_1(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{1}{\Delta y}\left(\dfrac{2}{\Delta y} - 0\right)$ which doesn't exist. Similarly, $f_{21}(0,0)$ doesn't exist.

66. $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^4+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$. $f_1(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0}\dfrac{0}{x}=0$

If $y\neq0$, $f_1(0,y)=\lim\limits_{x\to0}\dfrac{f(x,y)-f(0,y)}{x-0}=\lim\limits_{x\to0}\dfrac{1}{x}\left(\dfrac{x^2y^2}{x^4+y^4}-0\right)=0$

$f_{12}(0,0)=\lim\limits_{y\to0}\dfrac{f_1(0,y)-f_1(0,0)}{y-0}=\lim\limits_{y\to0}\dfrac{0-0}{y}=0$. Similarly, $f_{21}(0,0)=0$.

67. $f(x,y)=\begin{cases}x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y} & \text{if } x\neq0 \text{ and } y\neq0\\ 0 & \text{if either } x=0 \text{ or } y=0\end{cases}$. $f_1(0,0)=\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{0-0}{\Delta x}=0$

If $y\neq0$, $f_1(0,y)=\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,y)-f(0,y)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{(\Delta x)^2\tan^{-1}\frac{y}{\Delta x}-y^2\tan^{-1}\frac{\Delta x}{y}-0}{\Delta x}$

$=\lim\limits_{\Delta x\to0}(\Delta x)\tan^{-1}\dfrac{y}{\Delta x}-y^2\lim\limits_{\Delta x\to0}\dfrac{\tan^{-1}\frac{\Delta x}{y}}{\Delta x}=0-y^2\dfrac{\partial}{\partial x}\tan^{-1}\dfrac{x}{y}\Big|_{x=0}=-y^2\dfrac{1}{1+(x/y)^2}\cdot\dfrac{1}{y}\Big|_{x=0}=-y$

Hence $f_{12}(0,0)=\lim\limits_{\Delta y\to0}\dfrac{f_1(0,\Delta y)-f_1(0,0)}{\Delta y}=\lim\limits_{\Delta y\to0}\dfrac{-\Delta y-0}{\Delta y}=-1$.

In a similar way, we get $f_2(0,0)=0$ and if $x\neq0$, $f_2(x,0)=x$. Then

$f_{21}(0,0)=\lim\limits_{\Delta x\to0}\dfrac{f_2(\Delta x,0)-f_2(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{\Delta x-0}{\Delta x}=1$

68. Prove that if f is a function of two variables and all the partial derivatives of the fourth order are continuous on some open disk, then $D_{1122}f=D_{2121}f$.

▸ Theorem 12.3.3 says that we may interchange adjacent subscripts. Applying the theorem three times we get $D_{1122}f=D_{1212}f=D_{2112}f=D_{2121}f$

69. $S=2W^{0.4}H^{0.7}$. $\dfrac{\partial S}{\partial W}=0.8W^{-0.6}H^{0.7}$. $\dfrac{\partial S}{\partial H}=1.4W^{0.4}H^{-0.3}$. When $W=70$ and $H=1.8$,

$\dfrac{\partial S}{\partial W}=(0.8)(70^{-0.6})(1.8^{0.7})\approx0.0943$. The surface area changes 0.0943 m^2/kg of weight.

$\dfrac{\partial S}{\partial H}=(1.4)(70^{0.4})(1.8^{-0.3})\approx6.42$. The surface area changes 6.42 m^2/m of height.

12.4 DIFFERENTIABILITY AND THE TOTAL DIFFERENTIAL

12.4.1 Definition If f is a function of two variables x and y, then the *increment of f* at the point (x_0,y_0) denoted by $\Delta f(x_0,y_0)$, is given by
$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$

12.4.2 Definition If f is a function of two variables x and y and the increment of f at (x_0,y_0) can be written as
$\Delta f(x_0,y_0)=D_1f(x_0,y_0)\Delta x+D_2f(x_0,y_0)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$
where ϵ_1 and ϵ_2 are functions of Δx and Δy such that $\epsilon_1\to0$ and $\epsilon_2\to0$ as $(\Delta x,\Delta y)\to(0,0)$, then f is said to be *differentiable* at (x_0,y_0).
In Section 12.7 we show that a function is differentiable if and only if its graph has a tangent plane and whose equation is the total differential given below.

12.4.3 Definition If f is differentiable at (x,y), the *total differential* is the function
$df(x,y,\Delta x,\Delta y)=D_1f(x,y)\Delta x+D_2f(x,y)\Delta y$
or, if $z=f(x,y)$,
$dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$

12.4.3 Theorem If a function f of two variables is differentiable at a point, it is continuous at that point.

12.4.4 Theorem Let f be a function of two variables x and y. Suppose that D_1f and D_2f exist on an open disk $B(P_0;r)$ and D_1f and D_2f are continuous at P_0, then f is differentiable at P_0.
Note that continuity of D_1f and D_2f ⇒ differentiability of $f\begin{array}{l}\Rightarrow \text{existence of } D_1f \text{ and } D_2f\\ \Rightarrow \text{continuity of } f\end{array}$

but no other implications exist among these four properties.

The following definitions extend the preceding concepts to functions of n variables.

12.4.6 Definition If f is a function of n variables $x_1, x_2, \ldots, x_n$ and $\bar{P}$ is the point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, then the *increment of f* at $\bar{P}$ is given by

$$\Delta f(\bar{P}) = f(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2, \ldots, \bar{x}_n + \Delta x_n) - f(\bar{P})$$

12.4.7 Definition If f is a function of the n variables $x_1, x_2, \ldots, x_n$ and the increment of f at the point $\bar{P}$ can be written as

$$\Delta f(\bar{P}) = D_1 f(\bar{P})\Delta x_1 + D_2 f(\bar{P})\Delta x_2 + \cdots + D_n f(\bar{P})\Delta x_n \\ + \epsilon_1 \Delta x_1 + \epsilon_2 \Delta x_2 + \cdots + \epsilon_n \Delta x_n$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, …, $\epsilon_n \to 0$ as $(\Delta x_1, \Delta x_2, \ldots, \Delta x_n) \to (0, 0, \ldots, 0)$ then f is said to be *differentiable* at $\bar{P}$.

12.4.8 Definition If f is a function of the n variables $x_1, x_2, \ldots, x_n$ and f is differentiable at P, then the *total differential* of f is the function df having function values given by

$$df(P, \Delta x_1, \Delta x_2, \ldots, \Delta x_n) = D_1 f(P)\Delta x_1 + D_2 f(P)\Delta x_2 + \cdots + D_n f(P)\Delta x_n$$

or, if $w = f(x_1, x_2, \ldots, x_n)$

$$dw = \frac{\partial w}{\partial x_1}dx_1 + \frac{\partial w}{\partial x_2}dx_2 + \cdots + \frac{\partial w}{\partial x_n}dx_n$$

Exercises 12.4

---

1. $f(x, y) = 3x^2 + 2xy - y^2$

▹ (a) $\Delta f(1, 4) = f(1 + \Delta x, 4 + \Delta y) - f(1, 4) = 3(1 + \Delta x)^2 + 2(1 + \Delta x)(4 + \Delta y) - (4 + \Delta y)^2 - (3 + 8 - 16)$
$= 3 + 6\Delta x + 3(\Delta x)^2 + 8 + 8\Delta x + 2\Delta y + 2\Delta x\,\Delta y - 16 - 8\Delta y - (\Delta y)^2 + 5$
$= 3(\Delta x)^2 + 2\Delta x\,\Delta y - (\Delta y)^2 + 14\Delta x - 6\Delta y$
(b) When $\Delta x = 0.03$ and $\Delta y = -0.02$, we have
$\Delta f(1, 4) = 3(0.03)^2 + 2(0.03)(-0.02) - (0.02)^2 + 14(0.03) - 6(-0.02)$
$= 0.0027 - 0.0012 - 0.0004 + 0.42 + 0.12 = 0.5411$
(c) $df(1, 4, \Delta x, \Delta y) = D_1 f(1, 4)\Delta x + D_2 f(1, 4)\Delta y$. When $x = 1$ and $y = 4$,
$D_1 f(x, y) = 6x + 2y = 14$ and $D_2 f(x, y) = 2x - 2y = -6$. Thus $df(1, 4, \Delta x, \Delta y) = 14\Delta x - 6\Delta y$.
(d) $df(1, 4, 0.03, -0.02) = 14(0.03) - 6(-0.02) = 0.42 + 0.12 = 0.54$

2. $f(x, y) = 2x^2 + 5xy + 4y^2$

▹ (a) $\Delta f(2, -1) = f(2 + \Delta x, -1 + \Delta y) - f(2, -1) = 2(2 + \Delta x)^2 + 5(2 + \Delta x)(-1 + \Delta y) + 4(-1 + \Delta y)^2 - 2$
$= 2(\Delta x)^2 + 5\Delta x\,\Delta y + 4(\Delta y)^2 + 3\Delta x + 2\Delta y$
(b) When $\Delta x = -0.01$ and $\Delta y = 0.02$, we have
$\Delta f(2, -1) = 2(-0.01)^2 + 5(0.01)(0.02) + 4(0.02)^2 + 3(0.01) + 2(0.02) = 0.0108$
(c)$df(2, -1, \Delta x, \Delta y) = D_1 f(2, -1)\Delta x + D_2 f(2, -1)\Delta y$. When $x = 2$ and $y = -1$,
$D_1 f(x, y) = 4x + 5y = 3$ and $D_2 f(x, y) = 5x + 8y = 2$. Thus $df(2, -1, \Delta x, \Delta y) = 3\Delta x + 2\Delta y$
(d) $df(2, -1, -0.01, 0.02) = 3(-0.01) + 2(0.02) = 0.01$

3. $g(x, y) = xye^{xy}$ (a) $\Delta g(2, -4) = g(2 + \Delta x, -4 + \Delta y) - g(2, -4) = (2 + \Delta x)(-4 + \Delta y)e^{(2+\Delta x)(-4+\Delta y)} + 8e^{-8}$
(b) When $\Delta x = -0.1$ and $\Delta y = 0.2$, we have
$\Delta g(2, -4) = (2 - 0.1)(-4 + 0.2)e^{(2-0.1)(-4+0.2)} + 8e^{-8} = (1.9)(-3.8)e^{(1.9)(-3.8)} + 8e^{-8} = -0.0026$
(c) $dg(2, -4, \Delta x, \Delta y) = D_1 g(2, -4)\Delta x + D_2 g(2, -4)\Delta y$.
Because $D_1 g(x, y) = ye^{xy} + xy^2e^{xy}$ and $D_2 g(x, y) = xe^{xy} + x^2ye^{xy}$, we have
$dg(2, -4, \Delta x, \Delta y) = (-4e^{-8} + 32e^{-8})\Delta x + (2e^{-8} - 16e^{-8})\Delta y = 28e^{-8}\Delta x - 14e^{-8}\Delta y$
(d) $dg(2, 4, -0.1, 0.2) = 28e^{-8}(-0.1) - 14e^{-8}(0.2) = -5.6e^{-8} = -0.0019$

4. If $h(x, y) = (x + y)/(x - y)$, find: (a) $\Delta h(3, 0)$, the increment of h at $(3, 0)$; (b) $\Delta h(3, 0)$ when $\Delta x = 0.04$ and $\Delta y = 0.03$; (c) $dh(3, 0, \Delta x, \Delta y)$, the total differential of h at $(3, 0)$; (d) $dh(3, 0, 0.04, 0.03)$.

▹ (a) We apply Definition 12.4.1 to the given function h with $(x_0, y_0) = (3, 0)$. Thus,

$$\Delta h(3, 0) = h(3 + \Delta x, \Delta y) - h(3, 0) = \frac{3 + \Delta x + \Delta y}{3 + \Delta x - \Delta y} - 1 = \frac{2\Delta y}{3 + \Delta x - \Delta y}$$

(b) We substitute $\Delta x = 0.04$ and $\Delta y = 0.03$ into the result obtained in part (a) to get

$$\frac{2(0.03)}{3 + 0 + 0.04 - 0.03} = \frac{0.06}{3.01} = 0.0199$$

(c) We apply Definition 12.4.5 to the given function h.

$$dh(x,y,\Delta x,\Delta y)=\frac{\partial h}{\partial x}\Delta x+\frac{\partial h}{\partial y}\Delta y=\frac{(x-y)-(x+y)}{(x-y)^2}\Delta x+\frac{(x-y)-(x+y)(-1)}{(x-y)^2}=\frac{-2y\Delta x+2x\Delta y}{(x-y)^2}$$

Substituting $x=3$ and $y=0$, we obtain

$$dh(3,0,\Delta x,\Delta y)=\frac{0\Delta x+2(3)\Delta y}{(3-0)^2}=\frac{2\Delta y}{3}$$

(d) We substitute $\Delta x=0.04$ and $\Delta y=0.03$ into the result obtained in part (c).

$$dh(3,0,0.04,0.03)=\frac{2(0.03)}{3}=0.02$$

Observe that the result of part (d) is approximately equal to the result of part (b).

5. $f(x,y,z)=xy+\ln(yz)$

▷ (a) $\Delta f(4,1,5)=f(4+\Delta x,1+\Delta y,5+\Delta z)-f(4,1,5)=(4+\Delta x)(1+\Delta y)+\ln[(1+\Delta y)(5+\Delta z)]-(4+\ln 5)$
$=\Delta x+4\Delta y+\Delta x\,\Delta y+\ln(1+\Delta y)+\ln(5+\Delta z)-\ln 5$

(b) When $\Delta x=0.02$, $\Delta y=0.04$, and $\Delta z=-0.03$, we have
$\Delta f(4,1,5)=0.02+0.16+0.0008+\ln(1.04)+\ln(4.97)-\ln 5\approx 0.214003$

(c) $df(4,1,5,\Delta x,\Delta y,\Delta z)=D_1f(4,1,5)\Delta x+D_2f(4,1,5)\Delta y+D_3f(4,1,5)\Delta z$. When $x=4$, $y=1$, $z=5$, $D_1f(x,y,z)=y=1$, $D_2f(x,y,z)=x+\frac{1}{y}=5$, and $D_3f(x,y,z)=\frac{1}{z}=\frac{1}{5}$. Then
$df(4,1,5,\Delta x,\Delta y,\Delta z)=\Delta x+5\Delta y+\frac{1}{5}\Delta z$

(d) $df(4,1,5,0.02,0.04,-0.03)=0.02+5(0.04)+\frac{1}{5}(-0.03)=0.02+0.20-0.006=0.214$

6. $G(x,y,z)=x^2y+2xyz-z^3$. (a) $\Delta G(-3,0,2)=G(-3+\Delta x,\Delta y,2+\Delta z)-G(-3,0,2)$
$=(-3+\Delta x)^2\Delta y+2(-3+\Delta x)\Delta y(2+\Delta z)-(2+\Delta z)^3-8$
$=-3\Delta y-12\Delta z-2\Delta x\,\Delta z-6(\Delta z)^2+2\Delta x\,\Delta y\,\Delta z-(\Delta z)^3$

(b) When $\Delta x=0.01$, $\Delta y=0.03$, $\Delta z=-0.01$, we have
$\Delta G(-3,0,2)=-3(0.01)^2-12(-0.01)-2(0.01)(-0.01)-6(-0.01)^2+2(0.01)(0.03)(-0.01)-(-0.01)^3=0.031$

(c) $dG(-3,0,2,\Delta x,\Delta y,\Delta z)=D_1G(-3,0,2)\Delta x+D_2G(-3,0,2)\Delta y+D_3G(-3,0,2)\Delta z$. When $x=-3$, $y=0$, $z=2$, $D_1G(x,y,z)=2xy+2yz=0$, $D_2G(x,y,z)=x^2+2xz=-3$, $D_3G(x,y,z)=2xy-3z^2=-12$. Then
$dG(-3,0,2,\Delta x,\Delta y,\Delta z)=-3\Delta y-12\Delta z$

In Exercises 7–14, find the total differential dw.

7. $w=4x^3-xy^2+3y-7$. $dw=\frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy=(12x^2-y^2)dx+(3-2xy)dy$

8. $w=y\tan x^2-2xy$

▷ From Definition 12.4.5 we have

$$dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy=(2xy\sec^2x^2-2y)dx+(\tan x^2-2x)dy$$

9. $w=x\cos y-y\sin x$. $dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy=(\cos y-y\cos x)dx+(-x\sin y-\sin x)dy$

10. $w=xe^{2y}+e^{-y}$. $dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy=e^{2y}dx+(2xe^{2y}-e^{-y})dy$

11. $w=\ln(x^2+y^2+z^2)$. $dw=\frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy+\frac{\partial w}{\partial z}\,dz$

$$=\frac{2x}{x^2+y^2+z^2}\,dx+\frac{2y}{x^2+y^2+z^2}\,dy+\frac{2z}{x^2+y^2+z^2}\,dz=\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$$

12. $w=\dfrac{xyz}{x+y+z}$

▷ By Definition 12.4.8,

$$dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz=\frac{(x+y+z)(yz)-xyz}{(x+y+z)^2}dx+\frac{(x+y+z)(xz)-xyz}{(x+y+z)^2}dy+\frac{(x+y+z)(xy)-xyz}{(x+y+z)^2}dz$$

$$=\frac{yz(y+z)dx+xz(x+z)dy+xy(x+y)dz}{(x+y+z)^2}$$

13. $w=x\tan^{-1}z-\frac{y^2}{z}$. $dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz=\tan^{-1}z\,dx-\frac{2y}{z}\,dy+\left(\frac{x}{1+z^2}+\frac{y^2}{z^2}\right)dz$

14. $w=e^{yz}-\cos xz$. $dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz=z\sin xz\,dx+ze^{yz}dy+(ye^{yz}+x\cos xz)dz$

In Exercises 15–18, prove that f is differentiable at all points in its domain by doing each of the following: (a) Find $\Delta f(x_0, y_0)$; (b) find an ϵ_1 and an ϵ_2 so that the equation of Definition 12.4.2 holds; (c) show that the ϵ_1 and the ϵ_2 found in part (b) both approach zero as $(\Delta x, \Delta y) \to (0,0)$.

15. $f(x,y) = x^2y - 2xy$

▸ (a) $\Delta f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2(y+\Delta y) - 2(x+\Delta x)(y+\Delta y)$
$= 2(xy-y)\Delta x + (x^2-2x)\Delta y + (y\Delta x + \Delta x\Delta y)\Delta x + 2(x\Delta x - \Delta x)\Delta y$

(b) $D_1 f(x,y) = 2xy - 2y$ and $D_2 f(x,y) = x^2 - 2x$. Hence $\epsilon_1 = y\Delta x + \Delta x\Delta y$ and $\epsilon_2 = 2x\Delta x - 2\Delta x$

(c) $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1 = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} (y\Delta x + \Delta x\Delta y) = 0$; $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2 = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} (2x\Delta x - 2\Delta x) = 0$

16. $f(x,y) = 2x^2 + 3y^2$

▸ (a)

$$\begin{aligned}\Delta f(x_0,y_0) &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\\ &= [2(x_0+\Delta x)^2 + 3(y_0+\Delta y)^2] - [2{x_0}^2 + 3{y_0}^2]\\ &= 2{x_0}^2 + 4x_0\Delta x + 2(\Delta x)^2 + 3{y_0}^2 + 6y_0\Delta y + 3(\Delta y)^2 - 2{x_0}^2 - 3{y_0}^2\\ &= 4x_0\Delta x + 6y_0\Delta y + 2(\Delta x)^2 + 3(\Delta y)^2 \qquad (1)\end{aligned}$$

(b) The equation of Definition 12.4.2 states that

$$\Delta f(x_0,y_0) = D_1 f(x_0,y_0)\Delta x + D_2 f(x_0,y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \qquad (2)$$

Partial-differentiating the given function, we have

$$D_1 f(x_0,y_0) = 4x_0 \qquad (3)$$

$$D_2 f(x_0,y_0) = 6y_0 \qquad (4)$$

Substituting from Eqs. (1), (3), and (4) into (2), we get

$$4x_0\Delta x + 6y_0\Delta y + 2(\Delta x)^2 + 3(\Delta y)^2 = 4x_0\Delta x + 6y_0\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

$$2(\Delta x)^2 + 3(\Delta y)^2 = \epsilon_1\Delta x + \epsilon_2\Delta y \qquad (5)$$

Equation (5) will be an identity if

$\epsilon_1 = 2\Delta x$ and $\epsilon_2 = 3\Delta y$

(c) $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1 = \lim\limits_{\Delta x\to 0} 2\Delta x = 0$ $\qquad$ $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2 = \lim\limits_{\Delta y\to 0} 3\Delta y = 0$

17. $f(x,y) = \dfrac{x^2}{y}$

▸ (a) $\Delta f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y) = \dfrac{(x+\Delta x)^2}{y+\Delta y} - \dfrac{x^2}{y} = \dfrac{x^2y + 2xy\Delta x + y(\Delta x)^2 - x^2y - x^2\Delta y}{y(y+\Delta y)}$

$= \dfrac{2xy\Delta x + y(\Delta x)^2 - x^2\Delta y}{y(y+\Delta y)} = \dfrac{2xy + y\Delta x}{y(y+\Delta y)}\Delta x - \dfrac{x^2}{y(y+\Delta y)}\Delta y$

(b) $D_1 f(x,y) = \dfrac{2x}{y}$ and $D_2 f(x,y) = -\dfrac{x^2}{y^2}$. Thus,

$$\Delta f(x,y) - D_1 f(x,y)\Delta x - D_2 f(x,y)\Delta y = \left[\frac{2xy + y\Delta x}{y(y+\Delta y)} - \frac{2x}{y}\right]\Delta x + \left[-\frac{x^2}{y(y+\Delta y)} + \frac{x^2}{y^2(y+\Delta y)}\right]\Delta y$$

$$= \frac{2xy + y\Delta x - 2xy - 2x\Delta y}{y(y+\Delta y)}\Delta x + \frac{-x^2y + x^2y + x^2\Delta y}{y^2(y+\Delta y)}\Delta y = \frac{y\Delta x - 2x\Delta y}{y(y+\Delta y)}\Delta x + \frac{x^2\Delta y}{y^2(y+\Delta y)}$$

Thus, $\epsilon_1 = \dfrac{y\Delta x - 2x\Delta y}{y(y+\Delta y)}$ and $\epsilon_2 = \dfrac{x^2\Delta y}{y^2(y+\Delta y)}$.

(c) $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1 = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{y\Delta x - 2x\Delta y}{y(y+\Delta y)} = 0$; $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2 = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{x^2\Delta y}{y^2(y+\Delta y)} = 0$

18. $f(x,y) = \dfrac{y}{x}$ (a) $\Delta f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y) = \dfrac{y+\Delta y}{x+\Delta x} - \dfrac{y}{x} = \dfrac{x\Delta y - y\Delta x}{x(x+\Delta x)}$

(b) $D_1 f(x,y) = -\dfrac{y}{x^2}$ and $D_2 f(x,y) = \dfrac{1}{x}$. Thus, $\Delta f(x,y) - D_1 f(x,y)\Delta x - D_2 f(x,y)\Delta y$

$= \left(-\dfrac{y}{x(x+\Delta x)} + \dfrac{y}{x^2}\right)\Delta x + \left(\dfrac{1}{x+\Delta x} - \dfrac{1}{x}\right)\Delta y = \dfrac{y\Delta x}{x^2(x+\Delta x)}\Delta x - \dfrac{\Delta x}{x(x+\Delta x)}\Delta y$

Thus, $\epsilon_1 = \dfrac{y\Delta x}{x^2(x+\Delta x)}$ and $\epsilon_2 = \dfrac{\Delta x}{x(x+\Delta x)}$ and (c) $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

In Exercises 19–26, Use Theorem 12.4.4 to prove that the function is differentiable at all points in its domain.

19. $g(x,y) = 2x^4 - 3x^2y^2 + x^{-2}y^{-2}$. The domain of g is all points in R^2 except (0.0).
$\text{D}_1g(x,y) = 8x^3 - 6xy^2 - 2x^{-3}y^{-2}$; $\text{D}_2g(x,y) = -6x^2y - 2x^{-2}y^{-3}$
At all points in the domain of g, D_1g and D_2g exist and are continuous; so g is differentiable by Thm 12.4.4.

20. $f(x,y) = \dfrac{3x - 4y}{x^2 + 8y}$

▸ The domain of f is the set of all points (x,y) not on the parabola $y = -\frac{1}{8}x^2$. The partial derivatives of f are

$$\text{D}_1f(x,y) = \frac{3(x^2 + 8y) - (3x - 4y)(2x)}{(x^2 + 8y)^2} = \frac{-3x^2 + 24y + 8xy}{(x^2 + 8y)^2}$$

and

$$\text{D}_2f(x,y) = \frac{(-4)(x^2 + 8y) - (3x - 4y)8}{(x^2 + 8y)^2} = \frac{-4x^2 - 24x}{(x^2 + 8y)^2}$$

Both D_1f and D_2f are continuous at every point in the domain of f. Therefore, by Theorem 12.4.4, g is differentiable at all points in its domain.

21. $f(x,y) = 3 \ln xy + 5 \sin x$. The domain of f is the first and third quadrants in R^2.
$\text{D}_1f(x,y) = 3 \cdot \frac{1}{xy} \cdot y + 5 \cos x = \frac{3}{x} + 5 \cos x$; $\text{D}_2f(x,y) = 3 \cdot \frac{1}{xy} \cdot x = \frac{3}{y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

22. $f(x,y) = \sin \frac{y}{x} + \cos \frac{x}{y}$. The domain of f is all points not on either axis.
$\text{D}_1f(x,y) = -\frac{y}{x^2} \cos \frac{y}{x} - \frac{1}{y} \sin \frac{x}{y}$ and $\text{D}_2f(x,y) = \frac{1}{x} \cos \frac{y}{x} + \frac{x}{y^2} \sin \frac{x}{y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

23. $h(x,y) = \tan^{-1}(x + y) + \frac{1}{x - y}$. The domain of h is $\{(x,y) \mid x \neq y\}$.
$\text{D}_1h(x,y) = \dfrac{1}{1 + (x + y)^2} - \dfrac{1}{(x - y)^2}$; $\text{D}_2h(x,y) = \dfrac{1}{1 + (x + y)^2} + \dfrac{1}{(x - y)^2}$
At all points in the domain of h, D_1h and D_2h exist and are continuous; so h is differentiable by Thm. 12.4.4.

24. $g(x,y) = y \ln x - \frac{x}{y}$

▸ The domain of g is $\{(x,y) \mid x > 0 \text{ and } y \neq 0\}$. The partial derivatives of g are
$\text{D}_1g(x,y) = g_x(x,y) = \frac{y}{x} - \frac{1}{y}$ and $\text{D}_2g(x,y) = g_y(x,y) = \ln x + \frac{x}{y^2}$
Both D_1g and D_2g are continuous at every point in the domain of g. Therefore, by Theorem 12.4.4, g is differentiable at all points in its domain.

25. $f(x,y) = ye^{3x} - xe^{-3y}$. The domain of f is all of R^2.
$\text{D}_1f(x,y) = 3ye^{3x} - e^{-3y}$; $\text{D}_2f(x,y) = e^{3x} + 3xe^{-3y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

26. $f(x,y) = e^{2x}\sin y + e^{-2x}\cos y$. The domain of f is all of R^2.
$\text{D}_1f(x,y) = 2e^{2x}\sin y - 2e^{-2x}\cos y$ and $\text{D}_2f(x,y) = e^{2x}\cos y - e^{-2x}\sin y$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

27. $f(x,y) = \begin{cases} x + y - 2 & \text{if } x = 1 \text{ or } y = 1 \\ 2 & \text{if } x \neq 1 \text{ and } y \neq 1 \end{cases}$

$$\text{D}_1f(1,1) = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x, 1) - f(1,1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1 + \Delta x + 1 - 2) - 0}{\Delta x} = 1$$

$$\text{D}_2f(1,1) = \lim_{\Delta y \to 0} \frac{f(1, 1 + \Delta y) - f(1,1)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(1 + 1 + \Delta y - 2) - 0}{\Delta y} = 1$$

But $\lim\limits_{\substack{(x,y) \to (0,0) \\ (y = x)}} f(x,y) = \lim\limits_{x \to 1} 2 = 2 \neq 0 = f(1,1)$ so f is discontinuous at (1.1) and so not differentiable there.

28. Given $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Prove that $D_1f(0,0)$ and $D_2f(0,0)$ exist but D_1f and D_2f are not continuous at $(0,0)$.

▸ (a) $D_1f(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{0-0}{x} = 0$

If $(x,y) \neq (0,0)$, by the quotient rule,

$$D_1f(x,y) = \frac{(x^2+y^2)(6xy)-(3x^2y)(2x)}{(x^2+y^2)^2} = \frac{6xy^3}{(x^2+y^2)^2}$$

Because

$$\lim_{\substack{(x,y)\to(0,0)\\(y=x)}} D_1(x,y) = \lim_{x\to 0} \frac{6x^4}{4x^4} = \frac{3}{2} \neq D_1(0,0)$$

then D_1f is not continuous at $(0,0)$.

(b) $D_2f(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0} \dfrac{0-0}{y} = 0$

If $(x,y) \neq (0,0)$, by the quotient rule,

$$D_2f(x,y) = \frac{(x^2+y^2)(3x^2)-(3x^2y)(2y)}{(x^2+y^2)^2} = \frac{3x^4-3x^2y^2}{(x^2+y^2)^2}$$

Because

$$\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} D_2f(x,y) = \lim_{x\to 0} \frac{3x^4}{x^4} = 3 \neq D_2(0,0)$$

then D_2f is not continuous at $(0,0)$.

In Exercises 29 and 30, prove that $D_1f(0,0)$ and $D_2f(0,0)$ exist but f is not differentiable at $(0,0)$.

29. $f(x,y) = \begin{cases} \dfrac{3x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$D_1f(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{0-0}{x} = 0 \qquad D_2f(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0} \dfrac{0-0}{y} = 0$

But $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} f(x,y) = \lim\limits_{x\to 0} \dfrac{3x^4}{2x^4} = \dfrac{3}{2} \neq f(0,0)$; f is not continuous at $(0,0)$ and so not differentiable there.

30. $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$D_1f(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{0-0}{x} = 0 \qquad D_2f(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0} \dfrac{0-0}{y} = 0$

But $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=y^2)}} f(x,y) = \lim\limits_{y\to 0} \dfrac{y^4}{2y^4} = \dfrac{1}{2} \neq f(0,0)$; f is not continuous at $(0,0)$ and so not differentiable there.

In Exercises 31 and 32, prove that f is differentiable at all points in R^3 by doing each of the following: (a) Find $\Delta f(x,y,z)$; (b) find an ϵ_1, ϵ_2 and ϵ_3 so that the equation of Definition 12.4.7 holds; (c) show that the ϵ_1, ϵ_2, and ϵ_3 found in part (b) all approach zero as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$.

31. $f(x,y,z) = xy - xz + z^2$

(a) $\Delta f(x,y,z) = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

$= (x+\Delta x)(y+\Delta y) - (x+\Delta x)(z+\Delta z) + (z+\Delta z)^2 - (xy - xz + z^2)$

$= xy + x\Delta y + y\Delta x + (\Delta x)(\Delta y) - xz - x\Delta z - z\Delta x - (\Delta x)(\Delta z) + z^2 + 2z\Delta z + (\Delta z)^2 - xy + xz - z^2$

$= x\Delta y + y\Delta x - x\Delta z - z\Delta x + 2z\Delta z + (\Delta x)(\Delta y) - (\Delta x)(\Delta z) + (\Delta z)^2$

$= (y-z)\Delta x + x\Delta y + (2z-x)\Delta z - \Delta z\Delta x + \Delta x\Delta y + \Delta z\Delta z$

(b) $D_1f(x,y) = y - z$, $D_2f(x,y,z) = x$, $D_3f(x,y,z) = 2z - x$. Thus, $\epsilon_1 = -\Delta z$, $\epsilon_2 = \Delta x$, $\epsilon_3 = \Delta z$.

(c) $\lim\limits_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \epsilon_1 = \lim\limits_{\Delta z\to 0} (-\Delta z) = 0$; $\lim\limits_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \epsilon_2 = \lim\limits_{\Delta x\to 0} \Delta x = 0$;
$\lim\limits_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \quad = \lim\limits_{\Delta z\to 0} \Delta z = 0$

It follows from Definition 12.4.8 that f is differentiable at all points in R^3.

32. $f(x,y,z) = 2x^2z - 3yz^2$

▸ (a) $\Delta f(x,y,z)$

$$= f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$
$$= [2(x+\Delta x)^2(z+\Delta z) - 3(y+\Delta y)(z+\Delta z)^2 - [2x^2z - 3yz^2]$$
$$= 2x^2\Delta z + 4xz\Delta x + 4x\Delta x\Delta z + 2z(\Delta x)^2 + 2(\Delta x)^2\Delta z - 6yz\Delta z - 3y(\Delta z)^2 - 3z^2\Delta y - 6z\Delta y\Delta z - 3\Delta y(\Delta z)^2 \quad (1)$$

(b) From the equation of Definition 12.4.7 with $n = 3$

$$\epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z = \Delta f(x,y,z) - D_1f(x,y,z)\Delta x - D_2f(x,y,z)\Delta y - D_3f(x,y,z)\Delta z \quad (2)$$

Taking partial derivatives of f, we obtain

$D_1f(x,y,z)\Delta x = 4xz\Delta x$ $\qquad D_2f(x,y,z)\Delta y = -3z^2\Delta y$ $\qquad D_3f(x,y,z)\Delta z = (2x^2 - 6yz)\Delta z$ (3)

Substituting from (1) and (3) into (2), we get

$$\begin{aligned}\epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z &= 4x\Delta x\Delta z + 2z(\Delta x)^2 + 2(\Delta x)^2\Delta z - 3y(\Delta z)^2 - 6z\Delta y\Delta z - 3\Delta y(\Delta z)^2\\ &= (4x\Delta z + 2z\Delta x + 2\Delta x\Delta z)\Delta x + (-6z\Delta z - 3(\Delta z)^2)\Delta y + (-3y\Delta z)\Delta z\end{aligned}$$

Equating corresponding coefficients of Δx, Δy, and Δz in the above, we take

$\epsilon_1 = 4x\Delta z + 2z\Delta x + 2\Delta x\Delta z$ $\qquad \epsilon_2 = -6z\Delta z - 3(\Delta z)^2$ $\qquad \epsilon_3 = -3y\Delta z$

(c) Since each term contains at least one factor of Δx, Δy, or Δz, all approach zero as $(\Delta x, \Delta y, \Delta z)\to(0,0,0)$.

In Ex. 33 and 34, prove that $D_1f(0,0,0)$, $D_2f(0,0,0)$ and $D_3f(0,0,0)$ exist but f is not differentiable at $(0,0)$.

33. $f(x,y,z) = \begin{cases}\dfrac{xy^2z}{x^4+y^4+z^4} & \text{if } (x,y,z) \neq (0,0,0)\\ 0 & \text{if } (x,y,z) = (0,0,0)\end{cases}$

(a) $D_1f(0,0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0,0) - f(0,0,0)}{x - 0} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = \lim\limits_{x\to 0} 0 = 0$

$D_2f(0,0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y,0) - f(0,0,0)}{y - 0} = \lim\limits_{y\to 0}\dfrac{0-0}{y} = 0$; $D_3f(0,0,0) = \lim\limits_{z\to 0}\dfrac{f(0,0,z) - f(0,0,0)}{z - 0} = \lim\limits_{z\to 0}\dfrac{0-0}{z} = 0$

(b) $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(x=y=z)}} f(x,y,z) = \lim\limits_{x\to 0}\dfrac{x^4}{3x^4} = \lim\limits_{x\to 0}\dfrac{1}{3} = \dfrac{1}{3} \neq 0 = f(0,0,0)$.

Therefore f is discontinuous at $(0,0,0)$, and so f is not differentiable at $(0,0,0)$.

34. $f(x,y,z) = \begin{cases}\dfrac{3yz}{x^4+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0)\\ 0 & \text{if } (x,y,z) = (0,0,0)\end{cases}$

(a) $D_1f(0,0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0,0) - f(0,0,0)}{x - 0} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = \lim\limits_{x\to 0} 0 = 0$

$D_2f(0,0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y,0) - f(0,0,0)}{y - 0} = \lim\limits_{y\to 0}\dfrac{0-0}{y} = 0$; $D_3f(0,0,0) = \lim\limits_{z\to 0}\dfrac{f(0,0,z) - f(0,0,0)}{z - 0} = \lim\limits_{z\to 0}\dfrac{0-0}{z} = 0$

(b) $\lim\limits_{\substack{(x,y,z)\to(0,0,0)\\(y=z=x^2)}} f(x,y,z) = \lim\limits_{x\to 0}\dfrac{3x^4}{3x^4} = \lim\limits_{x\to 0} 1 = 1 \neq 0 = f(0,0,0)$.

Therefore f is discontinuous at $(0,0,0)$, and so f is not differentiable at $(0,0,0)$.

35. ℓ meters is the length, w m the width, and h m the height of the solid. If V m^3 is the volume of the solid, then

$V = \ell wh$; $dV = \dfrac{\partial V}{\partial \ell}\,d\ell + \dfrac{\partial V}{\partial w}\,dw + \dfrac{\partial V}{\partial h}\,dh = wh\,d\ell + \ell h\,dw + \ell w\,dh$

With $\ell = 8$, $d\ell = 0.08$, $w = 5$, $dw = 0.08$, $h = 4$, and $dh = 0.08$, we have

$dV = (5)(4)(0.08) + (8)(4)(0.08) + (8)(5)(0.08) = 1.60 + 2.56 + 3.20 = 7.36$. $\Delta V \approx 7.36$.

In fact, $\Delta V = (8.08)(5.08)(4.08) - (8)(5)(4) = 7.47$.

36. Use the total differential to find approximately the greatest error in calculating the area of a right triangle from the lengths of the legs if they are measured to be 6 cm and 8 cm respectively, with a possible error of 0.1 cm for each measurement. Also find the approximate percent error.

▸ Let x cm and y cm be the lengths of the legs of a right triangle, and let A cm^2 be the area. Then
$\mathrm{A} = \frac{1}{2}xy$
We take the total differential of A. Thus,

$$d\mathrm{A} = \frac{\partial \mathrm{A}}{\partial x}dx + \frac{\partial \mathrm{A}}{\partial y}dy = \tfrac{1}{2}y\ dx + \tfrac{1}{2}x\ dy$$

We are given that $x = 6$, $y = 8$, $|dx| \le 0.1$, and $|dy| \le 0.1$. Using the triangle inequality, we have

$$|d\mathrm{A}| \le \left|\tfrac{1}{2}y\right||dx| + \left|\tfrac{1}{2}x\right||dy| \le 4(0.1) + 3(0.1) = 0.7$$

Thus, the greatest error in calculating the area is approximately 0.7 cm^2. Because $\mathrm{A} = \frac{1}{2}(6)8 = 24$, the percent error is found by
$\frac{|d\mathrm{A}|}{\mathrm{A}} \le \frac{0.7}{24} = 0.029$
Thus, the error is approximately 2.9% of the calculated error.

37. Let ℓ cm be the length of the hypotenuse and x cm and y cm be the lengths of the legs.
$\ell = \sqrt{x^2 + y^2}$; $d\ell = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$. Let $x = 6$, $y = 8$, $dx = 0.1$, and $dy = 0.1$. Then
$d\ell = \frac{6}{10}(\pm 0.1) + \frac{8}{10}(\pm 0.1) = \pm 0.14$; $\Delta\ell \approx 0.14$. Hence the greatest error in calculating the length of the hypotenuse is approximately 0.14 cm. $\frac{d\ell}{\ell} = \frac{\pm 0.14}{10} = \pm 0.014$, and so the approximate percent error is 1.4%.

38. $\mathrm{PV} = k\mathrm{T}$; $\mathrm{P} = k\mathrm{TV}^{-1}$. $\left|\frac{d\mathrm{P}}{\mathrm{P}}\right| = \left|\frac{k\mathrm{V}^{-1}d\mathrm{T} - k\mathrm{TV}^{-2}d\mathrm{V}}{k\mathrm{TV}^{-1}}\right| = \left|\frac{d\mathrm{T}}{\mathrm{T}} - \frac{d\mathrm{V}}{\mathrm{V}}\right| \le \left|\frac{d\mathrm{T}}{\mathrm{T}}\right| + \left|\frac{d\mathrm{V}}{\mathrm{V}}\right| = 0.3\% + 0.8\% = 1.1\%$

39. $s = \frac{\mathrm{A}}{\mathrm{A} - \mathrm{W}}$. $ds = \frac{-\mathrm{W}}{(\mathrm{A}-\mathrm{W})^2}\,d\mathrm{A} + \frac{\mathrm{A}}{(\mathrm{A}-\mathrm{W})^2}\,d\mathrm{W}$. Let $\mathrm{A} = 20$, $\mathrm{W} = 12$, $d\mathrm{A} = \pm 0.01$ and $d\mathrm{W} = \pm 0.02$. Then
$s = \frac{20}{20-12} = 2.5$; $|ds| \le \left|\frac{-12}{64}\cdot 0.01\right| + \left|\frac{20}{64}\cdot 0.02\right| = \frac{0.12}{64} + \frac{0.40}{64} = 0.008125$
The approximate relative error is $\frac{ds}{s} = \pm\frac{0.00875}{2.5} = \pm 0.00325 = \pm 0.325\%$.

40. A wooden box is to be made of lumber that is $\frac{2}{3}$ in. thick. The inside length is to be 6 ft; the inside width is to be 3 ft; the inside depth is to be 4 ft, and the box is to have no top. Use the total differential to find the approximate amount of lumber to be used in the box.

▸ Let x ft be the length, y ft the width, z ft be the depth, and V ft^3 be the volume of a rectangular box. Then
$\mathrm{V} = xyz$
We have $x = 6$, $y = 3$, and $z = 4$. The thickness of the box is $\frac{2}{3}$ in $= \frac{1}{18}$ ft. Thus, take $dx = 2\cdot\frac{1}{18} = \frac{1}{9}$ and $dy = \frac{1}{9}$. Because there is no top for the box, we take $dz = \frac{1}{18}$. Hence,

$$\begin{aligned} d\mathrm{V} &= \frac{\partial \mathrm{V}}{\partial x}dx + \frac{\partial \mathrm{V}}{\partial y}dy + \frac{\partial \mathrm{V}}{\partial z}dz \\ &= yz\ dx + xz\ dy + xy\ dz = 3\cdot 4\cdot\tfrac{1}{9} + 6\cdot 4\cdot\tfrac{1}{9} + 6\cdot 3\cdot\tfrac{1}{18} = 5 \end{aligned}$$

Therefore, approximately 5 ft^3 of lumber is needed.

41. Let the dimensions of one crate be x m, y m, z m. If A m^2 is the total surface area,
$\mathrm{A} = 2(xy + xz + yz)$; $d\mathrm{A} = 2[(y + z)dx + (x + z)dy + (x + y)dz]$
Let $x = 3$, $dx = \pm 0.005$, $y = 4$, $dy = \pm 0.005$, $z = 5$, and $dz = \pm 0.005$. Then
$\Delta\mathrm{A} \approx d\mathrm{A} = 2[9(\pm 0.005) + 8(\pm 0.005) + 7(\pm 0.005)] = \pm 0.240$
The greatest possible error in the amount of wood used for one crate is approximately 0.24 m^2. Thus, the greatest possible error in the estimate of the cost of wood for manufacturing 10,000 crates is 10,000(0.24)($3) = $7200.

In Exercises 42–45, we show that a function f may be differentiable at a point even though it is not continuously differentiable there, where

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

42. Find $\Delta f(0,0)$. See (2) below.

43. $D_1 f(0,0) = 0$ and $D_2 f(0,0) = 0$; see (3) and (4) below. If $(x,y) \neq (0,0)$, the formulas for differentiation give

$$D_1 f(x,y) = \begin{cases} 2x \sin\dfrac{1}{\sqrt{x^2+y^2}} - \dfrac{x}{\sqrt{x^2+y^2}} \cos\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$D_2 f(x,y) = \begin{cases} 2y \sin\dfrac{1}{\sqrt{x^2+y^2}} - \dfrac{y}{\sqrt{x^2+y^2}} \cos\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

44. Prove that f is differentiable at $(0,0)$ by using Definition 12.4.2 and the results of Exercises 42 and 43.

▸ By Definition 12.4.2 we must find an ϵ_1 and an ϵ_2 where ϵ_1 and ϵ_2 are functions of Δx and Δy such that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ and where ϵ_1 and ϵ_2 satisfy the equation

$$\Delta f(0,0) = D_1 f(0,0)\Delta x + D_2 f(0,0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{1}$$

By Definition 12.4.1,

$$\Delta f(0,0) = f(\Delta x, \Delta y) - f(0,0) = [(\Delta x)^2 + (\Delta y)^2]\sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \tag{2}$$

and by Definition 16.4.1,

$$D_1 f(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin\dfrac{1}{|\Delta x|}}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \sin\frac{1}{|\Delta x|}$$

Because

$$-1 \leq \sin\frac{1}{|\Delta x|} < 1$$

then

$$-|\Delta x| \leq |\Delta x| \sin\frac{1}{|\Delta x|} \leq |\Delta x|$$

Therefore, by the squeeze theorem,

$$\lim_{\Delta x \to 0} |\Delta x| \sin\frac{1}{|\Delta x|} = 0$$

Thus,

$$D_1 f(0,0) = 0 \tag{3}$$

Interchanging x and y, we find also

$$D_2 f(0,0) = 0 \tag{4}$$

Substituting from Eqs. (2), (3), and (4) into (1), we obtain

$$[(\Delta x)^2 + (\Delta y)^2]\sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{5}$$

Equation (5) is an identity if

$$\epsilon_1 = \Delta x \sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \text{ and } \epsilon_2 = \Delta y \sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

Because

$$-1 \leq \sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq 1$$

then

$$-|\Delta x| \leq \epsilon_1 \leq |\Delta x|$$

and

$$-|\Delta y| \leq \epsilon_2 \leq |\Delta y|$$

so, by the squeeze theorem,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon_1 = 0 \quad \text{and} \quad \lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon_2 = 0$$

Therefore, by Definition 12.4.2, f is differentiable at $(0,0)$.

45. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0,\ x>0)}} D_1f(x,y) = \lim\limits_{x\to 0^+}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$. $\lim\limits_{x\to 0^+} 2x\sin\frac{1}{x} = 0$ by the squeeze theorem but $\lim\limits_{x\to 0^+}\cos\frac{1}{x}$ does not exist. Therefore, $\lim D_1f(x,y)$ does not exist, and so D_1f is discontinuous at $(0,0)$. In a similar way we prove that $\lim\limits_{(x,y)\to(0,0)} D_2f(x,y)$ does not exist by showing that $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0,\ y>0)}} D_2f(x,y)$ does not exist.

46. $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

If $(x,y) \neq (0,0)$, $D_1f(x,y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$ and $D_1f(0,0) = \lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x} = 0$.

If $(x,y) \neq (0,0)$, $D_2f(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$ and $D_2f(0,0) = \lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim\limits_{\Delta y\to 0}\dfrac{0-0}{\Delta y} = 0$.

Both D_1f and D_2f exist on every open disk centered at $(0,0)$. To show D_1f is continuous at $(0,0)$ we will show that $\lim\limits_{(x,y)\to(0,0)} D_1f(x,y) = 0$.

$$\left|\frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}\right| \le \frac{|x^4||y| + 4|x^2||y^3| + |y^5|}{(x^2+y^2)^2} \le \frac{6(\sqrt{x^2+y^2})^5}{(\sqrt{x^2+y^2})^4} = 6\sqrt{x^2+y^2}$$

Therefore, if $\delta = \frac{1}{6}\epsilon$, $|D_1f(x,y)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$ and so $\lim\limits_{(x,y)\to(0,0)} D_1f(x,y) = 0$. Hence D_1f is continuous at $(0,0)$. Similarly, D_2f is continuous at $(0,0)$. Hence, by Theorem 12.4.4 f is differentiable at $(0,0)$.

47. $f(x,y,z) = \begin{cases} \dfrac{xyz^2}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$. Let $r = \|(x,y,z) - (0,0,0)\| = \sqrt{x^2+y^2+z^2}$.

If $(x,y,z) \neq (0,0,0)$, then $D_1f(x,y,z) = \dfrac{yz^2(x^2+y^2+z^2) - xyz^2(2x)}{r^4} = \dfrac{-x^2yz^2 + y^3z^2 + yz^4}{r^4}$

$|D_1f(x,y,z)| \le \dfrac{3r^5}{r^4} = 3r \to 0$ as $r \to 0$ and $D_1f(0,0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0,0) - f(0,0,0)}{x-0} = \lim\limits_{x\to 0}\dfrac{0}{x} = 0$.

Similarly, $D_2f(x,y,z)$ and $D_3f(x,y,z)$ are continuous at $(0,0,0)$. By Thm 12.2.4, f is differentiable at $(0,0,0)$.

12.5 THE CHAIN RULE FOR FUNCTIONS OF MORE THAN ONE VARIABLE

12.5.1 Theorem (The Chain Rule). If u is a differentiable function of x and y, defined by $u = f(x,y)$, and $x = F(r,s)$, $y = G(r,s)$ and $\partial x/\partial r$, $\partial x/\partial s$, $\partial y/\partial r$, and $\partial y/\partial s$ all exist, then u is a function of r and s and

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

If in Theorem 12.5.1 both x and y are differentiable functions of the single variable t, then u is a function of t and the total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad (1)$$

The chain rule may be extended to functions of three or more variables as follows.

12.5.2 Theorem (The General Chain Rule) Suppose that u is a differentiable function of the n variables x_1, x_2, …, x_n and each of these variables is in turn a function of the m variables y_1, y_2, …, y_m. Suppose further that each of the mn partial derivatives $\partial x_i/\partial y_j$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) exists. Then u is a function of y_1, y_2, …, y_m, and

$$\frac{\partial u}{\partial y_1} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_1} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_1}$$
$$\frac{\partial u}{\partial y_2} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_2}$$
$$\vdots$$
$$\frac{\partial u}{\partial y_m} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_m} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_m} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_m}$$

If in Theorem 12.5.2 each x_i is a differentiable function of t, then u is a function of t, and the total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt} \tag{2}$$

If t also appears explicitly in u, then we must add the term $\partial u/\partial t$ to Eqs. (1) and (2). See Exercises 18, 21, and 22.

The next two theorems tell us how to calculate derivatives of functions defined implicitly.

12.5.3 Theorem Let $F(x, y)$ be a function defined in an open disk $B((x_0, y_0); r)$ such that F has continuous first partial derivatives in B and that $F(x_0, y_0) = 0$, $F_y(x_0, y_0) \neq 0$. Then inside some smaller disk $B((x_0, y_0); s)$ there is a unique function $y = f(x)$ such that $F(x, y) \equiv 0$. y has a continuous derivative given by

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

12.5.4 Theorem Let $F(x, y, z)$ be a function defined in an open ball $B((x_0, y_0, z_0); r)$ such that F has continuous first partial derivatives in B and that $F(x_0, y_0, z_0) = 0$. $F_z(x_0, y_0, z_0) \neq 0$. Then inside some smaller disk $B((x_0, y_0, z_0); s)$ there is a unique function $z = f(x, y)$ such that $F(x, y, z) \equiv 0$. z has continuous partial derivatives given by

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

Cauchy-Riemann If $u = f(x, y)$ and $v = g(x, y)$, the Cauchy-Riemann equations are $u_x = v_y$ and $v_x = -u_y$. See Exercises 41, 42, and 52.

Exercises 12.5

In Exercises 1–6, find the indicated partial derivative by two methods: (a) use the chain rule and (b) make the substitutions for x and y before differentiating.

1. $u = x^2 - y^2$; $x = 3r - s$; $y = r + 2s$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 2x(3) - 2y(1) = 6x - 2y$; $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = 2x(-1) - 2y(2) = -2x - 4y$

(b) $u = (3r - s)^2 - (r + 2s)^2 = 9r^2 - 6rs + s^2 - r^2 - 4rs - 4s^2 = 8r^2 - 10rs - 3s^2$. $\frac{\partial u}{\partial r} = 16r - 10s$; $\frac{\partial u}{\partial s} = -10r - 6s$

2. $u = 3x - 4y^2$; $x = 5pq$; $y = 3p^2 - 2q$

(a) $\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} = 3(5q) - 8y(6p) = 15q - 48py$; $\frac{\partial u}{\partial q} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial q} = 3(5p) - 8y(-2) = 15p + 16y$

(b) $u = 3(5pq) - 4(3p^2 - 2q)^2 = 15pq - 36p^4 + 48p^2q - 16q^2$. $\frac{\partial u}{\partial p} = 15q - 144p^3 + 96pq$. $\frac{\partial u}{\partial q} = 15p + 48p^2$

3. $u = 3x^2 + xy - 2y^2 + 3x - y$; $x = 2r - 3s$; $y = r + s$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (6x + y + 3)(2) + (x - 4y - 1)(1) = 13x - 2y + 5$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (6x + y + 3)(-3) + (x - 4y - 1)(1) = -17x - 7y - 10$

(b) $u = 3(2r - 3s)^2 + (2r - 3s)(r + s) - 2(r + s)^2 + 3(2r - 3s) - (r + s)$
$= 12r^2 - 36rs + 27s^2 + 2r^2 - rs - 3s^2 - 2r^2 - 4rs - 2s^2 + 6r - 9s - r - s = 12r^2 - 41rs + 22s^2 + 5r - 10s$

$\frac{\partial u}{\partial r} = 24r - 41s + 5$; $\frac{\partial u}{\partial s} = -41r + 44s - 10$.

4. $u = x^2 + y^2$; $x = \cosh r \cos t$; $y = \sinh r \sin t$; $\partial u/\partial r$; $\partial u/\partial t$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = 2x \sinh r \cos t + 2y \cosh r \sin t$

$= 2(\cosh r \cos t)\sinh r \cos t + 2(\sinh r \sin t)\cosh r \sin t$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} = -2x \cosh r \sin t + 2y \sinh r \cos t$

$= -2(\cosh r \cos t)\cosh r \sin t + 2(\sinh r \sin t)\sinh r \cos t$

(b) Substituting before differentiating, we have

$u = \cosh^2 r \cos^2 t + \sinh^2 r \sin^2 t$

$\partial u/\partial r = 2 \cosh r \sinh r \cos^2 t + 2 \sinh r \cosh r \sin^2 t$ $\quad\partial u/\partial t = -2 \cosh^2 r \cos t \sin t + 2 \sinh^2 r \sin t \cos t$

The results of (a) and (b) agree.

5. $u = e^{y/x}$; $x = 2r \cos t$; $y = 4r \sin t$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = e^{y/x}\left(-\frac{y}{x^2}\right)(2 \cos t) + e^{y/x}\left(\frac{1}{x}\right)(4 \sin t) = \frac{2e^{y/x}}{x^2}(2x \sin t - y \cos t)$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = e^{y/x}\left(-\frac{y}{x^2}\right)(-2r \sin t) + e^{y/x}\left(\frac{1}{x}\right)(4r \cos t) = \frac{2re^{y/x}}{x^2}(y \sin t + 2x \cos t)$

(b) $u = e^{4r \sin t/2r \cos t} = e^{2 \tan t}$. $\frac{\partial u}{\partial r} = 0$; $\frac{\partial u}{\partial t} = 2e^{2 \tan t} \sec^2 t$.

6. $V = \pi x^2 y$; $x = \cos z \sin t$; $y = z^2 e^t$

▸ (a) $\frac{\partial V}{\partial z} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial z} = 2\pi xy(-\sin z \sin t) + \pi x^2(2ze^t)$; $\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} = 2\pi xy(\cos z \cos t) + \pi x^2(z^2 e^t)$

(b) $V = \pi \cos^2 z \sin^2 t\, z^2 e^t$

$\frac{\partial V}{\partial z} = \pi \sin^2 t\, e^t(-2 \cos z \sin z\, z^2 + 2 \cos^2 z\, z)$; $\frac{\partial V}{\partial t} = \pi \cos^2 z\, z^2(2 \sin t \cos t\, e^t + \sin^2 t\, e^t)$

In Exercises 7–14, find the indicated partial derivatives by using the chain rule.

7. $u = x^2 + xy$; $x = r^2 + s^2$; $y = 3r - 2s$

▸ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (2x + y)(2r) + x(3) = 2r(2x + y) + 3x$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(2s) + x(-2) = 2s(2x + y) - 2x$

8. $u = xy + xz + yz$; $x = rs$; $y = r^2 - s^2$; $z = (r - s)^2$; $\partial u/\partial r$; $\partial u/\partial s$

▸ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r} = (y + z)s + (x + z)(2r) + (x + y)2(r - s)$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s} = (y + z)r + (x + z)(-2s) + (x + y)(-2)(r - s)$

9. $u = \sin^{-1}(3x + y)$; $x = r^2 e^s$; $y = \sin rs$

▸ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{3}{\sqrt{1-(3x+y)^2}}(2re^s) + \frac{1}{\sqrt{1-(3x+y)^2}}(s \cos rs) = \frac{6re^s + s \cos rs}{\sqrt{1-(3x+y)^2}}$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{3}{\sqrt{1-(3x+y)^2}}(r^2 e^s) + \frac{1}{\sqrt{1-(3x+y)^2}}(r \cos rs) = \frac{3r^2 e^s + r \cos rs}{\sqrt{1-(3x+y)^2}}$

10. $u = \sin xy$; $x = 2ze^t$; $y = t^2 e^{-z}$

▸ $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = y(\cos xy)2ze^t + x(\cos xy)2te^{-z}$; $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} = y(\cos xy)2e^t + x(\cos xy)(-t^2 e^{-z})$

11. $u = \cosh \frac{y}{x}$; $x = 3r^2 s$; $y = 6se^r$

▸ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \sinh \frac{y}{x}\left(-\frac{y}{x^2}\right)(6rs) + \sinh \frac{y}{x}\left(\frac{1}{x}\right)(6se^r) = \frac{6s}{x^2} \sinh \frac{y}{x}(xe^r - ry)$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \sinh \frac{y}{x}\left(-\frac{y}{x^2}\right)(3r^2) + \sinh \frac{y}{x}\left(\frac{1}{x}\right)(6e^r) = \frac{3}{x^2} \sinh \frac{y}{x}(2xe^r - yr^2) = 0$

12. $u = e^{-y}$; $x = \tan^{-1}(rst)$; $y = \ln(3rs + 5st)$; $\partial u/\partial r$; $\partial u/\partial s$; $\partial u/\partial t$

▸ We have $y = \ln s + \ln(3r + 5t)$. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = e^{-y}\frac{st}{1 + r^2s^2t^2} - xe^{-y}\frac{3}{3r + 5t}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = e^{-y}\frac{rt}{1 + r^2s^2t^2} - xe^{-y}\frac{1}{s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} = e^{-y}\frac{rs}{1 + r^2s^2t^2} - xe^{-y}\frac{5}{3r + 5t}$$

13. $u = x^2 + y^2 + z^2$; $x = r \sin\phi \cos\theta$; $y = r \sin\phi \sin\theta$; $z = r \cos\phi$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = 2x \sin\phi \cos\theta + 2y \sin\phi \sin\theta + 2z \cos\phi$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi} = 2xr \cos\phi \cos\theta + 2yr \cos\phi \sin\theta - 2zr \sin\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = -2xr \sin\phi \sin\theta + 2yr \sin\phi \cos\theta + 0$$

14. $u = x^2yz$; $x = rs^{-1}$; $y = re^s$; $z = re^{-s}$. $u_r = u_xx_r + u_yy_r + u_zz_r = 2xyzs^{-1} + x^2ze^s + x^2ye^{-s}$;

$$u_s = u_xx_s + u_yy_s + u_zz_s = 2xyz(-rs^{-2}) + x^2zre^s + x^2y(-re^{-s})$$

In Exercises 15–18, find the total derivative du/dt by two methods: (a) Use the chain rule; (b) make the substitutions for x and y (and z) before differentiating.

15. $u = ye^x + xe^y$; $x = \cos t$; $y = \sin t$

▸ (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = (ye^x + e^y)(-\sin t) + (e^x + xe^y)(\cos t) = e^x(\cos t - y \sin t) + e^y(x \cos t - \sin t)$

(b) $u = \sin t\, e^{\cos t} + \cos t\, e^{\sin t}$

$$\frac{du}{dt} = \cos t\, e^{\cos t} + \sin t\, e^{\cos t}(-\sin t) - \sin t\, e^{\sin t} + \cos t\, e^{\sin t}(\cos t) = e^{\cos t}(\cos t - \sin^2 t) + e^{\sin t}(\cos^2 t - \sin t)$$

16. $u = \ln xy + y^2$; $x = e^t$; $y = e^{-t}$

▸ We have $u = \ln x + \ln y + y^2$.

(a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt} = \dfrac{1}{x}e^t + \left(\dfrac{1}{y} + 2y\right)(-e^{-t}) = e^{-t}e^t + (e^t + 2e^{-t})(-e^{-t}) = -2e^{-2t}$

(b) Substituting for x and y, we have

$$u = \ln e^t + \ln e^{-t} + (e^{-t})^2 = t - t + e^{-2t} = e^{-2t}$$

$$\frac{du}{dt} = -2e^{-2t}$$

17. $u = \sqrt{x^2 + y^2 + z^2}$; $x = \tan t$; $y = \cos t$; $z = \sin t$; $0 < t < \frac{1}{2}\pi$. (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}(\sec^2 t) + \frac{y}{\sqrt{x^2 + y^2 + z^2}}(-\sin t) + \frac{z}{\sqrt{x^2 + y^2 + z^2}}(\cos t) = \frac{x \sec^2 t - y \sin t + z \cos t}{\sqrt{x^2 + y^2 + z^2}}$$

(b) $u = \sqrt{\tan^2 t + \cos^2 t + \sin^2 t} = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = |\sec t| = \sec t$. $\dfrac{du}{dt} = \sec t \tan t$.

18. $u = \dfrac{t + e^x}{y - e^t}$; $x = 3 \sin t$; $y = \ln t$.

▸ (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial t} = \dfrac{e^x}{y - e^t}(3 \cos t) - \dfrac{t + e^x}{(y - e^t)^2}\cdot\dfrac{1}{t} + \dfrac{1(y - e^t) + (t + e^x)e^t}{(y - e^t)^2}$

(b) $u = \dfrac{t + e^{3\sin t}}{\ln t - e^t}$. $\dfrac{du}{dt} = \dfrac{(1 + 3 \cos t\, e^{3\sin t})(\ln t - e^t) - (t + e^{3\sin t})(1/t - e^t)}{(\ln t - e^t)^2}$

In Exercises 19–22, find the total derivative du/dt by using the chain rule.

19. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$; $x = \ln t$; $y = e^t$

▸ $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(-\dfrac{y}{x^2}\right)\left(\dfrac{1}{t}\right) + \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(\dfrac{1}{x}\right)e^t = \dfrac{-y}{t(x^2 + y^2)} + \dfrac{xe^t}{x^2 + y^2} = \dfrac{txe^t - y}{t(x^2 + y^2)}$

20. $u = xy + xz + yz$; $x = t \cos t$; $y = t \sin t$; $z = t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = (y + z)(\cos t - t \sin t) + (x + z)(\sin t + t \cos t) + (x + y)$$

21. $u = \frac{x+t}{y+t}$; $x = \ln t$; $y = \ln \frac{1}{t} = -\ln t$. $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t} = \frac{1}{y+t}\left(\frac{1}{t}\right) + \left[-\frac{x+t}{(y+t)^2}\right]\left(-\frac{1}{t}\right) + \frac{(y+t)-(x+t)}{(y+t)^2}$
$= \frac{y+t+x+t}{t(y+t)^2} + \frac{y-x}{(y+t)^2} = \frac{x+y+2t+ty-tx}{t(y+t)^2}$

22. $u = \ln(x^2+y^2+t^2)$; $x = t \sin t$; $y = \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t} = \frac{2x}{x^2+y^2+t^2}(\sin t + t\cos t) + \frac{2y}{x^2+y^2+t^2}(-\sin t) + \frac{2t}{x^2+y^2+t^2}$$

In Exercises 23–26, use Theorem 12.5.3: $\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$. Compare with the indicated exercise in Exercises 2.9.

23. $x^3 + y^3 - 8xy = 0$. $\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{3x^2-8y}{3y^2-8x}$

24. $2x^3y + 3xy^3 = 5$; Exercise 24

▸ We have $F(x,y) = 2x^3y + 3xy^3 - 5 = 0$.

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{6x^2y+3y^3}{2x^3+9xy^2}$$

We get the same result as Exercise 2.9.24 with less work.

25. $x \sin y + y \cos x - 1 = 0$. $\frac{dy}{dx} = \frac{\sin y - y \sin x}{x \cos y + \cos x}$ **26.** $\cos(x+y) - y \sin x = 0$. $\frac{dy}{dx} = -\frac{-\sin(x+y) - y\cos x}{-\sin(x+y) - \sin x}$

In Exercises 27–30, assume that the equation defines z as a function of x and y. Find $\partial z/\partial x$ and $\partial z/\partial y$ by two methods: (a) use Theorem 12.5.4; (b) differentiate implicitly.

27. $3x^2 + y^2 + z^2 - 3xy + 4xz - 15 = 0$ (a) $\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{6x-3y+4z}{2z+4x}$; $\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{2y-3x}{2z+4x}$

(b) $6x + 2z\frac{\partial z}{\partial x} - 3y + 4z + 4x\frac{\partial z}{\partial x} = 0$; $\frac{\partial z}{\partial x} = \frac{3y-6x-4z}{4x+2z}$. $2y + 2z\frac{\partial z}{\partial y} - 3x + 4x\frac{\partial z}{\partial y} = 0$; $\frac{\partial z}{\partial y} = \frac{3x-2y}{4x+2z}$.

28. $z = (x^2+y^2)\sin xz$

▸ (a) We have $F(x,y,z) = (x^2+y^2)\sin xz - z = 0$. By Theorem 12.5.4.

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2x\sin xz + (x^2+y^2)z\cos xz}{(x^2+y^2)x\cos xz - 1}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{2y\sin xz}{(x^2+y^2)x\cos xz - 1}$$

(b) For $\partial z/\partial x$, we treat y as a constant and differentiate on both sides with respect to x. By the product rule,

$$\frac{\partial z}{\partial x} = (x^2+y^2)\frac{\partial(\sin xz)}{\partial x} + \sin xz\frac{\partial(x^2+y^2)}{\partial x} = (x^2+y^2)(\cos xz)\frac{\partial(xz)}{\partial x} + (\sin xz)2x$$
$$= (x^2+y^2)(\cos xz)\left(x\frac{\partial z}{\partial x} + z\right) + 2x\sin xz$$

Solving for $\partial z/\partial x$, we have

$$\frac{\partial z}{\partial x} = [x(x^2+y^2)\cos xz]\frac{\partial z}{\partial x} + z(x^2+y^2)\cos xz + 2x\sin xz$$
$$[1 - x(x^2+y^2)\cos xz]\frac{\partial z}{\partial x} = z(x^2+y^2)\cos xz + 2x\sin xz$$
$$\frac{\partial z}{\partial x} = \frac{z(x^2+y^2)\cos xz + 2x\sin xz}{1 - x(x^2+y^2)\cos xz}$$

For $\partial z/\partial y$, we regard x as a constant and differentiate on both sides with respect to y. Thus,

$$\frac{\partial z}{\partial y} = (x^2+y^2)(\cos xz)(x)\frac{\partial z}{\partial x} + (\sin xz)2y$$
$$[1 - x(x^2+y^2)\cos xz]\frac{\partial z}{\partial y} = 2y\sin xz$$
$$\frac{\partial z}{\partial y} = \frac{2y\sin xz}{1 - x(x^2+y^2)\cos xz}$$

29. $ye^{xyz}\cos 3xz - 5 = 0$. (a) $\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{y^2ze^{xyz}\cos 3xz - 3yz\sin 3xz}{xy^2e^{xyz}\cos 3xz - 3xy\sin 3xz} = -\frac{z}{x}$;

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{(e^{xyz} + xyze^{xyz}) + \cos 3xz}{xy^2e^{xyz}\cos 3xz - 3xy\sin 3xz}$$

(b) $ye^{xyz}\left(yz + xy\frac{\partial z}{\partial x}\right)\cos 3xz + ye^{xyz}(-\sin 3xz)\left(3z + 3x\frac{\partial z}{\partial x}\right) = 0$

$(y^2e^{xyz}\cos 3xz - 3ye^{xyz}\sin 3xz)\left(z + x\frac{\partial z}{\partial x}\right) = 0$; $\frac{\partial z}{\partial x} = -\frac{z}{x}$

$e^{xyz}\cos 3xz + ye^{xyz}\left(xz + xy\frac{\partial z}{\partial y}\right)\cos 3xz + ye^{xyz}(-\sin 3xz)\left(3x\frac{\partial z}{\partial y}\right) = 0$

$(xy^2\cos 3xz - 3xy\sin 3xz)\frac{\partial z}{\partial y} = -(xyz+1)\cos 3xz$; $\frac{\partial z}{\partial y} = \frac{-(xyz+1)\cos 3xz}{xy^2\cos 3xz - 3xy\sin 3xz} = \frac{xyz+1}{3xy\tan 3xz - xy^2}$

30. $ze^{yz} + 2xe^{xz} - 4e^{xy} - 3 = 0$. $\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2e^{xz} + 2xze^{xz} - 4ye^{xy}}{e^{yz} + yze^{yz}}$; $\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{z^2e^{yz} - 4xe^{xy}}{e^{yz}(1+yz)}$

31. $z = f(bx - ay)$. Let $u = bx - ay$. Then $\frac{\partial z}{\partial x} = f_u\frac{\partial u}{\partial x} = bf_u$ and $\frac{\partial z}{\partial y} = f_u\frac{\partial u}{\partial y} = -af_u$.

Therefore $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = a(bf_u) + b(-af_u) = 0$

32. If f is a differentiable function of two variables u and v, let $u = x - y$ and $v = y - x$ and prove that $f(x-y, y-x)$ satisfies the equation $\partial z/\partial x + \partial z/\partial y = 0$.

▸ We have $z = f(u,v)$. By the chain rule,

$$\frac{\partial z}{\partial x} = f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x} = f_1(u,v) - f_2(u,v) \tag{1}$$

and

$$\frac{\partial z}{\partial y} = f_1(u,v)\frac{\partial u}{\partial y} + f_2(u,v)\frac{\partial v}{\partial y} = -f_1(u,v) + f_2(u,v) \tag{2}$$

Adding the members of (1) and (2), we obtain the desired result

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

33. $u = f(x,y)$; $x = \cosh v\cos w$; $y = \sinh v\sin w$

▸ $\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = \sinh v\cos w\frac{\partial u}{\partial x} + \cosh v\sin w\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial w} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial w} = -\cosh v\sin w\frac{\partial u}{\partial x} + \sinh v\cos w\frac{\partial u}{\partial y}$

In Exercises 34–39, find the second partial derivative two ways: (a) first substitute; (b) use the chain rule.

34. $u = e^y\cos x$, $x = 2t$, $y = t^2$

▸ (a) $u = e^{t^2}\cos 2t$. $u_t = 2e^{t^2}(t\cos 2t - \sin 2t)$. $u_{tt} = 2e^{t^2}(2t^2\cos 2t - 4t\sin 2t - \cos 2t)$

(b) $\frac{du}{dt} = \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = -e^y\sin x(2) + e^y\cos x(2t) = 2e^y(t\cos x - \sin x)$

$\frac{d^2u}{dt^2} = \frac{\partial}{\partial x}\left(\frac{du}{dt}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{du}{dt}\right)\frac{dy}{dt} + \frac{\partial}{\partial t}\left(\frac{du}{dt}\right) = 2e^y(-t\sin x - \cos x)(2) + 2e^y(t\cos x - \sin x)(2) + 2e^y\cos x$

35. $u = 3xy - 4y^2$, $x = 2se^r$, and $y = re^{-s}$

(a) $u = 3(2se^r)(re^{-s}) - 4(re^{-s})^2 = 6rse^{r-s} - 4r^2e^{-2s}$, $\frac{\partial u}{\partial r} = 6se^{r-s} + 6rse^{r-s} - 8re^{-2s}$

$\frac{\partial^2 u}{\partial r^2} = 6se^{r-s} + 6se^{r-s} + 6rse^{r-s} - 8e^{-2s} = 6se^{r-s}(2+r) - 8e^{-2s} = 12se^{r-s} + 6rse^{r-s} - 8e^{-2s}$

(c) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 3y(2se^r) + (3x - 8y)e^{-s}$. $\frac{\partial^2 u}{\partial r^2} = 3e^{-s}(2se^r) + 3y(2se^r) + (3\cdot 2se^r - 8\cdot e^{-s})e^{-s}$

$= 6se^{r-s} + 3re^{-s}(2se^r) + 6se^{r-s} - 8e^{-2s} = 12se^{r-s} + 6rse^{r-s} - 8e^{-2s}$

36. $u = 3xy - 4y^2$, $x = 2se^r$, $y = re^{-s}$; $\partial^2 u.\partial s\partial r$

▸ (a) $u = 3(2se^r)(re^{-s}) - 4(re^{-s})^2 = 6rse^{r-s} - 4r^2e^{-2rs}$

Thus,

$$\frac{\partial u}{\partial r} = -6rse^{r-s} + 6se^{r-s} - 8re^{-2s}$$

Hence

$$\frac{\partial^2 u}{\partial s\partial r} = -6rse^{r-s} + 6re^{r-s} - 6se^{r-s} + 6e^{r-s} + 16re^{-2s}$$

(b) By using the chain rule, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = 3y\cdot 2se^r + (3x - 8y)e^{-s} = 6yse^r + (3x - 8y)e^{-s}$$

Partial-differentiating with respect to s and using the formula for the derivative of a product, we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial s\partial r} &= 6e^r\left(\frac{\partial y}{\partial s}s + 1\cdot y\right) + \frac{\partial(3x - 8y)}{\partial s}e^{-s} + (3x - 8y)(-e^{-s})\\ &= 6e^r(-rse^{-s} + y) + (6e^r + 8re^{-s})e^{-s} + (3x - 8y)(-e^{-s})\\ &= -6rse^{r-s} + 6ye^r + 6e^{r-s} + 8re^{-2s} - 3xe^{-s} + 8ye^{-s}\end{aligned}$$

In Exercises 37–39, $u = 9x^2 + 4y^2$; $x = r\cos\theta$; $y = r\sin\theta$

37. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta$; $\frac{\partial u}{\partial r} = 18r\cos^2\theta + 8r\sin^2\theta$; $\frac{\partial^2 u}{\partial r^2} = 18\cos^2\theta + 8\sin^2\theta = 10\cos^2\theta + 8$

(b) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 18x\cos\theta + 8y\sin\theta = 18r\cos^2\theta + 8r\sin^2\theta$; $\frac{\partial^2 u}{\partial r^2} = 18\cos^2\theta + 8\sin^2\theta = 10\cos^2\theta + 8$

38. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta = r^2[\frac{9}{2}(1 + \cos 2\theta) + 2(1 - \cos 2\theta)] = r^2(\frac{13}{2} + \frac{5}{2}\cos 2\theta)$;

$\frac{\partial u}{\partial \theta} = -5r^2\sin 2\theta$; $\frac{\partial^2 u}{\partial \theta^2} = -10r^2\cos 2\theta$

(b) $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = 18x(-r\sin\theta) + 8y(r\cos\theta) = r(-18x\sin\theta + 8y\cos\theta)$;

$\frac{\partial^2 u}{\partial \theta^2} = r[-18(-r\sin\theta)\sin\theta - 18x\cos\theta + 8(r\cos\theta)\cos\theta + 8y(-\sin\theta)]$

39. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta$; $\frac{\partial u}{\partial \theta} = -18r^2\cos\theta\sin\theta + 8r^2\sin\theta\cos\theta = -10r^2\sin\theta\cos\theta$

$\frac{\partial^2 u}{\partial r\partial \theta} = -20r\sin\theta\cos\theta = -10r\sin 2\theta$

(b) $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = 18x(-r\sin\theta) + 8y(r\cos\theta) = -18r\sin\theta(r\cos\theta) + 8r\cos\theta(r\sin\theta)$

$= -10r^2\sin\theta\cos\theta$. $\frac{\partial^2 u}{\partial r\partial \theta} = 20r\sin\theta\cos\theta = -10r\sin 2\theta$.

40. Suppose f is a differentiable function of x, y, and z and $u = f(x, y, z)$. If $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$, express $\partial u/\partial r$, $\partial u/\partial\phi$, and $\partial u/\partial\theta$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, *and* $\partial u/\partial z$.

▸
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}\sin\phi\cos\theta + \frac{\partial u}{\partial y}\sin\phi\sin\theta + \frac{\partial u}{\partial z}\cos\phi$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \phi} = \frac{\partial u}{\partial x}r\cos\phi\cos\theta + \frac{\partial u}{\partial y}r\cos\phi\sin\theta - \frac{\partial u}{\partial z}r\sin\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\phi\sin\theta + \frac{\partial u}{\partial y}r\sin\phi\cos\theta$$

41. $u = \frac{1}{2}\ln(x^2 + y^2)$; $v = \tan^{-1}(\frac{y}{x})$

▸ $\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$; $\frac{\partial v}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)}\cdot\frac{1}{x} = \frac{x}{x^2 + y^2}$. Therefore $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$; $\frac{\partial v}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)}\cdot\frac{-y}{x^2} = \frac{-y}{x^2 + y^2}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

42. (Cauchy-Riemann equation in polar coordinates) $x = r\cos\theta$, $y = r\sin\theta$.

▸ $u_r - r^{-1}v_\theta = (u_x x_r + u_y y_r) - r^{-1}(v_x x_\theta + v_y y_\theta) = (u_x\cos\theta + u_y\sin\theta) - r^{-1}(-v_x r\sin\theta + v_y r\cos\theta)$
$= (u_x - v_y)\cos\theta + (u_y + v_x)\sin\theta = 0 + 0 = 0$
$v_r + r^{-1}u_\theta = (v_x x_r + v_y y_r) + r^{-1}(u_x x_\theta + u_y y_\theta) = (v_x\cos\theta + v_y\sin\theta) + r^{-1}(-u_x r\sin\theta + u_y r\cos\theta)$
$= (v_x + u_y)\cos\theta + (v_y - u_x)\sin\theta = 0 + 0 = 0$

43. At t min, let x cm be the length of the first leg, y cm the length of the second leg, and θ radians the measure of the angle opposite the leg of length y cm. Then

$\theta = \tan^{-1}\frac{y}{x}$; $\frac{d\theta}{dt} = \frac{\partial\theta}{\partial x}\frac{dx}{dt} + \frac{\partial\theta}{\partial y}\frac{dy}{dt} = \frac{-y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}$

At the given instant $x = 10$, $\frac{dx}{dt} = 1$, $y = 12$, $\frac{dy}{dt} = -2$, so $\frac{d\theta}{dt} = \frac{-12}{244}(1) + \frac{10}{244}(-2) = -\frac{8}{61}$. Hence the measure of the angle is decreasing at a rate of $\frac{8}{61}$ rad/min.

44. Water is flowing into a tank in the form of a right circular cylinder at the rate of $\frac{4}{5}\pi$ m^3/min. The tank is stretching in such a way that even though it remains cylindrical, it radius is increasing at the rate of 0.2 cm/min. How fast is the surface of the water rising when the radius is 2 m and the volume of water in the tank is 20π m^3?

▸ After t minutes, the radius of the tank is r meters, the depth of the water is h meters, and the volume of the water is V m^3. We solve the volume formula for h:

$$V = \pi r^2 h$$
$$h = \frac{1}{\pi}r^{-2}V$$

Because 0.2 cm = 0.002 m, we are given

$\frac{dV}{dt} = \frac{4}{5}\pi$ and $\frac{dr}{dt} = 0.002$

We calculate dh/dt by the chain rule and substitute $r = 2$ and $V = 20\pi$. Thus,

$$\frac{dh}{dt} = \frac{\partial h}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial h}{\partial V}\cdot\frac{dV}{dt} = \frac{1}{\pi}\left(-2r^{-3}V\frac{dr}{dt} + r^{-2}\frac{dV}{dt}\right) = \frac{1}{\pi}\left[-2\left(\tfrac{1}{8}\right)20\pi(0.002) + \tfrac{1}{4}\left(\tfrac{4}{5}\pi\right)\right] = 0.19$$

Thus, the surface of the water is rising at the rate of 0.19 meters per minute.

45. At t min, let h cm be the height of the cylinder, r cm its radius and V cm^3 its volume.

$V = \pi r^2 h$; $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

Because $h = 50$, $\frac{dh}{dh} = -10$, $r = 16$, $\frac{dr}{dt} = 4$, then $\frac{dV}{dt} = 2\pi(16)(50)(4) + \pi(16)^2(-10) = 3840\pi$.

Thus at the given instant the volume is increasing at a rate of 3840π cm^3/min.

46. At t min, h cm and r cm are the height and radius of the cone. $\frac{dh}{dt} = 40$, $\frac{dr}{dt} = -5$, $h = 200$, $r = 60$. $V = \frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{1}{3}\pi[2(60)(200)(-15) + 60^2(40)] = -72{,}000\pi \text{ cm}^3/\text{min (decreasing)}$$

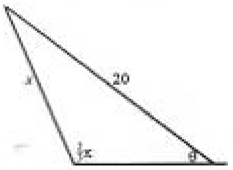
47. $PV = kT$; $V = \frac{kT}{P}$; $\frac{dV}{dt} = \frac{\partial V}{\partial T}\frac{dT}{dt} + \frac{\partial V}{\partial P}\frac{dP}{dt} = \frac{k}{P}\frac{dT}{dt} - \frac{kT}{P^2}\frac{dP}{dt}$.

Because $k = 1.2$, $T = 300$, $\frac{dT}{dt} = 3$, $P = 6$, $\frac{dP}{dt} = -0.1$, then $\frac{dV}{dt} = \frac{1.2}{6}(3) - \frac{1.2(300)}{6^2}(-0.1) = 1.6$.

Hence at the given instant the volume is increasing at a rate of 1.6 liters/min.

48. A wall makes an angle of radian measure $\frac{2}{3}\pi$ with the ground. A ladder of length 20 ft is leaning against the wall and its top is sliding down the wall at the rate of 3 ft/sec. How fast is the area of the triangle formed by the ladder, the wall and the ground changing when the ladder makes an angle of $\frac{1}{6}\pi$ radians with the ground?

▸ After t minutes, the top of the ladder is x feet from the ground and the bottom of the ladder makes an of θ with the ground and the area is A ft^2. See the figure. From the law of sines,



$\frac{x}{\sin\theta} = \frac{20}{\sin\frac{2}{3}\pi} = \frac{20}{\frac{1}{2}\sqrt{3}}$ and so $x = \frac{40}{\sqrt{3}}\sin\theta$

Because the angle between the wall and the ladder is $\pi - (\theta + \frac{2}{3}\pi) = \frac{1}{3}\pi - \theta$, then

$$A = \tfrac{1}{2}(20)x\sin\left(\tfrac{1}{3}\pi - \theta\right) = \frac{400}{\sqrt{3}}\sin\theta\sin\left(\tfrac{1}{3}\pi - \theta\right)$$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{400}{\sqrt{3}}\left[\cos\theta\sin\left(\tfrac{1}{3}\pi + \theta\right) - \sin\theta\cos\left(\tfrac{1}{3}\pi - \theta\right)\right]\frac{d\theta}{dt}$$

When $\theta = \frac{1}{6}\pi$, we have

$$\frac{dA}{dt} = \frac{400}{\sqrt{3}}\left(\cos\tfrac{1}{6}\pi\sin\tfrac{1}{6}\pi - \sin\tfrac{1}{6}\pi\cos\tfrac{1}{6}\pi\right)\frac{d\theta}{dt} = 0$$

Thus, the triangle is isosceles and its area is not changing.

49. $\frac{\partial \beta}{\partial P} + \frac{\partial \kappa}{\partial T} = \frac{\partial}{\partial P}\left(V^{-1}\frac{\partial V}{\partial T}\right) - \frac{\partial}{\partial T}\left(V^{-1}\frac{\partial V}{\partial P}\right) = -V^{-2}\frac{\partial V}{\partial P}\frac{\partial V}{\partial T} + V^{-1}\frac{\partial^2 V}{\partial P \partial T} + V^{-2}\frac{\partial V}{\partial T}\frac{\partial V}{\partial P} - V^{-1}\frac{\partial^2 V}{\partial T \partial P} = 0$

50. $(P + aV^{-2})(V - b) = RT$. $F(P, V, T) = P + aV^{-2} - RT(V - b)^{-1} = 0$

$\beta = \frac{1}{V}\frac{\partial V}{\partial T} = -\frac{1}{V}\frac{\partial F/\partial T}{\partial F/\partial V} = -\frac{1}{V}\cdot\frac{-R(V-b)^{-1}}{-2aV^{-3} + RT(V-b)^{-2}}\cdot\frac{V^2(V-b)^2}{V^2(V-b)^2} = \frac{RV^2(V-b)}{RTV^3 - 2a(V-b)^2}$

$\kappa = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{V}\frac{\partial F/\partial P}{\partial F/\partial V} = \frac{1}{V}\cdot\frac{1}{-2aV^{-3} + RT(v-b)^{-2}}\cdot\frac{V^2(V-b)^2}{V^2(V-b)^2} = \frac{V^2(V-b)^2}{RTV^3 - 2a(V-b)^2}$

If $a = b = 0$, $\beta = \frac{RV^3}{RTV^3} = \frac{1}{T}$, $k = \frac{V^4}{RTV^3} = \frac{V}{RT}$

51. (Changing to polar coordinates) $u = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Then

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$; $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$ (1)

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$; $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$ (2)

Multiplying (1) by $r\cos\theta$ and (2) by $(-\sin\theta)$ and adding, we obtain

$r\cos\theta\,\frac{\partial u}{\partial r} - \sin\theta\,\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(r\cos^2\theta + r\sin^2\theta)$; $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$

Multiplying (1) by $r\sin\theta$ and (2) by $\cos\theta$ and adding, we obtain

$r\sin\theta\,\frac{\partial u}{\partial r} + \cos\theta\,\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}(r\sin^2\theta + r\cos^2\theta)$; $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$

52. Suppose that $u = f(x, y)$, $v = g(x, y)$, and f and g and their first and second partial derivatives are continuous. Prove that if u and v satisfy the Cauchy Riemann equations, they also satisfy Laplace's equation (§12.3).

▷ The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (1)$$

Partial-differentiating with respect to x on both sides of the first equation in (1), we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad (2)$$

Partial-differentiating with respect to y on both sides of the second equation in (1), we obtain

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

Because g and its first and second partial derivatives are continuous, then $\partial^2 v/\partial y\partial x = \partial^2 v/\partial x\partial y$. Thus, Eq.(3) is equivalent to

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \qquad (4)$$

Adding the members of (2) and (4), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is Laplace's equation. Similarly, we may partial-differentiate with respect to y on both sides of the fist equation in (1) and partial-differentiate with respect to x on both sides of the second equation in (1). We obtain

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \qquad (5)$$

Because f and its first and second partial derivatives are continuous, then $\partial^2 u/\partial y\partial z = \partial^2 u/\partial x\partial y$. Thus, if we add the members of Eq.(5), the result is Laplace's equation for the function v:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

12.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

12.6.1 Definition Let f be a function of two variables x and y. If $\mathbf{U}$ is the unit vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, then the *directional derivative* of f in the direction of $\mathbf{U}$, denoted by $D_U f$, is given by

$$D_U f(x,y) = \lim_{h\to 0}\frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

if this limit exists.

12.6.2 Theorem If f is a differentiable function of x and y, and $\mathbf{U} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, then

$$D_U(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

12.6.3 Definition If f is a function of two variables x and y and f_x and f_y exist, then the *gradient* of f, denoted by ∇f (read:"del f") is defined by

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

The gradient of f can be used to find the directional derivative.

If f is a differentiable function of x and y and $\mathbf{U}$ is a unit vector, then

$$D_U f(x,y) = \mathbf{U}\cdot\nabla f(x,y)$$

12.6.4 Theorem Let f be a function of two variables and differentiable at (x_0, y_0) where $\nabla f(x_0, y_0) \neq \mathbf{0}$. Let $\mathbf{U}$ be any unit vector, so that $D_U f(x_0, y_0)$ is a function of $\mathbf{U}$.

(i) The maximum value of $D_U f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$, attained when $\mathbf{U}$ has the same direction as $\nabla f(x_0, y_0)$.

(ii) The minimum value of $D_U f(x_0, y_0)$ is $-\|\nabla f(x_0, y_0)\|$, attained when $\mathbf{U}$ has the opposite direction of $\nabla f(x_0, y_0)$.

We extend the definition of a directional derivative to a function of three variables.

12.6.5 Definition Let f be a function of three variables x, y, and z. If $\mathbf{U}$ is the unit vector $\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ then the *directional derivative* of f in the direction of $\mathbf{U}$, denoted by $D_U f$, is given by

$$D_U f(x,y,z) = \lim_{h\to 0}\frac{f(x + h\cos\alpha, y + h\cos\beta, z + h\cos\gamma) - f(x,y,z))}{h}$$

if this limit exists.

12.6.6 Theorem If f is a differentiable function of x, y, and z and $\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, then

$$D_U(x,y,z) = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma$$

12.6.7 Definition If f is a function of three variables x, y, and z and f_x, f_y, and f_z exist, then the *gradient* of f, denoted by ∇f is defined by

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

If $\mathbf{U}$ is a unit vector in R^3, then

$$D_U f(x,y,z) = \mathbf{U}\cdot\nabla f(x,y,z)$$

Exercises 12.6

In Exercises 1–6, find the directional derivative of the function in the direction of the unit vector $\mathbf{U}$ by using the appropriate Definition (12.6.1 or 12.6.5). Verify your result by applying Theorem 12.6.2 or Theorem 12.6.6.

1. $f(x,y) = 2x^2 + 5y^2$; $\mathbf{U} = \cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j}$

▸ $$D_U f(x,y) = \lim_{h\to 0}\frac{f(x + h\cos\frac{1}{4}\pi, y + h\sin\frac{1}{4}\pi) - f(x,y)}{h} = \lim_{h\to 0}\frac{2(x + \frac{1}{2}\sqrt{2}h)^2 + 5(y + \frac{1}{2}\sqrt{2}h)^2 - (2x^2 + 5y^2)}{h}$$

$$= \lim_{h\to 0}\frac{2x^2 + 2\sqrt{2}hx + h^2 + 5y^2 + 5\sqrt{2}hy + \frac{5}{2}h^2 - 2x^2 - 5y^2}{h} = \lim_{h\to 0}(2\sqrt{2}x + h + 5\sqrt{2}y + \frac{5}{2}h)$$

$$= 2\sqrt{2}x + 5\sqrt{2}y$$

$$D_U f(x,y) = f_x(x,y)\cos\tfrac{1}{4}\pi + f_y(x,y)\sin\tfrac{1}{4}\pi = 4x(\tfrac{1}{2}\sqrt{2}) + 10y(\tfrac{1}{2}\sqrt{2}) = 2\sqrt{2}x + 5\sqrt{2}y$$

2. $g(x,y) = 3x^2 - 4y^2$; $\mathbf{U} = \cos\frac{1}{3}\pi\mathbf{i} + \sin\frac{1}{3}\pi\mathbf{j}$

▸ $$D_U g(x,y) = \lim_{h\to 0}\frac{g(x + h\cos\frac{1}{3}\pi, y + h\sin\frac{1}{3}\pi) - g(x,y)}{h} = \lim_{h\to 0}\frac{3(x + \frac{1}{2}h)^2 - 4(y + \frac{1}{2}\sqrt{3}h)^2 - (3x^2 - 4y^2)}{h}$$

$$= \lim_{h\to 0}\frac{3x^2 + 3xh + \frac{3}{4}h^2 - 4y^2 - 4\sqrt{3}hy - 3h^2 - 3x^2 + 4y^2}{h} = \lim_{h\to 0}(3x + \frac{3}{4}h - 4\sqrt{3}y - 3h) = 3x - 4\sqrt{3}y$$

$$D_U g(x,y) = g_x(x,y)\cos\tfrac{1}{3}\pi + g_y(x,y)\sin\tfrac{1}{3}\pi = 6x(\tfrac{1}{2}) - 8y(\tfrac{1}{2}\sqrt{3}) = 3x - 4\sqrt{3}y$$

3. $h(x,y,z) = 3x^2 + y^2 - 4z^2$; $\mathbf{U} = \cos \frac{1}{3}\pi\mathbf{i} + \cos \frac{1}{4}\pi\mathbf{j} + \cos \frac{2}{3}\pi\mathbf{k}$

▸ $$\mathrm{D}_U h(x,y,z) = \lim_{t\to 0} \frac{h(x + t\cos\frac{1}{3}\pi, y + t\cos\frac{1}{4}\pi, z + t\cos\frac{2}{3}\pi) - h(x,y,z)}{t}$$

$$= \lim_{t\to 0} \frac{3tx + \frac{3}{4}t^2 + \sqrt{2}ty + \frac{1}{2}t^2 + 4tz - t^2}{t} = \lim_{t\to 0}(3x + \tfrac{3}{4}t + \sqrt{2}y + \tfrac{1}{2}t + 4z - t) = 3x + \sqrt{2}y + 4z$$

$$\mathrm{D}_U f(x,y,z) = h_x(x,y,z)\cos\tfrac{1}{3}\pi + h_y(x,y,z)\cos\tfrac{1}{4}\pi + h_z(x,y,z)\cos\tfrac{2}{3}\pi = 6x(\tfrac{1}{2}) + 2y(\tfrac{1}{2}\sqrt{2}) - 8z(-\tfrac{1}{2})$$

$$= 3x + \sqrt{2}y + 4z$$

4. $f(x,y,z) = 6x^2 - 2xy + yz$; $\mathbf{U} = \frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

▸ We use Definition 12.6.4 with

$\cos\alpha = \frac{3}{7}$ $\cos\beta = \frac{2}{7}$ $\cos\gamma = \frac{6}{7}$

Thus,

$$\mathrm{D}_U f(x,y,z) = \lim_{h\to 0} \frac{f(x + \frac{3}{7}h, y + \frac{2}{7}h, z + \frac{6}{7}h) - f(x,y,z)}{h}$$

$$= \lim_{h\to 0} \frac{[6(x + \frac{3}{7}h)^2 - 2(x + \frac{3}{7}h)(y + \frac{2}{7}h) + (y + \frac{2}{7}h)(z + \frac{6}{7}h)] - [6x^2 - 2xy + yz]}{h}$$

$$= \lim_{h\to 0} \frac{6x^2 + \frac{36}{7}xh + \frac{54}{49}h^2 - 2xy - \frac{4}{7}xh - \frac{6}{7}yh - \frac{12}{49}h^2 + yz + \frac{6}{7}yh + \frac{2}{7}hz + \frac{12}{49}h^2 - 6x^2 + 2xy - yz}{h}$$

$$= \lim_{h\to 0} \frac{\frac{32}{7}xh + \frac{54}{49}h^2 + \frac{2}{7}hz}{h} = \lim_{h\to 0}(\tfrac{32}{7}x + \tfrac{54}{49}h + \tfrac{2}{7}z) = \tfrac{32}{7}x + \tfrac{2}{7}z$$

Next, we apply Theorem 12.6.6. We have

$f_x(x,y,z) = 12x - 2y$ $f_y(x,y,z) = -2x + z$ $f_z(x,y,z) = y$

Substituting into the formula of the theorem, we get

$$\mathrm{D}_U f(x,y,z) = (12x - 2y)\tfrac{3}{7} + (-2x + z)\tfrac{2}{7} + \tfrac{6}{7}y = \tfrac{32}{7}x + \tfrac{2}{7}z$$

Therefore, the result of using Definition 12.6.5 agrees with the result of applying Theorem 12.6.6.

5. $g(x,y) = \frac{1}{x - y}$; $\mathbf{U} = -\frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$

▸ $$\mathrm{D}_U g(x,y) = \lim_{h\to 0} \frac{g(x - \frac{12}{13}h, y + \frac{5}{13}h) - g(x,y)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x - \frac{12}{13}h) - (y + \frac{5}{13}h)} - g(x,y)\right]$$

$$= \lim_{h\to 0} \frac{x - y - x + \frac{12}{13}h + y + \frac{5}{13}h}{h(x - y)(x - y - \frac{12}{13}h - \frac{5}{13}h)} = \lim_{h\to 0} \frac{\frac{17}{13}h}{h(x - y)(x - y - \frac{17}{13}h)} = \lim_{h\to 0} \frac{\frac{17}{13}}{(x - y)(x - y - \frac{17}{13}h)} = \frac{17}{13(x - y)^2}$$

$$\mathrm{D}_U g(x,y) = g_x(x,y)\left(-\frac{12}{13}\right) + g_y(x,y)\left(\frac{5}{13}\right) = -\frac{1}{(x - y)^2}\left(-\frac{12}{13}\right) + \frac{1}{(x - y)^2}\left(\frac{5}{13}\right) = \frac{17}{13(x - y)^2}$$

6. $f(x,y) = \frac{1}{x^2 + y^2}$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$. $\mathrm{D}_U f(x,y) = \lim\limits_{h\to 0} \frac{f(x + \frac{3}{5}h, y - \frac{4}{5}h) - f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2} - \frac{1}{x^2 + y^2}\right] = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x^2 + y^2) - (x^2 + \frac{6}{5}hx + \frac{9}{25}h^2 + y^2 - \frac{8}{5}yh + \frac{16}{25}h^2)}{[(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2](x^2 + y^2)}$$

$$= \lim_{h\to 0} \frac{-\frac{6}{5}x - \frac{9}{25}h + \frac{8}{5}y - \frac{16}{25}h}{[(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2](x^2 + y^2)} = \frac{-\frac{6}{5}x + \frac{8}{5}y}{(x^2 + y^2)^2}$$

$$\mathrm{D}_U f(x,y) = f_x(x,y)\tfrac{3}{5} + f_y(x,y)(-\tfrac{4}{5}) = -\frac{2x}{(x^2 + y^2)^2} \cdot \frac{3}{5} - \frac{2y}{(x^2 + y^2)^2}(-\tfrac{4}{5}) = \frac{-\frac{6}{5}x + \frac{8}{5}y}{(x^2 + y^2)^2}$$

In Exercises 7–14, find the gradient of the function.

7. $f(x,y) = 4x^2 - 3xy + y^2$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y,z)\mathbf{j} = (8x - 3y)\mathbf{i} + (2y - 3x)\mathbf{j}$

8. $g(x,y) = \frac{xy}{x^2 + y^2}$

▸ We apply Definition 17.1.3. Thus,

$$\nabla g(x,y) = g_x(x,y)\mathbf{i} + g_y(x,y)\mathbf{j}$$

$$= \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2}\mathbf{i} + \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2}\mathbf{j} = \frac{-x^2y + y^3}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^3 - xy^2}{(x^2 + y^2)^2}\mathbf{j}$$

9. $g(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$. $\nabla g(x,y) = g_x(x,y)\mathbf{i} + g_y(x,y)\mathbf{j} = \dfrac{x}{x^2+y^2}\mathbf{i} + \dfrac{y}{x^2+y^2}\mathbf{j}$

10. $f(x,y) = e^y \tan 2x$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2e^y \sec^2 2y\mathbf{i} + e^y \tan 2x\mathbf{j}$

11. $f(x,y,z) = \dfrac{x-y}{x+z}$. $\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = \dfrac{y+z}{(x+z)^2}\mathbf{i} - \dfrac{1}{x+z}\mathbf{j} - \dfrac{x-y}{(x+z)^2}\mathbf{k}$

12. $f(x,y,z) = 3z \ln(x+y)$

▹ We apply Definition 12.6.7. Thus,

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = \frac{3z}{x+y}\mathbf{i} + \frac{3z}{x+y}\mathbf{j} + 3\ln(x+y)\mathbf{k}$$

13. $g(x,y,z) = xe^{-2y}\sec z$.

▹ $\nabla g(x,y,z) = g_x(x,y,z)\mathbf{i} + g_y(x,y,z)\mathbf{j} + g_z(x,y,z)\mathbf{k} = e^{-2y}\sec z\mathbf{i} - 2xe^{-2y}\sec z\mathbf{j} + xe^{-2y}\sec z \tan z\mathbf{k}$

14. $g(x,y,z) = e^{2z}(\sin x - \cos y)$. $\nabla g(x,y,z) = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k} = e^{2z}\cos x\mathbf{i} + e^{2z}\sin y\mathbf{j} + 2e^{2z}(\sin x - \cos y)\mathbf{k}$

In Exercises 15–22, find the value of the directional derivative at point P_0 for the function in the direction of $\mathbf{U}$.

15. $f(x,y) = x^2 - 2xy^2$; $\mathbf{U} = \cos\pi\mathbf{i} + \sin\pi\mathbf{j} = -\mathbf{i}$

▹ $\nabla f(x,y) = (2x - 2y^2)\mathbf{i} - 4xy\mathbf{j}$; $\nabla f(1,-2) = -6\mathbf{i} + 8\mathbf{j}$. $D_U f(1,-2) = \mathbf{U}\cdot\nabla f(1,-2) = -\mathbf{i}\cdot(-6\mathbf{i}+8\mathbf{j}) = 6$

16. $g(x,y) = 3x^3y + 4y^2 - xy$; $\mathbf{U} = \cos\frac{1}{4}\pi\,\mathbf{i} + \sin\frac{1}{4}\pi\,\mathbf{j}$; $P_0 = (0,3)$

▹ We apply Definition 12.6.2. We have

$g_x(x,y) = 9x^2y - y$ $\qquad$ $g_y(x,y) = 3x^3 + 8y - x$

And because we are given that

$\mathbf{U} = \cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j}$

we take $\theta = \frac{1}{4}\pi$. Therefore, by Definition 12.6.2,

$$\begin{aligned} D_U g(x,y) &= g_x(x,y)\cos\theta + g_y(x,y)\sin\theta \\ &= (9x^2y - y)\cos\tfrac{1}{4}\pi + (3x^3 + 8y - x)\sin\tfrac{1}{4}\pi = (9x^2y - y)(\tfrac{1}{2}\sqrt{2}) + (3x^3 + 8y - x)(\tfrac{1}{2}\sqrt{2}) \end{aligned}$$

Substituting the coordinates of the given point (0, 3), we have

$D_U g(0,3) = (-3)(\frac{1}{2}\sqrt{2}) + (24)(\frac{1}{2}\sqrt{2}) = \frac{21}{2}\sqrt{2}$

17. $g(x,y) = y^2\tan^2 x$; $\mathbf{U} = -\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$. $\nabla g(x,y) = 2y^2 \tan x \sec^2 x\mathbf{i} + 2y\tan^2 x\mathbf{j}$.

$\nabla g(\frac{1}{3}\pi, 2) = 8\tan\frac{1}{3}\pi\sec^2\frac{1}{3}\pi\mathbf{i} + 4\tan^2\frac{1}{3}\pi\mathbf{j} = 8(\sqrt{3})(2)^2\mathbf{i} + 4(\sqrt{3})^2\mathbf{j} = 32\sqrt{3}\mathbf{i} + 12\mathbf{j}$

$D_U g(\frac{1}{3}\pi, 2) = \mathbf{U}\cdot\nabla g(\frac{1}{3}\pi, 2) = (-\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j})\cdot(32\sqrt{3}\mathbf{i} + 2\mathbf{j}) = -48 + 6 = -42$

18. $f(x,y) = xe^{2y}$; $\mathbf{U} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$. $\nabla f = e^{2y}\mathbf{i} + 2xe^{2y}\mathbf{j}$. $D_U f(2,0) = \nabla f(2,0)\cdot\mathbf{U} = (\mathbf{i} + 4\mathbf{j})\cdot(\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = \frac{1}{2} + 2\sqrt{3}$

19. $h(x,y,z) = \cos(xy) + \sin(xy)$; $\mathbf{U} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

▹ $\nabla h(x,y,z) = h_x(x,y,z)\mathbf{i} + h_y(x,y,z)\mathbf{j} + h_z(x,y,z)\mathbf{k} = -y\sin(xy)\mathbf{i} + [-x\sin(xy) + z\cos(yz)]\mathbf{j} + y\cos(yz)\mathbf{k}$

$\nabla h(2,0,-3) = 0\mathbf{i} + [-2\cdot 0 - 3\cdot 1]\mathbf{j} + 0\mathbf{k} = -3\mathbf{j}$. $D_U h(2,0,-3) = \mathbf{U}\cdot\nabla h(2,0,-3) = (-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})\cdot(-3\mathbf{j}) = -2$

20. $f(x,y,z) = \ln(x^2 + y^2 + z^2)$ $\quad \mathbf{U} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} - \dfrac{1}{\sqrt{3}}\mathbf{k}$ $\quad P_0 = (1,3,2)$

▹ We have

$f_x(x,y,z) = \dfrac{2x}{x^2+y^2+z^2}$ $\qquad f_y(x,y,z) = \dfrac{2y}{x^2+y^2+z^2}$ $\qquad f_z(x,y,z) = \dfrac{2z}{x^2+y^2+z^2}$

Thus,

$f_x(1,3,2) = \frac{1}{7}$ $\qquad f_y(1,3,2) = \frac{3}{7}$ $\qquad f_z(1,3,2) = \frac{2}{7}$

Applying Theorem 12.6.6, we have

$$D_U(1,3,2) = f_x(1,3,2)\cos\alpha + f_y(1,3,2)\cos\beta + f_z(1,3,2)\cos\gamma = \frac{1}{7}\cdot\frac{1}{\sqrt{3}} + \frac{3}{7}\left(-\frac{1}{\sqrt{3}}\right) + \frac{2}{7}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{4}{7\sqrt{3}}$$

21. $f(x,y) = e^{-3x}\cos 3y$; $\mathbf{U} = \cos(-\frac{1}{12}\pi)\mathbf{i} + \sin(-\frac{1}{12}\pi)\mathbf{j}$

$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = -3e^{-3x}\cos 3y\mathbf{i} - 3e^{-3x}\sin 3y\mathbf{j}$. $\nabla f(-\frac{1}{12}\pi, 0) = -3e^{\pi/4}\mathbf{i}$

$D_U f(-\frac{1}{12}\pi, 0) = [\cos(-\frac{1}{12}\pi)\mathbf{i} + \sin(-\frac{1}{12}\pi)\mathbf{j}]\cdot(-3e^{\pi/4}\mathbf{i}) = -3e^{\pi/4}\cos\frac{1}{12}\pi$

22. $g(x,y,z) = \cos 2x \cos 3y \sinh 4z$; $\mathbf{U} = \frac{1}{3}\sqrt{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

▹ $\nabla g(x,y,z) = -2\sin 2x\cos 3y\sinh 4z\mathbf{i} - 3\cos 2x\sin 3y\sinh 4z\mathbf{j} + 4\cos 2x\cos 3y\cosh 4z\mathbf{k}$

$D_U g(\frac{1}{2}\pi, 0, 0) = \nabla g(\frac{1}{2}\pi, 0, 0)\cdot\mathbf{U} = -4\mathbf{k}\cdot\frac{1}{3}\sqrt{3}(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -\frac{4}{3}\sqrt{3}$

In Exercises 23–26, find (a) the gradient of f at P and (b) the rate of change of f in the direction of **U** at P.

23. $f(x, y) = x^2 - 4y$; $\mathbf{U} = \cos\frac{1}{3}\pi\mathbf{i} + \sin\frac{1}{3}\pi\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$

(a) $\nabla f(x, y) = 2x\mathbf{i} - 4\mathbf{j}$; $\nabla f(-2, 2) = -4\mathbf{i} - 4\mathbf{j}$

(b) $D_U f(-2, 2) = (\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) \cdot (-4\mathbf{i} - 4\mathbf{j}) = -2 - 2\sqrt{3}$

24. $f(x, y) = e^{2xy}$; $\mathrm{P} = (2, 1)$; $\mathbf{U} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

▷ (a) $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2ye^{2xy}\mathbf{i} + 2xe^{2xy}\mathbf{j}$

Thus,

$\nabla f(2, 1) = 2e^4\mathbf{i} + 4e^4\mathbf{j}$

(b) The rate of change of the function in the direction of **U** at P is given by

$\mathbf{U} \cdot \nabla f(2, 1) = (\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}) \cdot (2e^4\mathbf{i} + 4e^4\mathbf{j}) = \frac{8}{5}e^4 - \frac{12}{5}e^4 = -\frac{4}{5}e^4$

25. $f(x, y, z) = y^2 + z^2 - 4xz$; $\mathbf{U} = \frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

▷ (a) $\nabla f(x, y, z) = -4z\mathbf{i} + 2y\mathbf{j} + (2z - 4x)\mathbf{k}$; $\nabla f(-2, 1, 3) = -12\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$

(b) $D_U f(-2, 1, 3) = \left(\frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}\right) \cdot (-12\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}) = -\frac{24}{7} - \frac{12}{7} + \frac{42}{7} = \frac{6}{7}$

26. $f(x, y, z) = 2x^3 + xy^2 + xz^2$; $\mathrm{P} = (1, 1, 1)$; $\mathbf{U} = \frac{1}{7}\sqrt{21}\mathbf{j} - \frac{2}{7}\sqrt{7}\mathbf{k}$

(a) $\nabla f(x, y, z) = (6x^2 + y^2 + z^2)\mathbf{i} + 2xy\mathbf{j} + 2xz\mathbf{k}$; $\nabla f(1, 1, 1) = 8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(b) $D_U f(1, 1, 1) = (\frac{1}{7}\sqrt{21}\mathbf{j} - \frac{2}{7}\sqrt{7}\mathbf{k}) \cdot (8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{2}{7}\sqrt{21} - \frac{4}{7}\sqrt{7}$

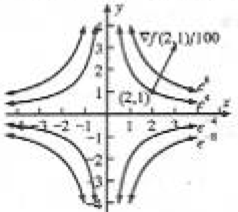
27. $f(x, y) = x^2 - 4y$. The level curves of f at 8, 4, 0, −4, and −8 are, respectively, the parabolas $x^2 - 4y = 8$, $x^2 - 4y = 4$, $x^2 - 4y = 0$, $x^2 - 4y = -4$, and $x^2 - 4y = -8$.

28. Draw a contour map showing the level curves of the function of Exercise 24 at e^8, e^4, 1, e^{-4}, e^{-8}. Also show the representation of $\nabla f(2, 1)$ having its initial point at $(2, 1)$.

▷ The function of Exercises 24 is defined by $f(x, y) = e^{2xy}$. If $f(x, y) = e^k$, then $e^{2xy} = e^k$ or equivalently, $2xy = k$, $k = 8, 4, 0, -4, -8$. If $k \neq 0$, these are equilateral hyperbolas passing through the point $(1, \frac{1}{2}k)$. If $k = 0$, then $xy = 0$ and the level curve consists of the lines $x = 0$ and $y = 0$, that is, the axes. Because

$\nabla f(2, 1) \approx 109.2\mathbf{i} + 218.4\mathbf{j}$

the figure shows the level curves and a representation of $\frac{1}{100}\nabla f(2, 1)$.



In Exercises 29–32, find $D_U f$ at the point P for which **U** is a unit vector on the direction of $\overrightarrow{\mathrm{PQ}}$. Also at P find the maximum of $D_U f$.

29. $f(x, y) = e^x \tan^{-1} y$; $\mathrm{P} = (0, 1)$; $\mathrm{Q} = (3, 5)$. $\mathbf{V}(\overrightarrow{\mathrm{PQ}}) = \langle 3 - 0, 5 - 1\rangle = \langle 3, 4\rangle$; $\left\|\mathbf{V}(\overrightarrow{\mathrm{PQ}})\right\| = \sqrt{9 + 16} = 5$

Therefore, a unit vector in the direction of $\overrightarrow{\mathrm{PQ}}$ is $\mathbf{U} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = e^x \tan^{-1} y\mathbf{i} + \dfrac{e^x}{1 + y^2}\mathbf{j}$; $\nabla f(0, 1) = \frac{\pi}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

$D_U f(0, 1) = \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) \cdot \left(\frac{1}{4}\pi\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{3}{20}\pi + \frac{2}{5} \approx 0.871$

The maximum of $D_U f(0, 1)$ is $\|\nabla f(0, 1)\| = \sqrt{\dfrac{\pi^2}{16} + \dfrac{1}{4}} = \frac{1}{4}\sqrt{\pi^2 + 4} \approx 0.931$.

30. $f(x, y) = e^x \cos y + e^y \sin x$; $\mathrm{P}(1, 0)$, $\mathrm{Q}(-3, 3)$. $\overrightarrow{\mathrm{PQ}} = \mathbf{q} - \mathbf{p} = \langle -3, 3\rangle - \langle 1, 0\rangle = -4\mathbf{i} + 3\mathbf{j}$. $\left\|\overrightarrow{\mathrm{PQ}}\right\| = \sqrt{4^2 + 3^2} = 5$.
$\mathbf{U} = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$. $\nabla f(x, y) = f_x\mathbf{i} + f_y\mathbf{j} = (e^x \cos y + e^y \cos x)\mathbf{i} + (-e^x \sin y + e^y \sin x)$;
$\nabla f(1, 0) = (e + \cos 1)\mathbf{i} + \sin 1\mathbf{j}$. $D_U f(0, 1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j}) \cdot [(e + \cos 1)\mathbf{i} + \sin 1\mathbf{j}] = \frac{1}{5}(-4e - 4\cos 1 + 3\sin 1)$

The maximum of $D_U f(0, 1)$ is $\|\nabla f(1, 0)\| = \sqrt{(e + \cos 1)^2 + \sin^2 1} = \sqrt{e^2 + 2e\cos 1 + 1}$.

31. $f(x, y, z) = x - 2y + z^2$; $\mathrm{P} = (3, 1, -2)$; $\mathrm{Q} = (10, 7, 4)$

$\mathbf{V}(\overrightarrow{\mathrm{PQ}}) = \langle 10 - 3, 7 - 1, 4 + 2\rangle = \langle 7, 6, 6\rangle$; $\left\|\mathbf{V}(\overrightarrow{\mathrm{PQ}})\right\| = \sqrt{49 + 36 + 36} = \sqrt{121} = 11$

Therefore, a unit vector in the direction of $\overrightarrow{\mathrm{PQ}}$ is $\mathbf{U} = \frac{7}{11}\mathbf{i} + \frac{6}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}$.
$\nabla f(x, y, z) = \mathbf{i} - 2\mathbf{j} + 2z\mathbf{k}$. Hence, $\nabla f(3, 1, -2) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$. Therefore

$D_U f(3, 1, -2) = \left(\frac{7}{11}\mathbf{i} + \frac{6}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}\right) \cdot (\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}) = \frac{7}{11} - \frac{12}{11} - \frac{24}{11} = -\frac{29}{11} \approx -2.64$

The maximum of $D_U f(3, 1, -2)$ is $\|\nabla f(3, 1, -2)\| = \sqrt{1 + 4 + 16} = \sqrt{21} \approx 4.58$.

32. $f(x,y,z) = x^2 + y^2 - 4xz$; $P(3,1,-2)$, $Q(-6,3,4)$

▹ We find the gradient vector. Thus,

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = (2x - 4z)\mathbf{i} + 2y\mathbf{j} - 4x\mathbf{k}$$

Hence,

$$\nabla f(3,1,-2) = 14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k} \tag{1}$$

Next, we find $\mathbf{U}$, a unit vector in the direction of $\overrightarrow{PQ}$. We have

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \langle -6,3,4\rangle - \langle 3,1,-2\rangle = -9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$\left\|\overrightarrow{PQ}\right\| = \sqrt{81+4+36} = \sqrt{121} = 11$

Thus,

$$\mathbf{U} = \overrightarrow{PQ}/\left\|\overrightarrow{PQ}\right\| = \tfrac{1}{11}(-9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) \tag{2}$$

From (1) and (2) we obtain

$D_U f(3,1,-2) = \mathbf{U}\cdot\nabla f(3,1,-2) = \tfrac{1}{11}(-9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})\cdot(14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}) = -\tfrac{194}{11}$

The maximum value of D_U, attained when $\mathbf{U}$ has the direction of $\nabla f(3,1,-2)$, is

$\|\nabla f(3,1,-2)\| = \|14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}\| = \sqrt{196+4+144} = 2\sqrt{86}$

33. $f(x,y) = e^{2y}\tan^{-1}\frac{y}{3x}$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = -\dfrac{3ye^{2y}}{9x^2+y^2}\,\mathbf{i} + \left(2e^{2y}\tan^{-1}\dfrac{y}{3x} + \dfrac{3xe^{2y}}{9x^2+y^2}\right)\mathbf{j}$

$\nabla f(1,3) = -\frac{1}{2}e^6\mathbf{i} + (\frac{1}{2}\pi e^6 + \frac{1}{6}e^6)\mathbf{j}$. Let $\mathbf{U} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. We seek the values of θ for which

$\nabla f(1,3)\cdot\mathbf{U} = 0$; $-\frac{1}{2}e^6\cos\theta + (\frac{1}{2}\pi e^6 + \frac{1}{6}e^6)\sin\theta = 0$; $-\cos\theta + (\pi + \frac{1}{3})\sin\theta = 0$; $\tan\theta = \dfrac{3}{3\pi+1}$

Therefore $\theta = \tan^{-1}\dfrac{3}{3\pi+1}$ and $\theta = \tan^{-1}\dfrac{3}{3\pi+1} - \pi$.

34. $\rho(x,y) = (x^2+y^2+3)^{-1/2}$. $\nabla\rho(x,y) = -(x^2+y^2+3)^{-3/2}(x\mathbf{i} + y\mathbf{j})$. $\nabla\rho(3,2) = -\frac{1}{64}(3\mathbf{i} + 2\mathbf{j})$.

(a) $\mathbf{U} = \cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j}$. $D_U\rho(3,2) = -\frac{1}{64}(3\mathbf{i} + 2\mathbf{j})\cdot(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = -\frac{1}{64}(-\frac{3}{2} + \sqrt{3})$ slugs/ft^2/ft

(b) The greatest magnitude is $\frac{1}{64}\|3\mathbf{i} + 2\mathbf{j}\| = \frac{1}{64}\sqrt{9+4} = \frac{1}{64}\sqrt{13}$ in the direction of $-(3\mathbf{i} + 2\mathbf{j})$.

35. $T = 3x^2 + 2xy$. (a) $\nabla T(x,y) = T_x(x,y)\mathbf{i} + T_y(x,y)\mathbf{j} = (6x+2y)\mathbf{i} + 2x\mathbf{j}$; $\nabla T(3,-6) = 6\mathbf{i} + 6\mathbf{j}$. The number of degrees per meter in the maximum rate of change of T at $(3,-6)$ is $\|\nabla T(3,-6)\| = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$
(b) The maximum rate of change of T occurs in the direction of $\nabla T(3,-6) = 6\mathbf{i} + 6\mathbf{j}$. A unit vector in this direction is $\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}$.

36. The temperature is $T(x,y,z)$ degrees at any point (x,y,z) in three-dimensional space and $T(x,y,z) = 60/(x^2+y^2+z^2+3)$. Distance is measured in inches. Find: (a) the rate of change of the temperature at the point $(3,-2,2)$ in the direction of the vector $-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, and (b) the direction and magnitude of the greatest rate of change of T at $(3,-1,2)$

▹ Because

$$T(x,y,z) = \frac{60}{x^2+y^2+z^2+3}$$

then

$$\begin{aligned}\nabla T(x,y,z) &= f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}\\ &= \frac{-120x}{(x^2+y^2+z^2+3)^2}\mathbf{i} + \frac{-120y}{(x^2+y^2+z^2+3)^2}\mathbf{j} + \frac{-120z}{(x^2+y^2+z^2+3)^2}\mathbf{k}\end{aligned}$$

Hence,

$$\nabla T(3,-2,2) = \frac{-120}{(9+4+4+3)^2}(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = -\tfrac{3}{10}(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

(a) Because $\|-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}\| = \sqrt{4+9+36} = 7$, a unit vector in the direction of $-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ is given by

$$\mathbf{U} = \tfrac{1}{7}(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \tag{2}$$

Hence, from (2) and (1), we obtain

$D_U T(3,-2,2) = \mathbf{U}\cdot\nabla T(3,-2,2) = \frac{1}{7}(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})\cdot(-\frac{3}{10})(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \frac{36}{35}$

Therefore, the temperature is increasing at the rate of $\frac{36}{35}$ degrees per inch.

(b) The greatest rate of change of T is in the direction of $\nabla T(3,-2,2)$, that is, in the direction of $-3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Its magnitude is $\frac{3}{10}\|3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\| = \frac{3}{10}\sqrt{17}$. Thus, the greatest rate of change in the temperature is about 1.24 degrees per inch.

37. $V = e^{-2x}\cos 2y$.

▸ (a) $\mathbf{U} = \cos\frac{1}{6}\pi\mathbf{i} + \sin\frac{1}{6}\pi\mathbf{j} = \frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$. $\nabla V(x,y) = V_x(x,y)\mathbf{i} + V_y(x,y)\mathbf{j} = -2e^{-2x}\cos 2y\mathbf{i} - 2e^{-2x}\sin 2y\mathbf{j}$ $\nabla V(0,\frac{1}{4}\pi) = -2e^0\cos\frac{1}{2}\pi\mathbf{i} - 2e^0\sin\frac{1}{2}\pi\mathbf{j} = -2\mathbf{j}$. The rate of change of V at the point $(0,\frac{1}{4}\pi)$ in the direction of $\mathbf{U}$ is $D_U V(0,\frac{1}{4}\pi) = \mathbf{U}\cdot\nabla V(0,\frac{1}{4}\pi) = (\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j})\cdot(-2\mathbf{j}) = -1$. (b) The greatest rate of change of V at $(0,\frac{1}{4}\pi)$ occurs in the direction of $\nabla V(0,\frac{1}{4}\pi)$. The direction is that of the unit vector $-\mathbf{j}$ and its magnitude is 2.

38. $z = 1200 - 3x^2 - 2y^2$. $\nabla z = -6x\mathbf{i} - 4y\mathbf{j}$. $\nabla z(-10,5) = 60\mathbf{i} - 20\mathbf{j}$ (a) Steepest ascent in direction $(3\mathbf{i} - \mathbf{j})/\sqrt{10}$
(b) $\nabla z\cdot\mathbf{i} = 60$: ascending 60 meters for each meter east (c) $\nabla z\cdot -(\mathbf{i}+\mathbf{j})/\sqrt{2} = -40/\sqrt{2} = -20\sqrt{2}$ m/m
(c) level is orthogonal: $\pm(\mathbf{i} + 3\mathbf{j})/\sqrt{10}$

12.7 TANGENT PLANES AND NORMALS TO SURFACES

Surfaces The graph of a continuous function $z = f(x,y)$ is a surface. f is differentiable at (x_0,y_0) if and only if the graph of $z = f(x,y)$ has a tangent plane at (x_0,y_0). The equation of the tangent plane is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

The graph of a function is a special case of the parametric surface $x = f(u,v)$, $y = g(u,v)$, $z = g(u,v)$ where f, g, and h are continuous and the representation is one-to-one, that is, distinct values of (u,v) give distinct values of (x,y,z). Thus, the cylinder $x^2 + y^2 = a^2$, which cannot be represented in the form $z = f(x,y)$, can be represented as $x = a\cos u$, $y = a\sin u$, $z = v$, $0 \le u < 2\pi$ and the sphere $x^2 + y^2 + z^2 = a^2$ can be represented as $x = a\cos u\sin v$, $y = a\sin u\sin v$, $z = a\cos v$, $0 \le u < 2\pi$, $0 \le v \le \pi$. (Compare with cylindrical and spherical coordinates in §13.1.) Representations such as these were used to prepare the computer graphics in the text. The grids seen in some figures represent equally spaced values of each parameter.

Suppose f, g, and h have partial derivatives with respect to u and v. If v is held constant, then $\mathbf{R} = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is a curve and $\mathbf{R}_u = f_u(u,v)\mathbf{i} + g_u(u,v)\mathbf{j} + h_u(u,v)\mathbf{k}$ is a vector tangent to the curve, and hence to the surface. Similarly, the vector $\mathbf{R}_v = f_v(u,v)\mathbf{i} + g_v(u,v)\mathbf{j} + h_v(u,v)\mathbf{k}$ is tangent to the surface. If, in addition, f, g, and h are differentiable, then all the tangents to the surface lie in a plane whose normal vector is $\mathbf{n} = \mathbf{R}_u \times \mathbf{R}_v$. If, in addition, $\mathbf{n}$ is continuous, then the representation is one-to-one if $\mathbf{n} \ne \mathbf{0}$.

Often we can eliminate the parameters u and v and express the surface in the form $F(x,y,z) = 0$.

12.7.1 Definition A vector that is orthogonal to a tangent vector of every curve C through a point P_0 on a surface S is called a *normal vector* to S at P_0.

12.7.2 Theorem If an equation of a surface S is $F(x,y,z) = 0$, where F is differentiable and F_x, F_y, and F_z are not all zero at the point $P_0(x_0,y_0,z_0)$ on S, then a normal vector to S at P_0 is $\nabla F(x_0,y_0,z_0)$.

12.7.6 Theorem If an equation of a curve C is $F(x,y) = 0$, where F is differentiable and F_x and F_y, are not both zero at the point $P_0(x_0,y_0)$ on C, then a normal vector to C at P_0 is $\nabla F(x_0,y_0)$.

12.7.3 Definition If an equation of a surface S is $F(x,y,z) = 0$, then the *tangent plane* of S at a point $P_0(x_0,y_0,z_0)$ is the plane through P_0 having as normal vector $\nabla F(x_0,y_0,z_0)$.

An equation of the tangent plane of Definition 12.7.3 is

$$\nabla F(x_0,y_0,z_0)\cdot[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or, equivalently,

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$$

Two surfaces are *tangent* at a point P if they have a common tangent plane at P or, equivalently, their normal vectors at P are parallel.

12.7.4 Definition The *normal line* to a surface S at a point P_0 on S is the line through P_0 parallel to a normal vector to S at P_0.

If an equation of the surface S is $F(x,y,z) = 0$, then parametric equations of the normal line to S at $P_0(x_0,y_0,z_0)$ are

$$x = x_0 + tF_x(x_0,y_0,z_0),\ y = y_0 + tF_y(x_0,y_0,z_0),\ z = z_0 + tF_z(x_0,y_0,z_0)$$

12.7.5 Definition The *tangent line* to a curve C at a point P_0 is the line through P_0 parallel to a tangent vector to C at P_0.
If the curve C is the intersection of two surfaces with normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$, then a tangent to the curve is parallel to $\mathbf{N}_1 \times \mathbf{N}_2$.

Exercises 12.7

In Exercises 1–12, find an equation of the tangent plane and equations of the normal line to the surface at the point.

1. The given surface is the sphere $x^2 + y^2 + z^2 = 17$. Let $F(x,y,z) = x^2 + y^2 + z^2 - 17$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$; $\nabla F(2,-2,3) = 4\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$. A normal vector is $2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. At $(2,-2,3)$, an equation of the tangent plane is $2(x-2) - 2(y+2) + 3(z-3) = 0$; $2x - 2y + 3z = 17$;
symmetric equations of the normal line are $\frac{x-2}{2} = \frac{y+2}{-2} = \frac{z-3}{3}$.

2. The given surface is ellipsoid $4x^2 + y^2 + 2z^2 = 26$. Let $F(x,y,z) = 4x^2 + y^2 + 2z^2 - 26$.
$\nabla F(x,y,z) = 8x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}$; $\nabla F(1,-2,3) = 8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. A normal vector is $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. At $(1,-2,3)$, an equation of the tangent plane is $2(x-1) - (y+2) + 3(z-3) = 0$; $2x - y + 3z - 13 = 0$;
symmetric equations of the normal line are $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{3}$.

3. The given surface is the paraboloid of revolution $x^2 + y^2 - 3z = 2$. Let $F(x,y,z) = x^2 + y^2 - 3z - 2$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$; $\nabla F(-2,-4,6) = -4\mathbf{i} - 8\mathbf{j} - 3\mathbf{k}$. At $(-2,-4,6)$, an equation of the tangent plane is $-4(x+2) - 8(y+4) - 3(z-6) = 0$; $4x + 8y + 3z + 22 = 0$;
symmetric equations of the normal line are $\frac{x+2}{4} = \frac{y+4}{8} = \frac{z-6}{3}$.

4. $x^2 + y^2 - z^2 = 6$; $(3,-1,2)$

▹ Let F be the function defined by
$F(x,y,z) = x^2 + y^2 - z^2 - 6$
Then
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$
A normal to the hyperboloid of one sheet $F(x,y,z) = 0$ at $(3,-1,2)$ is given by
$\nabla F(3,-1,2) = 6\mathbf{i} - 2\mathbf{j} - 4z\mathbf{k}$
or by
$3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
Thus, at $(3,-1,2)$ an equation of the tangent plane is
$3(x-3) - (y+1) - 2(z-2) = 0$
$3x - y - 2z = 0$
and symmetric equations of the normal line are
$\frac{x-3}{3} = \frac{y+1}{-1} = \frac{z-2}{-2}$

5. The given surface is $y = e^x \cos z$. Let $F(x,y,z) = e^x \cos z - y$.
$\nabla F(x,y,z) = e^x \cos z\mathbf{i} - \mathbf{j} - e^x \sin z\mathbf{k}$; $\nabla F(1,e,0) = e\mathbf{i} - \mathbf{j} + 0\mathbf{k}$. At $(1,e,0)$, an equation of the tangent plane is $e(x-1) - (y-e) + 0(z-0) = 0$; $ex - y = 0$;
symmetric equations of the normal line are $\frac{x-1}{e} = \frac{y-e}{-1}$, $z = 0$.

6. The given surface is $z = e^{3x} \sin 3y$. Let $F(x,y,z) = e^{3x} \sin 3y - z$.
$\nabla F(x,y,z) = 3e^{3x} \sin 3y\mathbf{i} + 3e^{3x} \cos 3y\mathbf{j} - \mathbf{k}$. $\nabla F(0,\frac{1}{6}\pi,1) = (3,0,-1)$. At $(0,\frac{1}{6}\pi,1)$, an equation of the tangent plane is $3x - (z-1) = 0$; $3x - z + 1 = 0$;
symmetric equations of the normal line are $\frac{x}{3} = \frac{z-1}{-1}$, $y = \frac{1}{6}\pi$

7. The given surface is the parabolic cylinder $x^2 = 12y$. Let $F(x,y,z) = x^2 - 12y$.
$\nabla(x,y,z) = 2x\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$; $\nabla F(6,3,3) = 12\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$. At $(6,3,3)$, an equation of the tangent plane is $12(x-3) - 12(y-3) + 0(z-3) = 0$; $x - y - 3 = 0$;
symmetric equations of the normal line are $\frac{x-6}{1} = \frac{y-3}{-1}$, $z = 3$.

8. $z = x^{1/2} + y^{1/2}$; $(1,1,2)$

▸ Let F be the function defined by
$F(x,y,z) = x^{1/2} + y^{1/2} - z$
$\nabla F(x,y,z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}y^{-1/2}\mathbf{j} - \mathbf{k}$
A normal to the graph of $F(x,y,z) = 0$ at $(1,1,2)$ is given by
$\nabla F(1,1,2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$
or
$\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
Thus, at $(1,1,2)$ an equation of the tangent plane is
$(x-1) + (y-1) - 2(z-2) = 0$
$x + y - 2z + 2 = 0$
and symmetric equations of the normal line are
$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-2}{-2}$

9. The given surface is $x^{1/2} + y^{1/2} + z^{1/2} = 4$. Let $F(x,y,z) = x^{1/2} + y^{1/2} + z^{1/2} - 4$.
$\nabla F(x,y,z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}y^{-1/2}\mathbf{j} + \frac{1}{2}z^{-1/2}\mathbf{k}$; $\nabla F(4,1,1) = \frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$. A normal vector is $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. At $(4,1,1)$, an equation of the tangent plane is $(x-4) + 2(y-1) + 2(z-1) = 0$; $x + 2y + 2z - 8 = 0$; symmetric equations of the normal line are $\frac{x-4}{1} = \frac{y-1}{2} = \frac{z-1}{2}$.

10. The given surface is $zx^2 - xy^2 - yz^2 = 18$. Let $F(x,y,z) = zx^2 - xy^2 - yz^2 - 18$.
$\nabla F(x,y,z) = (2xz - y^2)\mathbf{i} - (2xy + z^2)\mathbf{j} + (x^2 - 2zy)\mathbf{k}$; $\nabla F(0,-2,3) = -4\mathbf{i} - 9\mathbf{j} + 12\mathbf{j}$. At $(0,-2,3)$, an equation of the tangent plane is $-4x - 9(y+2) + 12(z-3) = 0$; $-4x - 9y + 12z - 54 = 0$; symmetric equations of the normal line are $\frac{x}{-4} = \frac{y+2}{-9} = \frac{z-3}{13}$.

11. The given surface is $x^{2/3} + y^{2/3} + z^{2/3} = 14$. Let $F(x,y,z) = x^{2/3} + y^{2/3} + z^{2/3} - 14$.
$\nabla F(x,y,z) = \frac{2}{3}x^{-1/3}\mathbf{i} + \frac{2}{3}y^{-1/3}\mathbf{j} + \frac{2}{3}z^{-1/3}\mathbf{k}$; $\nabla F(-8,27,1) = -\frac{1}{3}\mathbf{i} + \frac{2}{9}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Normal: $-3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. At $(-8,27,1)$, an equation of the tangent plane is $-3(x+8) + 2(y-27) + 6(z-1) = 0$; $3x - 2y - 6z + 84 = 0$; symmetric equations of the normal line are $\frac{x+8}{-3} = \frac{y-27}{2} = \frac{z-1}{6}$.

12. $x^{1/2} + z^{1/2} = 8$; $(25,2,9)$

▸ Let F be the function defined by
$F(x,y,z) = x^{1/2} + z^{1/2} - 8$
$\nabla F(x,y,z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}z^{-1/2}\mathbf{k}$
A normal to the parabolic cylinder $F(x,y,z) = 0$ at $(25,2,9)$ is given by
$\nabla F(25,2,9) = \frac{1}{10}\mathbf{i} + \frac{1}{6}\mathbf{k}$
or
$3\mathbf{i} + 5\mathbf{k}$
Thus, at $(25,2,9)$ an equation of the tangent plane is
$3(x-25) + 5(z-9) = 0$
$3x + 5z - 120 = 0$
and symmetric equations of the normal line are
$\frac{x-25}{3} = \frac{z-9}{5} \quad y = 2$

In Exercises 13–20, if the two surfaces intersect in a curve, find equations of the tangent line to the curve of intersection at the given point; if the two surfaces are tangent at the given point, prove it.

13. $x^2 + y^2 - z = 8$, $x - y^2 + z^2 = -2$; $(2,-2,0)$. Let $F(x,y,z) = x^2 + y^2 - 8$ and $G(x,y,z) = x - y^2 + z^2 + 2$. Then $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ and $G(x,y,z) = \mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$.

$\mathbf{n}_1 = \nabla F(2,-2,0) = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$; $\mathbf{n}_2 = \nabla G(2,-2,0) = \mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & -1 \\ 1 & 4 & 0 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} + 20\mathbf{k}$.

Equations of the tangent line of the curve of intersection are $\frac{x-2}{4} = \frac{y+2}{-1} = \frac{z}{20}$.

14. The given surfaces are $F(x,y,z) = x^2 + y^2 - 2z + 1 = 0$ and $G(x,y,z) = x^2 + y^2 - z^2 = 0$.
Then $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$.
$\mathbf{n}_1 = \nabla F(0,1,1) = 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{n}_2 = \nabla G(0,1,1) = 2\mathbf{j} - 2\mathbf{k}$. Because $\mathbf{n}_1 = \mathbf{n}_2$, the surfaces are tangent at $(0,1,1)$.

15. $y = x^2$ and $y = x^2$, $y = 16 - z^2$; $(4,16,0)$. Let $F(x,y,z) = x^2 - y$ and
$G(x,y,z) = y^2 + z^2 - 16$. Then $\nabla F(x,y,z) = 2x\mathbf{i} - \mathbf{j}$ and $\nabla G(x,y,z) = \mathbf{j} + 2z\mathbf{k}$.

$$\mathbf{n}_1 = \nabla F(4,16,0) = 8\mathbf{i} - \mathbf{j};\ \mathbf{n}_2 = \nabla G(4,16,0) = \mathbf{j}.\ \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 8\mathbf{k}.$$

Equations of the tangent line of the curve of intersection are $x = 4$, $y = 16$.

16. $x = 2 + \cos \pi yz$, $y = 1 + \sin \pi xz$; $(3,1,2)$

▸ Let F and G be the functions defined by
$F(x,y,z) = 2 + \cos \pi yz - x$ $\qquad$ $G(x,y,z) = 1 + \sin \pi xz - y$
We have
$\nabla F(x,y,z) = -\mathbf{i} - \pi z \sin \pi yz\mathbf{j} - \pi y \sin \pi yz\mathbf{k}$ $\qquad$ $\nabla G(x,y,z) = \pi z \cos \pi xz\mathbf{i} - \mathbf{j} + \pi x \cos \pi xz\mathbf{k}$
We take
$\mathbf{N}_1 = \nabla F(3,1,2) = -\mathbf{i}$ $\qquad$ $\mathbf{N}_2 = \nabla G(3,1,2) = 2\pi\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}$
Because the normal $\mathbf{N}_1$ to the surface $F(x,y,z) = 0$ and the normal $\mathbf{N}_2$ to the surface $G(x,y,z) = 0$ are not parallel, the surfaces intersect in a curve. A tangent to this curve is given by

$$\mathbf{N}_1 \times \mathbf{N}_2 = (-\mathbf{i}) \times (2\pi\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}) = 3\pi\mathbf{j} + \mathbf{k}$$

Symmetric equations of the tangent line at $(3,1,2)$ are

$$x = 3 \quad \frac{y-1}{3\pi} = \frac{z-2}{1}$$

17. $y = e^x \sin 2\pi z + 2$, $z = y^2 - \ln(x+1) - 3$; $(0,2,1)$

▸ Let $F(x,y,z) = e^x \sin 2\pi z - y + 2$ and $G(x,y,z) = y^2 - \ln(x+1) - z - 3$.
$\nabla F(x,y,z) = e^x \sin 2\pi z\mathbf{i} - \mathbf{j} + 2\pi e^x \cos 2\pi z\mathbf{k}$ and $\nabla G(x,y,z) = -\frac{1}{x+1}\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$.

$$\mathbf{n}_1 = \nabla F(0,2,1) = -\mathbf{j} + 2\pi\mathbf{k};\ \mathbf{n}_2 = \nabla G(0,2,1) = -\mathbf{i} + 4\mathbf{j} - \mathbf{k}.\ \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2\pi \\ -1 & 4 & -1 \end{vmatrix} = (1 - 8\pi)\mathbf{i} - 2\pi\mathbf{j} - \mathbf{k}.$$

Equations of the tangent line of the curve of intersection are $\frac{x}{1-8\pi} = \frac{y-2}{-2\pi} = \frac{z-1}{-1}$.

18. The given surfaces are $F(x,y,z) = x^2 - 3xy + y^2 - z = 0$ and $G(x,y,z) = 2x^2 + y^2 - 3z + 27 = 0$.
$\nabla F(x,y,z) = (2x - 3y)\mathbf{i} + (-3x + 2y)\mathbf{j} - \mathbf{k}$ and $\nabla G(x,y,z) = 4x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$

$$\mathbf{n}_1 = \nabla F(1,-2,11) = 8\mathbf{i} - 7\mathbf{j} - \mathbf{k};\ \mathbf{n}_2 = \nabla G(1,-2,11) = 4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}.\ \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -7 & -1 \\ 4 & -4 & -3 \end{vmatrix} = 17\mathbf{i} + 20\mathbf{j} - 4\mathbf{k}.$$

Equations of the tangent line of the curve of intersection are $\frac{x-1}{17} = \frac{y+2}{20} = \frac{z-11}{-4}$.

19. $x^2 + z^2 + 4y = 0$, $x^2 + y^2 + z^2 - 6z + 7 = 0$; $(0,-1,2)$

▸ Let $F(x,y,z) = x^2 + z^2 + 4y$ and $G(x,y,z) = x^2 + y^2 + z^2 - 6z + 7$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 4\mathbf{j} + 2z\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + (2z - 6)\mathbf{k}$.
$\mathbf{n}_1 = \nabla F(0,-1,2) = 0\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \nabla G(0,-1,2) = 0\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = -\frac{1}{2}\mathbf{n}_1$.
Hence $\mathbf{n}_1$ is parallel to $\mathbf{n}_2$, and so the surfaces are tangent at $(0,-1,2)$. Tangent surfaces may have a curve of intersection, but adding the first equation to twice the second, we get $3x^2 + 2y^2 + 4y + 3z^2 - 2z + 14 = 0$; $3x^2 + 2(y+1)^2 + 3(z-2)^2 = 0$, and so the surfaces intersect only at $(0,-1,2)$.

20. $x^2 + y^2 + z^2 = 8$, $yz = 4$; $(0,2,2)$

▸ Let F and G be the functions defined by
$F(x,y,z) = x^2 + y^2 + z^2 - 8$ and $\qquad$ $G(x,y,z) = yz - 4$
We have
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\qquad$ $\nabla G(x,y,z) = z\mathbf{j} + y\mathbf{k}$
We take
$\mathbf{N}_1 = \nabla F(0,2,2) = 4\mathbf{j} + 4\mathbf{k}$ $\qquad$ $\mathbf{N}_2 = \nabla G(0,2,2) = 2\mathbf{j} + 2\mathbf{k}$
Because $\mathbf{N}_1 = 2\mathbf{N}_2$, the normal vectors to the surfaces are parallel and so the surfaces are tangent at $(0,2,2)$.

In Exercises 21–24, use the gradient to find an equation of the tangent line to the curve at the indicated point.

21. $F(x,y) = 9x^3 - y^3 - 1 = 0$. $\nabla F(x,y) = 27x^2\mathbf{i} - 3y^2\mathbf{j}$. $\nabla F(1,2) = 27\mathbf{i} - 12\mathbf{j}$. $27(x-1) - 12(y-2) = 0$; $9x - 4y = 1$

22. $F(x,y) = 16x^4 + y^4 - 32 = 0$. $\nabla F(x,y) = 64x^3\mathbf{i} + 4y^3\mathbf{j}$. $\nabla F(1,2) = 64\mathbf{i} + 32\mathbf{j}$. $2(x-1) + (y-2) = 0$; $2x + y = 4$

23. $F(x,y) = 2x^3 + 2y^3 - 9xy = 0$. $\nabla F(x,y) = (6x^2 - 9y)\mathbf{i} + (6y^2 - 9x)\mathbf{j}$. $\nabla F(1,2) = -12\mathbf{i} + 15\mathbf{j}$.
$-4(x-1) + 5(y-2) = 0$; $-4x + 5y = 6$

24. $x^4 + 2xy - y^2 = 4$; $(2,-2)$
▸ Let F be the function defined by
$F(x,y) = x^4 + 2xy - y^2 - 4$
We have
$\nabla F(x,y) = (4x^3 + 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$
A normal vector is
$\nabla F(2,-2) = 28\mathbf{i} + 8\mathbf{j}$
An equation of the tangent line is
$7(x-2) + 2(y+2) = 0$
$7x + 2y - 10 = 0$

25. The given spheres are $x^2 + y^2 + z^2 = a^2$ and $(x-b)^2 + y^2 + z^2 = (b-a)^2$.
Let $F(x,y,z) = x^2 + y^2 + z^2 - a^2$ and $G(x,y,z) = (x-b)^2 + y^2 + z^2 - (b-a)^2$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla G(x,y,z) = 2(x-b)\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. $\mathbf{n}_1 = \nabla F(a,0,0) = 2a\mathbf{i}$ and $\mathbf{n}_2 = \nabla G(a,0,0) = 2(a-b)\mathbf{i} = \frac{a-b}{a}\mathbf{n}_1$. Hence $\mathbf{n}_1$ is parallel to $\mathbf{n}_2$, and so the spheres are tangent at $(a,0,0)$. Tangent spheres cannot have a curve of intersection.

26. $F(x,y,z) = xyz - 36 = 0$; $G(x,y,z) = 4x^2 + y^2 + 9z^2 - 108 = 0$. $\nabla F(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.
$\nabla G(x,y,z) = 8x\mathbf{i} + 2y\mathbf{j} + 18z\mathbf{k}$. $\mathbf{n}_1 = \nabla F(3,6,2) = 12\mathbf{i} + 6\mathbf{j} + 18\mathbf{k}$; $\mathbf{n}_2 = \nabla G(3,6,2) = 24\mathbf{i} + 12\mathbf{j} + 36\mathbf{k} = 2\mathbf{n}_1$

27. The surface $x^2 - 2yz + y^3 = 4$ is perpendicular to every member of the family of surfaces $x^2 + (4c-2)y^2 - cz^2 + 1 = 0$ at the point $(1,-1,2)$ if their normal vectors are orthogonal.
Let $F(x,y,z) = x^2 - 2yz + y^3 - 4$ and $G(x,y,z) = x^2 + (4c-2)y^2 - cz^2 + 1$.
$\nabla F(x,y,z) = 2x\mathbf{i} + (3y^2 - 2z)\mathbf{j} - 2y\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 4(2c-1)y\mathbf{j} - 2cz\mathbf{k}$.
Let $\mathbf{n}_1 = \nabla F(1,-1,2) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = \nabla G(1,-1,2) = 2\mathbf{i} - 4(2c-1)\mathbf{j} - 4c\mathbf{k}$. Then
$\mathbf{n}_1 \cdot \mathbf{n}_2 = 4 + 4(2c-1) - 8c = 0$ so $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal.

28. Prove that every normal line to the sphere $x^2 + y^2 + z^2 = a^2$ passes through the center of the sphere.
▸ Let $F(x,y,z) = x^2 + y^2 + z^2 - a^2$. Then $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and so a normal line to the sphere at the point $P(x,y,z)$ has the same direction as the position vector $V(\overrightarrow{OP}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence the normal line passes through O, the center of the sphere.

12.8 EXTREMA OF FUNCTIONS OF TWO VARIABLES

12.8.1 Definition The function f of two variables is said to have an *absolute maximum value* on its domain D in the xy plane if there is some point (x_0, y_0) in D such that $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) in D. f is said to have an *absolute minimum value* on its domain D in the xy plane if there is some point (x_0, y_0) in D such that $f(x_0, y_0) \leq f(x,y)$ for all points (x,y) in D.

12.8.2 Definition The function f of two variables is said to have a *relative maximum value* at the point (x_0, y_0) if there exists an open disk $B((x_0, y_0); r)$ such that $f(x_0, y_0) \geq f(x,y)$ for all (x,y) in the open disk that are in the domain of f. f is said to have a *relative minimum value* at the point (x_0, y_0) if there exists an open disk $B((x_0, y_0); r)$ such that $f(x_0, y_0) \leq f(x,y)$ for all (x,y) in the open disk that are in the domain of f.

12.8.3 Theorem If $f(x,y)$ exists at all points in some open disk $B((x_0, y_0); r)$, f has a relative extremum at (x_0, y_0), and $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then
$f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

12.8.4 Definition If $f(x,y)$ exists at all points in some open disk $B((x_0, y_0); r)$, the point (x_0, y_0) is a *critical point* of f if one of the following conditions holds:
(i) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$;
(ii) $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

It is not possible to test a critical point of $f(x,y)$ for relative extrema by using only the first-order partial derivatives of f.

12.8.5 Theorem (Second Derivative Test) Let f be a differentiable function of two variables x and y such that f_x and f_2 are differentiable in some open disk $\mathrm{B}((a,b);r)$. Suppose further that $f_x(a,b)=0$ and $f_y(a,b)=0$. Let the *Hessian* of f be

$$\mathrm{D}(a,b)=f_{xx}(a,b)f_{yy}(a,b)-f_{xy}{}^2(a,b)=\begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b)\end{vmatrix}$$

Then

(i) f has a relative minimum value at (a,b) if
$\mathrm{D}(a,b)>0$ and $f_{xx}(a,b)>0$ or $f_{yy}>0$

(ii) f has a relative maximum value at (a,b) if
$\mathrm{D}(a,b)>0$ and $f_{xx}(a,b)<0$ or $f_{yy}<0$

(iii) $f(a,b)$ is not a relative extremum if $\mathrm{D}(a,b)<0$.

(iv) No conclusion regarding relative extrema can be made if $\mathrm{D}(a,b)=0$.

The level curves of f near (a,b) are given approximately by
$z=\frac{1}{2}[f_{xx}h^2+2f_{xy}hk+f_{yy}k^2]$ where $h=x-a$ and $k=y-b$.
If $\mathrm{D}(a,b)<0$, then the graph of $f(x,y)$ lies on both sides of its tangent plane at (a,b). We call such a point a *hyperbolic point* of the surface. Every point of a hyperbolic paraboloid is a hyperbolic point. If $\mathrm{D}(a,b)<0$ at a critical point, as in (iii) above, the hyperbolic point is called a *saddle point.*

P is a *boundary point* of a set R if every open disk centered at P contains at least one point of R and one point not of R. A set is *closed* if it contains all of its boundary points. A set is *bounded* if it is contained in some open disk. The interior of an ellipse is a region that is bounded but not closed; a parabola with its interior is a region that is closed but not bounded. The continuous function $f(x,y)=x^2+y^2$ does not have an absolute maximum value on either of these two sets. Thus both hypotheses on R are essential for the following theorem.

12.8.6 Theorem (Extreme Value Theorem for functions of Two Variables) Let R be a closed and bounded region in the xy plane, and let f be a function of two variables that is continuous on R. Then there is at least one point in R where f has an absolute maximum value and at least one point in R where f has an absolute minimum value.

Because an absolute extremum is a relative extremum, it follows from Theorems 12.8.3 and 12.8.8 that a continuous function f on a closed and bounded domain D has a maximum and a minimum which are attained either

(i) at a point of the boundary of D;

(ii) at an interior point of D which is a critical point.

Suppose $f(x,y)$ can be expressed as $g(x)+h(y)$ or $g(x)h(y)$ where g and h are positive. If $g(x_0)\geq g(x)$ and $h(y_0)\geq h(y)$ then $g(x_0)+h(x_0)\geq g(x)+h(y)$ and $g(x_0)h(y_0)\geq f(x)h(y)$, that is, the absolute maximum of f is the sum or product of the maxima of g and h. See Exercises 24 and 32. In Exercise 26 we prove the following theorems stated in §1.3.

Maximum Product of a set of positive numbers of constant sum is when factors are equal.

Minimum Sum of a set of positive numbers of constant product is when terms are equal.

Regression Line The regression line of y on x for the set of data points (x_1,y_1), (x_2,y_2), …, (x_n,y_n) obtained from the method of least squares is $y=mx+b$, where

$$m=\frac{n\sum\limits_{i=1}^{n}x_iy_i-\sum\limits_{i=1}^{n}x_i\sum\limits_{i=1}^{n}y_i}{n\sum\limits_{i=1}^{n}x_i{}^2-\left(\sum\limits_{i=1}^{n}x_i\right)^2} \tag{I}$$

and

$$b=\frac{1}{n}\left[\sum_{i=1}^{n}y_i-m\sum_{i=1}^{n}x_i\right] \tag{II}$$

The regression for x on y for the same set of data points is not the same line.

Exercises 12.8

In Exercises 1–6, find any relative extrema by finding the critical points and applying Definition 12.8.2.

1. $f(x,y)=\sqrt{16-x^2-y^2}$. $f_x=\dfrac{-x}{\sqrt{16-x^2-y^2}}=0$, $x=0$. $f_y=\dfrac{-y}{\sqrt{16-x^2-y^2}}=0$, $y=0$

$f(0,0)=\sqrt{16}\geq\sqrt{16-x^2-y^2}$ for any (x,y). f has an absolute maximum at $(0,0)$.

2. $f(x,y)=\sqrt{x^2+y^2+9}$. $f_x=\dfrac{x}{\sqrt{x^2+y^2+9}}=0$, $x=0$. $f_y=\dfrac{y}{\sqrt{x^2+y^2+9}}=0$, $y=0$

$f(0,0)=\sqrt{9}\leq\sqrt{x^2+y^2+9}$ for any (x,y). f has an absolute minimum at $(0,0)$.

3. $f(x,y)=x^2+y^2-4x-8y+16$. $f_x=2x-4=0$, $x=2$. $f_y=2y-8=0$, $y=4$.
$f(x,y)=(x-2)^2+(y-4)^2-4\geq-4=f(2,4)$. f has an absolute minimum at $(2,4)$.

4. $f(x,y)=2+2x+6y-x^2-y^2$

▸ To locate the critical point, we set the partial derivatives to 0.
$f_x(x,y)=2-2x=0$
$f_y(x,y)=6-2y=0$
Solving for x and y, we find
$x=1\quad y=3$
To apply Definitions 12.8.1 or 12.8.2, we calculate

$$\begin{aligned}f(3,1)-f(x,y)&=12-(2+2x+6y-x^2-y^2)\\&=x^2-2x+y^2-6y+10\\&=(x-1)^2+(y-3)^2\geq0\end{aligned}$$

Because $f(3,1)\geq f(x,y)$ for any (x,y), we conclude that f has an absolute maximum value of 12 at $(3,1)$.

5. $f(x,y)=9-\sqrt{x^2+y^2-2x+1}$. $f_x=-\dfrac{x-1}{\sqrt{x^2+y^2-2x+1}}=0$, $x=1$. $f_y=-\dfrac{y}{\sqrt{x^2+y^2-2x+1}}=0$, $y=0$

$f(x,y)=9-\sqrt{(x-1)^2+y^2}\leq9=f(1,0)$ for any (x,y). f has an absolute minimum at $(1,0)$.

6. $f(x,y)=\sqrt{x^2+y^2}+1$. $f_x=\dfrac{x}{\sqrt{x^2+y^2}}=0$, $x=0$. $f_y=\dfrac{y}{\sqrt{x^2+y^2}}=0$, $y=0$

$f(x,y)=\sqrt{x^2+y^2}+1\geq1=f(0,0)$ for any (x,y). f has an absolute minimum value at $(0,0)$

In Exercises 7–18, determine the relative extrema of f, if there are any, and locate any saddle points.

7. $f(x,y)=x^3+y^2-6x^2+y-1$. $\quad f_x(x,y)=3x^2-12x \quad f_y(x,y)=2y+1$
$f_{xx}(x,y)=6x-12 \quad f_{yy}(x,y)=2 \quad f_{xy}(x,y)=0$
$f_x(x,y)=0$: $3x^2-12x=0$; $3x(x-4)=0$; $x=0$, $x=4$. $f_y(x,y)=0$: $2y+1=0$; $y=-\frac{1}{2}$
The critical points of f are $(0,-\frac{1}{2})$ and $(4,-\frac{1}{2})$. The results of applying the second-derivative test at these points are summarized in the following table.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy}-f_{xy}{}^2$	f	Conclusion
$(0,-\frac{1}{2})$	-12	2	0	-24	$-\frac{5}{4}$	saddle point
$(4,-\frac{1}{2})$	12	2	0	24	$-\frac{133}{4}$	relative minimum value

8. $f(x,y)=18x^2-32y^2-36x-128y-110$

▸ First we apply Definition 12.8.4 to find each critical point.
$f_x(x,y)=36x-36=36(x-1)$
$f_y(x,y)=-64y-128=-64(y+2)$
If $f_x(x,y)=0$, then $x=1$.
If $f_y(x,y)=0$, then $y=-2$.
Therefore, $(1,-2)$ is the only critical point. By Theorem 12.8.3, if f has a relative extremum, it must occur at the critical point $(1,-2)$. We apply the second-derivative test to determine if f has a relative extremum at $(1,-2)$.
$f_{xx}(x,y)=36\quad f_{yy}(x,y)=-64\quad f_{xy}(x,y)=0$
Because

$$f_{xx}(x,y)f_{yy}(x,y)-f_{xy}{}^2(x,y)=36(-64)-0^2=-2048<0$$

by Theorem 12.8.5(iii) we conclude that f does not have a relative extremum at $(1,-2)$. Therefore, the function f does not have a relative extremum at any point.

9. $f(x,y) = y^2 - x^2 + 2x - 4y + 3$ $\quad f_x(x,y) = -2x + 2 = 0,\ x = 1$ $\quad f_y(x,y) = 2y - 4 = 0,\ y = 2$
$f_{xx}(x,y) = -2$ $\quad f_{yy}(x,y) = 2$ $\quad f_{xy}(x,y) = 0$
$D(x,y) = (-2)2 - 0^2 - 4 < 0$. The critical point $(1,2)$ is a saddle point.

10. $f(x,y) = x^2 - y^2 + 6x - 8y + 25$ $\quad f_x(x,y) = 2x + 6 = 0,\ x = -3$ $\quad f_y(x,y) = -2y - 8 = 0,\ y = -4$
$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = -2$ $\quad f_{xy}(x,y) = 0$
$D(x,y) = 2(-2) - 0^2 = -4 < 0$. The critical point $(-3,-4)$ is a saddle point.

11. $f(x,y) = \frac{1}{x} - \frac{64}{y} + xy.$ $\quad f_x(x,y) = -\frac{1}{x^2} + y$ $\quad f_y(x,y) = \frac{64}{y^2} + x$

$f_{xx}(x,y) = \frac{2}{x^3}$ $\quad f_{yy}(x,y) = -\frac{128}{y^3}$ $\quad f_{xy}(x,y) = 1$

$f_x(x,y) = 0$: $y = \frac{1}{x^2}$. $f_y = 0$: $64 + xy^2 = 0$; $64 + \frac{1}{x^3} = 0$; $x = -\frac{1}{4}$, $y = 16$. Critical point: $(-\frac{1}{4}, 16)$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(-\frac{1}{4}, 16)$	-128	$-\frac{1}{32}$	1	3	-12	relative maximum value

12. $f(x,y) = x^2 - 4xy + y^3 + 4y$

▷ We calculate the partial derivatives and set them to 0.

$$f_x(x,y) = 2x - 4y = 0 \tag{1}$$
$$f_y(x,y) = -4x + 3y^2 + 4 = 0 \tag{2}$$

Solving (1) for x and substituting in (2), we get

$$x = 2y$$
$$3y^2 - 8y + 4 = 0$$
$$(3y - 2)(y - 2) = 0$$

$y = \frac{2}{3},\ x = \frac{4}{3}$
$y = 2,\ x = 4$
$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = 6y$ $\quad f_{xy}(x,y) = -4$

The critical points of f are $(\frac{4}{3}, \frac{2}{3})$ and $(4,2)$. The results of applying the second-derivative test at these points are summarized in the following table.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(\frac{4}{3}, \frac{2}{3})$	2	4	-4	$2(4) - (-4)^2 = -8$	$\frac{32}{27}$	saddle point
$(4,2)$	2	12	-4	$2(12) - (-4)^2 = 8$	0	relative minimum

13. $f(x,y) = 4xy^2 - 2x^2y - x.$ $\quad f_x(x,y) = 4y^2 - 4xy - 1$ $\quad f_y = 8xy - 2x^2$
$f_{xx}(x,y) = -4y$ $\quad f_{yy}(x,y) = 8x$ $\quad f_{xy}(x,y) = 8y - 4x$
$f_x(x,y) = 0$: $8xy - 2x^2 = 0$; $2x(4y - x) = 0$; $x = 0$ or $x = 4y$
$f_y(x,y) = 0$: $4y^2 - 4xy - 1 = 0$. If $x = 0$, $4y^2 - 1 = 0$; $y = \pm\frac{1}{2}$.
If $x = 4y$, $4y^2 - 16y^2 - 1 = 0$; $12y^2 = -1$; no solution. Critical points: $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	Conclusion
$(0, \frac{1}{2})$	-2	0	4	-16	saddle point
$(0, -\frac{1}{2})$	2	0	-4	-16	saddle point

14. $f(x,y) = y^4 - 4y^3 + 2x^2 + 8xy$ $\quad f_x(x,y) = 4x + 8y = 0,\ x = -2y$ $\quad f_y(x,y) = 4y^3 - 12y^2 + 8x = 0$
$0 = 4y^3 - 12y^2 - 16y = 4y(y^2 - 3y - 4) = 4y(y - 4)(y + 1)$
$f_{xx}(x,y) = 4$ $\quad f_{yy}(x,y) = 12y^2 - 24y = 12y(y - 2)$ $\quad f_{xy}(x,y) = 8$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(0,0)$	4	0	8	$0 - 8^2 = -64$	0	saddle point
$(-8,4)$	4	96	8	320	-128	absolute minimu
$(2,-1)$	4	20	8	16	-3	relative minimum

15. $f(x,y) = x^3 + y^3 + 3y^2 - 3x - 9y + 2$ $\quad f_x(x,y) = 3x^2 - 3$ $\quad f_y(x,y) = 3y^2 + 6y - 9$
$f_{xx}(x,y) = 6x$ $\quad f_{yy}(x,y) = 6y + 6$ $\quad f_{xy}(x,y) = 0$
$f_{xx}(x,y) = 0$: $x^2 - 1 = 0$; $x = \pm 1$. $f_{yy}(x,y) = 0$: $y^2 + 2y - 3 = 0$; $(y-1)(y+3) = 0$; $y = 1, -3$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(1,1)$	6	12	0	72	-5	relative minimum value
$(1,-3)$	6	-12	0	-72	27	saddle point
$(-1,1)$	-6	12	0	-72	-1	saddle point
$(-1,-3)$	-6	-12	0	72	31	relative maximum value

16. $f(x,y) = e^x \sin y$
▷ $f_x(x,y) = e^x \sin y$ $\quad f_y(x,y) = e^x \cos y$
Because
$f_x(x,y)^2 + f_y(x,y)^2 = e^{2x}(\sin^2 y + \cos^2) = e^{2x} > 0$
then f_x and f_y are never both 0. We conclude that there are no critical points for the function f. Because f is continuous at every point in R^2, by Theorem 12.8.3 we conclude that f does not have a relative extremum. See Figure 12.1.14b in the text.

17. $f(x,y) = e^{xy}$. $\quad f_x(x,y) = ye^{xy}$ $\quad f_y(x,y) = xe^{xy}$
$f_{xx}(x,y) = y^2 e^{xy}$ $\quad f_{yy}(x,y) = x^2 e^{xy}$ $\quad f_{xy}(x,y) = e^{xy} + xye^{xy}$
$f_x = 0$: $ye^{xy} = 0$; $y = 0$. $f_y = 0$: $xe^{xy} = 0$; $x = 0$. Critical point: $(0,0)$
At $(0,0)$, $f_{xx}f_{yy} - f_{xy}{}^2 = -1$; hence $(0,0)$ is a saddle point.

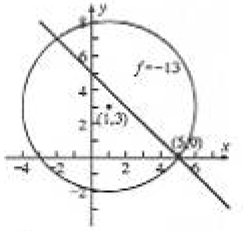
18. $f(x,y) = x^3 + y^3 - 18xy$ $\quad f_x(x,y) = 3x^2 - 18y = 0$, $y = \frac{1}{6}x^2$ $\quad f_y(x,y) = 3y^2 - 18x = 0$
$0 = 3(\frac{1}{6}x^2)^2 - 18x = \frac{1}{12}x(x^3 - 216)$, $x = 0, 6$
$f_{xx}(x,y) = 6x$ $\quad f_{yy}(x,y) = 6y$ $\quad f_{xy}(x,y) = -18$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(0,0)$	0	0	-18	-324	0	saddle point
$(6,6)$	36	36	-18	972	-216	relative minimum value

In Exercises 19–24, find the absolute extrema of the function on the closed bounded region R.

19. The function of Ex. 3; R is the triangular region with vertices $A(0,0)$, $B(4,0)$, $C(0,8)$.
▷ The absolute minimum is -4 at $(2,4)$, the only interior critical point. We look for the absolute maximum on the boundary. $f(x,y) = (x-2)^2 + (y-4)^2 - 4$ is greatest at the point furthest from $(2,4)$; it has the value 16 at each of the vertices.

20. The function of Exercise 4; R is the triangular region whose sides are the x axis, the y axis, and the line $x + y = 5$.
▷ From Exercise 4, the absolute maximum is 12 at $(1,3)$, the only interior critical point. We look for the absolute minimum on the boundary. Because
$f(x,y) = 12 - [(x-1)^2 + (y-3)^2]$
f is least at the point furthest from $(1,3)$. This is the vertex $(5,0)$. See the figure. The minimum value is
$12 - [4^2 + 3^2] = -13$.



21. $f(x,y) = 3x^2 + xy$; R is the region bounded by the parabola $y = x^2$ and the line $y = 4$.
▷ $f_x = 6x + y = 0$, $f_y = x = 0$. The only critical point is $(0,0)$. $f_{xx} = 6$, $f_{yy} = 0$, $f_{xy} = 1$, $D = 6(0) - 1^2 < 0$, a saddle point. Thus the extrema are on the boundary.
On $y = 4$, $f = 3x^2 + 4x$, $-2 \le x \le 2$. $f' = 6x + 4$. $f(-2) = 4$, $f(-\frac{2}{3}) = -\frac{4}{3}$, $f(2) = 20$.
On $y = x^2$, $f = 3x^2 + x^3$, $-2 \le x \le 2$. $f' = 6x + 3x^2 = 3x(2+x)$. $f(-2) = 4$, $f(0) = 0$, $f(2) = 20$.
Thus the absolute maximum is 20 at $(2,4)$ and the absolute minimum is $-\frac{4}{3}$ at $(-\frac{2}{3}, 4)$.

22. $f(x,y) = x^2 - 2xy + 2y$; R is the region bounded by the parabola $y = 4 - x^2$ and the x axis.
▷ $f_x = 2x - 2y = 0$, $f_y = -2x + 2$. The critical point is $(1,1)$. $f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = -2$, $D = 2(0) - (-2)^2 < 0$ a saddle point. Thus the extrema are on the boundary.
On $y = 0$, $f = x^2$, $-2 \le x \le 2$. f has a minimum of 0 at 0 and a maximum of 4 at ± 2.
On $y = 4 - x^2$, $f = 2x^3 - x^2 - 8x + 8, -2 \le x \le -2$. $f' = 6x^2 - 2x - 8 = 2(x+1)(3x-4)$. $f(-2) = f(2) = 4$, $f(-1) = 13$, $f(\frac{4}{3}) = \frac{8}{27}$. Thus, the absolute maximum is 13 at $(-1,3)$ and the absolute minimum is 0 at $(0,0)$.

23. $f(x,y) = y^3 - x^2 - 3y$; R is the region bounded by the circle $x^2 + (y-1)^2 = 1$.
▸ $f_x = -2x = 0$, $f_y = 3y^2 - 3 = 3(y-1)(y+1)$. The critical points are $(0,1)$ and $(0,-1)$. $f_{xx} = 0$, $f_{yy} = 6y$, $f_{xy} = 0$, $D = 0$ at both points. $f(0,1) = -2$, $f(0,-1) = 2$. If $x^2 = 1-(y-1)^2$, $f = y^3 - y^2 - y$, $0 \leq y \leq 2$
$f^{\prime} = 3y^2 - 2y - y = (y+1)(3y-1)$. $f(0) = 0$, $f(\frac{1}{3}) = -\frac{11}{27}$, $f(2) = 2$.
The absolute maximum is 2 at $(0,-1)$ and the absolute minimum is -2 at $(0,-1)$.

24. $f(x,y) = \sin x + \sin y$; R is the region bounded by the square having vertices at $(0,0)$, $(\pi,0)$, $(0,\pi)$ and (π,π).
▸ Because on R
$0 \leq \sin x \leq 1$ and $0 \leq \sin y \leq 1$
then the absolute maximum value of f is $1+1=2$ at $(\frac{1}{2}\pi, \frac{1}{2}\pi)$ and the absolute minimum value of f is $0+0=0$ at $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π).

25. Let x, y, and z be the three positive numbers whose sum is 24. Let P be their product.
$x+y+z=24$; $z = 24-x-y$. Then $P(x,y) = xyz = xy(24-x-y) = 24xy - x^2y - xy^2$.
If $x \geq 24$ or $y \geq 24$ or $x = 0$ or $y = 0$, then $P \leq 0$. Hence the maximum of P occurs at a critical point inside the square $0 \leq x \leq 24$, $0 \leq y \leq 24$.
$P_x(x,y) = 24y - 2xy - y^2$ $\qquad$ $P_y(x,y) = 24x - x^2 - 2xy$
$P_x(x,y) = 0$: $y(24-2x-y) = 0$. $\qquad$ $P_y(x,y) = 0$: $x(24-x-2y) = 0$.
Because $x > 0$, $y > 0$, the only critical point is $(8,8)$ which gives the absolute maximum.
Then $z = 8$, and so the three positive numbers are 8, 8, and 8. Alternatively, see Exercise 26.

26. Find the n positive numbers whose product is a^n such that their sum is as small as possible.
▸ Suppose the numbers are not all equal. We replace the smallest, say x, and the largest, say y by two numbers a and xy/a with the same product. We show that the sum is decreased. In fact
$(x+y) - \left(a + \frac{xy}{a}\right) = \frac{ax + ay - a^2 - xy}{a} = \frac{(a-x)(y-a)}{a}$. Because x is the smallest and y is the largest, then $a - x > 0$ and $y - a > 0$. Since each step introduces a new term of a, in at most n steps of decreasing sum we arrive at the case of all terms equal. In particular, if the product of three positive numbers is 24, the sum is least when each number is $\sqrt[3]{24}$. To prove the other theorem, we replace x and y by two numbers a and $x+y-a$ with the same sum and show the product is increased. In fact, $a(x+y-a) - xy = (a-x)(y-a) > 0$.

27. Let w units be the distance from the point $(1,-2,3)$ to a point (x,y,z) in the plane
$3x + 2y - z - 5$; $z = 3x + 2y - 5$. Then
$w^2 = (x-1)^2 + (y+2)^2 + (z-3)^2$; $w^2 = (x-1)^2 + (y+2)^2 + (3x+2y-8)^2$
Because w will be a minimum when w^2 is a minimum, we seek the absolute minimum value of
$f(x,y) = (x-1)^2 + (y+2)^2 + (3x+2y-8)^2$. Because $f(x,y) \geq 100$ when $(x-1)^2 + (y+2)^2 \geq 100$
the minimum must occur at a critical point inside the circle $(x-1)^2 + (y+2)^2 = 100$.
$f_x(x,y) = 2(x-1) + 2(3x+2y-8)(3) = 20x + 12y - 50$. $f_x(x,y) = 0$: $10x + 6y = 25$
$f_y(x,y) = 2(y+2) + 2(3x+2y-8)(2) = 12x + 10y - 28$. $f_y(x,y) = 0$: $6x + 5y = 14$
The only critical point is $(\frac{41}{14}, -\frac{5}{7})$ which must be the absolute minimum. Then $z = \frac{33}{14}$.
Therefore, the point in the plane closest to $(1,-2,3)$ is $(\frac{41}{14}, -\frac{5}{7}, \frac{33}{14})$ and the minimum distance is
$\sqrt{(\frac{41}{14} - 1)^2 + (-\frac{5}{7} + 2)^2 + (\frac{33}{14} - 3)^2} = \frac{9}{14}\sqrt{14}$

28. Find the points on the surface $y^2 - xz = 4$ that are closest to the origin and find the minimum distance.
▸ Let F be the square of the number of units between the origin and any point (x,y,z) on the hyperboloid of one sheet $y^2 - xy = 4$. The distance is a minimum when F is a minimum. We have

$$F = x^2 + y^2 + z^2 = x^2 + z^2 + xz + 4$$

$F_x = 2x + z$ $\qquad$ $F_y = 2z + x$

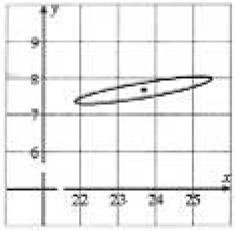
Therefore, $x = 0$ and $z = 0$ is the only critical point of the function, and when $x = z = 0$, we have $F = 4$. We show that 2 is the absolute minimum value of F. Consider a sphere of radius 3 which intersects the xz plane in a circle of radius 3. This circle with its interior is a closed and bounded set R and so, by Theorem 12.8.8, F has an absolute minimum value on R. Because F has the value 3 on the boundary of R and 3 is greater than 2, then the absolute minimum of F on R cannot occur on the boundary. Hence, the absolute minimum of F on R must occur at the critical point. Moreover, $F > 3$ at all points outside R. Therefore, the critical point gives the absolute minimum value of F for all points. Furthermore, if $x = 0$ and $z = 0$, we have $y^2 = 4$. Thus the minimum distance of 2 occurs at the points. $(0,2,0)$ and $(0,-2,0)$.
ALTERNATE SOLUTION: Completing the square on x, we have

$$F = (x + \tfrac{1}{2}z)^2 + \tfrac{3}{4}z^2 + 4 \geq 4$$

and equality holds if and only if $x + \frac{1}{2}z = 0$ and $z = 0$, that is, if and only if $x = 0$ and $z = 0$. Therefore F has an absolute minimum value of 4 when $x = 0$ and $z = 0$.

29. Let w units be the distance from the point $P(x, y, z)$ of the ellipsoid $x^2 + 4y^2 + 4z^2 = 4$ to the origin. Then $y^2 + z^2 = 1 - \frac{1}{4}x^2$ so that $w^2 = x^2 + y^2 + z^2 = x^2 + (1 - \frac{1}{4}x^2) = \frac{3}{4}x^2 + 1$ which has an absolute minimum value of 1 when $x = 0$. Because P lies on the ellipsoid and on the plane $x - 4y - z = 0$, we find the y and z coordinates of P by solving the system $y^2 + z^2 = 1$, $4y + z = 0$. Then $z = -4y$; $y^2 + 16y^2 = 1$; $y = \pm\frac{1}{\sqrt{17}}$, $z = \mp\frac{4}{\sqrt{17}}$. The points are $\left(0, \frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right)$ and $\left(0, -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$.

30. $C = 14x + 13y + y^3 + x^2 - 8xy + 600$, $x \geq 3$, $y \geq 3$, x and y integers. $C_x = 14 + 2x - 8y = 0$, $x = 4y - 7$
$C_y = 13 + 3y^2 - 8x = 0$, $0 = 3y^2 - 8(4y - 7) + 13 = 3y^2 - 32y + 69 = (3y - 23)(y - 3)$. $y = 3$, $x = 7$ or $y = \frac{23}{3}$, $x = \frac{71}{3}$. $C_{xx} = 2$, $C_{yy} = 6y$, $C_{xy} = -8$, $D = 12y - 64$.
$C_{yy}(7, 3) = 18$, $D(7, 3) = -24$, saddle point.



$C_{yy}(\frac{71}{3}, \frac{23}{3}) = 46$, $D(\frac{71}{3}, \frac{23}{3}) = 28$, relative minimum. At $(\frac{71}{3}, \frac{23}{3})$, $C = 590.2$ and the level curves are approximately $z = \frac{1}{2}(2h^2 - 16hk + 46k^2) = h^2 - 8hk + 23k^2$.
We choose $z = 1$ and complete the square on h to get a parametric representation. Thus, $1 = (h - 4k)^2 + 7k^2$, $k = \sin\theta/\sqrt{7}$, $h = 4k + \cos\theta$
$x = \frac{71}{3} + 4\sin\theta/\sqrt{7} + \cos\theta$, $y = \frac{23}{3} + \sin\theta/\sqrt{7}$. We see that the nearest integer point $(24, 8)$ lies outside the curve, but $(25, 8)$ lies inside. In fact, $C(24, 8) = 592$ but $C(25, 8) = 591$ is the minimum.

31. $R = x^2y^3(c - x - y) = cx^2y^3 - x^3y^3 - x^2y^4$, $x \geq 0$, $y \geq 0$. If $x \geq c$, $y \geq c$, $x = 0$, or $y = 0$, then $R \leq 0$. Hence the maximum we seek must occur at a critical point inside the square $0 \leq x \leq c$, $0 \leq y \leq c$.
$R_x = 2cxy^3 - 3x^2y^3 - 2xy^4$ and $R_y = 3cx^2y^2 - 3x^3y^2 - 4x^2y^3$.
$R_x = 0$: $xy^3(2c - 3x - 2y) = 0$; $3x + 2y = c$. $R_y = 0$: $x^2y^2(3c - 3x - 4y) = 0$; $3x + 4y = 3c$
The only critical point is when $x = \frac{1}{3}c$, $y = \frac{1}{2}c$. Hence the maximum response is when $\frac{1}{3}c$ mg of drug A and $\frac{1}{2}c$ mg of drug B is injected.

32. Suppose that t hr after the injection of x mg of adrenalin the response is R units, and $R = te^{-t}(c - x)x$ where c is a positive constant. What values of x and t will cause the maximum response?

▸ Let
$R(t, x) = te^{-t}(c - x)x$
Partial-differentiating, we obtain
$R_t(t, x) = (1 - t)e^{-t}(c - x)x$ and $R_x(t, x) = te^{-t}(c - 2x)$
If $R_t(t, x) = 0$, then either $t = 1$, $x = 0$ or $x = c$. If $R_x(t, x) = 0$, then either $t = 0$ or $x = \frac{1}{2}c$. The critical points are $(0, 0)$, $(0, c)$, and $(1, \frac{1}{2}c)$.
The domain of R is the closed but unbounded set $\{(t, x): t \geq 0, 0 \leq x \leq c\}$. To show that

$$R(1, \tfrac{1}{2}c) = \tfrac{1}{4}e^{-1}c^2 \approx 0.09c^2$$

is an absolute maximum value, consider the closed and bounded set

$$D = \{(t, x): 0 \leq t \leq 2, 0 \leq x \leq c\}$$

On the three sides of the boundary, $t = 0$, $x = 0$, and $x = c$, F has the value 0. Because

$$(c - x)x = \tfrac{1}{4}c^2 - (x - \tfrac{1}{2}c)^2$$

then on the side $t = 2$, we have

$$R(2, x) = 2e^{-2}(c - x)x \leq \tfrac{1}{2}e^{-2}c^2 \approx 0.07c^2 < R(1, \tfrac{1}{2}c)$$

Thus R has an absolute maximum value on D which must occur at the interior critical point. Furthermore, because $D_t(te^{-t}) = (1 - t)e^{-t} < 0$ for $x > 1$, then te^{-t} is decreasing for $t > 1$. Therefore, for any point (t, x) outside of D we have

$$R(t, x) = te^{-t}(c - x)x < 2e^{-2}(c - x)x < R(2, x)$$

Thus the absolute maximum on D is an absolute maximum for the entire domain of R. Therefore, the maximum response occurs when $t = 1$ and $x = \frac{1}{2}c$.
Alternatively, $R(t, x) = g(t)h(x)$, where $g(t) = te^{-t}$, $t \geq 0$ and $h(x) = (c - x)x$, $0 \leq x \leq c$. Because g and h are positive, we maximize each function separately.
$g'(t) = (1 - t)e^{-t}$
Because $g'(t) > 0$ when $0 \leq t < 1$ and $g'(t) < 0$ if $t > 1$, then g has an absolute maximum value when $t = 1$.
$h'(x) = c - 2x$
Because $h'(x) > 0$ if $0 \leq x < \frac{1}{2}c$ and $h'(x) < 0$ if $\frac{1}{2}c < x \leq c$, then h has an absolute maximum value when $x = \frac{1}{2}c$. Thus, R has an absolute maximum value when $t = 1$ and $x = \frac{1}{2}c$.

33. Let 2ℓ, $2w$ and $2h$ be the number of units in the length, width, and height, respectively, of the parallelepiped P. Then in the first octant the vertex of P is at (ℓ, w, h). Since this point is on the ellipsoid, $36\ell^2 + 9w^2 + 4h^2 = 36$; $h^2 = \frac{9}{4}(4 - 4\ell^2 - w^2)$. If V cubic units is the the volume of P, then $V = (2\ell)(2w)(2h) = 8\ell wh$. Let
$f(\ell, w) = V^2 = 64\ell^2w^2h^2 = 144\ell^2w^2(4 - 4\ell^2 - w^2) = 144(4\ell^2w^2 - 4\ell^4w^2 - \ell^2w^4)$
The domain of f is the closed region bounded by the ellipse $4\ell^2 + w^2 = 4$, and $f(\ell, w) = 0$ on the boundary. Hence the maximum we seek must occur at an interior critical point.
$f_\ell(\ell, w) = 144(8\ell w^2 - 16\ell^3w^2 - 2\ell w^4)$. $f_w(\ell, w) = 144(8\ell^2w - 8\ell^4w - 4\ell^2w^3)$
$f_\ell(\ell, w) = 0$: $288\ell w^2(4 - 8\ell^2 - w^2) = 0$; $8\ell^2 + w^2 = 4$ ($\ell = 0$ or $w = 0$ gives a boundary point)
$f_w(\ell, w) = 0$: $576\ell^2w(2 - 2\ell^2 - w^2) = 0$; $2\ell^2 + w^2 = 2$
Hence $\ell^2 = \frac{1}{3}$ and $w^2 = \frac{4}{3}$. The only critical point, and thus the absolute maximum value, is when $\ell = \dfrac{1}{\sqrt{3}}$ and $w = \dfrac{2}{\sqrt{3}}$. Then $h^2 = \frac{9}{4}(4 - \frac{4}{3} - \frac{4}{3}) = 3$; $h = \sqrt{3}$ and the maximum volume is $8\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)(\sqrt{3}) = \dfrac{16}{\sqrt{3}} = \frac{16}{3}\sqrt{3}$

Alternatively, because the sum $36\ell^2 + 9w^2 + 4h^2$ is a constant, the product $\frac{81}{4}V^2 = (36\ell^2)(9w^2)(4h^2)$ is greatest when the terms are equal, and so $36\ell^2 = 9w^2 = 4h^2 = 12$, $\ell = 1/\sqrt{3}$, $w = 2/\sqrt{3}$, $h = \sqrt{3}$.

34. Let ℓ ft, w ft, and h ft be the length, width, and height, of the box, and V ft^3 the volume. Then $.15\ell w + .30(2\ell h + 2wh) = 10$, $3\ell w + 12\ell h + 12wh = 200$. Because the sum is constant, the product $432V^2 = (3\ell w)(12\ell h)(12wh)$ is greatest when $3\ell w = 12\ell h = 12wh = \frac{200}{3}$, $432\ell^2w^2h^2 = (\frac{200}{3})^3$, $\ell wh = \frac{500}{27}\sqrt{2}$.
$\ell = \ell wh/\ell w = \frac{10}{3}\sqrt{2} = w$, $h = \ell wh/wh = \frac{5}{6}\sqrt{2}$.

35. Let ℓ ft, w ft, and h ft be the length, width, and height, respectively, of the box. Let C cents be the cost of the materials. Then
$\ell wh = 16$; $h = \dfrac{16}{\ell w}$ and $C = 18\ell w + 16wh + 12\ell h = 18\ell w + \dfrac{256}{\ell} + \dfrac{192}{w}$
If $\ell \geq 1000$ and $w \geq 0.1$, or $\ell \geq 0.1$ and $w \geq 1000$, then $C \geq 18(1000)(0.1) = 1800$. If $\ell \leq 0.1$, then $C \geq 256/(0.1) = 2560$; if $w \leq 1$, then $C \geq 192/(0.1) = 1920$. Hence the absolute minimum value of C must occur at a critical point in the interior of the square $0.1 \leq \ell \leq 1000$, $0.1 \leq w \leq 1000$.
$\dfrac{\partial C}{\partial \ell} = 18w - \dfrac{256}{\ell^2} = 0$; $w = \dfrac{64}{9\ell^2}$. $\dfrac{\partial C}{\partial w} = 18\ell - \dfrac{192}{w^2} = 0$; $18\ell - 192\left(\dfrac{9\ell^2}{64}\right)^2$; $\ell^3 = \dfrac{512}{27}$
The only critical point, and hence the absolute minimum value, is when $\ell = \frac{8}{3}$ and $w = 2$. Then $h = 3$ and $C = 288$.

36. Suppose that T degrees is the temperature at any point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 4$, and $T = 100xy^2z$. Find the greatest and least temperatures and the points on the sphere where they occur.

▸ Because $y^2 = 4 - x^2 - z^2$, T is a function of x and z and

$$T(x, z) = 100xz(4 - x^2 - z^2);\quad x^2 + z^2 \leq 4 \tag{1}$$

Thus,

$$T_x(x, z) = 100[xz(-2x) + z(4 - x^2 - z^2)] = -100z(3x^2 + z^2 - 4)$$

and

$$T_z(x, z) = 100[xz(-2z) + x(4 - x^2 - z^2)] = -100x(3z^2 + x^2 - 4)$$

If $T_x(x, z) = 0$ and $T_z(x, z) = 0$, we have

$$z(3x^2 + z^2 - 4) = 0 \tag{2}$$

$$x(3z^2 + x^2 - 4) = 0 \tag{3}$$

If $z = 0$, then from Eq. (3) we get $x = 0$ or $x = \pm 2$.
If $x = 0$, then from Eq. (2) we get $z = 0$ or $z = \pm 2$
If $x \neq 0$ and $z \neq 0$, then we have

$$3x^2 + z^2 - 4 = 0 \quad\text{and}\quad 3z^2 + x^2 - 4 = 0 \tag{4}$$

The solutions of Eqs. (4) are $x = \pm 1$ and $z = \pm 1$. We use (1) to find the value of T at each critical point. Thus,

$T(0, 0) = 0 \quad T(0, 2) = 0 \quad T(0, -2) = 0 \quad T(2, 0) = 0 \quad T(-2, 0) = 0$

and

$T(1, 1) = T(-1, -1) = 200$
$T(1, -1) = T(-1, 1) = -200$

Because $x^2 + z^2 \leq 4$ is a closed and bounded set and T has the value 0 on the boundary, the absolute maximum and minimum values of T are found at the critical points. Furthermore, $y^2 = 2$ when $x = \pm 1$ and $z = \pm 1$. Therefore, 200 degrees is the greatest temperature, and this temperature occurs at the points

$(1, \pm\sqrt{2}, 1)$ and $(-1, \pm\sqrt{2}, -1)$. The least temperature is -200 degrees, which occurs at the points $(1, \pm\sqrt{2}, -1)$ and $(-1, \pm\sqrt{2}, 1)$.
Alternatively, the extrema of T occur at the maximum of T^2.
$\mathrm{T}^2 = 2500(2x^2)y^2y^2(2z^2)$
Because the sum of the factors
$2x^2 + y^2 + y^2 + 2z^2 = 8$
is a constant, the product is greatest when the factors are equal, that is when
$2x^2 = y^2 = 2z^2 = \frac{1}{4}(8) = 2$
$x = \pm 1, y = \pm\sqrt{2}, z = \pm 1$
which leads to the same conclusion.

37. When the production of the commodity requires x machine hours and y person-hours, the cost of production is given by $f(x, y) = 2x^3 - 6xy + y^2 + 500$, $x \geq 0$, $y \geq 0$.
$f_x(x, y) = 6x^2 - 6y = 0$; $y = x^2$ and $f_y(x, y) = -6x + 2y = 0$; $-6x + 2x^2 = 0$; $x = 0, 3$
The critical points are $(0, 0)$ and $(3, 9)$. $f(0, 0) = 500$ and $f(3, 9) = 473$. Now $f(x, y) = x^2(2x - 9) + (3x - y)^2 + 500$. If $x \geq 10$, $f(x, y) \geq 100(11) + 500 = 1600 > 473$; if $0 \leq x \leq 10$ and $y \geq 60$ then $f(x, y) \geq -9(100) + (30)^2 + 500 = 500 > 473$. Hence the absolute minimum must occur at a critical point inside the rectangle $0 \leq x \leq 10$, $0 \leq y \leq 60$, that is, at $(3, 9)$.

38. The unit profits are $x - 40$ and $y - 50$.
$\mathrm{P} = (x - 40)(3200 - 50x + 25y) + (y - 50)(25x - 25y) = 3950x + 250y - 50x^2 + 50xy - 25y^2 - 128{,}000$
$\mathrm{P}_x = 3950 - 100x + 50y = 0$. $\mathrm{P}_y = 250 + 50x - 50y = 0$.
$\mathrm{P}_x + \mathrm{P}_y = 4200 - 50x = 0$, $x = 84$. $\mathrm{P}_x + 2\mathrm{P}_y = 4450 - 50y = 0$, $y = 89$
$\mathrm{P}_{xx} = -100$, $\mathrm{P}_{yy} = -50$, $\mathrm{P}_{xy} = 50$, $\mathrm{D} = 2500 > 0$. The critical point is a relative maximum. Because P is a quadric, it is also an absolute maximum. Sell the first at \$84, the second at \$89 for a profit of \$49,025.

In Exercises 39–46, only parentheses actually needed on a calculator will be used. Every multiplication will be explicitly indicated. Negative numbers are entered by following the number with the $\boxed{+/-}$ key. Answers are given to 3 digits.

39. Let x be the number of 20-year periods since 1921, and let y be the value of the painting $20x$ years after 1921 in hundreds of dollars.

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 1 & 46 & 110 & 200 \end{array}, \quad \sum_{i=1}^{4} x_i y_i = 866, \ \sum_{i=1}^{4} x_i = 6, \ \sum_{i=1}^{4} y_i = 357, \ \sum_{i=1}^{4} x_i^2 = 14, \ \mathrm{m} = \frac{(4 \times 866 - 6 \times 357)}{(4 \times 14 - 6^2)} = 66.1,$$

$b = \dfrac{(357 - 66.1 \times 6)}{4} = -9.9$. The regression line has the equation $y = 66.1x - 9.9$.

When $x = 4$, $y = 66.1 \times 4 - 9.9 = 254.8$, so in 2001 the painting will be worth \$25,450.

40. A 1991 model car was sold as a used car in 1992 for \$6,800. Its value was \$6,200 in 1993, \$5,700 in 1984 and \$4,800 in 1996. Use the method of least squares to estimate what its value was in 1995.

▸ Let y dollars be the value of the car x years after 1991. We have $n = 4$ and
$x_1 = 1 \qquad x_2 = 2 \qquad x_3 = 3 \qquad x_4 = 5$
$y_1 = 6{,}800 \qquad y_2 = 6{,}200 \qquad y_3 = 5{,}700 \qquad y_4 = 4{,}800$
$\sum_{i=1}^{n} x_i = 11 \qquad \sum_{i=1}^{n} y_i = 23{,}500 \qquad \sum_{i=1}^{n} x_i^2 = 39 \qquad \sum_{i=1}^{n} x_i y_i = 60{,}300$
Substituting into equations (I) and (II), we obtain
$m = \dfrac{(4 \times 60300 - 11 \times 23500)}{4 \times 39 - 11^2} = -494$ and $b = \frac{1}{4}[23500 - (-494) \times 11] = 7233$
Thus the regression line for the data points is
$y = -494x + 7234$
Because $x = 4$ in 1989, then
$y = -494 \times 4 + 7234 = 5258$
Our estimate of the value of the car in 1995 is \$5300.

41.

x (Week number)	1	2	3	4	5
y (Attendance/100)	50	45	41	39	35

$\sum_{i=1}^{5} x_i y_i = 594$, $\sum_{i=1}^{5} x_i = 15$, $\sum_{i=1}^{5} y_i = 210$, $\sum_{i=1}^{5} {x_i}^2 = 55$.

(a) $m = \frac{(5 \times 594 - 15 \times 210)}{(5 \times 55 - 15^2)} = -3.6$, $b = \frac{(210 + 3.6 \times 15)}{5} = 52.8$.

The regression line has the equation $y = -3.6x + 52.8$. When $x = 6$, $y = -3.6 \times 6 + 52.8 = 31.2$. The expected attendance for the sixth week is $100y = 3120$.

(b) We find the regression line for predicting x. $\sum_{i=1}^{5} {y_i}^2 = 8952$. $m = \frac{(5 \times 594 - 15 \times 210)}{(5 \times 8952 - 210^2)} = -0.273$, $b = (15 + 0.273 \times 210)/5 = 14.5$. The regression line has the equation $x = -0.273y + 14.5$. This is not the same line as in part (a). When $y = 22.5$, $x = -0.273 \times 22.5 + 14.5 = 8.35$. The film will play out the ninth week.

42.

x (μg hormone)	28	57	38	75	82
y (leaves detached)	206	350	300	620	719

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 142{,}632$, $\sum_{i=1}^{n} x_i = 280$, $\sum_{i=1}^{n} y_i = 2197$, $\sum_{i=1}^{n} {x_i}^2 = 17{,}826$. (a) $m = \frac{(5 \times 142632 - 280 \times 2197)}{(5 \times 17826 - 280^2)} = 9.133$, $b = \frac{(2197 - 9.133 \times 280)}{5} = -72.06$

The regression line has the equation $y = 9.133x - 72.1$. (b) When $x = 100$, $y = 9.133 \times 100 - 72.06 = 841$.

43.

x (sec)	300.5	350.6	407.3	326.2	512.8
y (mi/min/kg)	418.5	375.6	350.2	400.2	325.8

$\sum_{i=1}^{5} x_i y_i = 697{,}696.55$, $\sum_{i=1}^{5} x_i = 1897.4$, $\sum_{i=1}^{n} y_i = 1870.3$, $\sum_{i=1}^{5} {x_i}^2 = 748{,}484.18$. $m = \frac{(5 \times 697696.55 - 1897.4 \times 1870.3)}{(5 \times 748484.18 - 1897.4^2)} = -0.423$, $b = \frac{(1870.3 + 0.423 \times 1897.4)}{5} = 534.7$.

The regression line has the equation $y = -0.423x + 534.7$. If $x = 340.4$, $y = -0.423 \times 340.4 + 534.7 = 390.6$.

44. The score on a student's entrance examination was used to predict the student's grade-point average at the end of the freshman year. The following table gives the data for six students, where x is the test score and y is the grade point average. (a) Find an equation of the regression line for the data. (b) Predict the grade-point average average of a student whose scores 88.

	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6
x	92	81	73	98	79	85
y	3.4	2.7	3.1	3.8	2.2	3.0

▸ We have $n = 6$ and

$\sum_{i=1}^{n} x_i = 508 \quad \sum_{i=1}^{n} y_i = 18.1 \quad \sum_{i=1}^{n} {x_i}^2 = 43{,}424 \quad \sum_{i=1}^{n} x_i y_i = 1559.0$

Substituting into equations (I) and (II), we obtain

$m = \frac{6 \times 1559 - 508 \times 18.1}{6 \times 43424 - 508^2} = 0.0642$ and $b = (18.1 - 0.0642 \times 508)/6 = -2.419$

Therefore, the regression line is

$y = 0.0642x - 2.419$

(b) Substituting $x = 88$, we get

$y = 0.0642 \times 88 - 2.419 = 3.23$

We estimate the student's grade-point average to be 3.2.

45.

x (thousand units)	65	72	82	90	100
y (thousand dollars)	30	35	42	48	60

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 18234$, $\sum_{i=1}^{n} x_i = 409$, $\sum_{i=1}^{n} y_i = 215$, $\sum_{i=1}^{n} {x_i}^2 = 34233$. $m = \frac{(5 \times 18234 - 409 \times 215)}{(5 \times 34233 - 409^2)} = 0.833$, $b = \frac{(215 - 0.833 \times 409)}{5} = -25.13$.

The regression line has the equation $y = 0.833x - 25.13$. $y(105) = 62.32$. Profit will be about $62,300.

46.

w kg	20	30	35	40	50
$x = \ln w$	2.996	3.401	3.555	3.689	3.912
y mm hg	70	85	90	96	100

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 1564.12$, $\sum_{i=1}^{n} x_i = 17.55$, $\sum_{i=1}^{n} y_i = 441$, $\sum_{i=1}^{n} {x_i}^2 = 62.09$. $m = \frac{(5 \times 1564.12 - 17.55 \times 441)}{(5 \times 62.09 - 17.55^2)} = 33.75$, $b = \frac{(441 - 33.75 \times 17.55)}{5} = -30.28$

The regression curve has equation $y = 33.75(\ln w) - 30.28$. $y(45) = 33.75 \times \ln 45 - 30.28 = 98.2$

47. $P = R - C = x(100 - 2x) + y(120 - 3y) - (12x + 12y + 4xy) = -2x^2 - 4xy - 3y^2 + 88x + 108y$
$P_x = -4x - 4y + 88 = 0$, $P_y = -4x - 6y + 108 = 0$. $3P_x - 2P_y = -4x + 48 = 0$, $x = 12$. $P_x - P_y = 2y - 20 = 0$,
$y = 10$. $P_{xx} = -4$, $P_{yy} = -6$, $P_{xy} = -4$, $D = 8$. $(12, 10)$ is a relative maximum; because P is a quadric, it is
an absolute maximum. $P(12, 10) = 1064$. Produce 12 of type 1, 10 of type 2, with profit of \$1064.

48. Prove that the box having the largest volume that can be placed inside a sphere is in the shape of a cube.
▸ Let a units be the diameter of the sphere, where a is a constant. We assume that the box is a rectangular solid and that the box with greatest volume is inscribed in the sphere. If the sides of the box have lengths x, y, and z units, and the volume is V cubic units, then $v = xyz$. Because the box is inscribed in the sphere, a diagonal of the box is a diameter of the sphere. Thus, $x^2 + y^2 + z^2 = a^2$. We eliminate the variable z. Then

$$V^2 = x^2y^2z^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let f be the function defined by

$$f(x, y) = x^2y^2(a^2 - x^2 - y^2) = a^2x^2y^2 - x^4y^2 - x^2y^4$$

and the maximum value of V occurs at the point where f has a maximum value. To find the critical points of f we set the partial derivatives equal to 0. Thus,

$$f_x(x, y) = 2a^2xy^2 - 4x^3y^2 - 2xy^4 = 0 \qquad (1)$$

$$f_y(x, y) = 2a^2x^2y - 2x^4y - 4x^2y^3 = 0 \qquad (2)$$

We eliminate a from the system of Eqs. (1) and (2). Because $x \neq 0$ and $y \neq 0$, we divide Eq. (1) by y and Eq. (2) by x. This results in

$$2a^2xy - 4x^3y - 2xy^3 = 0 \qquad (3)$$

$$2a^2xy - 2x^3y - 4xy^3 = 0 \qquad (4)$$

Subtracting Eq. (4) from Eq. (3), we have

$$-2x^3y + 2xy^3 = 0$$

Dividing on both sides by $2xy$, we get

$$-x^2 + y^2 = 0$$
$$y^2 = x^2$$
$$y = x$$

Substituting $y = x$ into Eq. (1), we have

$$2a^2x^3 - 4x^5 - 2x^5 = 0$$
$$2a^2 = 6x^2$$
$$x = \tfrac{1}{3}\sqrt{3}a$$
$$y = \tfrac{1}{3}\sqrt{3}a$$

We show that $f(\frac{1}{3}\sqrt{3}a, \frac{1}{3}\sqrt{3}a) = \frac{1}{27}a^6$ is an absolute maximum value. Because the domain of f is the closed and bounded set $x^2 + y^2 \leq a^2$ and f has the value 0 on the boundary, then f has an maximum value which must occur at the interior critical point.
Because $x^2 + y^2 + z^2 = a^2$, then by substituting the values found for x and y, we obtain

$$z^2 = a^2 - x^2 - y^2 = a^2 - \tfrac{1}{3}a^2 - \tfrac{1}{3}a^2 = \tfrac{1}{3}a^2$$

Hence,

$$z = \tfrac{1}{3}\sqrt{3}a$$

Therefore, $x = y = z$ and we conclude that V has a maximum volume if the box is in the shape of a cube.
Alternatively, because the sum of the factors of
$V^2 = x^2y^2z^2$
is the constant
$x^2 + y^2 + z^2 = a^2$
the product is greatest when the factors are equal. Therefore, $x = y = z$ and we conclude that V has a maximum volume if the box is in the shape of a cube.

49. Let ℓ, w and h be the number of units in the length, width and height of the box. Let S square units be its surface area (S is a constant) and V cubic units its volume.
$S = \ell w + 2\ell h + 2wh$; $h = \dfrac{S - \ell w}{2(\ell + w)}$. $V = \ell wh = \dfrac{S\ell w - \ell^2w^2}{2(\ell + w)}$, $\ell > 0$, $w > 0$.
$\dfrac{\partial V}{\partial \ell} = \dfrac{Sw^2 - \ell^2w^2 - 2\ell w^3}{2(\ell + w)} = 0$; $S - \ell^2 - 2\ell w = 0$ and $\dfrac{\partial V}{\partial w} = \dfrac{S\ell^2 - \ell^2w^2 - 2\ell^3 w}{2(\ell + w)^2} = 0$; $S - w^2 - 2\ell w = 0$
Subtracting, we get $\ell^2 = w^2$; $\ell = w$ and so $(\sqrt{S/3}, \sqrt{S/3})$ is the only critical point. Then
$h = \frac{1}{2}\sqrt{S/3}$ and $V = \frac{1}{2}(S/3)^{3/2}$. If $\ell \geq 10\sqrt{S}$ and $w \geq 0.1\sqrt{S}$ or $\ell \geq 0.1\sqrt{S}$ and $w \geq 10\sqrt{S}$, then

$V = \frac{\ell w(S - \ell w)}{2(\ell + w)} \leq 0$. If $w \leq 0.1\sqrt{S}$, $V \leq \frac{\ell(0.1\sqrt{S})S}{2\ell} = \frac{S^{3/2}}{20} < \frac{1}{2}\left(\frac{S}{3}\right)^3$ and similarly if $\ell \leq 0.1\sqrt{S}$. Hence the absolute maximum volume occurs at a critical point inside the square $0.1\sqrt{S} \leq \ell \leq 10\sqrt{S}$, $0.1\sqrt{S} \leq w \leq 10\sqrt{S}$, that is when $\ell{:}w{:}h = 1{:}1{:}\frac{1}{2}$.
Alternatively, because the sum $\ell w + 2\ell h + 2wh$ is a constant, the product $4V^2 = (\ell w)(2\ell h)(2wh)$ is greatest when the terms are equal. Thus, $\ell w = 2\ell h = 2wh$, $1/h = 2/w = 2/\ell$, $\ell{:}w{:}h = 2{:}2{:}1$.

50. Unit profit: $p - 2$, $q - 1$. $P = (p-2)(11 - 2p - 2q) + (q-1)(19 - 2p - 3q) = 17p + 26q - 2p^2 - 4pq - 3q^2 - 41$
$P_p = 17 - 4p - 4q = 0, P_q = 26 - 4p - 6q = 0$. $3P_p - 2P_q = -1 - 4p = 0, p = -\frac{1}{4}$. $P_p - P_q = -9 + 2q = 0$, $q = \frac{9}{2}$
Because the only critical point is not in the domain, the maximum is on the boundary. If $p = 0$, $P = 26q - 3q^2 - 41$, $P' = 26 - 6q = 0$, $q = \frac{13}{3}$, $P = \frac{46}{3}$. If $q = 0$, $17p - 2p^2 - 41 \leq 0$. Thus, the maximum profit occurs if the staplers are free and a box of staples sells for \$4.33.

51. If $\nabla f = 0\mathbf{i} + 0\mathbf{j}$, then a normal vector to the plane is $(0, 0, 1)$, that is the plane is horizontal.

52. Obtain Equation (I) from $\left(\sum_{i=1}^{n} x_i^2\right)m + \left(\sum_{i=1}^{n} x_i\right)b = \sum_{i=1}^{n} x_i y_i$ and $\left(\sum_{i=1}^{n} x_i\right)m + nb = \sum_{i=1}^{n} y_i$.

▸ We omit the limits. Multiply the first equation by n and the second by $\sum x_i$ to get
$n(\sum x_i^2)m + n(\sum x_i)b = n\sum x_i y_i$
$(\sum x_i)^2 m + n(\sum x_i)b = \sum x_i \sum y_i$
Subtract to get
$[n(\sum x_i^2) - (\sum x_i)^2]m = [n\sum x_i y_i - \sum x_i \sum y_i]$
Divide to get equation (4).

53. We give a complete proof that (3) and (4) give an *absolute* minimum value of f.
$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $X = \sum_{i=1}^{n} x_i^2$, $Y = \sum_{i=1}^{n} y_i^2$, $Z = \sum_{i=1}^{n} x_i y_i$, $\sigma_x^2 = \frac{1}{n}X - \bar{x}^2$, $\sigma_y^2 = \frac{1}{n}Y - \bar{y}^2$.
Then $f = \sum_{i=1}^{n}[m(x_i - \bar{x}) + b - y_i]^2 = n\sigma_x{}^2[m - \frac{Z - n\bar{x}\bar{y}}{n\sigma_x{}^2}]^2 + n(b - \bar{y})^2 + n\sigma_y{}^2 - \frac{(Z - n\bar{x}\bar{y})^2}{n\sigma_x{}^2}$.
Hence f is least when $b = \bar{y}$ and $m = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i{}^2 - n\bar{x}^2} = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i{}^2 - (\sum_{i=1}^{n} x_i)^2}$

12.8 SUPPLEMENT

1. Prove part (ii) of Theorem 12.8.5.

▸ We are given $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}{}^2(a,b) > 0$ and $f_{xx}(a,b) < 0$. Let $g(x,y) = -f(x,y)$.
Then $g_{xx}(a,b)g_{yy}(a,b) - g_{xy}{}^2 = [-f_{xx}(a,b)][-f_{yy}(a,b)] - [-f_{xy}(a,b)]^2 = D > 0$ and $g_{xx}(a,b) > 0$. By part (i) of the theorem, g has a relative minimum value at (a,b). Hence f has a relative maximum value at (a,b).

2. If $D(a,b) < 0$ then the discriminant of the right side of Eq.(10), $(2f_{xy})^2 - 4f_{xx}f_{yy} = -4D(a,b)$, is positive, and so the right side factors into two distinct linear factors, which determine two lines through (a,b). In two of the sectors, the expression is positive and in two it is negative. Thus, (a,b) is neither a relative maximum nor a relative minimum.

12.9 LAGRANGE MULTIPLIERS

If f and g are differentiable functions of x, y, and z, then the critical numbers of f subject to the constraint $g(x,y,z) = 0$, are among the critical points of the function F defined as follows.
$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$
Note that the condition $F_\lambda(x,y,z,\lambda) = 0$ is equivalent to the constraint.
Suppose that at the critical point $g_z \neq 0$. Then we can find $\partial x/\partial z$ and $\partial y/\partial z$ either by differentiating g implicitly or by applying Theorem 12.5.3 to g. We can now evaluate f_x, f_y, f_{xx}, f_{yy}, and f_{xy} and apply the second derivative test (Theorem 12.8.5). A direct test for relative extrema using determinants of 3rd and 4th order can be found in books on mathematical economics. It may be difficult to prove that absolute extrema exist. See Exercise 16.
If we use F to find the critical numbers of f, we are using the method of *Lagrange multipliers*. The method may be extended to include more than one constraint. Thus, the critical numbers of f subject to the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ are among the critical numbers of G defined as follows.
$G(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$

Exercises 12.9

In Exercises 1–4, use Lagrange multipliers to find the critical points of the function subject to the constraint.

1. $f(x,y) = 25 - x^2 - y^2$, and $x^2 + y^2 - 4y = 0$. $F(x,y,\lambda) = 25 - x^2 - y^2 + \lambda(x^2 + y^2 - 4y)$.
$F_x(x,y,\lambda) = 2x + 2\lambda x$; $F_y(x,y,\lambda) = -2y + \lambda(2y - 4)$; $F_\lambda(x,y,\lambda) = x^2 + y^2 - 4y$
Setting $F_x(x,y,\lambda) = 0$, we get $\lambda = 1$ or $x = 0$. Setting $F_y(x,y,\lambda) = 0$ we see that $\lambda = 1$ is impossible. Setting $F_\lambda(x,y,\lambda) = 0$ and with $x = 0$ we get $y = 0$ and $y = 4$. Hence the critical points are $(0,0)$ and $(0,4)$.

2. $f(x,y) = 4x^2 + 2y^2 + 5$ and $g(x,y) = x^2 + y^2 - 2y = 0$. $F(x,y,\lambda) = 4x^2 + 2y^2 + 5 + \lambda(x^2 + y^2 - 2y)$.
$F_x(x,y,\lambda) = 8x + 2\lambda x = 2x(4 + \lambda)$; $F_y(x,y,\lambda) = 4y + 2\lambda y = 2y(2 + \lambda) = 0$. From F_x, either $x = 0$ and from g, $y = 0$ or $y = 2$ which satisfy F_y with $\lambda = -2$; or $\lambda = -4$, and from F_y, $y = 0$ so $x = 0$ from g. Thus $(0,0)$ and $(0,2)$ are the critical points.

3. $f(x,y,z) = x^2 + y^2 + z^2$, and $3x - 2y + z - 4 = 0$. $F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(3x - 2y + z - 4)$.
$F_x = 2x + 3\lambda = 0$; $F_y = 2y - 2\lambda = 0$; $F_z = 2z + \lambda = 0$; $F_\lambda = 3x - 2y + z - 4 = 0$
Solving simultaneously, we get $\lambda = -\frac{4}{7}$, $x = \frac{6}{7}$, $y = -\frac{4}{7}$, and $z = \frac{2}{7}$. Hence the critical point is $(\frac{6}{7}, -\frac{4}{7}, \frac{2}{7})$.

4. $f(x,y,z) = x^2 + y^2 + z^2$ with constraint $y^2 - x^2 = 1$

▸ Let $g(x,y,z) = x^2 - y^2 + 1$. Let F be the function defined by

$$\begin{aligned} F(x,y,z,\lambda) &= f(x,y,z) + \lambda g(x,y,z) \\ &= x^2 + y^2 + z^2 + \lambda x^2 - \lambda y^2 + \lambda \end{aligned}$$

We find the critical points of the function F. Let

$$F_x(x,y,z,\lambda) = 2x + 2\lambda x = 2x(1 + \lambda) = 0 \tag{1}$$
$$F_y(x,y,z,\lambda) = 2y - 2\lambda y = 2y(1 - \lambda) = 0 \tag{2}$$
$$F_z(x,y,z,\lambda) = 2z = 0 \tag{3}$$
$$F_\lambda(x,y,z,\lambda) = x^2 - y^2 + 1 = 0 \tag{4}$$

From (3) we have $z=0$. From (2) we have either $\lambda = 1$ or $y = 0$. We reject $y = 0$ because then Eq. (4) would require that $x^2 + 1 = 0$, which is impossible. If $\lambda = 1$ in Eq. (1), we get $x = 0$. Substituting $x = 0$ in Eq. (4), we have $-y^2 + 1 = 0$, $y = \pm 1$. Therefore, $(0,1,0)$ and $(0,-1,0)$ are the critical points.

In Exercise 5–8, use Lagrange multipliers to find the absolute extrema of f subject to the constraint. Also find the points at which the extrema occur. In Exercises 9–12, find the absolute maximum and minimum in the region.

5. $f(x,y) = x^2 + y$, and $x^2 + y^2 = 9$. $F(x,y,\lambda) = x^2 + y + \lambda(x^2 + y^2 - 9)$.
$F_x(x,y,\lambda) = 2x + 2\lambda x = 0$; $F_y(x,y,\lambda) = 1 + 2\lambda y = 0$; $F_\lambda(x,y,\lambda) = x^2 + y^2 - 9 = 0$
Solving these equations simultaneously, we get the solutions:
$x = 0$, $y = 3$, $\lambda = -\frac{1}{6}$; $x = 0$, $y = -3$, $\lambda = \frac{1}{6}$; $x = \frac{1}{2}\sqrt{35}$, $y = \frac{1}{2}$, $\lambda = -1$; $x = -\frac{1}{2}\sqrt{35}$, $y = \frac{1}{2}$, $\lambda = -1$

$f(0,3) = 3$; $f(0,-3) = -3$, absolute minimum; $f(\pm\frac{1}{2}\sqrt{35}, \frac{1}{2}) = \frac{37}{4}$, absolute maximum.

9. $f(x,y) = x^2 + y$, $x^2 + y^2 \le 9$. $f_x = 2x = 0$, $f_y = 1 = 0$. There are no critical points. The absolute extrema are those on the boundary given in Ex. 5.

6. $f(x,y) = x^2 y$ and $g(x,y) = x^2 + 8y^2 - 24$. $F(x,y,\lambda) = x^2 y + \lambda(x^2 + 8y^2 - 24)$.
$F_x(x,y,\lambda) = 2xy + 2\lambda x = 2x(y + \lambda) = 0$; $F_y(x,y,\lambda) = x^2 + 16\lambda y = 0$. If $x = 0$, then from g, $y = \pm\sqrt{3}$.
If $\lambda = -y$, then from F_y, $x^2 = 16y^2$, and from g, $y = \pm 1$, $x = \pm 4$. $f(0, \pm\sqrt{3}) = 0$; $f(\pm 4, 1) = 4$, absolute maximum; $f(\pm 4, -1) = -4$, absolute minimum.

10. $f(x,y) = x^2 y$ and $x^2 + 8y^2 \le 24$. $f_x = 2xy = 0$; $f_y = x^2 = 0$. The only critical point is $(0,0)$. $f(0,0) = 0$. The absolute extrema are those on the boundary given in Ex. 6.

7. $f(x,y,z) = xyz$, and $x^2 + 2y^2 + 4z^2 = 4$ $f(x,y,z,\lambda) = xyz + \lambda(x^2 + 2y^2 + 4z^2 - 4)$.
$F_x = yz + 2x\lambda = 0$; $xyz = -2x^2\lambda$. $F_y = xz + 4y\lambda = 0$; $xyz = -4y^2\lambda$. $F_z = xy + 8z\lambda = 0$; $xyz = -8z^2\lambda$.
Hence $x^2 = 4z^2$, $y^2 = 2z^2$, and $4z^2 + 2(2z^2) + 4z^2 = 4$; $z^2 = \frac{1}{3}$, $z = \pm\sqrt{\frac{1}{3}}$. Then $x = \pm\sqrt{\frac{4}{3}}$, $y = \pm\sqrt{\frac{2}{3}}$.

$$f(-\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = -\tfrac{2}{9}\sqrt{6}$$
$$f(\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = \tfrac{2}{9}\sqrt{6}$$

If $\lambda = 0$, we have 6 additional critical points $(\pm 1,0,0)$, $(0,\pm 1,0)$, $(0,0,\pm 1)$ at which $f = 0$.
Because the constraint set is closed and bounded, f has extrema, taken at the critical points. Hence f has a maximum value of $\frac{2}{9}\sqrt{6}$ and a minimum value of $-\frac{2}{9}\sqrt{6}$.

11. $f(x,y,z) = xyz$, $x^2 + 2y^2 + 4z^2 \le 4$. $f_x = yz = 0$; $f_y = xz = 0$; $f_z = xy = 0$. Each point of each axis is a critical point and $f = 0$. The absolute extrema are those on the boundary given in Ex. 7.

8. $f(x,y,z) = y^3 + xz^2$ with constraint $x^2 + y^2 + z^2 = 1$.

▸ Let $g(x,y,z) = x^2 + y^2 + z^2 - 1$. Let F be the function defined by

$$\begin{aligned} F(x,y,z,\lambda) &= f(x,y,z) + \lambda g(x,y,z) \\ &= y^3 + xz^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 - \lambda \end{aligned}$$

We find the critical points of the function F. Let

$$F_x(x,y,z,\lambda) = z^2 + 2\lambda x = 0 \tag{1}$$
$$F_y(x,y,z,\lambda) = 3y^2 + 2\lambda y = y(3y + 2\lambda) \tag{2}$$
$$F_z(x,y,z,\lambda) = 2xz + 2\lambda z = 2z(x + \lambda) \tag{3}$$
$$F_\lambda(x,y,z,\lambda) = x^2 + y^2 + z^2 - 1 = 0 \tag{4}$$

From Eq. (3), either $z = 0$ or $x = -\lambda$.

If $z = 0$, then from Eq. (1), either

$x = 0$: then from Eq. (4), $y = \pm 1$, or

$\lambda = 0$: then from (2), $y = 0$ and from (4) $x = \pm 1$

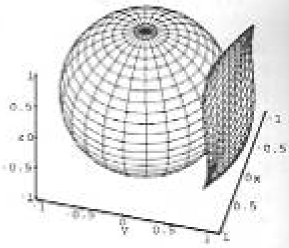
If $x = -\lambda$, then from Eq. (1) $z^2 = 2\lambda^2$ and from Eq. (2) either

$y = 0$: then from (4), $3\lambda^3 - 1 = 0$, $\lambda = \pm\frac{1}{3}\sqrt{3}$ so $x = \mp\frac{1}{3}\sqrt{3}$ and $z^2 = \frac{2}{3}$ or

$y = -\frac{2}{3}\lambda$: then from (4), $\lambda^2 + \frac{4}{9}\lambda^2 + 2\lambda^2 - 1 = 0$, $\lambda^2 = \frac{9}{31}$

We list the critical points, the values of f and our conclusion. The figure shows the constraint and the level surface for $f = 1$ tangent at $(0,1,0)$.

x	y	z	λ	$f(x,y,z)$	
0	1	0	$-\frac{3}{2}$	1	absolute maximum
0	-1	0	$\frac{3}{2}$	-1	absolute minimum
1	0	0	0	0	
-1	0	0	0	0	
$\frac{1}{3}\sqrt{3}$	0	$\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{3}$	$\frac{2}{9}\sqrt{3}$	
$\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{3}$	$\frac{2}{9}\sqrt{3}$	
$-\frac{1}{3}\sqrt{3}$	0	$\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{3}$	$-\frac{2}{9}\sqrt{3}$	
$-\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{3}$	$-\frac{2}{9}\sqrt{3}$	
$\frac{3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	
$\frac{3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	$\frac{-3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	
$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	
$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	



12. $f(x,y,z) = y^3 + xz^2$, $x^2 + y^2 + z^2 \le 1$.

▸ We find the critical points of f.

$f_x(x,y,z) = z^2 = 0$
$f_y(x,y,z) = 3y^2 = 0$
$f_z(x,y,z) = xz = 0$

Thus, any point of the x axis is a critical point. Because $f(x,y,z) = 0$ if $z = z = 0$, the absolute extrema are those on the boundary given in Exercise 8.

In Exercises 13 and 14, find the absolute minimum value of f, and in Ex. 15 and 16, find the absolute maximum.

13. $f(x,y,z) = x^2 + y^2 + z^2$, and $xyz = 1$. $f(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(xyz - 1)$.

$F_x(x,y,z,\lambda) = 2x + \lambda yz = 0$; $2x^2 = -\lambda xyz = -\lambda$. $F_y(x,y,z,\lambda) = 2y + \lambda xz = 0$; $2y^2 = -\lambda xyz = -\lambda$.

$F_z(x,y,z,\lambda) = 2z + \lambda xy = 0$; $2z^2 = -\lambda xyz = -\lambda$.

Hence $x^2 = y^2 = z^2$. From $xyz = 1$ we have the four critical points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$. At each of these points $f(x,y,z) = 3$ which is the minimum of f.

14. $f(x,y,z) = xyz$, and $x^2+y^2+z^2=4$ $f(x,y,z,\lambda) = xyz + \lambda(x^2+y^2+z^2-1)$.
$F_x = yz + 2x\lambda = 0$; $xyz = -2x^2\lambda$. $F_y = xz + 2y\lambda = 0$; $xyz = -2y^2\lambda$. $F_z = xy + 2z\lambda = 0$; $xyz = -2z^2\lambda$
Hence $x^2 = y^2 = z^2 = \frac{4}{3}$, $x = \pm\frac{2}{3}\sqrt{3}$. At $+++$, $+--$, $-+-$, $+--$, $f = \frac{8}{9}\sqrt{3}$ and at $---$, $-++$, $+-+$, $++-$, $f = -\frac{8}{9}\sqrt{3}$. If $\lambda = 0$, we have $(\pm 2,0,0)$, $(0,\pm 2,0)$, $(0,0,\pm 2)$ at which $f = 0$. The minimum is $-\frac{8}{9}\sqrt{3}$.

15. $f(x,y,z) = x+y+z$, and $x^2+y^2+z^2 = 9$. $f(x,y,z,\lambda) = x + + z + \lambda(x^2+y^2+z^2-9)$.
$F_x = 1 + 2\lambda x = 0$; $x = -1/2\lambda$. $F_y = 1 + 2\lambda y = 0$; $y = -1/2\lambda$. $F_z = 1 + 2\lambda z = 0$; $z = -1/2\lambda$. $x = y = z$.
$F_\lambda = x^2+y^2+z^2-9 = 0$; $x^2+y^2+z^2-9=0$; $x = \pm\sqrt{3}$
Critical points are $(\sqrt{3},\sqrt{3},\sqrt{3})$, $(-\sqrt{3},-\sqrt{3},-\sqrt{3})$. $f(\sqrt{3},\sqrt{3},\sqrt{3}) = 3\sqrt{3}$, $f(-\sqrt{3},-\sqrt{3},-\sqrt{3}) = -3\sqrt{3}$.
Because the constraint set is closed and bounded, f has extreme values, assumed at the critical points. Therefore, the maximum value of f is $3\sqrt{3}$.

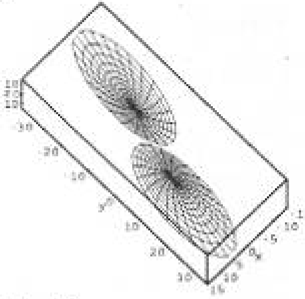
16. $f(x,y,z) = xyz$ with constraint $2xy + 3xz + yz = 72$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

▹ The constraint set is a hyperboloid of two sheets (see below) and meets the coordinate planes in the hyperbolas $zy = 72$, $xz = 24$, and $xy = 36$. Because it is not bounded, we must prove there is an absolute maximum. If $x = y = z$, then $6x^2 = 72$, $x = 2\sqrt{3}$ and $f = x^3 = 24\sqrt{3} > 41$. Now we show that f has smaller values outside a certain closed and bounded set. We solve the constraint for z and substitute in f. Then $z = \dfrac{72-2xy}{3x+y}$ and $f = xy\dfrac{72-2xy}{3x+y}$. Because the maximum is positive, we have $xy \leq 36$ and $f \leq 36 \cdot \frac{72}{3x+y}$. If (x,y) is on or outside the square $S = [0,20]\times[0,20]$, then $f \leq \frac{36 \cdot 72}{60+20} = 32.4$. Therefore, f has a maximum at an interior point of S, that is, at a critical point.

Let F be the function defined by
$F(x,y,z,\lambda) = xyz + \lambda(2xy + 3xz + yz - 72)$
We find the critical points of the function F. Let

$$F_x(x,y,z,\lambda) = yz + \lambda(2y+3z) = 0 \tag{1}$$
$$F_y(x,y,z,\lambda) = xz + \lambda(2x+z) = 0 \tag{2}$$
$$F_z(x,y,z,\lambda) = xy + \lambda(3x+y) = 0 \tag{3}$$
$$F_\lambda(x,y,z,\lambda) = 2xy + 3xz + yz - 72 = 0 \tag{4}$$

We multiply of Eq. (1) by x, Eq. (2) by y, and Eq. (3) by z. This results in



$$xyz + \lambda(2xy + 3xz) = 0 \tag{5}$$
$$xyz + \lambda(2xy + yz) = 0 \tag{6}$$
$$xyz + \lambda(3xz + yz) = 0 \tag{7}$$

From Eqs. (5) and (6) we conclude that

$$3xz = yz$$
$$x = \tfrac{1}{3}y \tag{8}$$

From Eqs. (6) and (7) we conclude that

$$2xy = 3xz$$
$$z = \tfrac{2}{3}y \tag{9}$$

Substituting from (8) and (9) into (4), we have

$$2(\tfrac{1}{3}y)y + 3(\tfrac{1}{3}y)(\tfrac{2}{3}y) + y(\tfrac{2}{3}y) - 72 = 0$$
$$2y^2 = 72$$

Then $y = 6$, $x = 2$, and $z = 4$ and the maximum value is $f(2,6,4) = (2)(6)(4) = 48$.
Alternatively, consider $6f^2 = 6x^2y^2z^2 = (2xy)(3xz)(yz)$. Because the sum of the factors is constant (72) the product is maximum when they are equal, leading immediately to Eqs. (8) and (9).
We show how to graph the constraint. Start with the simpler surface
$xy + yz + zx = 1$
Substitute $x = u + v$, $y = u - v$, and complete the square to show that we have a hyperboloid of two sheets.
$u^2 + 2uz - v^2 = 1$
$(u+z)^2 - z^2 - v^2 = 1$
Now let $(u+z)^2 = \sec^2 r$, $z^2 + v^2 = \tan^2 r$ which leads to
$u + z = \sec r$, $z = \tan r \sin s$, $v = \tan r \cos s$
The parametric equations are
$x = (u+z) + v - z = \sec r + \tan r \cos s - \tan r \sin s$
$y = (u+z) - v - z = \sec r - \tan r \cos s - \tan r \sin s$
$z = \tan r \sin s$
For surface of our problem, multiply the formulas for x, y, z by $2\sqrt{3}$, $6\sqrt{3}$, $4\sqrt{3}$.

17. $f(x,y,z) = x^2 + 4y^2 + 16z^2$, and (a) $xyz = 1$; (b) $xy = 1$; (c) $x = 1$.
Because z, and hence $f(x,y,z)$, can be be arbitrarily large, f has no maximum so we expect the critical points to give minima.
(a) $xyz = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(xyz - 1)$.
$F_x = 2x + yz\lambda = 0$; $2x^2 = -xyz\lambda$. $F_y = 8y + xz\lambda = 0$; $8y^2 = -xyz\lambda$. $F_z = 32z + xy\lambda = 0$; $32z^2 = -xyz\lambda$.
Hence $x^2 = 16z^2$; $x = \pm 4z$ and $y^2 = 4z^2$; $y = \pm 2z$. From $xyz = 1$ we have $(\pm 4z)(\pm 2z)z = 1$; $z = \pm\frac{1}{2}$.
Because $xyz = 1$, the 4 critical points are $(2, 1, \frac{1}{2})$, $(2, -1, -\frac{1}{2})$, $(-2, 1, -\frac{1}{2})$ and $(-2, -1, \frac{1}{2})$.
At each of these, $f(x,y,z) = 4 + 4 + 4 = 12$. Therefore, the minimum value of f is 12.
(b) $xy = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(xy - 1)$.
$F_x = 2x + \lambda y = 0$; $2x^2 = -\lambda xy$. $F_y = 8y + \lambda x = 0$; $8y^2 = -\lambda xy$. $x^2 = 4y^2$; $x = \pm 2y$. $F_z = 32z = 0$; $z = 0$.
From $xy = 1$, if $x = 2y$, $2y^2 = 1$, $y = \pm\frac{1}{2}\sqrt{2}$; $x = -2y$ is impossible. The 2 critical points are $(\sqrt{2}, \frac{1}{2}\sqrt{2}, 0)$ and $(-\sqrt{2}, -\frac{1}{2}\sqrt{2}, 0)$. At each, $f(x,y,z) = 4$. Hence, the minimum value of f is 4.
(c) $x = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(x - 1)$
$F_x = 2x + \lambda = 0$. $F_y = 8y = 0$; $y = 0$. $F_z = 32z = 0$; $z = 0$. Thus the critical point is $(1, 0, 0)$.
Because $f(1, 0, 0) = 1$, the minimum value of f is 1.

18. Let w units be the distance from the point $(1, 3, 0)$ to the point (x, y, z) in the plane $4x + 2y - z = 5$. w has a minimum when w^2 does, and has no maximum. Let $f(x,y,z) = w^2 = (x-1)^2 + (y-3)^2 + z^2$.
$F(x,y,z,\lambda) = (x-1)^2 + (y-3)^2 + z^2 + \lambda(4x + 2y - z - 5)$
$F_x = 2(x-1) + 4\lambda = 0$, $x = 1 - 2\lambda$. $F_y = 2(y-3) + 2\lambda = 0$, $y = 3 - \lambda$. $F_z = 2z - \lambda = 0$, $z = \frac{1}{2}\lambda$.
$4(1 - 2\lambda) + 2(3 - \lambda) - \frac{1}{2}\lambda = 5$, $10 - \frac{21}{2}\lambda = 5$, $\lambda = \frac{10}{21}$. $x = \frac{1}{21}$, $y = \frac{53}{21}$. $z = \frac{5}{21}$. Therefore, f has the critical point $(\frac{1}{21}, \frac{53}{21}, \frac{5}{21})$ and at this point $w = \sqrt{(\frac{1}{21} - 1)^2 + (\frac{53}{21} - 3)^2 + (\frac{5}{21})^2} = \frac{5}{21}\sqrt{21}$.

19. Let w units be the distance from the point $(1, -1, -1)$ to the point (x, y, z) in the plane $x + 4y + 3z = 2$. w has a minimum when w^2 does, and has no maximum. Let $f(x,y,z) = w^2 = (x-1)^2 + (y+1)^2 + (z+1)^2$.
$F(x,y,z,\lambda) = (x-1)^2 + (y+1)^2 + (z+1)^2 + \lambda(x + 4y + 3z - 2)$
$F_x = 2(x-1) + \lambda = 0$; $x = 1 - \frac{1}{2}\lambda$. $F_y = 2(y+1) + 4\lambda = 0$; $y = -1 - 2\lambda$. $F_z = 2(z+1) + 3\lambda = 0$; $z = -1 - \frac{3}{2}\lambda$
$(1 - \frac{1}{2}\lambda) + 4(-1 - 2\lambda) + 3(-1 - \frac{3}{2}\lambda) = 2$; $\lambda = -\frac{8}{13}$. $x = \frac{17}{13}$. $y = \frac{3}{13}$. $z = -\frac{1}{13}$. Therefore, f has the critical point $(\frac{17}{13}, \frac{3}{13}, -\frac{1}{13})$ and at this point $w = \sqrt{(\frac{17}{13} - 1)^2 + (\frac{3}{13} + 1)^2 + (-\frac{1}{13} + 1)^2} = \sqrt{(\frac{4}{13})^2 + (\frac{16}{13})^2 + (\frac{12}{13})^2} = \frac{4}{13}\sqrt{26}$

20. Find the least and greatest distance from the origin to a point on the ellipse $x^2 + 4y^2 = 16$.
▶ Because the ellipse is a closed and bounded set, the distance has a maximum and a minimum which occurs at a critical point. The distance from the origin to any point (x, y) is $\sqrt{x^2 + y^2}$ units. Because the least and greatest distances occur at the points where the square of the distance has its extrema, we let
$f(x,y) = x^2 + y^2$
We must find the points where the function f has its extrema, subject to the constraint
$x^2 + 4y^2 = 16$ (1)
Let F be the function defined by
$F(x,y,\lambda) = x^2 + y^2 + \lambda(x^2 + 4y^2 - 16)$
We find the critical points of the function F. Let
$F_x(x,y,\lambda) = 2x + 2\lambda x = 2x(1 + \lambda) = 0$ (2)
$F_y(x,y,\lambda) = 2y + 8\lambda y = 2y(1 + 4\lambda) = 0$ (3)
From Eq.(2) either $x = 0$ or $\lambda = -1$.
If $x = 0$, then from Eq. (1) we have $y = \pm 2$ and from Eq. (3) we have $\lambda = -\frac{1}{4}$.
If $\lambda = -1$, then from Eq. (3) we have $y = 0$ and from Eq. (1) we have $x = \pm 4$.
We list each critical point of F, the value of $\sqrt{x^2 + y^2}$ and our conclusion. Note that the extrema occur at the ends of the axes.

x	y	λ	$\sqrt{x^2 + y^2}$	
0	2	$-\frac{1}{4}$	2	absolute minimum
0	−2	$-\frac{1}{4}$	2	absolute minimum
4	0	−1	4	absolute maximum
−4	0	−1	4	absolute maximum

21. Let w units be the distance from the origin to a point (x, y, z) on the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. The ellipsoid is a closed and bounded set so there are extrema for w attained at the extrema of w^2. Let
$f(x, y, z) = w^2 = x^2 + y^2 + z^2$. $F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(9x^2 + 4y^2 + z^2 - 36)$
$F_x(x, y, z, \lambda) = 2x + 18\lambda x = 0$; $x = 0$ or $\lambda = -\frac{1}{9}$. $F_y(x, y, z, \lambda) = 2y + 8\lambda y = 0$; $y = 0$ or $\lambda = -\frac{1}{4}$.
$F_z(x, y, z, \lambda) = 2z + 2\lambda z = 0$; $z = 0$ or $\lambda = -1$. $F_\lambda(x, y, z, \lambda) = 9x^2 + 4y^2 + z^2 - 36 = 0$.
The three possible solutions are:
$\lambda = -\frac{1}{9}$, $y = 0$, $z = 0$; $9x^2 - 36 = 0$; $x = \pm 2$. Critical points $(2, 0, 0)$ and $(-2, 0, 0)$. $w = 2$.
$x = 0$, $\lambda = -\frac{1}{4}$, $z = 0$; $4y^2 - 36 = 0$; $y = \pm 3$. Critical points $(0, 3, 0)$ and $(0, -3, 0)$. $w = 3$.
$x = 0$, $y = 0$, $\lambda = -1$. $z^2 - 36 = 0$; $z = \pm 6$. Critical points $(0, 0, 6)$ and $(0, 0, -6)$. $w = 6$.
Therefore, the least distance from the origin to a point on the ellipsoid is 2 and the greatest distance is 6. The points $(0, \pm 3, 0)$ are saddle points of f.

22. $f = 2x^2 + 3y^2 + z^2$, $g = x + y + z = 5$. $F(x, y, z, \lambda) = 2x^2 + 3y^2 + z^2 + \lambda(x + y + z - 5)$
$F_x = 4x + \lambda = 0$, $x = -\frac{1}{4}\lambda$; $F_y = 6y + \lambda = 0$, $y = -\frac{1}{6}\lambda$; $F_z = 2z + \lambda = 0$, $z = -\frac{1}{2}\lambda$
$-\frac{1}{4}\lambda - \frac{1}{6}\lambda - \frac{1}{2}\lambda = 5$, $\lambda = -\frac{60}{11}$, $x = \frac{15}{11}$, $y = \frac{10}{11}$, $z = \frac{30}{11}$. Although the constraint set is a plane which is unbounded, the level surfaces are ellipsoids and it is clear that some smallest ellipsoid touches the plane.

23. $f(x, y, z) = x^2 + y^2 + z^2$, and $x + 2y + 3z = 6$, and $x - y - z = -1$.
$f(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 6) + \mu(x - y - z + 1)$
The constraint set is a line, closed but unbounded, so f has a minimum but no maximum.
$F_x = 2x + \lambda + \mu = 0$; $x = -\frac{1}{2}\lambda - \frac{1}{2}\mu$. $F_y = 2y + 2\lambda - \mu = 0$; $y = -\lambda + \frac{1}{2}\mu$. $F_z = 2z + 3\lambda - \mu = 0$; $z = -\frac{3}{2}\lambda + \frac{1}{2}\mu$.
$F_\lambda = x + 2y + 3z - 6 = 0$; $-7\lambda + 2\mu = 6$. $F_\mu = x - y - z + 1 = 0$; $2\lambda - \frac{3}{2}\lambda = -1$.
Therefore $\lambda = -\frac{14}{13}$, $\mu = -\frac{10}{13}$, $x = \frac{12}{13}$, $y = \frac{9}{13}$, $z = \frac{16}{13}$. The critical point is $(\frac{12}{13}, \frac{9}{13}, \frac{16}{13})$.
$f(\frac{12}{13}, \frac{9}{13}, \frac{16}{13}) = (\frac{12}{13})^2 + (\frac{9}{13})^2 + (\frac{16}{13})^2 = \frac{481}{169} = \frac{37}{13}$, which is the minimum value of f.

24. Use Lagrange multipliers to find the absolute minimum function value of f if $f(x, y, z) = x^2 + y^2 + z^2$, with the two constraints
$x + y + z = 1$ (1)
$3x - 2y + z = -4$ (2)

▸ The constraint is the line formed by the intersection of two planes; its distance from the origin attains an absolute minimum at a critical point. Let F be the function defined by
$F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x + y + 2z - 1) + \mu(3x - 2y + z + 4)$
Then
$F_x(x, y, z, \lambda, \mu) = 2x + \lambda + 3\mu = 0$ (3)
$F_y(x, y, z, \lambda, \mu) = 2y + \lambda - 2\mu = 0$ (4)
$F_z(x, y, z, \lambda, \mu) = 2z + 2\lambda + \mu = 0$ (5)
We solve Eq. (3) for x, Eq. (4) for y, and Eq. (5) for z.
$x = -\frac{1}{2}(\lambda + 3\mu)$ (6)
$y = -\frac{1}{2}(\lambda - 2\mu)$ (7)
$z = -\frac{1}{2}(2\lambda + \mu)$ (8)
Substituting from Eqs. (6), (7), and (8) into (1), we get

$$\begin{aligned} -\tfrac{1}{2}(\lambda + 3\mu) - \tfrac{1}{2}(\lambda - 2\mu) - (2\lambda + \mu) - 1 &= 0 \\ -\lambda - 3\mu - \lambda + 2\mu - 4\lambda - 2\mu - 2 &= 0 \\ 6\lambda + 3\mu &= -2 \qquad (9) \end{aligned}$$

Substituting from Eqs. (6), (7), and (8) into (2), we get

$$\begin{aligned} -\tfrac{3}{2}(\lambda + 3\mu) + (\lambda - 2\mu) - \tfrac{1}{2}(2\lambda + \mu) + 4 &= 0 \\ -3\lambda - 9\mu + 2\lambda - 4\mu - 2\lambda - \mu + 8 &= 0 \\ 3\lambda + 14\mu &= 8 \qquad (10) \end{aligned}$$

Solving,
14(Eq. 9) − 3(Eq. 10): $75\lambda + 52 = 0$, $\lambda = -\frac{52}{75}$
(Eq. 9) − 2(Eq. 10): $-25\mu + 18 = 0$, $\mu = \frac{18}{25}$

Substituting into (6), (7), and (8), we obtain
$x = -\frac{1}{2}(-\frac{52}{75} + \frac{54}{25}) = -\frac{11}{15}$ $y = -\frac{1}{2}(-\frac{52}{75} - \frac{36}{25}) = \frac{16}{15}$ $z = -\frac{1}{2}(-\frac{104}{75} + \frac{18}{25}) = \frac{1}{3}$
Evaluating f at the critical point gives the absolute minimum value $f(-\frac{11}{15}, \frac{16}{15}, \frac{1}{3}) = (-\frac{11}{15})^2 + (\frac{16}{15})^2 + (\frac{1}{3})^2 = \frac{134}{75}$

25. $f(x,y,z) = xyz$, $x+y+z=4$, $x-y-z=3$. $F(x,y,z,\lambda,\mu) = xyz + \lambda(x+y+z-4) + \mu(x-y-z-3)$.
$F_x = yz + \lambda + \mu = 0$ (1). $F_y = xz + \lambda - \mu = 0$ (2). $F_z = xy + \lambda - \mu = 0$ (3). $F_\lambda = x+y+z-4 = 0$ (4)
$F_\mu = x - y - z - 3 = 0$ (5)
From (4) and (5) we have $2x - 7 = 0$; $x = \frac{7}{2}$. From (2) and (3) we get $y = z$. From (4) we get
$y = \frac{1}{4}$, $z = \frac{1}{4}$. The critical point is $P(\frac{7}{2}, \frac{1}{4}, \frac{1}{4})$. $f(\frac{7}{2}, \frac{1}{4}, \frac{1}{4}) = (\frac{7}{2})(\frac{1}{4})(\frac{1}{4}) = \frac{7}{32}$.
Consider x and y as functions of z and denote $D_z x$ by x' and $D_z y$ by y'. From the constraints, $x' + y' + 1 = 0$, $x' - y' - 1 = 0$. Hence $x' = 0$, $y' = -1$. $f' = x'yz + xy'z + xy = -xz + xy$.
$f'' = -x'z - x + x'y + xy' = -2x = -7 < 0$ at P. Hence f has a relative maximum value at P.

26. $f(x,y,z) = x^3 + y^3 + z^3$, $x + y + z = 1$ (1), $x + y - z = 0$ (2). $F = x^3 + y^3 + z^3 + \lambda(x+y+z-1) + \mu(x+y-z)$
$F_x = 3x^2 + \lambda + \mu = 0$ (3). $F_y = 3y^2 + \lambda + \mu = 0$ (4). $F_z = 3z^2 + \lambda - \mu = 0$ (5) . From (1) and (2) we have $2z = 1$, $z = \frac{1}{2}$. From (3) and (4) we get $x^2 = y^2$. From (1), if $y = x$: $x + x + \frac{1}{2} = 1$, $x = \frac{1}{4}$; if $y = -x$, $x - x + \frac{1}{2} = 1$, impossible. At the critical point P, $f(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) = \frac{1}{64} + \frac{1}{64} + \frac{1}{8} = \frac{5}{32}$. Let $'$ be the derivative with respect to x. Because $z = \frac{1}{2}$, we have from (1) or (2), $1 + y' = 0$, $y' = -1$. $f' = 3x^2 + 3y^2y' = 3x^2 - 3y^2$.
$f'' = 6x - 6yy' = 6x + 6y = 3 > 0$ at P. Thus, P is an absolute minimum value.

In Exercises 27–36, use Lagrange multipliers to solve the indicated exercise of Exercises 12.8.

27. Let x, y, and z be the numbers. Then $f(x,y,z) = xyz$ and $x + y + z = 24$, $x \geq 0$, $y \geq 0$, $z \geq 0$. The constraint set is a triangle, which is closed and bounded so f has a maximum and a minimum. The minimum is 0 attained at all points of the boundary so any critical point is a maximum. Let $f(x,y,z,\lambda) = xyz + \lambda(x+y+z-24)$.
$F_x = yz + \lambda = 0$; $xyz = -\lambda x$. $F_y = xz + \lambda = 0$; $xyz = -\lambda y$. $F_z = xy + \lambda = 0$; $xyz = -\lambda z$. $x = y = z$.
$F_\lambda = x + y + z - 24 = 0$; $3x = 24$. Hence $x = 8$, $y = 8$, $z = 8$ gives the maximum $xyz = 512$ of f.

28. Find three positive numbers whose product is 24 such that their sum is as small as possible.

▸ Let $x \geq 0$, $y \geq 0$, $z \geq 0$ be the numbers. We want to determine x, y, and z such that the function f defined by
$f(x,y,z) = x + y + z$
has a minimum value, subject to the constraint
$xyz = 24$ (1)
If $x > 3$, $y > 3$, $z > 3$, then $f > 3 + 3 + 3 = 9$. Therefore, if the relative extremum on the closed and bounded cube $0 \leq x \leq 3$, $0 \leq y \leq 3$, $0 \leq z \leq 3$ is less than 9, it must be the absolute minimum. Let F be the function defined by
$F(x,y,z,\lambda) = x + y + z + \lambda(xyz - 24)$
Then
$F_x(x,y,z,\lambda) = 1 + \lambda yz = 0$ (2)
$F_y(x,y,z,\lambda) = 1 + \lambda xz = 0$ (3)
$F_z(x,y,z,\lambda) = 1 + \lambda xy = 0$ (4)
We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in
$x + \lambda xyz = 0$ (5)
$y + \lambda xyz = 0$ (6)
$z + \lambda xyz = 0$ (7)
From Eqs. (5), (6), (7) we conclude that
$x = y = z$ (8)
Substituting from Eqs. (8) into (1) we get

$$x^3 = 24$$
$$x = \sqrt[3]{24} < \sqrt[3]{27} = 3$$

Therefore each number is $\sqrt[3]{24}$. Because the sum is less than 9, we have the absolute minimum.

29. Let w units be the distance from the point $(1, -2, 3)$ to a point (x,y,z) in the plane $3x + 2y - z - 5 = 0$. w will be a minimum when w^2 is a minimum.
$f(x,y,z) = w^2 = (x-1)^2 + (y+2)^2 + (z-3)^2$
$F(x,y,z,\lambda) = (x-1)^2 + (y+2)^2 + (z-3)^2 + \lambda(3x + 2y - z - 5)$
$F_x = 2(x-1) + 3\lambda = 0$; $x = 1 - \frac{3}{2}\lambda$. $F_y = 2(y+2) + 2\lambda = 0$; $y = -2 - \lambda$. $F_z = 2(z-3) - \lambda = 0$; $z = 3 + \frac{1}{2}\lambda$.
$F_\lambda = 3x + 2y - z - 5 = 0$; $3(1 - \frac{3}{2}\lambda) + 2(-2 - \lambda) - (3 + \frac{1}{2}\lambda) - 5 = 0$; $7\lambda = -9$.
Hence $\lambda = -\frac{9}{7}$, $x = \frac{41}{14}$, $y = -\frac{5}{7}$, $z = \frac{33}{14}$. At the critical point $P(\frac{41}{14}, -\frac{5}{7}, \frac{33}{14})$,
$w = \sqrt{(\frac{41}{14} - 1)^2 + (-\frac{5}{7} + 2)^2 + (\frac{33}{14} - 3)^2} = \frac{9}{14}\sqrt{14}$
Since x, and hence $f(x,y,z)$, can be arbitrarily large, w has a minimum at P.

30. Find the points on the surface $g = y^2 - xz = 4$ that are closest to the origin and find the minimum distance.

▹ Because $g = y^2 - \frac{1}{4}[(x+z)^2 - (x-z)^2] = y^2 - \frac{1}{4}(x+z)^2 + \frac{1}{4}(x-z)^2$, the surface is a hyperboloid of one sheet. $F(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(y^2 - xz - 4)$. $F_x = 2x + \lambda z = 0$, $z = -(2/\lambda)x$. $F_y = 2y - 2\lambda y = 2y(1-\lambda) = 0$. $F_z = 2z + \lambda x = 0$, $z = -(\lambda/2)x$. If $\lambda \neq \pm 2$, then $x = z = 0$, $y^2 = 4$, $y = \pm 2$, $d = 2$. If $\lambda = 2$, $z = -x$, $y = 0$, $x^2 = 4$, $x = \pm 2$, $z = \mp 2$, $d = 2\sqrt{2}$. If $\lambda = 2$, $z = x$, $y = 0$, $-x^2 = 4$, impossible. The minimum distance is 2 at $(0, \pm 2, 2)$.

31. Let w units be the distance to the origin of a point (x,y,z) on $x^2 + 4y^2 + 4z^2 = 4$ and $x - 4y - z = 0$. w will be a minimum when w^2 is a minimum. Let $f(x,y,z) = w^2 = x^2 + y^2 + z^2$.

$F(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 + \lambda(x^2 + 4y^2 + 4z^2 - 4) + \mu(x - 4y - z)$

$F_x = 2x + 2\lambda x + \mu = 0$. $F_y = 2y + 8\lambda y - 4\mu = 0$. $F_z = 2z + 8\lambda z - \mu = 0$.

$F_y - 4F_z$: $(2 + 8\lambda)(y - 4z) = 0$; $y = 4z$ or $\lambda = -\frac{1}{4}$. If $\lambda = -\frac{1}{4}$, from F_y we get $\mu = 0$ and from F_x we get $x = 0$.

$F_\lambda = x^2 + 4y^2 + 4z^2 - 4 = 0$. $F_\mu = x - 4y - z = 0$.

If $y = 4z$, then $x - 16z - \ = 0$; $x = 17z$ and $289z^2 + 64z^2 + 4z^2 - 4 = 0$; $357z^2 = 4$. Therefore

$z = \pm\frac{2}{\sqrt{357}}$, the critical points are $\left(\frac{34}{\sqrt{357}}, \frac{8}{\sqrt{357}}, \frac{2}{\sqrt{357}}\right)$ and $\left(-\frac{34}{\sqrt{357}}, -\frac{8}{\sqrt{357}}, -\frac{2}{\sqrt{357}}\right)$ so $f = \frac{1124}{357}$.

If $x = 0$, then $0 - 4y - z = 0$; $z = -4y$ and $0 + 4y^2 + 64y^2 - 4 = 0$; $17y^2 = 1$. Therefore

$y = \pm\frac{1}{\sqrt{17}}$, the critical points are $P\left(0, \frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right)$ and $Q\left(0, -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$ so $f = 1$.

Because the constraint set is closed and bounded, f attains a minimum at a critical point. Hence P and Q are closest to the origin and the minimum distance is 1.

32. Exercise 34. A rectangular box without a top is to be made at a cost of \$10 for the material. If the material for the bottom of the box costs \$0.15 per square foot and material for the sides costs \$0.30 per square foot, find the dimensions of the box of greatest volume that can be made.

▹ Let x ft, y ft, and z ft be the length, width, and height of the box, $x \geq 0$, $y \geq 0$, $z \geq 0$. We must find x, y, z so that the volume function defined by

$V(x,y,z) = xyz$

is a maximum, subject to the constraint

$$0.15xy + 0.30(2xz + 2yz) = 10.00 \tag{1}$$

As in Exercise 16, the constraint set is ubounded but there is an absolute maximum occuring at a critical point. Let F be the function defined by

$F(x,y,z,\lambda) = xyz + \lambda(0.15xy + 0.60xz + 0.60yz - 10)$

Then

$$F_x(x,y,z,\lambda) = yz + 0.15\lambda y + 0.60\lambda z = 0 \tag{2}$$
$$F_y(x,y,z,\lambda) = xz + 0.15\lambda x + 0.60\lambda z = 0 \tag{3}$$
$$F_z(x,y,z,\lambda) = xy + 0.60\lambda x + 0.60\lambda y = 0 \tag{4}$$

We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in

$$xyz + \lambda(0.15xy + 0.60xz) = 0 \tag{5}$$
$$xyz + \lambda(0.15xy + 0.60yz) = 0 \tag{6}$$
$$xyz + \lambda(0.60xz + 0.60yz) = 0 \tag{7}$$

From Eqs (5) and (6), we conclude that

$x = y$

From Eqs. (6) and (7), we have

$$0.15y = 0.60z$$
$$y = 4z$$

Replacing x and y by $4z$ in Eq. (1), we obtain

$$\begin{aligned} 0.15(4z)(4z) + 0.60(4z)z + 0.60(4z)z &= 10 \\ 7.2z^2 &= 10 \\ z &= \tfrac{5}{6}\sqrt{2} \\ x = y &= \tfrac{10}{3}\sqrt{2} \end{aligned}$$

Thus, the largest box has a square base of side $\frac{10}{3}\sqrt{2}$ ft and a height of $\frac{5}{6}\sqrt{2}$ ft.

33. Let ℓ ft, w ft, h ft be the length, width and height of the box. C cents is the cost.
$C = 18\ell w + 16wh + 12\ell h$ and $\ell wh = 16$. $F(\ell, w, h, \lambda) = 18\ell w + 16wh + 12\ell h + \lambda(\ell wh - 16)$.
$\frac{\partial F}{\partial \ell} = 18w + 12h + \lambda wh = 0$; $\frac{18}{h} + \frac{12}{w} = -\lambda$
$\frac{\partial F}{\partial w} = 18\ell + 16h + \lambda \ell h = 0$; $\frac{18}{h} + \frac{16}{\ell} = -\lambda$; $\frac{12}{w} = \frac{16}{\ell}$; $w = \frac{3}{4}\ell$
$\frac{\partial F}{\partial h} = 16w + 12\ell + \lambda \ell w = 0$; $\frac{16}{\ell} + \frac{12}{w} = -\lambda$; $\frac{18}{h} = \frac{16}{\ell}$; $h = \frac{9}{8}\ell$
$\frac{\partial F}{\partial \lambda} = \ell wh - 16 = 0$; $\ell(\frac{3}{4}\ell)(\frac{9}{8}\ell) = 16$; $\ell^3 = \frac{512}{27}$; $\ell = \frac{8}{3}$. The critical point is $P(\frac{8}{3}, 2, 3)$.
Because wh, and hence C, can be arbitrarily large, C has a minimum at P.

34. $T = 100xy^2z$, $g = x^2 + y^2 + z^2 = 4$. $F = 100xy^2z - \lambda(x^2 + y^2 + z^2 - 4)$. $F_x = 100y^2z - 2\lambda x = 0$.
$F_y = 200xyz - 2\lambda y = 2y(100xz - \lambda) = 0$. $F_z = 100xy^2 - 2\lambda z = 0$. If $x = 0$, $y = 0$, or $z = 0$, then $T = 0$.
Otherwise, F_y: $\lambda = 100xz$, F_x: $100y^2z - 200x^2z = 0$, $x^2 = \frac{1}{2}y^2$. F_z: $100xy^2 - 200xz^2 = 0$, $z^2 = \frac{1}{2}y^2$.
$\frac{1}{2}y^2 + y^2 + \frac{1}{2}y^2 = 4$, $y = \pm\sqrt{2}$, $x = \pm 1$, $z = \pm 1$ (8 points). Max: $T = 200$ if $x = z$, min: -200 if $x = -z$.

35. Let ℓ, w, and h be the number of units in the length, width and height of the box. Let V cubic units be its volume and S square units its surface area (S is a constant).
$V = \ell wh$ and $S = \ell w + 2\ell h + 2wh$, $\ell > 0$, $w > 0$, $h > 0$. $f(\ell, w, h, \lambda) = \ell wh + \lambda(\ell w + 2\ell h + 2wh - S)$.
$wh < \frac{1}{2}S$, so $V < \frac{1}{2}\ell S$, and V can be arbitrarily small. Any critical point is a maximum.
$\frac{\partial F}{\partial \ell} = wh + \lambda(w + 2h) = 0$; $\frac{1}{h} + \frac{2}{w} = -\frac{1}{\lambda}$. $\frac{\partial F}{\partial w} = \ell h + \lambda(\ell + 2h) = 0$; $\frac{1}{h} + \frac{2}{\ell} = -\frac{1}{\lambda}$.
$\frac{\partial F}{\partial h} = \ell w + \lambda(2\ell + 2w) = 0$; $\frac{2}{w} + \frac{2}{\ell} = -\frac{1}{\lambda}$. Hence $\frac{1}{\ell} = -\frac{1}{4\lambda}$, $\frac{1}{w} = -\frac{1}{4\lambda}$, $\frac{1}{h} = -\frac{1}{2\lambda}$.
Therefore V has a maximum when $\ell{:}w{:}h = 1{:}1{:}\frac{1}{2}$.

36. Prove that the box having the largest volume that can be placed inside a sphere is in the shape of a cube.

▸ Let x, y, z be the number of units in the dimensions of the box. If a units is the diameter of the sphere, because the largest box is inscribed in the sphere, the diagonal of the box is a units. Thus we we want to show that the volume function defined by
$V(x, y, z) = xyz$
subject to the constraint
$$x^2 + y^2 + z^2 = a^2 \tag{1}$$
has an absolute maximum value if $x = y = z$.
Let F be the function defined by
$F(x, y, z, \lambda) = xyz + \lambda(0.15xy + 0.60xz + 0.60yz - 10)$
Then
$$F_x(x, y, z, \lambda) = yz + 2\lambda x = 0 \tag{2}$$
$$F_y(x, y, z, \lambda) = xz + 2\lambda y = 0 \tag{3}$$
$$F_z(x, y, z, \lambda) = xy + 2\lambda z = 0 \tag{4}$$
We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in
$xyz + 2\lambda x^2 = 0$
$xyz + 2\lambda y^2 = 0$
$xyz + 2\lambda z^2 = 0$
and we conclude that $x^2 = y^2 = z^2$, and thus, $x = y = z$. Because the constraint set is closed and bounded, V has an absolute maximum value which must occur at the critical point. Therefore, the box having the largest volume that can be placed in a sphere is a cube.

In Exercises 37 and 38 we wish to maximize U subject to $g = 2x + 3y + 4z = 90$.

37. $U = \exp(x^2yz)$. $12x^2yz = xx(3y)(4z)$. Because $x + x + 3y + 4z$ is constant, the product is maximum if $x = 3y = 4z = \frac{90}{4} = \frac{45}{2}$, $y = \frac{15}{2}$, $z = \frac{45}{8}$. This point maximizes U.

38. $U = x^2y^3z$. $4x^2y^3z = xxyyy(4z)$. Because $x + x + y + y + y + 4z$ is constant, the product is maximum if $x = y = 4z = \frac{90}{6} = 15$, $z = \frac{15}{4}$. This point maximizes U.

39. $T = 2x^2 + y^2 - y$. Because the disk $x^2 + y^2 \le 1$ is closed and bounded, extrema are attained at a critical point or a boundary point. At a critical point $T_x = 4x = 0$; $x = 0$ and $T_y = 2y - 1 = 0$; $y = \frac{1}{2}$. $T(0, \frac{1}{2}) = -\frac{1}{4}$.
On the boundary, $x^2 = 1 - y^2$, $-1 \le y \le 1$, $T = 2(1 - y^2) + y^2 - y = 2 - y^2 - y$. $T_y = -2y - 1$; $y = \frac{1}{2}$.
When $y = -1$, $x = 0$ and $T = 2$. When $y = -\frac{1}{2}$, $x = \pm\frac{1}{2}\sqrt{3}$ and $T = \frac{9}{4}$. When $y = 1$, $x = 0$ and $T = 0$.

Thus the temperature is hottest at the points $(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and coldest at the point $(0, \frac{1}{2})$.

40. If $T(x, y)$ degrees is the temperature at any point (x, y) of the top half of a circular disk, defined by $x^2 + y^2 \le 1$ and $y \ge 0$ and $T(x, y) = 2x^2 - 3xy + 5y^2$, find the hottest and coldest points in the region and the temperature at those points.

▹ Because the region is closed and bounded, extrema are attained at a critical point or a boundary point.
$T_x = 4x - 3y = 0$
$T_y = -3x + 10y = 0$
Thus the only critical point is $(0, 0)$ and $T(0, 0) = 0$, which turns out to be the absolute minimum.
If $y = 0$, $-1 \le x \le 1$, then $T = 2x^2$, which has a minimum of 0 at 0 and a maximum of 1 at ± 1.
If $x^2 + y^2 = 1$, $y \ge 0$, let
$F(x, y, \lambda) = 2x^2 - 3xy + 5y^2 + \lambda(x^2 + y^2 - 1)$
$F_x(x, y, \lambda) = 4x - 3y + 2\lambda x = (4 + 2\lambda)x - 3y = 0$ (1)
$F_y(x, y, \lambda) = -3x + 10y + 2\lambda y = -3x + (10 + 2\lambda)y = 0$ (2)
This homogeneous linear system has the solution $(0, 0)$. Otherwise, solving Eqs (1) and (2) for y, we obtain

$y = \frac{4 + 2\lambda}{3}x$ and $y = \frac{3}{10 + 2\lambda}$

and so

$$\frac{4 + 2\lambda}{3} = \frac{3}{10 + 2\lambda}$$
$$4\lambda^2 + 28\lambda + 40 = 9$$
$$4\lambda^2 + 28\lambda + 31 = 0 \qquad (3)$$
$$\lambda = \tfrac{1}{2}(-7 \pm 3\sqrt{2})$$
$$\frac{4 + 2\lambda}{3} = -1 \pm \sqrt{2} \qquad (4)$$

Substituting into the constraint, we get
$$1 = x^2 + y^2 = x^2 + (-1 \pm \sqrt{2})x^2 = (4 \pm 2\sqrt{2})x^2$$
and so
$$x^2 = \frac{1}{4 \pm 2\sqrt{2}} = \tfrac{1}{4}(2 \pm \sqrt{2})$$
$$x = \pm\sqrt{2 \pm \sqrt{2}}$$
If we select the positive sign in (4), we choose the positive root. Then

$x = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $y = \frac{1}{2}\sqrt{2 - \sqrt{2}}$ and $T = \frac{7}{2} - \frac{3}{2}\sqrt{2}$

(after much work). If we choose the negative sign in (3), we choose the negative root. Then

$x = -\frac{1}{2}\sqrt{2 - \sqrt{2}}$, $y = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $T = \frac{7}{2} + \frac{3}{2}\sqrt{2}$
which is the absolute maximum value.
The critical points are the *eigenvectors* of the system $4x - 3y = 0$, $-3x + 10y = 0$, and Eq. (3) is the *characteristic equation*. The TI-85 can find these by using the eigVc function in the MATRX MATH menu; other advanced graphics calculators can too.
Alternatively, let $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le \pi$.
$$T = 2\cos^2\theta - 3\sin\theta\cos\theta + 5\cos^2\theta = (1 + \cos 2\theta) - \tfrac{3}{2}\sin 2\theta + \tfrac{5}{2}(1 - \cos 2\theta) = \tfrac{7}{2} - \tfrac{3}{2}\sin 2\theta - \tfrac{3}{2}\cos 2\theta$$
$$= \tfrac{7}{2} - \tfrac{3}{2}\sqrt{2}(\tfrac{1}{2}\sqrt{2}\sin 2\theta + \tfrac{1}{2}\sqrt{2}\cos 2\theta) = \tfrac{7}{2} - \tfrac{3}{2}\sqrt{2}\sin(2\theta + \tfrac{1}{4}\pi)$$

The maximum value of T on the arc, and the absolute maximum value, is $\frac{7}{2} + \frac{3}{2}\sqrt{2}$ if
$$\sin(2\theta + \tfrac{1}{4}\pi) = -1$$
$$2\theta + \tfrac{1}{4}\pi = \tfrac{3}{2}\pi$$
$$2\theta = \tfrac{5}{4}\pi$$
$x = \cos\theta = -\sqrt{\frac{1}{2}(1 + \cos\frac{5}{4}\pi)} = -\frac{1}{2}\sqrt{2 - \sqrt{2}}$ $y = \sin\theta = \sqrt{\frac{1}{2}(1 - \cos\frac{5}{4}\pi)} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

The minimum on the arc is $\frac{7}{2} - \frac{3}{2}\sqrt{2} > 0$ when $x = \frac{1}{2}\sqrt{2+\sqrt{2}}$ and $y = \frac{1}{2}\sqrt{2-\sqrt{2}}$.

A third alternative is to simplify T without changing the constraint by using the rotation of axes formulas A.10.3.

41. Given $2x + 3y + 4z + s + t = 150$, we maximize $24\mathrm{U} = (2x)(3y)(4z)st$ when $2x = 3y = 4z = s = t = \frac{150}{5} = 30$, $x = 15$, $y = 10$, $z = 7.5$.

42. Minimize $\mathrm{C} = (3x^2 + 200) + (y^2 + 400) + (2z^2 + 300) = 3x^2 + y^2 + 2z^2 + 1100$ subject to $x + y + z = 1100$. $\mathrm{F} = 3x^2 + y^2 + 2z^2 + 1100 - \lambda(x + y + z - 100)$. $\mathrm{F}_x = 6x - \lambda = 0$, $x = \frac{1}{6}\lambda$. $\mathrm{F}_y = 2y - \lambda = 0$, $y = \frac{1}{2}\lambda$. $\mathrm{F}_z = 4z - \lambda = 0$, $z = \frac{1}{4}\lambda$. $\frac{1}{6}\lambda + \frac{1}{2}\lambda + \frac{1}{4}\lambda = \frac{11}{12}\lambda = 1100$, $\lambda = 1200$. $x = 200$, $y = 600$, $z = 300$.

43. See Exercise 12.8.26.

Miscellaneous Exercises for Chapter 12

In Exercises 1–4, determine the domain of f and sketch as a region in R^2 the set of points in the domain.

1. $f(x,y) = \sqrt{x^2 + 4y^2 - 16}$. The domain of f is $\{(x,y) \mid x^2 + 4y^2 \geq 16\}$, which is the set of all points in R^2 outside and on the ellipse $x^2/16 + y^2/4 = 1$.

2. $f(x,y) = 6/\sqrt{16 - x^2 - y^2}$. $\operatorname{dom}(f) = \{(x,y) \mid x^2 + y^2 < 16\}$, the points in R^2 inside the circle $x^2 + y^2 = 16$.

3. $f(x,y) = \ln(y - x^2)$. $\operatorname{dom}(f) = \{(x,y) \mid y > x^2\}$, the set of all points in R^2 inside the parabola $y = x^2$.

4. $f(x,y) = \sin^{-1}(5 - x^2 - y^2)$

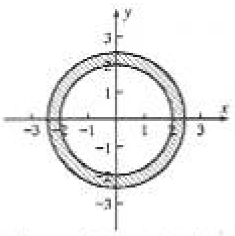
▸ Because the domain of $\sin^{-1} x$ is $[-1,1]$, then the domain of f is the set of all points (x,y) such that

$$-1 \leq 5 - x^2 - y^2 \leq 1$$

$$-6 \leq -x^2 - y^2 \leq -4$$

$$6 \geq x^2 + y^2 \geq 4$$

Thus the domain is the set of all points that lie either on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 6$ of radii 2 and $\sqrt{6}$, as shown in the figure.



5. $f(x,y,z) = \frac{x}{|y| - |z|}$. $f(x,y,z)$ is defined if $|y| - |z| \neq 0$. Thus the domain of f is the set of all points (x,y,z) in R^3 except those on the planes $y = \pm z$. The range of f is $(-\infty, +\infty)$.

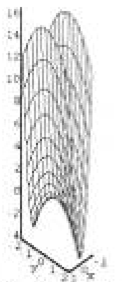
6. $f(x,y,z) = \ln(x^2 + y^2 + z^2 - 4)$. $\operatorname{dom}(f) = \{(x,y,z) \mid x^2 + y^2 + z^2 > 4\}$, the points in R^3 outside the sphere $x^2 + y^2 + z^2 = 4$.

In Exercises 7 and 8, determine the domain of f and sketch the graph of f.

7. $f(x,y) = \sqrt{36 - 4x^2 - 9y^2}$. The domain of f is $\{(x,y) \mid 4x^2 + 9y^2 \leq 36\}$, the set of all points (x,y) in R^2 inside and on the ellipse $x^2/9 + y^2/4 = 1$. The graph of $z = \sqrt{36 - 4x^2 - 9y^2}$ is the upper half of the ellipsoid $x^2/9 + y^2/4 + z^2/36 = 1$.

8. $f(x,y) = 16x^2 - y^2$

▸ The domain of f is all points of R^2. The graph of f is the graph of $z = 16x^2 - y^2$, a hyperbolic paraboloid, as shown in the figure.



9. $f(x,y) = 4x^{1/2}y$. If $f = k$ we have $k = 4x^{1/2}y$. The constant product curves are $y = k/4x^{1/2}$, $k = 16, 8, 4, 2$.

10. $t(x,y) = x^2 + 2y$. If $t = k$ then $k = x^2 + 2y$. The isothermals are the parabolas $y = \frac{1}{2}(k - x^2)$, $k = 0, 2, 4, 6, 8$.

In Exercises 11–24, find the indicated partial derivatives.

11. $f(x,y) = 2x^2y - 3xy^2 + 4x - 2y$

(a) $\mathrm{D}_1 f(x,y) = 4xy - 3y^2 + 4$

(b) $\mathrm{D}_2 f(x,y) = 2x^2 - 6xy - 2$

(c) $\mathrm{D}_{11} f(x,y) = \frac{\partial}{\partial x}(4xy - 3y^2 + 4) = 4y$

(d) $\mathrm{D}_{22} f(x,y) = \frac{\partial}{\partial y}(2x^2 - 6xy - 2) = -6x$

(e) $\mathrm{D}_{12} f(x,y) = \frac{\partial}{\partial y}(4xy - 3y^2 + 4) = 4x - 6y$

(f) $\mathrm{D}_{21} f(x,y) = \frac{\partial}{\partial x}(2x^2 - 6xy - 2) = 4x - 6y$

12. $f(x,y) = (4x^2 - 2y)^3$; (a) $f_1(x,y)$; (b) $f_2(x,y)$; (c) $f_{11}(x,y)$; (d) $f_{22}(x,y)$; (e) $f_{12}(x,y)$; (f) $f_{21}(x,y)$

▸ (a) $f_1(x,y) = \mathrm{D}_x(4x^2-2y)^3 = 3(4x^2-2y)^2\mathrm{D}_x(4x^2-2y) = 3(4x^2-2y)^2(8x) = 96x(2x^2-y)^2$

(b) $f_2(x,y) = \mathrm{D}_y(4x^2-2y)^3 = 3(4x^2-2y)^2\mathrm{D}_y(4x^2-2y) = 3(4x^2-2y)^2(-2) = -24(2x^2-y)^2$

(c) Applying the result of part (a), we obtain

$$\begin{aligned} f_{11}(x,y) &= \mathrm{D}_x f_1(x,y) = \mathrm{D}_x[96x(2x^2-y)^2] = 96[(2x^2-y)^2 + 2x(2x^2-y)(4x)] \\ &= 96(2x^2-y)[2x^2-y+8x^2] = 96(2x^2-y)(10x^2-y) \end{aligned}$$

(d) Applying the result of part (b), we obtain

$$f_{22}(x,y) = \mathrm{D}_y f_2(x,y) = \mathrm{D}_y[-24(2x^2-y)^2] = (-24)(2)(2x^2-y)\mathrm{D}_y(2x^2-y) = 48(2x^2-y)$$

(e) Applying the result of part (a), we obtain

$$f_{12}(x,y) = \mathrm{D}_y[96x(2x^2-y)^2] = (96x)(2)(2x^2-y)(-1) = -192x(2x^2-y)$$

(f) Applying the result of part (b), we obtain

$$f_{12}(x,y) = \mathrm{D}_x f_2(x,y) = \mathrm{D}_x[-24(2x^2-y)^2] = (-24)(2)(2x^2-y)(4x) = -192x(2x^2-y)$$

13. $f(x,y) = \dfrac{x^2-y}{3y^2}$ (a) $f_x(x,y) = \dfrac{2x}{3y^2}$ (b) $f_y(x,y) = \dfrac{1}{3}\cdot\dfrac{(-1)(y^2)-(x^2-y)(2y)}{(y^2)^2} = \dfrac{1}{3}\cdot\dfrac{y^2-2x^2y}{y^4} = \dfrac{y-2x^2}{3y^3}$

(c) $f_{xy}(x,y) = \dfrac{(-2)2x}{3y^3} = -\dfrac{4x}{3y^3}$ (d) $f_{yx}(x,y) = \dfrac{2(-2x)}{3y^3} = -\dfrac{4x}{3y^3}$

14. $f(r,s) = re^{2rs}$ (a) $\mathrm{D}_r f = e^{2rs} + 2rse^{2rs} = (1+2rs)e^{2rs}$ (b) $\mathrm{D}_s f = 2r^2e^{2rs}$

(c) $\mathrm{D}_{rs} f = 2re^{2rs} + (1+2rs)2re^{2rs} = (4r+4r^2s)e^{2rs}$ (d) $\mathrm{D}_{sr} f = \mathrm{D}_{rs}$.

15. $g(s,t) = \sin(st^2) + te^s$. (a) $\mathrm{D}_s g(s,t) = t^2\cos(st^2) + te^s$; (b) $\mathrm{D}_t g(s,t) = 2st\cos(st^2) + e^s$

(c) $\mathrm{D}_{st} g(s,t) = 2t\cos(st^2) + t^2[-2st\sin(st^2)] + e^s = 2t\cos(st^2) - 2st^3\sin(st^2) + e^s$

(d) $\mathrm{D}_{ts} g(s,t) = 2t\cos(st^2) + 2st[-t^2\sin(st^2)] + e^s = 2t\cos(st^2) - 2st^3\sin(st^2) + e^s$

16. $h(x,y) = \tan^{-1}\dfrac{x^3}{y^2}$; (a) $\mathrm{D}_1 h(x,y)$; (b) $\mathrm{D}_2 h(x,y)$; (c) $\mathrm{D}_{11} h(x,y)$; (d) $\mathrm{D}_{22} h(x,y)$

▸ (a) $\mathrm{D}_1 h(x,y) = \dfrac{1}{1+\left(\dfrac{x^3}{y^2}\right)^2}\cdot\mathrm{D}_x\left(\dfrac{x^3}{y^2}\right) = \dfrac{y^4}{x^6+y^4}\cdot\dfrac{3x^2}{y^2} = \dfrac{3x^2y^2}{x^6+y^4}$

(b) $\mathrm{D}_2 h(x,y) = \dfrac{1}{1+\left(\dfrac{x^3}{y^2}\right)^2}\cdot\mathrm{D}_x\left(\dfrac{x^3}{y^2}\right) = \dfrac{y^4}{x^6+y^4}\cdot\dfrac{-2x^3}{y^3} = \dfrac{-2x^3y}{x^6+y^4}$

(c) $\mathrm{D}_{11} h(x,y) = \mathrm{D}_x\left(\dfrac{3x^2y^2}{x^6+y^4}\right) = \dfrac{6xy^2(x^6+y^4) - 3x^2y^2\cdot 6x^5}{(x^6+y^4)^2} = \dfrac{-6xy^2(2x^6-y^4)}{(x^6+y^4)^2}$

(d) $\mathrm{D}_{22} h(x,y) = \mathrm{D}_y\left(\dfrac{-2x^3y}{x^6+y^4}\right) = \dfrac{-2x^3(x^6+y^4) - (-2x^3y)(4y^3)}{(x^6+y^4)^2} = \dfrac{-2x^3(x^6-3y^4)}{(x^6+y^4)^2}$

17. $f(x,y) = e^{x/y} + \ln\dfrac{x}{y} = e^{x/y} + \ln|x| - \ln|y|$

(a) $f_x(x,y) = \dfrac{1}{y}e^{x/y} + \dfrac{1}{x}$ (b) $f_y(x,y) = -\dfrac{x}{y^2}e^{x/y} - \dfrac{1}{y}$ (c) $f_{xx}(x,y) = \dfrac{1}{y^2}e^{x/y} - \dfrac{1}{x^2}$

(d) $f_{yy}(x,y) = \dfrac{2x}{y}e^{x/y} - \dfrac{x}{y^2}e^{x/y}\left(-\dfrac{x}{y^2}\right) + \dfrac{1}{y^2} = \left(\dfrac{2x}{y^3} + \dfrac{x^2}{y^4}\right)e^{x/y} + \dfrac{1}{y^2}$

18. $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$ (a) $f_1 = \dfrac{x}{x^2+y^2}$ (b) $f_{11} = \dfrac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

(c) $f_{12} = \dfrac{-2xy}{(x^2+y^2)^2}$ (d) $f_{121} = \dfrac{-2y(x^2+y^2)^2 + 2xy\cdot 4x(x^2+y^2)}{(x^2+y^2)^4} = \dfrac{-2y(x^2+y^2)+8x^2y}{(x^2+y^2)^3} = \dfrac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

19. $f(x,y,z) = \dfrac{x}{x^2+y^2+z^2}$. (a) $\mathrm{D}_1 f(x,y,z) = \dfrac{x^2+y^2+z^2-x(2x)}{(x^2+y^2+z^2)^2} = \dfrac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$

(b) $\mathrm{D}_2 f(x,y,z) = \dfrac{-x(2y)}{(x^2+y^2+z^2)^2} = \dfrac{-2xy}{(x^2+y^2+z^2)^2}$ (c) $\mathrm{D}_3 f(x,y,z) = \dfrac{-x(2z)}{(x^2+y^2+z^2)^2} = \dfrac{-2xz}{(x^2+y^2+z^2)^2}$

20. $f(x,y,z)=\sqrt{x^2+3yz-z^2}$; (a) $f_x(x,y,z)$; (b) $f_y(x,y,z)$; (c) $f_z(x,y,z)$

▸ (a) $f_x(x,y,z)=\frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_x(x^2+3yz-z^2)=\dfrac{x}{\sqrt{x^2+3yz-z^2}}$

(b) $f_y(x,y,z)=\frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_y(x^2+3yz-z^2)=\dfrac{3z}{2\sqrt{x^2+3yz-z^2}}$

(c) $f_z(x,y,z)=\frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_z(x^2+3yz-z^2)=\dfrac{3y-2z}{2\sqrt{x^2+3yz-z^2}}$

21. $f(u,v,w)=\ln(u^2+4v^2-5w^2)$. $f_u(u,v,w)=\dfrac{2u}{u^2+4v^2-5w^2}$.

(a) $f_{uw}(u,v,w)=\dfrac{-(2u)(-10w)}{(u^2+4v^2-5w^2)^2}=\dfrac{20uw}{(u^2+4v^2-5w^2)^2}$. $f_{uwv}(u,v,w)=\dfrac{-2(20uw)(8v)}{(u^2+4v^2-5w^2)^3}=\dfrac{-320uvw}{(u^2+4v^2-5w^2)^3}$

(b) $f_{uv}=\dfrac{-(2u)(8v)}{(u^2+4v^2-5w^2)^2}=\dfrac{-16uv}{(u^2+4v^2-5w^2)^2}$

$$f_{uvv}=-16u\left[\frac{1}{(u^2+4v^2-5w^2)^2}-\frac{2v(8v)}{(u^2+4v^2-5w^2)^3}\right]=-16u\frac{u^2+4v^2-5w^2-16v^2}{(u^2+4v^2-5w^2)^3}=\frac{16u(12v^2+5w^2-u^2)}{(u^2+4v^2-5w^2)^3}$$

22. $f(r,s,t)=t^2e^{4rst}$ (a) $f_r=4st^3e^{4rst}$ (b) $f_{rt}=4s(3t^2e^{4rst}+t^3\cdot 4rse^{4rst})=4(3st^2+rs^2t^3)e^{4rst}$

$f_{rts}=4[(3t^2+2rst^3)e^{4rst}+(3st^2+rs^2t^3)\cdot 4rte^{4rst}]=4(3+t^2+20rst^3+16r^2s^2t^4)e^{4rst}$

23. $f(r,s,t)=\dfrac{\ln 4rs}{t^2}=\dfrac{\ln r+\ln 4s}{t^2}$ (a) $\mathrm{D}_1f(r,s,t)=\dfrac{1}{rt^2}$ (b) $\mathrm{D}_{13}f(r,s,t)=\dfrac{-2}{rt^3}$ (c) $\mathrm{D}_{131}f(r,s,t)=\dfrac{2}{r^2t^3}$

24. $f(u,v,w)=w\cos 2v+3v\sin u-2uv\tan w$;
(a) $\mathrm{D}_2f(u,v,w)$; (b) $\mathrm{D}_1f(u,v,w)$; (c) $\mathrm{D}_{131}f(u,v,w)$

▸ (a) $\mathrm{D}_2f(u,v,w)=\mathrm{D}_v(w\cos 2v+3v\sin u-2uv\tan w)=-2w\sin 2v+3\sin u-2u\tan w$
(b) $\mathrm{D}_1f(u,v,w)=\mathrm{D}_u(w\cos 2v+3v\sin u-2uv\tan w)=3v\cos u-2v\tan w$
(c) $\mathrm{D}_{13}f(u,v,w)=\mathrm{D}_w\mathrm{D}_1f(u,v,w)=\mathrm{D}_w(3v\cos u-2v\tan w)=-2v\sec^2 w$ and
$\mathrm{D}_{131}f(u,v,w)=\mathrm{D}_u(-2v\sec^2 w)=0$

25. $w=x^2y-xy^2+y^2z-z^2y+z^2x-x^2z$

$\dfrac{\partial w}{\partial x}=2xy-y^2+z^2-2xz$; $\dfrac{\partial w}{\partial y}=x^2-2yx+2yz-z^2$; $\dfrac{\partial w}{\partial z}=y^2-2zy+2zx-x^2$. Thus $\dfrac{\partial w}{\partial x}+\dfrac{\partial w}{\partial y}+\dfrac{\partial w}{\partial z}=0$.

26. Show that $u=(x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation.

▸ $u_x=-x(x^2+y^2+z^2)^{-3/2}$. $u_{xx}=-(x^2+y^2+z^2)^{-3/2}+3x^2(x^2+y^2+z^2)^{-5/2}$
$=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$. Similarly, $u_{yy}=\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$, $u_{zz}=\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ and so $u_{xx}+u_{yy}+u_{zz}=0$

In Exercises 27 and 28, find $\partial u/\partial t$ and $\partial u/\partial s$ by two methods.

27. $u=y\ln(x^2+y^2)$, $x=2s+3t$, and $y=3t-2s$.

(a) $\dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}=\dfrac{2xy}{x^2+y^2}(3)+\left[\ln(x^2+y^2)+\dfrac{2y^2}{x^2+y^2}\right](3)=\dfrac{6y(x+y)}{x^2+y^2}+3\ln(x^2+y^2)$

$\dfrac{\partial u}{\partial s}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}=\dfrac{2xy}{x^2+y^2}(2)+\left[\ln(x^2+y^2)+\dfrac{2y^2}{x^2+y^2}\right](-2)=\dfrac{4y(x-y)}{x^2+y^2}-2\ln(x^2+y^2)$

(b) $u=(3t-2s)\ln(8s^2+18t^2)$

$\dfrac{\partial u}{\partial t}=(3t-2s)\dfrac{18t}{4s^2+9t^2}+3\ln(8s^2+18t^2)$; $\dfrac{\partial u}{\partial s}=(3t-2s)\dfrac{8s}{4s^2+9t^2}-2\ln(8s^2+18t^2)$

28. $u = e^{2x+y}\cos(2y - x)$, $x = 2s^2 - t^2$, $y = s^2 + 2t^2$

▸ We use the chain rule. Thus,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\\ &= [e^{2x+y}\sin(2y-x) + 2\cos(2y-x)e^{2x+y}](-2t) + [-2e^{2x+y}\sin(2y-x) + e^{2x+y}\cos(2y-x)](4t)\\ &= -10te^{2x+y}\sin(2y-x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\\ &= [e^{2x+y}\sin(2y-x) + 2\cos(2y-x)e^{2x+y}]4s + [-2e^{2x+y}\sin(2y-x) + e^{2x+y}\cos(2y-x)]2s\\ &= 10se^{2x+y}\cos(2y-x)\end{aligned}$$

For the second method, we eliminate x and y before partial-differentiating. Because
$2x + y = 5s^2$ and $2y - x = 5t^2$
we have
$u = e^{5s^2}\cos 5t^2$
Thus,
$u_t = -10te^{5s^2}\sin 5t^2$
and
$u_s = 10se^{5s^2}\cos 5t^2$

29. $u = 3x^2y + 2xy - 3yz - 2z^2$; $x = e^{3rs}$; $y = r^3s^2$; $z = \ln 4$

(a) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = (6xy + 2y)(3se^{3rs}) + (3x^2 + 2x - 3z)(3r^2s^2) + (-3y - 4z)(0)$
$= 18xyse^{3rs} + 6yse^{3rs} + 9x^2r^2s^2 + 6xr^2s^2 - 9zr^2s^2$

(b) $u = 3r^3s^2e^{6rs} + 2r^3s^2e^{3rs} - 3(\ln 4)r^3s^2 - 2\ln^2 4$

$\dfrac{\partial u}{\partial r} = 9r^2s^2e^{6rs} + 18r^3s^3e^{6rs} + 6r^2s^2e^{3rs} + 6r^3s^3e^{3rs} - 9(\ln 4)r^2s^2 = [9(1 + 2rs)e^{6rs} + 6(1 + rs)e^{3rs} - 9\ln 4]r^2s^2$

30. $u = e^{x^2+y^2} - \dfrac{3x}{y} + 3z$, $x = \sin\theta$, $y = \cos\theta$, $z = \tan\theta$

▸ (a) $\dfrac{du}{d\theta} = \dfrac{\partial u}{\partial x}\dfrac{dx}{d\theta} + \dfrac{\partial u}{\partial y}\dfrac{dy}{d\theta} + \dfrac{\partial u}{\partial z}\dfrac{dz}{d\theta} = \left(2xe^{x^2+y^2} - \dfrac{3}{y}\right)\cos\theta - \left(2ye^{x^2+y^2} + \dfrac{3x}{y^2}\right)\sin\theta + 3\sec^2\theta$

(b) $u = e - \tan\theta + 3\tan\theta = e + 2\tan\theta$. $\dfrac{du}{d\theta} = 2\sec^2\theta$

31. $u = xy + x^2$; $x = 4\cos t$; $y = 3\sin t$

(a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = (y + 2x)(-4\sin t) + x(3\cos t) = 3x\cos t - 4(y + 2x)\sin t$
At $t = \frac{1}{4}\pi$, $x = 2\sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$ and $\dfrac{du}{dt} = 3(2\sqrt{2})(\frac{1}{2}\sqrt{2}) - 4(\frac{3}{2}\sqrt{2} + 4\sqrt{2})(\frac{1}{2}\sqrt{2}) = -16.$

(b) $u = 12\sin t\cos t + 16\cos^2 t = 6\sin 2t + 8 + 8\cos 2t$

$\dfrac{du}{dt} = 12\cos 2t - 16\sin 2t$. At $t = \frac{1}{4}\pi$, $\dfrac{du}{dt} = 12\cos\frac{1}{2}\pi - 16\sin\frac{1}{2}\pi = -16$

32. If $f(x,y) = x^2 + ye^x$, find: (a) $\Delta f(0,2)$, the increment of f at $(0,2)$; (b) $\Delta f(0,2)$ when $\Delta x = -0.1$ and $\Delta y = 0.2$; (c) $df(0,2,\Delta x,\Delta y)$, the total differential of f at $(0,2)$; (d) $df(0,2,-0.1,0.2)$.

▸ (a) $\Delta f(0,2) = f(\Delta x, 2 + \Delta y) - f(0,2) = (\Delta x)^2 + (2 + \Delta y)e^{\Delta x} - 2$

(b) If $\Delta x = -0.1$ and $\Delta y = 0.2$, then from the result given part (a) we have
$\Delta f(0,2) = (-0.1)^2 + (2 + 0.2)^2e^{-0.1} - 2 = 0.01 + 2.2e^{-0.1} - 2 \approx 0.000642$

(c) $df(x,y,\Delta x,\Delta y) = D_1f(x,y)\Delta x + D_2f(x,y)\Delta y = (2x + ye^x)\Delta x + e^x\Delta y$
Thus, $df(0,2,\Delta x,\Delta y) = 2\Delta x + \Delta y$

(d) From the result given in part (c), we obtain
$df(0,2,-0.1,0.2) = 2(-0.1) + 0.2 = 0$

33. $f(x,y,z) = 3xy^2 - 5xz^2 - 2xyz$. (a) $\Delta f(-1,3,2) = f(-1 + \Delta x, 3 + \Delta y, 2 + \Delta z) - f(-1,3,2)$
$= 3(-1 + \Delta x)(3 + \Delta y)^2 - 5(-1 + \Delta x)(2 + \Delta z)^2 - 2(-1 + \Delta x)(3 + \Delta y)(2 + \Delta z) - 5$

(b) When $\Delta x = 0.02$, $\Delta y = -0.01$, and $\Delta z = -0.02$, then
$\Delta f(-1,3,2) = 3(-0.98)(2.99)^2 - 5(-0.98)(1.98)^2 - 2(-0.98)(2.99)(1.98) - 5 = -0.470$

(c) At $(-1,3,2)$, $D_1f(x,y,z) = 3y^2 - 5z^2 - 2yz = -5$, $D_2f(x,y,z) = 6xy - 2xz = -14$,
$D_3f(x,y,z) = -10xz - 2xy = 26$. Therefore
$df(-1,3,2,\Delta x,\Delta y,\Delta z) = D_1f(-1,3,2)\Delta x + D_2f(-1,3,2)\Delta y + D_3f(x,y,z)\Delta z = -5\Delta x - 14\Delta y + 26\Delta z.$

(d) $df(-1,3,2,0.02,-0.01,-0.02) = -5(0.02) - 14(0.01) + 26(-0.02) = -0.48$

34. $f(x)=x^2+1$, $g(x,y)=\frac{2x}{3y}$, $h(x)=\frac{1}{x}$ (a) $(h\circ g)(-3,4)=\frac{3y}{2x}=\frac{12}{-6}=-2$ (b) $g(f(3),h(\frac{1}{4}))=g(10,4)=\frac{20}{12}=\frac{5}{3}$
(c) $g(f(x),h(y))=g\left(x^2+1,\frac{1}{y}\right)=\frac{2(x^2+1)}{3(1/y)}=\frac{2}{3}y(x^2+1),\ y\neq 0$ (d) $f(h\circ g)=f\left(\frac{3y}{2x}\right)=\frac{9y^2}{4x^2}+1$

In Exercises 35–37, evaluate the limit by the use of limit theorems.

35. $\lim\limits_{(x,y)\to(e,0)}\ln\left(\frac{x^2}{y+1}\right)=\ln\left(\lim\limits_{(x,y)\to(e,0)}\frac{x^2}{y+1}\right)=\ln e^2=2$

36. $\lim\limits_{(x,y)\to(0,\pi/2)}\frac{xy^2+e^x}{\cos x+\sin y}$

▸ Because the limit of the denominator is not zero, we may apply the limit of a quotient theorem after obtaining the limits of the numerator and denominator.

$$\lim_{(x,y)\to(0,\pi/2)}\frac{xy^2+e^x}{\cos x+\sin y}=\frac{0(\frac{1}{2}\pi)^2+e^0}{\cos 0+\sin\frac{1}{2}\pi}=\frac{1}{2}$$

37. $\lim\limits_{(x,y)\to(1,3)}\sin^{-1}\left(\frac{3x}{2y}\right)=\sin^{-1}\left(\lim\limits_{(x,y)\to(1,3)}\frac{3x}{2y}\right)=\sin^{-1}\frac{1}{2}=\frac{1}{6}\pi$

In Exercises 38–40, establish the limit by finding a $\delta>0$ for any $\epsilon>0$ such that Definition 12.2.5 holds.

38. $\lim\limits_{(x,y)\to(4,-1)}(4x-5y)=21$

▸ $|4x-5y-21|=|4(x-4)-5(y+1)|\leq\sqrt{4^2+5^2}\sqrt{(x-4)^2+(y+1)^2}<\sqrt{41}\delta\leq\epsilon$ whenever $\delta<\epsilon/\sqrt{41}$

39. To prove $\lim\limits_{(x,y)\to(2,-2)}(3x^2-4y^2)=-4$, we show that for any $\epsilon>0$ there is a $\delta>0$ such that

$$\left|(3x^2-4y^2)+4\right|<\epsilon \text{ whenever } 0<\sqrt{(x-2)^2+(y+2)}<\delta \tag{1}$$

$$\left|3x^2-4y^2+4\right|=\left|3(x^2-4)-4(y^2-4)\right|\leq 3|x-2||x+2|+4|y+2||y-2| \tag{2}$$

To get an upper bound for $|x+2|$ and $|y-2|$, we restrict $\delta\leq 1$. Then whenever $|x-2|\leq\sqrt{(x-2)^2+(y+2)^2}<\delta\leq 1$, we have $-1<x-2<1$, and so $3<x+2<5$. Also whenever $|y+2|\leq\sqrt{(x-2)^2+(y+2)^2}<\delta\leq 1$, we have $-1<y+2<1$; $-5<y-2<-3$; $3<|y-2|<5$.
Therefore, whenever $0<\sqrt{(x-2)^2+(y+2)^2}<\delta\leq 1$, we have

$$3|x-2||x+2|+4|y+2||y-2|<3\cdot\delta\cdot 5+4\cdot\delta\cdot 5=35\delta \tag{3}$$

Hence, if $\delta=\min(1,\frac{1}{35}\epsilon)$ then from (2) and (3) we have statement (1).

40. $\lim\limits_{(x,y)\to(3,1)}(x^2-y^2+2x-4y)=10$

▸ For any $\epsilon>0$, we must find a $\delta>0$ such that
$\left|x^2-y^2+2x-4y-10\right|<\epsilon$ whenever $0<\sqrt{(x-3)^2+(y-1)^2}<\delta$.
We use the triangle inequality and then the Cauchy-Schwarz inequality.

If $0<\sqrt{(x-3)^2+(y-1)^2}<\delta$ then
$\left|x^2-y^2+2x-4y-10\right|$
$=\left|(x^2-6x+9)-(y^2-2y+1)+(8x-24)-(6y-6)\right|=\left|(x-3)^2-(y-1)^2+8(x-3)-6(y-1)\right|$
$\leq(x-3)^2+(y-1)^2+\left|8(x-3)-6(y-1)\right|<\delta^2+\sqrt{8^2+6^2}\sqrt{(x-3)^2+(y-1)^2}<\delta^2+10\delta\leq 11\delta$ if $\delta\leq 1$
$=\epsilon$
if $\delta=\min(1,\frac{1}{11}\epsilon)$

In Exercises 41–44, determine if the limit exists.

41. $0<\left|\frac{x^3y^3}{x^2+y^2}\right|=\frac{|x|^3|y|^3}{x^2+y^2}\leq\frac{(x^2+y^2)^{3/2}(x^2+y^2)^{3/2}}{x^2+y^2}=(x^2+y^2)^2$. By the squeeze theorem $\lim\limits_{(x,y)\to(0,0)}\frac{x^3y^3}{x^2+y^2}=0$.

42. See Exercise 48.

43. $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}}\frac{x^9y}{(x^6+y^2)^2}=\lim\limits_{y\to 0}0=0$; $\lim\limits_{\substack{(x,y)\to(0,0)\\y=x^3}}\frac{x^9y}{(x^6+y^2)^2}=\lim\limits_{x\to 0}\frac{x^9x^3}{(x^6+x^6)^2 2}=\lim\limits_{x\to 0}\frac{x^{12}}{4x^{12}}=\lim\limits_{x\to 0}\frac{1}{4}=\frac{1}{4}$
Because these two limits are not equal, then $\lim\limits_{(x,y)\to(0,0)}\frac{x^9y}{(x^6+y^2)^2}$ does not exist.

44. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+4x^2y}{x^2+y^2}$

▹ $0 \le \left|\dfrac{2x^3+4x^2y}{x^2+y^2}\right| = \dfrac{x^2}{x^2+y^2}|2x+4y| \le |2x+4y|$

Because $\displaystyle\lim_{(x,y)\to(0,0)} |2x+4y| = 0$, then by the squeeze theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+4x^2y}{x^2+y^2} = 0$.

In Exercises 45–48, determine all points at which f is continuous.

45. $f(x,y) = \dfrac{x^2+4y^2}{x^2-4y^2}$. Because f is a rational function it is continuous at all points in its domain. Thus, f is continuous at all points (x,y) in R^2 not on the lines $x = \pm 2y$.

46. $f(x,y) = 1/(\cos^2 \frac{1}{2}\pi x + \cos^2 \frac{1}{2}\pi y)$. f is continuous unless both cosines are 0, ie. unless x and y are odd integers.

47. $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \ne (0,0)$, f is a rational function and hence is continuous. Furthermore, $f(0,0) = 0$ and from Exercise 41 $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$, and so f is continuous at $(0,0)$. Therefore, f is continuous at all points (x,y) in R^2.

48. $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^4+y^4} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▹ f is continuous at all points $(x,y) \ne (0,0)$ because then $x^4+y^4 \ne 0$. We wish to determine whether $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4} = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$$

Because the restricted limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, so f is discontinuous at $(0,0)$.

In Exercises 49–53, find the value of the directional derivative at P_0 for the function in the direction of **U**.

49. $f(x,y) = 3x^2 - 2xy + 1$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

▹ $\mathrm{D}_U f(x,y) = f_x(x,y)\left(\frac{3}{5}\right) + f_y(x,y)\left(-\frac{4}{5}\right) = \frac{3}{5}(6x-2y) - \frac{4}{5}(-2x) = \frac{26}{5}x - \frac{6}{5}y$. $\mathrm{D}_U f(5,10) = \frac{26}{5}(5) - \frac{6}{5}(10) = 14$

51. $h(x,y) = e^x + y^2\cos x$; $\mathbf{U} = \frac{1}{2}\sqrt{2}\mathbf{i} - \frac{1}{2}\sqrt{2}\mathbf{j}$

$\mathrm{D}_U h(x,y) = h_x(x,y)(\frac{1}{2}\sqrt{2}) + h_y(x,y)(-\frac{1}{2}\sqrt{2}) = (e^x - y^2\sin x)(\frac{1}{2}\sqrt{2}) + 2y\cos x(-\frac{1}{2}\sqrt{2})$

$\mathrm{D}_U h(0,3) = \frac{1}{2}\sqrt{2}(1 - 9\cdot 0) - \frac{1}{2}\sqrt{2}(6\cdot 1) = \frac{1}{2}\sqrt{2} - 3\sqrt{2} = -\frac{5}{2}\sqrt{2}$

52. $f(x,y) = x^2 - 2x^2y + \ln x$; $\mathbf{U} = \cos\pi\mathbf{i} + \sin\pi\mathbf{j}$; $\mathrm{P}_0 = (1,-2)$

▹ First, we find the value of the gradient of f at the point P_0.

$\nabla f(x,y) = \langle 2x - 4xy + x^{-1}, -2x^2\rangle$

$\nabla f(1,-2) = \langle 2+8+1, -2\rangle = \langle 11,-2\rangle$

We are given that

$\mathbf{U} = \langle\cos\pi, \sin\pi\rangle = \langle -1, 0\rangle$

Thus, the value of the required directional derivative is given by

$\mathrm{D}_U f(1,-2) = \nabla f(1,-2)\cdot\mathbf{U} = \langle 11,-2\rangle\cdot\langle -1,0\rangle = -11$

53. $f(x,y,z) = xy^2z - 3xyz + 2xz^2$; $\mathbf{U} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

$\mathrm{D}_U f(x,y,z) = [f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}]\cdot\mathbf{U}$

$= [(y^2z - 3yz + 2z^2)\mathbf{i} + (2xyz - 3xz)\mathbf{j} + (xy^2 - 3xy + xz)\mathbf{k}]\cdot\mathbf{U}$

$\mathrm{D}_U f(2,1,1) = [(1-3+2)\mathbf{i} + (4-6)\mathbf{j} + (2-6+8)\mathbf{k}]\cdot\mathbf{U} = (-2\mathbf{j} + 4\mathbf{k})(-\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}) = -2\cdot\frac{2}{3} - 4\cdot\frac{1}{3} = -\frac{8}{3}$

In Exercises 54–57, find (a) the gradient of f at P_0; (b) the rate of change of f in the direction of $\mathbf{U}$ at P_0.

54. $f(x,y) = 3x^2 - 2xy^3$; $\mathbf{U} = \cos \frac{1}{6}\pi\mathbf{i} + \sin \frac{1}{6}\pi\mathbf{j} = \frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$ (a) $\nabla f(x,y) = (6x - 2y^3)\mathbf{i} - 6xy^2\mathbf{j}$;
$\nabla f(-3,1) = -20\mathbf{i} + 18\mathbf{j}$ (b) $D_U f(-3,1) = \nabla f(-3,1) \cdot \mathbf{U} = (-20\mathbf{i} + 18\mathbf{j}) \cdot (\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}) = -10\sqrt{3} + 9$

55. $f(x,y) = \frac{1}{2}\ln(x^2 + y^2)$; $\mathbf{U} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$ (a) $\nabla f(x,y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j}$; $\nabla f(1,1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
(b) The rate of change of $f(x,y)$ in the direction of $\mathbf{U}$ is
$D_U f(1,1) = \nabla f(1,1) \cdot \mathbf{U} = (\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}) \cdot (\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = \frac{1}{4} + \frac{1}{4}\sqrt{3}$

56. $f(x,y,z) = yz - y^2 - xz$; $\mathbf{U} = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$; $P_0 = (1,2,3)$
▸ (a) $\nabla f(x,y,z) = -z\mathbf{i} + (z - 2y)\mathbf{j} + (y - x)\mathbf{k}$
$\nabla f(1,2,3) = -3\mathbf{i} - \mathbf{j} + \mathbf{k}$
(b) We have
$D_U f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{U} = (-3\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}) = -\frac{19}{7}$
The function value is decreasing at the rate of $\frac{19}{7}$ units in the direction of $\mathbf{U}$ at P_0.

57. $f(x,y,z) = x^3 + y^3 + 2xyz$; $\mathbf{U} = \dfrac{3}{\sqrt{14}}\mathbf{i} - \dfrac{2}{\sqrt{14}}\mathbf{j} + \dfrac{1}{\sqrt{14}}\mathbf{k}$
(a) $\nabla f(x,y,z) = (3x^2 + 2yz)\mathbf{i} + (3y^2 + 2xz)\mathbf{j} + 2xy\mathbf{k}$; $\nabla f(2,-1,0) = 12\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
(b) The rate of change of $f(x,y,z)$ in the direction of $\mathbf{U}$ at $(2,-1,0)$ is
$D_U f(2,-1,0) = \nabla f(2,-1,0) \cdot \mathbf{U} = (12\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot \left(\dfrac{3}{\sqrt{14}}\mathbf{i} - \dfrac{2}{\sqrt{14}}\mathbf{j} + \dfrac{1}{\sqrt{14}}\mathbf{k}\right) = \dfrac{36}{\sqrt{14}} - \dfrac{6}{\sqrt{14}} - \dfrac{4}{\sqrt{14}} = \frac{13}{7}\sqrt{14}$

In Exercises 58 and 59, determine the relative extrema of f, if there are any.

58. $f(x,y) = 2x^2 - 3xy + 2y^2 + 10x - 11y$. $f_x = 4x - 3y + 10 = 0$; $f_y = -3x + 4y - 11 = 0$. $4f_x + 3f_y = 7x + 7 = 0$,
$x = -1$. $3f_x + 4f_y = 7y - 14 = 0$, $y = 2$. $f_{xx} = 4 > 0$. $f_{yy} = 4$, $f_{xy} = -3$. $D = 4 \cdot 4 - (-3)^2 = 7 > 0$. rel. min.

59. $f(x,y) = x^3 + y^3 + 3xy$. $f_x(x,y) = 3x^2 + 3y$; $f_y(x,y) = 3y^2 + 3x$.
$f_{xx}(x,y) = 6x \quad f_{yy}(x,y) = 6y \quad f_{xy}(x,y) = 3$
$f_x(x,y) = 0$: $x^2 + y = 0$; $y = -x^2$. $f_y(x,y) = 0$: $y^2 + x = 0$; $x^4 + x = 0$; $x(x^3 + 1) = 0$; $x = 0, -1$.
The critical points are $(0,0)$ and $(-1,-1)$. We apply the second-derivative test.
$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0) = 0 \cdot 0 - 3^2 < 0$. Hence $f(0,0)$ is not a relative extremum.
$f_{xx}(-1,-1) = -6 < 0$ and $f_{xx}(-1,-1)f_{yy}(-1,-1) - f_{xy}{}^2(-1,-1) = (-6)(-6) - 3^2 = 27 > 0$
Therefore $f(-1,-1) = 1$ is a relative maximum value.

In Exercises 60 and 61, prove that f is differentiable at all points in its domain by showing Definition 12.4.2. holds.

60. $f(x,y) = 3xy^2 - 4x^2 + y^2$
▸ By Definition 16.5.2, we must find ϵ_1 and ϵ_2 such that

$$\epsilon_1\Delta x + \epsilon_2\Delta y = \Delta f(x_0,y_0) - D_1 f(x_0,y_0)\Delta x - D_2 f(x_0,y_0)\Delta y \tag{1}$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. We have

$$\begin{aligned}\Delta f(x_0,y_0)\\ &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0)\\ &= [3(x_0 + \Delta x)(y_0 + \Delta y)^2 - 4(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2] - [3x_0y_0{}^2 - 4x_0{}^2 + y_0{}^2]\\ &= 6x_0y_0\Delta y + 3x_0(\Delta y)^2 + 3y_0{}^2\Delta x + 6y_0\Delta x\Delta y + 3\Delta x(\Delta y)^2 - 8x_0\Delta x - 4(\Delta x)^2 + 2y_0\Delta y + (\Delta y)^2\end{aligned} \tag{2}$$

Next, we find the following.

$$D_1 f(x_0,y_0)\Delta x = 3y_0{}^2\Delta x - 8x_0\Delta x \tag{3}$$

$$D_2 f(x_0,yq)\Delta y = 6x_0y_0\Delta y + 2y_0\Delta y \tag{4}$$

Substituting from (2), (3), and (4) into (1), we obtain
$\epsilon_1\Delta x + \epsilon_2\Delta y = [6y_0\Delta y - 4\Delta x + 3(\Delta y)^2]\Delta x + (3x_0\Delta y + \Delta y)\Delta y$
Therefore, we take $\epsilon_1 = 6y_0\Delta y - 4\Delta x + 3(\Delta y)^2$ and $\epsilon_2 = 3x_0\Delta y + \Delta y$ and $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.
Thus, f is differentiable at all points in R^2.

61. $f(x,y) = \dfrac{2x + y}{y^2}$. $D_1 f(x,y) = \dfrac{2}{y^2}$ and $D_2 f(x,y) = \dfrac{-4x - y}{y^3}$. Therefore

$$\Delta f(x,y) - D_1 f(x,y)\Delta x - D_2 f(x,y)\Delta y = \frac{2(x + \Delta x) + (y + \Delta y)}{(y + \Delta y)^2} - \frac{2x + y}{y^2} - \frac{2}{y^2}\Delta x + \frac{4x + y}{-y^3}\Delta y$$

$$= \frac{-4y^2\Delta x - 2y(\Delta y)^2}{y^3(y + \Delta y)^2}\Delta x + \frac{y^2\Delta y + 6xy\Delta y + 4x(\Delta y)^2 + y(\Delta y)^2}{y^3(y + \Delta y)^2}\Delta y$$

In Definition 12.4.2 let $\epsilon_1 = \dfrac{-4y^2\Delta x - 2y(\Delta y)^2}{y^3(y + \Delta y)^2}$ and $\epsilon_2 = \dfrac{y^2\Delta y + 6xy\Delta y + 4x(\Delta y)^2 + y(\Delta y)^2}{y^3(y + \Delta y)^2}$.
As $(\Delta x, \Delta y) \to (0,0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ so f is differentiable at all points in its domain.

62. $\sin\alpha = ac^{-1}$. $|d\sin\alpha| = |c^{-1}da - ac^{-2}dc| \le c^{-1}da + ac^{-2}dc = (7.14)^{-1}(0.01) + 3.52(7.14)^{-2}(0.01) = 0.00209$

63. Let x meters be the length of the ceiling, y meters the width of the ceiling, and z meters the height of the room. If C dollars is the cost of the job, then
$C = 2(2xz + 2yz + xy) = 4xz + 4yz + 2xy$; $dC = (4z + 2y)dx + (4z + 2x)dy + (4x + 4y)dz$
Let $x = 5$, $y = 4$, $z = 3$, $dx = dy = dz = \pm 0.005$. Then
$\Delta C \approx dC = 20(\pm 0.005) + 22(\pm 0.005) + 36(\pm 0.005) = \pm 0.39$
Therefore, the greatest error in estimating the cost of the job is 39 cents.

64. At a given instant, the length of one side of a rectangle is 6 cm and it is increasing at the rate of 1 cm/s; the length of another side of the rectangle is 10 cm and it is decreasing at the rate of 2 cm/s. fins the rate of change of the area of the rectangle at the given instant.

▸ At t seconds, let x cm and y cm be the lengths of the sides of the rectangle and A cm^2 be its area. We are given that $dx/dt = 1$ and $dy/dt = -2$ when $x = 6$ and $y = 10$. To find dA/dt at this moment, we calculate the total differential and evaluate it. Thus,

$$A = xy$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = 10 \cdot 1 + 6(-2) = -2$$

Therefore, the area is decreasing at the rate of 2 cm^2/s at the given moment.

65. Let r cm be the radius of a right-circular cylinder and let h cm be its height. If V cm^3 is the volume, then
$V = \pi r^2 h$; $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = 2\pi rh\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt}$
When $r = 20$, $\dfrac{dr}{dt} = -5$, $h = 40$, $\dfrac{dh}{dt} = 12$, we have $\dfrac{dv}{dt} = 2\pi(20)(40)(-5) + \pi(20)^2(12) = -3200\pi$.

Thus, at the given instant the volume is decreasing at the rate of 3200π cm^3/min.

66. If $x = 4$ we have $f(y,z) = 396 - 16y^2 + 9z^2 = 0$. $\partial z/\partial y = -f_y/f_z = 32y/18z$. $y = 9$ and $z = 10$ gives 8/5

67. The ideal gas law: if P atm is the pressure, V liters is the volume, and T degrees is the temperature, then
$P = \dfrac{kT}{V}$; $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{k}{V}\dfrac{dT}{dt} - \dfrac{kT}{V^2}\dfrac{dV}{dt}$
If $k = 1.4$, $V = 20$, $T = 75$, $\dfrac{dV}{dt} = 0.3$ and $\dfrac{dT}{dt} = 0.5$, then $\dfrac{dP}{dt} = \dfrac{1.4}{20}(0.5) - \dfrac{1.4(75)}{20^2}(0.3) = -0.044$.

Thus at the given instant the pressure is decreasing at the rate of 0.044 atm/min.

In Ex. 68–70, find an equation of the tangent plane and equations of the normal line to the surface at the point.

68. $z = x^2 + 2xy$; $(1,3,7)$

▸ Let F be the function defined by
$F(x,y,z) = x^2 + 2xy - z$
Thus, the hyperbolic paraboloid is the graph of $F(x,y,z) = 0$. We have

$$\nabla F(x,y,z) = (2x + 2y)\mathbf{i} + 2x\mathbf{j} - \mathbf{k}$$
$$\nabla F(1,3,7) = 8\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

Thus, an equation of the tangent plane to the surface at $(1,3,7)$ is

$$8(x-1) + 2(y-3) - (z-7) = 0$$
$$8x + 2y - z - 7 = 0$$

and equations of the normal line to the surface at $(1,3,7)$ are

$$\frac{x-1}{8} = \frac{y-3}{2} = \frac{z-7}{-1}$$

69. Let $f(x,y,z) = x^2 + 2y + z - 8$. Then $\nabla f(x,y,z) = 2x\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Hence $\nabla f(2,1,2) = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
An equation of the tangent plane at $(2,1,2)$ is
$4(x-2) + 2(y-1) + (z-2) = 0$; $4x + 2y + z - 12 = 0$

Equations of the normal line at $(2,1,2)$ are $\dfrac{x-2}{4} = \dfrac{y-1}{2} = \dfrac{z-2}{1}$.

70. Let $f(x,y,z) = 3x^2 + 2xy - y^2 - 15$. Then $\nabla f = (6x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$. $\nabla f(2,3,4) = 18\mathbf{i} - 2\mathbf{j}$.
Tangent plane: $9(x-2) - (y-3) = 0$; $9x - y = 15$. Normal line: $(x-2)/9 = -(y-3)$, $z = 4$

71. Let $f(x,y,z)=x^2-3xy+y^2-z$ and $\mathrm{G}(x,y,z)=2x^2+y^2-3z+27$.
$\nabla f(x,y,z)=(2x-3y)\mathbf{i}+(-3x+2y)\mathbf{j}-\mathbf{k}$ and $\nabla\mathrm{G}(x,y,z)=4x\mathbf{i}+2y\mathbf{j}-3\mathbf{k}$.
Let $\mathbf{n}_1=\nabla f(1,-2,11)=8\mathbf{i}-7\mathbf{j}-\mathbf{k}$ and $\mathbf{n}_2=\nabla\mathrm{G}(1,-2,11)=4\mathbf{i}-4\mathbf{j}-3\mathbf{k}$.

$$\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\8&-7&-1\\4&-4&-3\end{vmatrix}=17\mathbf{i}+20\mathbf{j}-4\mathbf{k}$$

Equations of the tangent line to the curve of intersection at $(1,-2,11)$ are $\frac{x-1}{17}=\frac{y+2}{20}=\frac{z-11}{-4}$.

72. Find equations of the tangent line to the curve of intersection of the surface $z=3x^2+y^2+1$ with the plane $x=2$ at the point $(2,-1,14)$.

▸ Because $x=2$, we have
$z=y^2+13$
$\left.\frac{dz}{dy}\right|_{y=-1}=\left.2y\right|_{y=-1}=-2$
Thus the equations of the tangent line at $(2,-1,14)$ are
$x=2$ and $z-14=-2(y+1)$ or, equivalently, $z+2y=12$

73. $z=900-3xy$. Let $f(x,y)=900-3xy$. $\nabla f(x,y)=-3y\mathbf{i}-3x\mathbf{j}$.
(a) The direction of steepest ascent at $(50,4,300)$ is $\nabla f(50,4)=-12\mathbf{i}-150\mathbf{j}$.
(b) The y axis points to the south so the direction of north is $-\mathbf{j}$. If $\mathbf{U}=-\mathbf{j}$, then
$\mathrm{D}_{\mathbf{U}}f(x,y)=(-12\mathbf{i}-150\mathbf{j})\cdot(-\mathbf{j})=150>0$. Therefore the climber is ascending.
(c) Let $\mathbf{U}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$. The climber is traveling a level path when his direction is that for which $\mathrm{D}_{\mathbf{U}}f(x,y)=0$, that is,
$(-12\mathbf{i}-150\mathbf{j})\cdot(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})=0$; $-12\cos\theta-150\sin\theta=0$; $\tan\theta=-\frac{2}{25}$
Then either $\cos\theta=\frac{25}{\sqrt{629}}$ and $\sin\theta=-\frac{2}{\sqrt{629}}$ or $\cos\theta=-\frac{25}{\sqrt{629}}$ and $\sin\theta=\frac{2}{\sqrt{629}}$.
A level path lies in either of the directions $\frac{25}{\sqrt{629}}\mathbf{i}-\frac{2}{\sqrt{629}}\mathbf{j}$ or $-\frac{25}{\sqrt{629}}\mathbf{i}+\frac{2}{\sqrt{629}}\mathbf{j}$.

74. $f(x,y)=x^2+6xy+3y^2$ (a) $f(15,8)=15^2+6(15)8+3(8)^2=1137$ (b) $f_x(x,y)=2x+6y$,
$f_x(15,8)=2(15)+6(8)=78$ (c) $f_y(x,y)=6x+6y$, $f_y(15,8)=6(15)+6(8)138$

In Exercises 75–78, use Lagrange multipliers to find the critical point(s) of the function subject to the constraint and determine their nature.

75. $f(x,y)=5+x^2-y^2$, and $x^2-2y^2=5$. $f(x,y,\lambda)=5+x^2-y^2+\lambda(x^2-2y^2-5)$.
$\mathrm{F}_x(x,y,\lambda)=2x+2\lambda x=0$; $x=0$ or $\lambda=-1$. $\mathrm{F}_y(x,y,\lambda)=-2y-4\lambda y=0$; $y=0$ or $\lambda=-\frac{1}{2}$.
$\mathrm{F}_\lambda(x,y,\lambda)=x^2-2y^2-5=0$. If $x=0$, $-2y^2-5=0$ is impossible. If $y=0$, $x^2-5=0$; $x=\pm\sqrt{5}$.
The critical points are $(\sqrt{5},0)$ and $(-\sqrt{5},0)$. Consider y as a function of x.
From the constraint, $2x-4y\frac{dy}{dx}=0$; $\frac{dy}{dx}=\frac{x}{2y}$. $\frac{df}{dx}=2x-2y\frac{dy}{dx}=2x-2y\cdot\frac{x}{2y}=x$, $\frac{d^2f}{dx^2}=1>0$.
Therefore f has a relative minimum at each critical point and $f(\pm\sqrt{5},0)=10$.

76. $f(x,y,z)=x^2+y^2+z^2$ with constraint $x^2-y^2=1$

▸ Let F be the function defined by
$\mathrm{F}(x,y,z)=x^2+y^2+z^2+\lambda(x^2-y^2-1)$
Then
$\mathrm{F}_x(x,y,z,\lambda)=2x+2\lambda x=2x(1+\lambda)=0$ (1)
$\mathrm{F}_y(x,y,z,\lambda)=2y-2\lambda y=2y(1-\lambda)=0$ (2)
$\mathrm{F}_z(x,y,z,\lambda)=2z=0$ (3)
$\mathrm{F}_\lambda(x,y,z,\lambda)=g(x,y)=x^2-y^2-1=0$ (4)
If $\lambda\neq\pm1$, then from (1), $x=0$; from (2), $y=0$; from (4), $-1=0$, impossible.
If $\lambda=1$, then from (1), $x=0$; from (4) $y^2=-1$, impossible.
If $\lambda=-1$, then from (2), $y=0$; from (4) $x^2=1$, $x=\pm1$.
Thus the critical points are $(1,0,0)$ and $(-1,0,0)$.
Because $g(x,y)=0$ defines x as a function of y, then
$\frac{dx}{dy}=-\frac{\partial g/\partial y}{\partial g/\partial x}=\frac{y}{x}$
Then

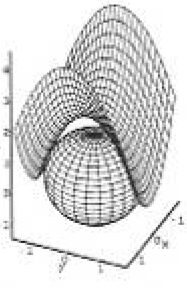
$f_y = 2x \cdot \frac{y}{x} + 2y = 4y$
$f_z = 2z$
$f_{yy} = 4$
$f_{zz} = 2$
$f_{yz} = 0$
$\text{D} = f_{yy}f_{zz} - f_{yz} = 4 \cdot 2 - 0^2 = 8$
and so each critical point is a relative minimum; the function value is 1. In fact, each is an absolute minimum.

77. $f(x, y, z) = y + xz - 2x^2 - y^2 - z^2$, and $z = 35 - x - y$.
$f(x, y, z, \lambda) = y + xz - 2x^2 - y^2 - z^2 + \lambda(35 - x - y - z)$.
$\text{F}_x(x, y, z, \lambda) = z - 4x - \lambda = 0$; $z = 4x + \lambda$. $\text{F}_y(x, y, z, \lambda) = 1 - 2y - \lambda = 0$; $y = \frac{1}{2}(1 - \lambda)$.
$\text{F}_z(x, y, z, \lambda) = x - 2z - \lambda = 0$; $x - 8x - 2\lambda - \lambda = 0$; $x = -\frac{3}{7}\lambda$, $z = -\frac{5}{7}\lambda$.
$\text{F}_\lambda(x, y, z, \lambda) = 35 - x - y - z = 0$; $-\frac{3}{7}\lambda + \frac{1}{2}(1 - \lambda) - \frac{5}{7}\lambda = 35$; $-6\lambda + 7 - 7\lambda - 0\lambda = 490$; $-23\lambda = 463$.
$\lambda = -21$. $x = -\frac{3}{7}\lambda = -\frac{3}{7}(-21) = 9$. $y = \frac{1}{2}(1 - \lambda) = \frac{1}{2}(1 + 21) = 11$, $z = -\frac{5}{7}\lambda = -\frac{5}{7}(-21) = 15$.
The critical point is $\text{P}(9, 11, 15)$ and $f(9, 11, 15) = -362$.
Consider z as a function of x and y. From the constraint, $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -1$.
$f_x = z + x\frac{\partial z}{\partial x} - 4x - 2z\frac{\partial z}{\partial x} = z - x - 4x + 2z = 3z - 5x$; $f_y = 1 + x\frac{\partial z}{\partial y} - 2y - 2z\frac{\partial z}{\partial y} = 1 - x - 2y + 2z$.
$f_{xx} = 3\frac{\partial z}{\partial x} - 5 = -3 - 5 = -8 < 0$; $f_{yy} = -2 + 2\frac{\partial z}{\partial y} = -2 - 2 = -4$; $f_{xy} = 3\frac{\partial z}{\partial y} = -3$.
$f_{xx}f_{yy} - {f_{xy}}^2 = (-8)(-4) - (-3)^2 = 23$. Hence f has a relative maximum at P.

78. See Exercise 12.9.9

79. Let w units be the distance from the point $(4, 1, 2)$ to a point (x, y, z) in the plane $x - y + 2z = 0$. w will be a minimum when w^2 is a minimum. Let
$f(x, y, z) = w^2 = (x - 4)^2 + (y - 1)^2 + (z - 2)^2$
$f(x, y, z, \lambda) = (x - 4)^2 + (y - 1)^2 + (z - 2)^2 + \lambda(x - y + 2z)$
$\text{F}_x = 2x - 8 + \lambda = 0$; $x = \frac{1}{2}(8 - \lambda)$. $\text{F}_y = 2y - 2 - \lambda = 0$; $y = \frac{1}{2}(2 + \lambda)$. $\text{F}_z = 2z - 4 + 2\lambda = 0$; $z = 2 - \lambda$.
$\text{F}_\lambda = x - y + 2z = 0$; $\frac{1}{2}(8 - \lambda) - \frac{1}{2}(2 + \lambda) + 2(2 - \lambda) = 0$. $\lambda = \frac{7}{3}$, $x = \frac{17}{3}$, $y = \frac{13}{3}$, $z = -\frac{1}{3}$. The critical point is
$\text{P}(\frac{17}{6}, \frac{13}{6}, -\frac{1}{3})$ and at P, $w = \sqrt{(\frac{17}{6} - 4)^2 + (\frac{13}{6} - 1)^2 + (-\frac{1}{3} - 2)^2} = \sqrt{\frac{49}{36} + \frac{49}{36} + \frac{49}{9}} = \frac{7}{6}\sqrt{6}$.
Because $f(x, y, z) \geq 10$ on or outside the sphere $(x - 4)^2 + (y - 1)^2 + (z - 2)^2 = 10$, f has a minimum at a critical point inside the closed set bounded by the intersection of this sphere and the given plane, that is at P.

80. Use Lagrange multipliers to find the point on the surface $z = x^2 - y^2 + 2$ that is closest to the origin.
▹ Because $\sqrt{x^2 + y^2 + z^2}$ is the distance from the origin to any point (x, y, z), we wish to find the absolute minimum value of the function f defined by
$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
subject to the constraint
$z = x^2 - y^2 + 2$ (1)
Because f has a minimum value whenever f^2 has a minimum value, we let F be the function defined by
$\text{F}(x, y, z) = x^2 + y^2 + z^2 + \lambda(x^2 - y^2 - z + 2)$
The critical points of f are among those of the auxiliary function F. Then
$\text{F}_x(x, y, z, \lambda) = 2x + 2\lambda x = 2x(1 + \lambda) = 0$ (2)
$\text{F}_y(x, y, z, \lambda) = 2y - 2\lambda y = 2y(1 - \lambda) = 0$ (3)
$\text{F}_z(x, y, z, \lambda) = 2z - \lambda = 0$
so that
$z = \frac{1}{2}\lambda$ (4)



If $\lambda \neq \pm 1$, then from (2), $x = 0$; from (3), $y = 0$; and from (1), $z = 2$. Then $f(0, 0, 2) = 2$.
If $\lambda = 1$, then from (2), $x = 0$; from (4), $z = \frac{1}{2}$; and from (1), $y = \pm\frac{1}{2}\sqrt{6}$. Then $f(0, \pm\frac{1}{2}\sqrt{6}, \frac{1}{2}) = \frac{1}{2}\sqrt{7} \approx 1.32$.
If $\lambda = -1$, then from (3), $y = 0$; from (4), $z = -\frac{1}{2}$; and from (1), $x^2 = -\frac{5}{2}$, which is impossible.
Thus, the points on the hyperbolic paraboloid closest to the origin are $(0, \frac{1}{2}\sqrt{6}, \frac{1}{2})$ and $(0, -\frac{1}{2}\sqrt{6}, \frac{1}{2})$. The figure shows the surface and a sphere centered at the origin of radius $\frac{1}{2}\sqrt{7}$ tangent to the surface at the extrema.

81. Let the three numbers be x, y, and $100 - x - y$. Let $f(x,y) = x^2 + y^2 + (100 - x - y)^2$.
$f_x(x,y) = 4x + 2y - 200 = 0$; $f_y(x,y) = 4y + 2x - 200 = 0$. Then $x = \frac{100}{3}$, $y = \frac{100}{3}$, $100 - x - y = \frac{100}{3}$.
$f(\frac{100}{3}, \frac{100}{3}) = \frac{10{,}000}{3}$. If $|x| \geq 100$ or $|y| \geq 100$, then $f(x,y) \geq 10{,}000$, so f has an absolute minimum at a critical point inside the square $|x| \leq 100$, $|y| \leq 100$, i.e. at $(\frac{100}{3}, \frac{100}{3})$. Thus the three numbers are $\frac{100}{3}$, $\frac{100}{3}$, and $\frac{100}{3}$.

82. $P(x,y) = 33x + 66y - x^2 + xy - 3y^2$. $P_x = 33 - 2x + y = 0$. $P_y = 66 + x - 6y = 0$. $6P_x + P_y = 264 - 11x = 0$, $x = 24$. $P_x + 2P_y = 165 - 11y = 0$, $y = 15$. $P_{xx} = -2$, $P_{yy} = -6$, $P_{xy} = 1$, $D = (-2)(-6) - 1^2 = 11$. maximum

83. Let 2r, 2s and 2t be the measures of the edges of the rectangular parallelpiped P parallel to the x, y, and z axes. If V cubic units is the volume of P, then $V = 8rst$. Let $f = V^2/64 = r^2s^2t^2$, $r \geq 0$, $s \geq 0$, $t \geq 0$. Because the vertex (r,s,t) is on the ellipsoid $x^2 + 9y^2 + z^2 = 9$, then $r^2 + 9s^2 + t^2 = 9$; $r^2 = 9 - 9s^2 - t^2$. Therefore
$f = (9 - 9s^2 - t^2)s^2t^2 = 9s^2t^2 - 9s^4t^2 - s^2t^4$
If $s \geq 1$, $s = 0$, $t \geq 3$, or $t = 0$, then $f \leq 0$. Hence f has an absolute maximum value at a critical point inside the rectangle $0 \leq s \leq 1$, $0 \leq t \leq 3$.
$f_s = 18st^2 - 36s^3t^2 - 2st^4 = 2st^2(9 - 18s^2 - 2t^2) = 0$; $18s^2 + 2t^2 = 9$
$f_t = 18s^2t - 18s^4t - 4s^2t^3 = 2s^2t(9 - 9s^2 - 4t^2) = 0$; $9s^2 + 4t^2 = 9$
Solving simultaneously for s^2 and t^2 we get $s^2 = \frac{1}{3}$, $t^2 = 3$, so the only critical point in R is $(\frac{1}{3}\sqrt{3}, \sqrt{3})$ which must give the maximum. Then $r^2 = 3$; $r = \sqrt{3}$ so the dimensions of the rectangular parallepiped of greatest volume are $2\sqrt{3}$ by $\frac{2}{3}\sqrt{3}$ by $2\sqrt{3}$.

84. The temperature is T degrees at any point (x,y) of the curve $4x^2 + 12y^2 = 1$ and $T(x,y) = 4x^2 + 24y^2 - 2x$. Find the points on the curve where the temperature is the greatest and where it is least. Also find the temperature at these points.

▸ Let F be the function defined by
$F(x,y,\lambda) = 4x^2 + 24y^2 - 2x + \lambda(4x^2 + 12y^2 - 1)$
Then

$$F_x(x,y,\lambda) = 8x - 2 + 8x\lambda = 8x(1 + \lambda) - 2 = 0 \qquad (1)$$
$$F_y(x,y,\lambda) = 48y + 24y\lambda = 24y(2 + \lambda) = 0 \qquad (2)$$
$$F_\lambda(x,y,\lambda) = 4x^2 + 12y^2 - 1 = 0 \qquad (3)$$

From (2) we have either $y = 0$ or $\lambda = -2$. If $y = 0$, from (3) we get $x = \pm\frac{1}{2}$. If $\lambda = -2$, from (1) we get $x = -\frac{1}{4}$, and from (3) we get $y = \pm\frac{1}{4}$. Thus, the critical points are $(\pm\frac{1}{2}, 0)$ and $(-\frac{1}{4}, \pm\frac{1}{4})$. Then

$T(\frac{1}{2}, 0) = 0$ $T(-\frac{1}{2}, 0) = 2$ $T(-\frac{1}{4}, \frac{1}{4}) = \frac{9}{4}$ $T(-\frac{1}{4}, -\frac{1}{4}) = \frac{9}{4}$
Because the curve is an ellipse which a closed and bounded set, the extrema are among the critical points. Therefore the maximum temperature is $\frac{9}{4}$°, which occurs at the points $(-\frac{1}{4}, \pm\frac{1}{4})$, and the minimum temperature is 0°, which occurs at the points $(\pm\frac{1}{2}, 0)$.

85. The temperature is T degrees at any point (x,y) and $T(x,y) = \dfrac{44}{x^2 + y^2 + 9}$, where distance is in centimeters from the origin. (a) We wish to find $D_U T(3,2)$ where $U = \cos\frac{1}{6}\pi\mathbf{i} + \sin\frac{1}{6}\pi\mathbf{j}$.
$T_x(x,y) = \dfrac{-88x}{(x^2 + y^2 + 9)^2}$; $T_x(3,2) = \dfrac{-88 \cdot 3}{22^2} = -\frac{6}{11}$. $T_y(x,y) = \dfrac{-88y}{(x^2 + y^2 + 9)^2}$; $T_y(3,2) = \dfrac{-88 \cdot 2}{22^2} = -\dfrac{4}{11}$
$\nabla T(3,2) = -\frac{6}{11}\mathbf{i} - \frac{4}{11}\mathbf{j}$. Hence $D_U T(3,2) = \mathbf{U} \cdot \nabla T(3,2) = (\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}) \cdot (-\frac{6}{11}\mathbf{i} - \frac{4}{11}\mathbf{j}) = -\frac{3}{11}\sqrt{3} - \frac{2}{11}$.
The rate of change of the temperature at the point (3,2) in the direction of **U** is $-\frac{3}{11}\sqrt{3} - \frac{2}{11}$ degrees per cm.
(b) $D_U T(3,2)$ is maximum when **U** has the direction of $\nabla T(3,2)$. $|\nabla T(3,2)| = \frac{1}{11}\sqrt{36 + 16} = \frac{1}{11}\sqrt{52} = \frac{2}{11}\sqrt{13}$
The greatest rate of change of T at (3,2) is $\frac{2}{11}\sqrt{13}$ degrees per cm in the direction of $-\frac{3}{13}\sqrt{13}\mathbf{i} - \frac{2}{13}\sqrt{13}\mathbf{j}$.

86. The base is x ft by y ft, the height is z ft. Then $xy + 2xz + 2yz = 216$ (1). WE wish maximize the volume $V = xyz$. Let $F = xyz + \lambda(xy + 2xz + 2yz - 216)$. $F_x = yz + \lambda(y + 2z) = 0$, $1 + \lambda(1/z + 2/y) = 0$ (2).
$F_y = xz + \lambda(x + 2z) = 0$, $1 + \lambda(1/z + 2/x) = 0$ (3). $F_z = xy + \lambda(2x + 2y) = 0$, $1 + \lambda(2/y + 2/x) = 0$ (4).
From (2) and (3), $y = x$. From (2) and (4), $z = \frac{1}{2}x$. From (1), $x^2 + x^2 + x^2 = 216$, $x = 6\sqrt{2} = y$, $z = 3\sqrt{2}$

87. Let x ft, y ft, z ft be the length, width, and height of the crate. Then
$4x + 4y + 4z = 216$; $x + y + z = 54$; $z = 54 - x - y$
If V ft^3 is the volume of the box, then $V = xyz = xy(54 - x - y) = 54xy - x^2y - xy^2$.
If $x \geq 54$, $x = 0$, $y \geq 54$, or $y = 0$, then $V \leq 0$. Hence V has an absolute maximum value at a critical point inside the square $0 \leq x \leq 54$, $0 \leq y \leq 54$.
$V_x = 54y - 2xy - y^2 = y(54 - 2x - y) = 0$; $2x + y = 54$
$V_y = 54x - x^2 - 2xy = x(54 - x - 2y) = 0$; $x + 2y = 54$
The only critical point is (18,18), which gives the desired maximum. The dimensions of the crate of greatest volume are 18 ft by 18 ft by 18 ft.

88. A piece of wire L feet long is cut into three pieces. One piece is bent into the shape of a circle; a second piece is bent into the shape of a square; and the third piece is bent into the shape of an equilateral triangle. How should the wire be cut so that the combined area of the three figures is (a) as small as possible, and (b) as large as possible.

▹ Let r ft be the radius of the circle, x ft the side of the square, y ft the side of the triangle, and A ft^2 the combined area of the three figures. Then
$A = \pi r^2 + x^2 + \frac{1}{4}\sqrt{3}y^2$ (1)
Because the circumference of the circle is $2\pi r$ ft, the perimeter of the square is $4x$ ft, and the perimeter of the triangle is $3y$ ft, we also have
$L = 2\pi r + 4x + 4y$, $r \geq 0$, $x \geq 0$, $y \geq 0$ (2)
We want to find the extrema of A subject to the constraint (2). We use the method of Lagrange multipliers to find the critical points. Let F be the function defined by
$F(r, x, y, \lambda) = \pi r^2 + x^2 + \frac{1}{4}\sqrt{3} + \lambda(2\pi r + 4x + 3y - L)$
Then
$F_r(r, x, y, \lambda) = 2\pi r + 2\pi\lambda = 0$, $r = -\lambda$
$F_x(r, x, y, \lambda) = 2x + 4\lambda = 0$, $x = -2\lambda$
$F_y(r, x, y, \lambda) = \frac{1}{2}\sqrt{3}y + 3\lambda = 0$, $y = -2\sqrt{3}\lambda$
Substituting the values for r, x, y in (2), we obtain
$L = -2\pi\lambda - 8\lambda - 6\sqrt{3}\lambda$
Solving for λ gives
$$\lambda = \frac{-L}{2\pi + 8 + 6\sqrt{3}} \approx -0.04L$$
Thus,
$r = -\lambda \approx 0.04L$
$x = -2\lambda \approx 0.08L$
$y = -2\sqrt{3}\lambda \approx 0.14L$
Substituting these values into (1), we have
$$\begin{aligned} A &= \pi(-\lambda)^2 + (-2\lambda)^2 + \tfrac{1}{4}\sqrt{3}(-2\sqrt{3}\lambda)^2 = (-\lambda)^2(\pi + 4 + 3\sqrt{3}) = \left(\frac{L}{2\pi + 8 + 6\sqrt{3}}\right)^2(\pi + 4 + 3\sqrt{3}) \\ &= \frac{L^2}{4(\pi + 4 + 3\sqrt{3})} \approx 0.02L^2 \end{aligned} \tag{3}$$
Because the level surfaces of A are ellipsoids, we need only compare the value of A at the critical point with the value of A at the corners of the triangle which is the graph of (2).
If $x = 0$ and $y = 0$, then from (2) we have $r = L/2\pi$ and from (1) we obtain
$A = \pi\left(\frac{L}{2\pi}\right)^2 \approx 0.08L^2$ (4)
If $r = 0$ and $y = 0$, then $x = \frac{1}{4}L$ and
$A = (\frac{1}{4}L)^2 \approx 0.06L^2$ (5)
If $r = 0$ and $x = 0$, then $y = \frac{1}{3}L$ and
$A = \frac{1}{3}\sqrt{3}(\frac{1}{3}L)^2 \approx 0.05L^2$ (6)
Comparing the values of A given in Eqs. (3), (4), (5) and (6), we conclude that the minimum combined area is $0.02L^2$ ft^2 and that it occurs when the radius is 0.04L ft, the side of the square is 0.08L ft long, a side of the triangle is 0.14L ft long. The maximum combined area is $0.08L^2$ ft^2, and it occurs when all the wire is used to make the circle.

89. See Exercises 12.8.49 and 12.9.35 for two different solutions.

90. Find the greatest and least distances from the origin to the curve of intersection of the ellipsoid $x^2 + 3y^2 + 2z^2 = 30$ and the elliptical cone $x^2 = 2yz$.

▸ We seek the extrema of $d^2 = x^2 + y^2 + z^2 = y^2 + 2yz + z^2 = (y + z)^2$ subject to $3y^2 + 2yz + 2z^2 = 30$, $yz \geq 0$. Let $F = y + z + \lambda(3y^2 + 2yz + 2z^2)$. $F_y = 1 + \lambda(6y + 2z) = 0$. $F_z = 1 + \lambda(2y + 4z) = 0$. Thus $6y + 2z = 2y + 4z$, $z = 2y$, and so $3y^2 + 4y^2 + 8y^2 = 30$, $y^2 = 2$, $y = \pm\sqrt{2}$, $z = \pm 2\sqrt{2}$, $x = \pm 2\sqrt{2}$, $d = \sqrt{8 + 2 + 8} = 3\sqrt{2}$, the absolute maximum. If $y = 0$, then $x = 0$ and $d = z = \sqrt{15}$. If $z = 0$ then $x = 0$ and $d = y = \sqrt{10}$, the min.

91.

x (years)	54	46	40	36	30
y (days)	15	12	9	10	8

$\sum_{i=1}^{5} x_i y_i = 2322$, $\sum_{i=1}^{5} x_i = 206$, $\sum_{i=1}^{5} y_i = 54$, $\sum_{i=1}^{5} x_i^2 = 8828$.

$m = \dfrac{(5 \times 2322 - 206 \times 54)}{(5 \cdot 8828 - 206^2)} = 0.285$, $b = \dfrac{(54 - 0.285 \times 206)}{5} = -0.951$. The regression line has the equation $y = 0.285x - 0.951$. When $x = 42$, $y = 0.285 \times 42 - .951 = 11.0$. The stay is 11 days.

92. In the following table, a patients systolic blood pressure and corresponding heart rate are given, where x millimeters of mercury is the systolic blood pressure and y beats per minute is the heart rate.

	Patient 1	Patient 2	Patient 3	Patient 4	Patient 5	Patient 6
x mm hg	110	117	133	146	115	127
y beats/min	70	74	80	85	60	77

(a) Find an equation of the regression line for the data in the table. (b) Use the regression line to estimate a patients heart rate if the systolic blood pressure is 85 mm of mercury.

▸ (a) The sums required to find the regression line of y on x are

$$\sum_{i=1}^{6} x_i = 748 \quad \sum_{i=1}^{6} y_i = 446 \quad \sum_{i=1}^{6} x_i^2 = 94{,}148 \quad \sum_{i=1}^{6} x_i y_i = 56{,}087$$

Then, with $n = 6$,

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{(6 \times 56087 - 748 \times 446)}{(6 \times 94148 - 748^2)} = 0.5412$$

$$b = \frac{1}{n}\left[\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right] = \frac{1}{6}(446 - 0.5412 \times 748) = 6.863$$

Therefore, the regression line is

$y = 0.5412x + 6.863$

(b) If $x = 85$, then

$y = 0.5412 \times 85 + 6.863 = 52.87$

The estimate of the heart rate is 53 beats per minute when the blood pressure is 85 mm of mercury.

93.

x (mm/yr/100)	1	2	4	5	6	6.5
y (kg/ha/100)	10	19	32	44	58	64

$\sum_{i=1}^{6} x_i y_i = 1160$, $\sum_{i=1}^{6} x_i = 24.5$, $\sum_{i=1}^{6} y_i = 227$, $\sum_{i=1}^{6} x_i^2 = 124.25$. $m = \dfrac{(6 \times 1160 - 24.5 \times 227)}{(6 \cdot 124.25 - 24.5^2)} = 9.628$, $b = \dfrac{(227 - 9.628 \times 24.5)}{6} = -1.482$.
The regression line has the equation $y = 9.628x - 1.482$. When $x = 3$, $100y = 100 \times (9.628 \times 3 - 1.482) = 2740$.

94.

x (cents)	130	140	150	160
y (M cases)	100	85	75	63

$n = 4$. $\sum_{i=1}^{n} x_i y_i = 46230$, $\sum_{i=1}^{n} x_i = 580$, $\sum_{i=1}^{n} y_i = 323$, $\sum_{i=1}^{n} x_i^2 = 84600$

$m = \dfrac{(4 \times 46230 - 580 \times 323)}{(4 \times 84600 - 580^2)} = -1.21$. $m = \dfrac{(323 + 1.21 \times 580)}{4} = 256.2$. Regression line: $y = -1.21x + 256.2$

$y(120) = 256.2 - 1.21 \times 120 = 111$ thousand cases. $y(170) = 256.2 - 1.21 \times 170 = 50.5$ thousand cases.

95. $f(x,y)=\begin{cases}\dfrac{12x^2y-3y^2}{x^2+y} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

(a) $f_2(x,0)=\lim\limits_{\Delta y\to 0}\dfrac{f(x,\Delta y)-f(x,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{\dfrac{12x^2\Delta y-3(\Delta y)^2}{x^2+\Delta y}-0}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{12x^2-3\Delta y}{x^2+\Delta y}=\dfrac{12x^2}{x^2}=12$

(b) $f_2(0,0)=\lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{\dfrac{0-3(\Delta y)^2}{0+\Delta y}-0}{\Delta y}=\lim\limits_{\Delta y\to 0}(-3)=-3$

96. Verify that $u(x,y)=\sinh x\sin y$ satisfies Laplace's equation in R^2: $u_{xx}+u_{yy}=0$

▸ $u_x=\cosh y\sin y$
$u_{xx}=\sinh x\sin y$ (1)
Furthermore
$u_y=\sinh x\cos y$
$u_{yy}=-\sinh x\sin y$ (2)
Adding Eqs. (1) and (2), we obtain
$u_{xx}+u_{yy}=\sinh x\sin y+(-\sinh x\sin y)=0$

97. $z=xy+f(x^2+y^2)$ and $u=x^2+y^2$. $\dfrac{\partial z}{\partial x}=y+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}=y+\dfrac{\partial f}{\partial u}(2x)$; $\dfrac{\partial z}{\partial y}=x+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}=x+\dfrac{\partial f}{\partial u}(2y)$.

$y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=y\left[y+\dfrac{\partial f}{\partial u}(2x)\right]-x\left[x+\dfrac{\partial f}{\partial u}(2y)\right]=y^2+2xy\dfrac{\partial f}{\partial u}-x^2-2xy\dfrac{\partial f}{\partial u}=y^2-x^2$

98. $u(r,\theta)=r^n\sin n\theta$. $u_r=nr^{n-1}\sin n\theta$. $u_\theta=nr^n\cos n\theta$. $u_{rr}=n(n-1)r^{n-2}\sin n\theta$. $u_{\theta\theta}=-n^2r^n\sin n\theta$
$r^2u_{rr}+ru_r+u_{\theta\theta}=n(n-1)r^n\sin n\theta+nr^n\sin n\theta-n^2r^n\sin n\theta=0$, Laplace's equation in polar coordinates.

99. $u(x,y,z)=e^{3x+4y}\sin 5z$. $\dfrac{\partial u}{\partial x}=3e^{3x+4y}\sin 5z$; $\dfrac{\partial u}{\partial y}=4e^{3x+4y}\sin 5z$; $\dfrac{\partial u}{\partial z}=5e^{3x+4y}\cos 5z$

$\dfrac{\partial^2 u}{\partial x^2}=9e^{3x+4y}\sin 5z$; $\dfrac{\partial^2 u}{\partial y^2}=16e^{3x+4y}\sin 5z$; $\dfrac{\partial^2 u}{\partial z^2}=-25e^{3x+4y}\sin 5z$

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=e^{3x+4y}\sin 5z(9+16-25)=0$

100. Verify that $u(x,t)=\mathrm{A}\cos(kat)\sin(kx)$, where A and k are arbitrary constants, satisfies the partial differential equation for a vibrating string: $u_{tt}=a^2u_{xx}$.

▸ Partial-differentiating u with respect to t, we obtain
$u_t=-ka\mathrm{A}\sin(kat)\sin(kx)$
$u_{tt}=-k^2a^2\mathrm{A}\cos(kat)\sin(kx)$ (1)
Partial-differentiating u with respect to x, we obtain
$u_x=k\mathrm{A}\cos(kat)\cos(kx)$
$u_{xx}=-k^2\mathrm{A}\cos(kat)\sin(kx)$
Thus,
$a^2u_{xx}=-k^2a^2\mathrm{A}\cos(kat)\sin(kx)$ (2)
Comparing Eqs. (1) and (2), we obtain the desired result
$u_{tt}=a^2u_{xx}$

101. $u(x,t)=\sin\dfrac{n\pi x}{\mathrm{L}}\exp\left(-\dfrac{n^2\pi^2k^2}{\mathrm{L}^2}t\right)$. $\dfrac{\partial u}{\partial x}=\dfrac{n\pi}{\mathrm{L}}\cos\dfrac{n\pi x}{\mathrm{L}}\exp\left(-\dfrac{n^2\pi^2k^2}{\mathrm{L}^2}t\right)$;

$\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{n^2\pi^2}{\mathrm{L}^2}\sin\dfrac{n\pi x}{\mathrm{L}}\exp\left(-\dfrac{n^2\pi^2k^2}{\mathrm{L}^2}t\right)$. Then $\dfrac{\partial u}{\partial t}=-\dfrac{n^2\pi^2k^2}{\mathrm{L}^2}\sin\dfrac{n\pi x}{\mathrm{L}}\exp\left(-\dfrac{n^2\pi^2k^2}{\mathrm{L}^2}t\right)=k^2\dfrac{\partial^2 u}{\partial x^2}$.

102. Interchange x and y in Exercises 12.4.30.

103. $f(x,y,z) = \begin{cases} \dfrac{x^2y^2z^2}{(x^2+y^2+z^2)^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$. $\mathrm{D}_1 f(0,0,0) = \lim\limits_{\Delta x \to 0} \dfrac{0-0}{\Delta x} = 0 = \mathrm{D}_2 f(0,0,0) = \mathrm{D}_3 f(0,0,0)$.

Therefore $\Delta f(0,0,0) - \mathrm{D}_1 f(0,0,0) - \mathrm{D}_2 f(0,0,0) - \mathrm{D}_3 f(0,0,0)$

$$= \frac{(\Delta x)^2(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2} = \frac{(\Delta x)(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}\,\Delta x + 0\Delta y + 0\Delta z$$

In Definition 12.4.7 let $\epsilon_1 = \dfrac{(\Delta x)(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}$, $\epsilon_2 = 0$, $\epsilon_3 = 0$. Then

$$0 \le |\epsilon_1| < \frac{\Delta x[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2][(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2} = |\Delta x|$$

Hence as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$, $\epsilon_1 \to 0$. Also $\epsilon_2 \to 0$ and $\epsilon_3 \to 0$. Thus f is differentiable

In Exercises 104 and 105, f is the function defined by $f(x,y) = \begin{cases} \dfrac{e^{-1/x^2}y}{e^{-2/x^2}+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

104. Prove that f is discontinuous at the origin.

▸ We wish to determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. Because $\lim\limits_{x\to 0} e^{-1/x^2} = 0$, then $(0,0)$ is an accumulation point of the curve $y = me^{-1/x^2}$. Furthermore,

$$\lim_{\substack{(x,y)\to(0,0) \\ y=me^{-x^2}}} f(x,y) = \lim_{x\to 0} \frac{e^{-1/x^2}(me^{-1/x^2})}{e^{-1/x^2}+m^2e^{-1/x^2}} = \frac{m}{1+m^2}$$

Thus the restricted limit depends on m, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and f is discontinuous at the origin.

105. $\mathrm{D}_1 f(0,0) = \lim\limits_{\Delta x\to 0} \dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{0-0}{\Delta x} = 0$; $\mathrm{D}_2 f(0,0) = \lim\limits_{\Delta y\to 0} \dfrac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim\limits_{\Delta y\to 0} \dfrac{0-0}{\Delta y} = 0$

106. $u = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

▸ $u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta$. $u_\theta = u_x x_\theta + u_y y_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$

$(u_r)^2 + r^{-2}(u_\theta)^2 = (u_x{}^2\cos^2\theta + 2u_xu_y\sin\theta\cos\theta + u_y{}^2\sin^2\theta) + (u_x{}^2\sin^2\theta - 2u_xu_y\sin\theta\cos\theta + u_y{}^2\cos^2\theta)$

$= (\cos^2\theta + \sin^2\theta)u_x{}^2 + (\sin^2\theta + \cos^2\theta)u_y{}^2 = u_x{}^2 + u_y{}^2$

107. We are given $\dfrac{d^2f}{dx^2} = -\lambda^2 f(x)$, $\dfrac{dg}{dt} = -k^2\lambda^2 g(t)$ and $u = f(x)g(t)$. Then $\dfrac{\partial u}{\partial x} = g(t)\dfrac{df}{dx}$, and $\dfrac{\partial^2 u}{\partial x^2} = g(t)\dfrac{d^2f}{dx^2} = g(t)[-\lambda^2 f(x)]$. Thus $\dfrac{\partial u}{\partial t} = f(x)\dfrac{dg}{dt} = f(x)[-k^2\lambda^2 g(t)] = k^2 g(t)[-\lambda^2 f(t)] = k^2\dfrac{\partial^2 u}{\partial x^2}$.

108. The partial differential equation for a vibrating string is $u_{tt} = a^2 u_{xx}$. Show that if f is a function of x satisfying the equation $f_{xx} + \lambda^2 f = 0$ and g is a function of t satisfying the equation $g_{tt} + a^2\lambda^2 g = 0$, where a and λ are constants, then $u = f(x)g(t)$ satisfies the differential equation.

▸ We have

$u_t = f(x)g_t(t)$

$u_{tt} = f(x)g_{tt}(t)$

and

$u_x = f_x(x)g(t)$

$u_{xx} = f_{xx}(x)g(t)$

Furthermore,

$f_{xx} = -\lambda^2 f$ and $g_{tt} = -a^2\lambda^2 g$

Thus,

$$u_{tt} - a^2u_{xx} = f(x)g_{tt}(t) - a^2 f_{xx}g(t) = f(x)[-a^2\lambda^2 g(t)] - a^2[-\lambda^2 f(x)]g(t) = 0$$

Therefore, $u_{tt} = a^2u_{xx}$ and the differential equation is satisfied.

109. $u = f(x + at) + g(x - at)$

$$\frac{\partial u}{\partial x} = f'(x + at)\frac{\partial}{\partial x}(x + at) + g'(x - at)\frac{\partial}{\partial x}(x - at) = f'(x + at) + g'(x - at)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x + at)\frac{\partial}{\partial x}(x + at) + g''(x - at)\frac{\partial}{\partial x}(x - at) = f''(x + at) + g''(x - at)$$

$$\frac{\partial u}{\partial t} = f'(x + at)\frac{\partial}{\partial t}(x + at) + g'(x - at)\frac{\partial}{\partial t}(x - at) = af'(x + at) - ag'(x - at)$$

$$\frac{\partial^2 u}{\partial t^2} = af''(x + at)\frac{\partial}{\partial t}(x + at) - ag''(x - at)\frac{\partial}{\partial t}(x - at) = a^2 f''(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 u}{dx^2}$$

110. $\mathrm{V} = r^{-1}\phi(r,t)$ (a) $r^2\mathrm{V}_r = r^2[-r^{-2}\phi(r,t) + r^{-1}\phi_r(r,t)] = r\phi_r(r,t) - \phi(r,t)$. Let $k^2 = \mu c$.

$$0 = r^{-2}(r^2\mathrm{V}_r)_r - k^2\mathrm{V}_{tt} = r^{-2}[\phi_r(r,t) + r\phi_{rr} - \phi_r(r,t)] - k^2 r^{-1}\phi_{tt}(r,t) = r^{-1}\phi_{rr} - k^2\phi_{tt}(r,t) \rightarrow \phi_{rr} - k^2\phi_{tt} = 0$$

(b) $\phi = f(t - kr)$. $\phi_t = f'(t - kr)$; $\phi_{tt} = f''(t - kr)$. $\phi_r = f'(t - kr)(-k)$; $\phi_{rr} = f''(t - kr)k^2 = k^2\phi_{tt}$

111. $h = \mathrm{H}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y)$. $h_x = -\mathrm{H}m\pi a^{-1}\sin(m\pi a^{-1}x)\cos(n\pi b^{-1}y)$. $h_y = -\mathrm{H}n\pi b^{-1}\cos(m\pi a^{-1}x)\sin(n\pi b^{-1}y)$

$$h_{xx} + h_{yy} + \mathrm{K}^2 h = -\mathrm{H}m^2\pi^2 a^{-2}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) - \mathrm{H}n^2\pi^2 b^{-2}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) + (m^2\pi^2 a^{-2} + n^2\pi^2 b^{-2})\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) = 0$$

THIRTEEN

MULTIPLE INTEGRATION

13.1 CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical We take the polar axis as the positive x axis with the polar plane as the xy plane. If cylindrical coordinates (r,θ,z) of a point P are given, we use the equations

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z \tag{I}$$

to uniquely determine the cartesian coordinates (x,y,z) of P. If the Cartesian coordinates of P are given, then cylindrical coordinates satisfy equations (II) where the rule for determining θ is given by (VI)

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z \tag{II}$$

Spherical If spherical coordinates (ρ,θ,ϕ) of a point P are given, we use the equations

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

to uniquely determine the cartesian coordinates of P. Furthermore, a set of cylindrical coordinates of P is given by

$$r = \rho\sin\phi \quad \theta = \theta \quad z = \rho\cos\phi \tag{IV}$$

If the Cartesian coordinates of P are given and P is not the origin, then the spherical coordinates satisfy

$$\rho = \sqrt{x^2+y^2+z^2} \quad \tan\theta = \frac{y}{x} \quad \phi = \cos^{-1}\frac{z}{\rho} \tag{V}$$

In spherical and cylindrical coordinates we have

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \frac{1}{2}\pi\,\operatorname{sgn}(y) & \text{if } x = 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases} \tag{VI}$$

Arc Length In cylindrical coordinates, $L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$. See Exercise 33.

In spherical coordinates, $L = \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\,dt$. See Exercise 34.

Exercises 13.1

The cartesian, cylindrical and spherical coordinates of a point P will be denoted by $P_{\text{cartesian}}$, $P_{\text{cylindrical}}$, and $P_{\text{spherical}}$.

1. (a) $P_{\text{cylindrical}} = (3,\frac{1}{2}\pi,5)$. Then $x = 3\cos\frac{1}{2}\pi = 0$, $y = 3\sin\frac{1}{2}\pi = 3$, $z = 5$. $P_{\text{cartesian}} = (0,3,5)$.

   (b) $P_{\text{cylindrical}} = (7,\frac{2}{3}\pi,-4)$. $x = 7\cos\frac{2}{3}\pi = -\frac{7}{2}$, $y = 7\sin\frac{2}{3}\pi = \frac{7}{2}\sqrt{3}$, $z = -4$. $P_{\text{cartesian}} = (-\frac{7}{2},\frac{7}{2}\sqrt{3},-4)$.

   (c) $P_{\text{cylindrical}} = (1,1,1)$. $x = 1\cos 1 = \cos 1$, $y = 1\sin 1 = \sin 1$, $z = 1$. $P_{\text{cartesian}} = (\cos 1,\sin 1,1)$.

2. (a) $P_{\text{cartesian}} = (4,4,-2)$. $r = \sqrt{4^2+4^2} = 4\sqrt{2}$, $\theta = \tan^{-1}\frac{4}{4} = \frac{1}{4}\pi$, $z = -2$. $P_{\text{cylindrical}} = (4\sqrt{2},\frac{1}{4}\pi,-2)$.

   (b) $P_{\text{cartesian}} = (-3\sqrt{3},3,6)$. $r = \sqrt{27+9} = 6$, $\theta = \dfrac{3}{-3\sqrt{3}} + \pi = -\frac{1}{6}\pi + \pi = \frac{5}{6}\pi$, $z = 6$. $P_{\text{cylindrical}} = (6,\frac{5}{6}\pi,6)$

   (c) $P_{\text{cartesian}} = (1,1,1)$. $r = \sqrt{1+1} = \sqrt{2}$, $\theta = \tan^{-1}\frac{1}{1} = \frac{1}{4}\pi$, $z = 1$. $P_{\text{cylindrical}} = (\sqrt{2},\frac{1}{4}\pi,1)$

3. (a) $P_{\text{spherical}} = (4,\frac{1}{6}\pi,\frac{1}{4}\pi)$. Then $x = 4\sin\frac{1}{4}\pi\cos\frac{1}{6}\pi = 4(\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{3}) = \sqrt{6}$,
   $y = 4\sin\frac{1}{4}\pi\sin\frac{1}{6}\pi = 4(\sqrt{2})(\frac{1}{2}) = \sqrt{2}$, $z = 4\cos\frac{1}{4}\pi = 4(\frac{1}{2}\sqrt{2}) = 2\sqrt{2}$. $P_{\text{cartesian}} = (\sqrt{6},\sqrt{2},2\sqrt{2})$.

   (b) $P_{\text{spherical}} = (4,\frac{1}{2}\pi,\frac{1}{3}\pi)$. Then $x = 4\sin\frac{1}{3}\pi\cos\frac{1}{2}\pi = 4(\frac{1}{2}\sqrt{3})(0) = 0$,
   $y = 4\sin\frac{1}{3}\pi\sin\frac{1}{2}\pi = 4(\frac{1}{2}\sqrt{3})(1) = 2\sqrt{3}$, $z = 4\cos\frac{1}{3}\pi = 4(\frac{1}{2}) = 2$. $P_{\text{cartesian}} = (0,2\sqrt{3},2)$.

   (c) $P_{\text{spherical}} = (\sqrt{6},\frac{1}{3}\pi,\frac{3}{4}\pi)$. Then $x = \sqrt{6}\sin\frac{3}{4}\pi\cos\frac{1}{3}\pi = \sqrt{6}(\frac{1}{2}\sqrt{2})(\frac{1}{2}) = \frac{1}{2}\sqrt{3}$,
   $y = \sqrt{6}\sin\frac{3}{4}\pi\sin\frac{1}{3}\pi = \sqrt{6}(\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{3}) = \frac{3}{2}$, $z = \sqrt{6}\cos\frac{3}{4}\pi = \sqrt{6}(-\frac{1}{2}\sqrt{2}) = -\sqrt{3}$. $P_{\text{cartesian}} = (\frac{1}{2}\sqrt{3},\frac{3}{2},-\sqrt{3})$

4. Find a set of spherical coordinates of the point having the given Cartesian coordinates: (a) $(1, -1, -\sqrt{2})$; (b) $(-1, \sqrt{3}, 2)$; (c) $(2, 2, 2)$

▹ (a) We use formulas (V) and (VI) where $x > 0$. Thus,

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-1)^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$

$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{1}{4}\pi$

$\phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3}{4}\pi$

Spherical coordinates are $(2, -\frac{1}{4}\pi, \frac{3}{4}\pi)$.

(b) We use formulas (V) and (VI) where $x < 0$. Thus,

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + (\sqrt{3})^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1} \frac{y}{x} + \pi = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) + \pi = -\frac{1}{3}\pi + \pi = \frac{2}{3}\pi$

$\phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1} \frac{2}{2\sqrt{2}} = \frac{1}{4}\pi$

Spherical coordinates are $(2\sqrt{2}, \frac{2}{3}\pi, \frac{1}{4}\pi)$.

(c) We use formulas (V) and (VI) where $x > 0$. Thus,

$\rho = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

$\theta = \tan^{-1} \frac{2}{2} = \frac{1}{4}\pi$

$\phi = \cos^{-1} \frac{2}{2\sqrt{3}} = \cos^{-1} \frac{1}{3}\sqrt{3}$

Spherical coordinates are $(2\sqrt{3}, \frac{1}{4}\pi, \cos^{-1} \frac{1}{3}\sqrt{3})$.

5. From (IV), $r = \rho \sin \phi$. (a) $P_{\text{spherical}} = (4, \frac{2}{3}\pi, \frac{5}{6}\pi)$.

$r = 4 \sin \frac{5}{6}\pi = 4(\frac{1}{2}) = 2$, $\theta = \frac{2}{3}\pi$, $z = 4 \cos \frac{5}{6}\pi = 4(-\frac{1}{2}\sqrt{3}) = -2\sqrt{3}$. $P_{\text{cylindrical}} = (2, \frac{2}{3}\pi, -2\sqrt{3})$.

(b) $P_{\text{spherical}} = (\sqrt{2}, \frac{3}{4}\pi, \pi)$. Then $r = \sqrt{2} \sin \pi = \sqrt{2}(0) = 0$, $\theta = \frac{3}{4}\pi$, $z = \sqrt{2} \cos \pi = \sqrt{2}(-1) = -\sqrt{2}$.

$P_{\text{cylindrical}} = (0, \frac{3}{4}\pi, -\sqrt{2})$.

(c) $P_{\text{spherical}} = (2\sqrt{3}, \frac{1}{3}\pi, \frac{1}{4}\pi)$. Then

$r = 2\sqrt{3} \sin \frac{1}{4}\pi = 2\sqrt{3}(\frac{1}{2}\sqrt{2}) = \sqrt{6}$, $\theta = \frac{1}{3}\pi$, $z = 2\sqrt{3} \cos \frac{1}{4}\pi = 2\sqrt{3}(\frac{1}{2}\sqrt{2}) = \sqrt{6}$. $P_{\text{cylindrical}} = (\sqrt{6}, \frac{1}{3}\pi, \sqrt{6})$.

6. (a) $P_{\text{cylindrical}} = (3, \frac{1}{6}\pi, 3)$. $\rho = \sqrt{3^2 + 3^2} = 3\sqrt{2}$, $\theta = \frac{1}{6}\pi$, $\phi = \cos^{-1} \frac{3}{3\sqrt{2}} = \frac{1}{4}\pi$. $P_{\text{spherical}} = (3\sqrt{2}, \frac{1}{6}\pi, \frac{1}{4}\pi)$

(b) $P_{\text{cylindrical}} = (3, \frac{1}{2}\pi, 2)$. $\rho = \sqrt{3^2 + 2^2} = \sqrt{13}$, $\theta = \frac{1}{2}\pi$, $\phi = \cos^{-1} \frac{2}{\sqrt{13}}$. $P_{\text{spherical}} = (\sqrt{13}, \frac{1}{2}\pi, \cos^{-1} \frac{2}{\sqrt{13}})$

(c) $P_{\text{cylindrical}} = (2, \frac{5}{6}\pi, -4)$. $\rho = \sqrt{2^2 + 4^2} = 2\sqrt{5}$, $\theta = \frac{5}{6}\pi$, $\phi = \cos^{-1} \frac{-4}{2\sqrt{5}}$. $P_{\text{spherical}} = (2\sqrt{5}, \frac{5}{6}\pi, \cos^{-1} \frac{2}{5}\sqrt{5})$

In Exercises 7–12, find an equation in cylindrical coordinates of the surface and identify the surface. The cartesian, cylindrical and spherical equation of a surface S will be denoted by $S_{\text{cartesian}}$, $S_{\text{cylindrical}}$, and $S_{\text{spherical}}$.

7. $S_{\text{cartesian}}$ is $x^2 + y^2 + 4z^2 = 16$. S is an ellipsoid. Because $x^2 + y^2 = r^2$, $S_{\text{cylindrical}}$ is $r^2 + 4z^2 = 16$

8. $x^2 - y^2 = 9$

▹ The surface is a hyperbolic cylinder with rulings parallel to the z axis, The vertices of the hyperbola–given by the xy trace of the cylinder–are at the points $(\pm 3, 0)$, and the lines $y = \pm x$ are the asymptotes of the hyperbola. To find an equation in cylindrical coordinates for the cylinder, we replace x by $r \cos \theta$ and y by $r \sin \theta$ in the given equation. Thus,

$$\begin{aligned}
(r \cos \theta)^2 - (r \sin \theta)^2 &= 9 \\
r^2(\cos^2\theta - \sin^2\theta) &= 9 \\
r^2 \cos 2\theta &= 9 \\
r^2 &= 9 \sec 2\theta \\
r &= \pm 3\sqrt{\sec 2\theta}
\end{aligned}$$

9. $S_{\text{cartesian}}$ is $x^2 + y^2 = 3z$. S is a paraboloid of revolution. Because $x^2 + y^2 = r^2$, $S_{\text{cylindrical}}$ is $r^2 = 3z$

10. $S_{\text{cartesian}}$ is $9x^2+4y^2=36$. S is an elliptic cylinder. Because $x = r\cos\theta$ and $y = r\sin\theta$,
$S_{\text{cylindrical}}$ is $36 = 9(r\cos\theta)^2+4(r\sin\theta)^2 = r^2(9\cos^2\theta+4\sin^2\theta) = r^2(5\cos^2\theta+4)$

11. $S_{\text{cartesian}}$ is $x^2-y^2=3z^2$. S is an elliptic cone, Because $x = r\cos\theta$ and $y = r\sin\theta$,
$S_{\text{cylindrical}}$ is $r^2\cos^2\theta - r^2\sin^2\theta = 3z^2$; $r^2(\cos^2\theta-\sin^2\theta) = 3z^2$; $r^2\cos 2\theta = 3z^2$.

12. $x^2+y^2=z^2$

▹ Because the given equation is equivalent to $x^2+y^2-z^2=0$, the equation is of type III in Section 15.7. Thus, the graph is a right circular cone with axis the z axis. Because $x^2+y^2=r^2$, a cylindrical-coordinate equation is $r^2=z^2$. Because the point (r,θ,z) is the same as the point $(-r,\theta\pm\pi,z)$, the graph of the equation $r=z$ is the same as the graph of $r=-z$. Thus, the graph of $r^2=z^2$ is the same as the graph of $r=z$, which is a cylindrical-coordinate equation of the cone $x^2+y^2=z^2$.

In Exercises 13–17, find an equation in spherical coordinates of the surface, and identify the surface.

13. $S_{\text{cartesian}}$ is $x^2+y^2+z^2-9z=0$. S is a sphere. Because $x^2+y^2+z^2=\rho^2$ and $z=\rho\cos\phi$,
$S_{\text{spherical}}$ is $\rho^2-9\rho\cos\phi=0$; $\rho=9\cos\phi$. The origin is not lost by discarding the factor ρ.

14. $S_{\text{cartesian}}$ is $x^2+y^2=z^2$ is a circular cone; $x^2+y^2+z^2=2z^2$. $S_{\text{spherical}}$ is $\rho^2=2\rho^2\cos^2\phi$; $\cos\phi=\pm\frac{1}{2}\sqrt{2}$
$\phi=\frac{1}{4}\pi$ and $\phi=\frac{3}{4}\pi$, two nappes of the same cone.

15. $S_{\text{cartesian}}$ is $x^2+y^2=9$. S is a right circular cylinder. Because $x^2+y^2=r^2=(\rho\sin\phi)^2$,
$S_{\text{spherical}}$ is $(\rho\sin\phi)^2=9$; $\rho\sin\phi=3$.

16. $x^2+y^2=2z$

▹ The graph of the equation is a paraboloid of revolution whose axis is the z axis. The given equation is equivalent to
$r^2=2z$
Substituting from (IV), we obtain
$\rho^2\sin^2\phi=2\rho\cos\phi$
Thus, either $\rho=0$ or
$$\rho\sin^2\phi=2\cos\phi \tag{1}$$
Because Eq. (1) contains the origin $(0,0,\frac{1}{2}\pi)$, we do not need $\rho=0$. Thus, (1) is spherical-coordinate equation of the paraboloid of revolution.

17. $S_{\text{cartesian}}$ is $x^2+y^2+z^2-8x=0$. S is a sphere. Because $x^2+y^2+z^2=\rho^2$ and $x=\rho\sin\phi\cos\theta$,
$S_{\text{spherical}}$ is $\rho^2-8\rho\sin\phi\cos\theta$; $\rho=8\sin\phi\cos\theta$. The origin is not lost by discarding the factor ρ.

In Exercises 18–22, find an equation in Cartesian coordinates for the surface whose equation is given in cylindrical coordinates. In Exercises 18–20, identify the surface.

18. $S_{\text{cylindrical}}$ is $r=3\cos\theta$. S is a circular cylinder containing the origin. $r^2=3r\cos\theta$, $x^2+y^2=3x$

19. (a) $S_{\text{cylindrical}}$ is $r=4$. S is a right circular cylinder. Because $x^2+y^2=r^2$, $S_{\text{cartesian}}$ is $x^2+y^2=16$
(b) $S_{\text{cylindrical}}$ is $\theta=\frac{1}{4}\pi$. S is a plane through the z axis. Because $\tan\theta=y/x$, and $\tan\theta=\tan\frac{1}{4}\pi=1$, $S_{\text{cartesian}}$ is $x=y$.

20. $r=3+2\cos\theta$

▹ The surface is a cylinder with rulings parallel to the$z$ axis, and whose generatrix is a limaçon with a dent. Because $\cos\theta\geq-1$, then $r\geq 3-1=2$. Hence, multiplying by r adds the origin as an isolated point. Thus,
$$r^2-2r\cos\theta=3r,\quad r\neq 0$$
$$(r^2-2r\cos\theta)^2=9r^2,\quad r\neq 0$$
Substituting $r^2=x^2+y^2$, $r\cos\theta=x$, we obtain
$$(x^2+y^2-2x)^2=9(x^2+y^2),\ z^2+y^2\neq 0$$
which is an equation in Cartesian coordinates for the cylinder.

21. $S_{\text{cylindrical}}$ is $r^2\cos 2\theta=z^3$; $r^2(\cos^2\theta-\sin^2\theta)=z^3$. Because $r\cos\theta=x$, $r\sin\theta=y$, $S_{\text{cartesian}}$ is $x^2-y^2=z^3$.

In Exercises 23–28, find an equation in Cartesian coordinates for the surface whose equation is given in spherical coordinates. In Exercises 23–25, identify the surface.

22. $S_{\text{cylindrical}}$ is $z^2\sin^3\theta=r^3$; $z^2(r^3\sin^3\theta)=(r^2)^3$. $S_{\text{cartesian}}$ is $z^2y^3=(x^2+y^2)^3$

23. (a) $S_{\text{spherical}}$ is $\rho = 9$. S is a sphere. Because $\rho^2 = x^2 + y^2 + z^2$, $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 = 81$.
(b) $S_{\text{spherical}}$ is $\theta = \frac{1}{4}\pi$. S is a plane through the z axis. Because $\tan\theta = y/x$ and $\tan\theta = \frac{1}{4}\pi = 1$, $S_{\text{cartesian}}$ is $x = y$.
(c) $S_{\text{spherical}}$ is $\phi = \frac{1}{4}\pi$. S is a cone with vertex at the origin. $r = z\tan\phi = z\tan\frac{1}{4}\pi = 1$. Because $r = \sqrt{x^2 + y^2}$, $S_{\text{cartesian}}$ is $z = \sqrt{x^2 + y^2}$.

24. $\rho = 9\sec\phi$

▹ The given equation is equivalent to
$\rho\cos\theta = 9$
Because $\rho\cos\phi = z$, a Cartesian equation of the surface is
$z = 9$
The graph is a plane perpendicular to the z axis at the point where $z = 9$.

25. $S_{\text{spherical}}$ is $\rho = 6\csc\phi$; $\rho\sin\phi = 6$. Because $\rho\sin\theta = r$ and $r^2 = x^2 + y^2$, $S_{\text{cartesian}}$ is $x^2 + y^2 = 36$. S is a right circular cylinder whose axis is the z axis.

26. $S_{\text{spherical}}$ is $\rho = 3\cos\phi$; $\rho^2 = 3\rho\cos\phi$. $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 = 3z$; $x^2 + y^2 + (z - \frac{3}{2})^2 = \frac{9}{4}$, a sphere.

27. $S_{\text{spherical}}$ is $\rho = 2\tan\theta$. Because $\rho = \sqrt{x^2 + y^2 + z^2}$ and $\tan\theta = y/x$, $S_{\text{cartesian}}$ is
$\sqrt{x^2 + y^2 + z^2} = \frac{2y}{x}$; $x\sqrt{x^2 + y^2 + z^2} = 2y$

28. $\rho = 6\sin\phi\sin\theta + 3\cos\phi$

▹ Because $(0, 0, \frac{1}{2}\pi)$ satisfies the given equation, the graph contains the origin. Thus, we may multiply on both sides by ρ. We have
$\rho^2 = 6\rho\sin\phi\sin\theta + 3\rho\cos\phi$
Substituting from (III) and (V) into the above, we get

$$x^2 + y^2 + z^2 = 6y + 3z$$
$$x^2 + (y^2 - 6y + 9) + (z^2 - 3z + \tfrac{9}{4}) = \tfrac{45}{4}$$
$$x^2 + (y - 3)^2 + (z - \tfrac{3}{2})^2 = \tfrac{45}{4}$$

which is the cartesian equation of a sphere.

In Exercises 29–32, match the equation, which is in either cylindrical or spherical coordinates, with one of the surfaces shown in (i)–(x).

29. (a) $r = 4$. A circular cylinder of radius 4. Fig. (iii). (b) $\rho = 4$. A sphere of radius 4. Fig (vi). (c) $r = 2\sin\theta$; $r^2 = 2r\sin\theta$; $x^2 + y^2 = 2y$; $x^2 + (y - 1)^2 = 1$, a circular cylinder of radius 1, centered at $(0, 1)$. Fig. (viii).

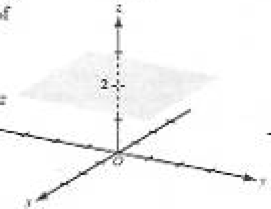
30. (a) $\theta = \frac{1}{3}\pi$. A plane perpendicular to the xy plane; Fig. (ii) (b) $\phi = \frac{1}{3}\pi$, a half-cone; Fig (iv). (c) $r = 2\cos\theta$; $r^2 = 2r\cos\theta$; $x^2 + y^2 = 2x$; $(x - 1)^2 + y^2 = 1$, a circular cylinder of radius 1, centered at $(1, 0)$. Fig. (x).

31. (a) $\rho\sin\phi = 2$; $r = 2$. Circular cylinder of radius 2; Fig. (ix) . (b) $r^2 = 4z$; $x^2 + y^2 = 4z$; paraboloid of revolution; Fig. (v).

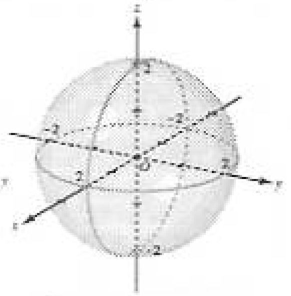
32. (a) $\rho\cos\phi = 2$; (b) $z^2 + r^2 = 4$

▹ (a) We have $z = 2$, a plane perpendicular to the z axis at the point $z = 2$ as shown in Figure (vii).
(b) We have $z^2 + x^2 + y^2 = 4$, a sphere of radius 2 as shown in Figure (i).

(vii)



(i)



33. Prove the formula for arc length in cylindrical coordinates.

▹ Because $x = r\cos\theta$, $D_t x = \cos\theta\, D_t r - r\sin\theta\, D_t\theta$,
$(D_t x)^2 = \cos^2\theta(D_t r)^2 - 2r\sin\theta\cos\theta(D_t r)(D_t\theta) = r^2\sin^2\theta(D_t\theta)^2$
Because $y = r\sin\theta$, $D_t y = \sin\theta\, D_t r + r\cos\theta\, D_t\theta$,
$(D_t y)^2 = \sin^2\theta(D_t r)^2 + 2r\sin\theta\cos\theta(D_t r)(D_t\theta) + r^2\cos^2\theta(D_t\theta)^2$
Thus $(D_t x)^2 + (D_t y)^2 = (D_t r)^2(\cos^2\theta + \sin^2\theta) + r^2(D_t\theta)^2(\sin^2\theta + \cos^2\theta) = (D_t r)^2 + r^2(D_t\theta)^2$.
If L units is the length of arc of C from $t = a$ to $t = b$, then from the formula of Theorem 11.2.1,
$\mathrm{L} = \int_a^b \sqrt{(D_t x)^2 + (D_t y)^2 + (D_t z)^2}\,dt = \int_a^b \sqrt{(D_t r)^2 + r^2(D_t\theta)^2 + (D_t z)^2}$

34. Prove the formula for arc length in spherical coordinates.

▸ $x'^2 + y'^2 + z'^2 = (\rho \sin \phi \cos \theta)'^2 + (\rho \sin \phi \sin \theta)'^2 + (\rho \cos \phi)'^2$

$= (\rho' \sin \phi \cos \theta + \rho \cos \phi \cos \theta\, \phi' - \rho \sin \phi \sin \theta\, \theta')^2 + (\rho' \sin \phi \sin \theta + \rho \cos \phi \sin \theta\, \phi' + \rho \sin \phi \cos \theta\, \theta')^2$

$+ (\rho' \cos \phi - \rho \sin \phi\, \phi')^2$

$= \rho'^2(\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi) + \rho^2\phi'^2(\cos^2\phi \cos^2\theta + \cos^2\phi \sin^2\theta + \sin^2\phi)$

$+ \rho^2\theta'^2(\sin^2\phi \sin^2\theta + \sin^2\phi \cos^2\theta) + 2\rho\rho'\phi'(\sin \phi \cos \phi \cos^2\theta + \sin \phi \cos \phi \sin^2\theta - \sin \phi \cos \phi)$

$+ 2\rho\rho'\theta'(-\sin^2\phi \sin \theta \cos \theta + \sin^2\phi \sin \theta \cos \theta) + \rho^2\phi'\theta'(-\sin \phi \cos \phi \sin \theta \cos \theta + \sin \phi \cos \phi \sin \theta \cos \theta)$

$= \rho'^2 + \rho^2\phi'^2 + \rho^2\theta'^2 \sin^2\phi + 0 + 0 + 0$

35. (a) A set of parametric equations of the circular helix H is $x = a \cos t$, $y = a \sin t$, $z = t$.
If the cylindrical-coordinate representation of a point is (r, θ, z) then $r^2 = x^2 + y^2 = a^2\cos^2 t + a^2 \sin^2 t = a^2$ and $\tan \theta = y/x = \tan t$. Hence a set of parametric equations of H is $r = a$, $\theta = t$, $z = t$.

(b) Using the parametric equations from part (a), $D_t r = 0$, $D_t \theta = 1$, and $D_t z = 1$. Hence if L units is the length of arc of H from $t = 0$ to $t = 2\pi$, we have from Exercise 23

$$L = \int_0^{2\pi} \sqrt{0 + r^2(1)}\, dt = \int_0^{2\pi} \sqrt{a^2+1}\, dt = \sqrt{a^2+1}\, t \Big]_0^{2\pi} = 2\pi\sqrt{a^2+1}$$

36. Use the formula for arc length in spherical coordinates to find the length of arc from $t = 0$ to $t = 2\pi$ of the helix having parametric equation $\rho = t$, $\theta = t$, $\phi = \frac{1}{4}\pi$.

▸ We have

$$\frac{d\rho}{dt} = 1 \quad \frac{d\theta}{dt} = 1 \quad \frac{d\phi}{dt} = 0 \quad \sin \phi = \tfrac{1}{2}\sqrt{2}$$

Therefore,

$$L = \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2 \sin^2\phi \left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{1 + \tfrac{1}{2}t^2}\, dt$$

Let $u = \tan^{-1} \frac{1}{2}\sqrt{2}t$; $c = \tan^{-1}\sqrt{2}\pi$. Then $\sqrt{1 + \frac{1}{2}t^2} = \sec u$; $dt = \sqrt{2} \sec^2 u\, du$ and so

$$L = \sqrt{2}\int_0^c \sec^3 u\, du = \tfrac{1}{2}\sqrt{2}\Big[\sec u \tan u + \ln|\sec u + \tan u|\Big]_0^c$$

We have $\tan c = \sqrt{2}\pi$ and $\sec c = \sqrt{1 + \tan^2 c} = \sqrt{1 + 2\pi^2}$. Thus,

$$L = \tfrac{1}{2}\sqrt{2}[\sqrt{1+2\pi^2}\sqrt{2}\pi + \ln(\sqrt{1+2\pi^2} + \sqrt{2}\pi)]$$

13.2 DOUBLE INTEGRALS

13.2.1 Definition We partition a rectangle with lines parallel to its sides. The *norm* of a partition Δ, denoted by $\|\Delta\|$, is the length of the longest diagonal of a rectangular subregion of Δ.

Let f be a function defined on a closed rectangular region R. The number L is said to be the *limit* of sums of the form $\sum_{i=1}^n f(u_i, v_i)\Delta_i A$ if L satisfies the property that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition Δ for which $\|\Delta\| < \delta$ and for all possible selections of the point (u_i, v_i) in the ith rectangle, $i = 1, 2, \ldots, n$,

$$\left|\sum_{i=1}^n f(u_i, v_i)\Delta_i A - L\right| < \epsilon$$

If such a number exists, we write

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^n f(u_i, v_i)\Delta_i A = L$$

13.2.2 Definition A function f of two variables is said to be *integrable* on a rectangular region R if f is defined on R and the number L of Definition 13.2.1 exists. This number L is called the *double integral* of f on R, and we write

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^n f(u_i, v_i)\Delta_i A = \iint_R f(x, y)\, dA$$

13.2.3 Theorem If a function f of two variables is continuous on a closed rectangular region R, then f is integrable on R.

A *closed region* is one whose boundary consists of a finite number of arcs of smooth curves that are joined together to form a simple closed curve.

13.2.3′ Theorem If a function f of two variables is bounded and continuous on a closed region R except for a set of points which, for any $\epsilon > 0$, is contained in a union of rectangles the sum of the measures of whose area is less than ϵ, then f is integrable on R.

13.2.4 Theorem Let f be a function of two variables that is continuous on a closed region R in the xy plane and $f(x,y) \geq 0$ for all (x,y) in R. If V(S) is the measure of the solid S having the region R as its base and having an altitude of measure $f(x,y)$ at the point (x,y) in R, then

$$V(S) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\Delta_i A = \iint_R f(x,y)dA$$

In particular, if $f(x,y) \equiv 1$, then the measure A of the area of R is given by

$$A = \iint_R dA$$

13.2.5 Theorem If c is a constant and the function f is integrable on a closed region R, then cf is integrable on R and

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$$

13.2.6 Theorem If the functions f and g are integrable on a closed region R, then the function $f+g$ is integrable on R and

$$\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$$

13.2.7 Theorem If the functions f and g are integrable on the closed region R and $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

13.2.8 Theorem Let the function f be integrable on a closed region R, and suppose that m and M are two numbers such that $m \leq f(x,y) \leq M$ for all (x,y) in R. If A is the measure of the area of region R, then

$$mA \leq \iint_R f(x,y)dA \leq MA$$

13.2.9 Theorem Suppose that the function f is continuous on the closed region R and that region R is composed of the two subregions R_1 and R_2 which have no points in common except for points on parts of their boundaries. Then

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

Evaluation Let f be a continuous function of two variables x and y. Let R be a region in the xy plane that is bounded on the left by the line $x = x_1$, bounded on the right by the line $x = x_2$, bounded below by the smooth curve $y = g_1(x)$ and bounded above by the smooth curve $y = g_2(x)$. Then the double integral of f on R is equivalent to an iterated integral, and

$$\iint_R f(x,y)dA = \int_{x_1}^{x_2}\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

Let R be a region in the xy plane that is bounded below by the line $y = y_1$, bounded above by the line $y = y_2$, bounded on the left by the smooth curve $x = g_1(y)$ and bounded on the right by the smooth curve $x = g_2(y)$. Then

$$\iint_R f(x,y)dA = \int_{y_1}^{y_2}\int_{g_1(y)}^{g_2(y)} f(x,y)dx\,dy$$

If R can be described both ways, we can reverse the order of integration. See Ex. 55 and 56.

If the limits of integration are constants, so that R is a rectangle, and f can be expressed as the product $f(x,y) = g(x)h(y)$, then the iterated integral can be expressed as a product

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dy\,dx = \int_{x_1}^{x_2} g(x)dx \int_{y_1}^{y_2} h(y)dy$$

Improper Suppose f or R is unbounded. Let $R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots \subset R$ be such that each point of R, except possibly those in a set of arbitrarily small area, is in R_n for some n. Let $f(x,y) \geq 0$ be continuous in R_n for each n. Then $\iint_R f(x,y)dA = \lim_{n \to \infty} \iint_{R_n} f(x,y)dA$ if this limit exists.

Exercises 13.2

1. Let $f(x,y) = 3x - 2y + 1$. $\iint_R (3x - 2y + 1)dA$

$\approx f(\frac{1}{2}, -\frac{1}{2}) \cdot 1 + f(\frac{3}{2}, -\frac{1}{2}) \cdot 1 + f(\frac{5}{2}, -\frac{1}{2}) \cdot 1 + f(\frac{1}{2}, -\frac{3}{2}) \cdot 1 + f(\frac{3}{2}, -\frac{3}{2}) \cdot 1 + f(\frac{5}{2}, -\frac{3}{2}) \cdot 1$

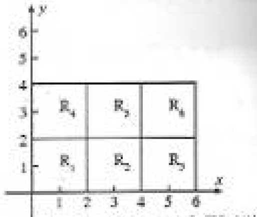
$= \frac{7}{2} + \frac{13}{2} + \frac{19}{2} + \frac{11}{2} + \frac{17}{2} + \frac{23}{2} = 45$

2. Let $f(x,y) = y^2 - 4x$. $\Delta A = 1$. $u_i = -0.5, 0.5$. $v_i = 0.5, 1.5, 2.5$. $\iint_R (y^2 - 4x)dA$
$\approx f(-0.5,0.5) + f(-0.5,1.5) + f(-0.5,2.5) + f(0.5,0.5) + f(0.5,1.5) + f(0.5,2.5) = 17.5$ using
sum seq$(y^2+2, y, .5, 2.5, 1)$ + sum seq$(y^2-2, y, .5, 2.5, 1)$ENTER on the TI-85.

In Exercises 3–8, find an approximate value of the given double integral where R is the rectangular region having the vertices P and Q. Δ is a partition of R, and (u_i, v_i) is the midpoint of each subregion.

3. Let $f(x,y) = x^2 + y$. $\Delta A = 1$. $\iint_R (x^2 + y)dA \approx$
$f(\frac{1}{2},\frac{1}{2}) + f(\frac{3}{2},\frac{1}{2}) + f(\frac{5}{2},\frac{1}{2}) + f(\frac{7}{2},\frac{1}{2}) + f(\frac{1}{2},\frac{3}{2}) + f(\frac{3}{2},\frac{3}{2}) + f(\frac{5}{2},\frac{3}{2}) + f(\frac{7}{2},\frac{3}{2}) = \frac{3}{4} + \frac{11}{4} + \frac{27}{4} + \frac{51}{4} + \frac{7}{4} + \frac{15}{4} + \frac{31}{4} + \frac{55}{4} = 50$

4. $\iint_R (2 - x - y)dA$; P(0,0); Q(6,4); Δ: $x_1 = 2$, $x_2 = 4$, $y_1 = 2$



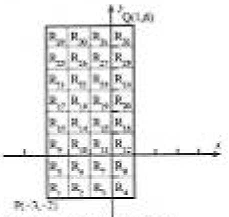
▹ The figure shows the region R which is partitioned into 6 subregions of equal area $\Delta A = 2 \times 2 = 4$ square units and the midpoint of each region. If f is the function defined by $f(x,y) = 2 - x - y$, then we have, approximately
$$\iint_R (2 - x - y)dA$$
$$\approx 4[f(1,1) + f(3,1) + f(5,1) + f(1,3) + f(3,3) + f(5,3)$$
$$= 4[0 + (-2) + (-4) + (-2) + (-4) + (-6)]$$
$$= 4(-18) = -72$$
You can use the sum and sequence capability of some graphics calculators (for example the TI-82 and TI-85) to automatically calculate and add the first three terms where $f(x,1) = 1 - x$ and the last three terms where $f(x,3) = -x - 1$:
4 × (sum seq$(1 - x, x, 1, 5, 2)$ + sum seq$(-1 - x, x, 1, 5, 2)$)ENTER–72

5. Let $f(x,y) = xy + 3y^2$. $\iint_R (xy + 3y^2)dA$
$\approx f(-1,1)\cdot 4 + f(1,1)\cdot 4 + f(3,1)\cdot 4 + f(3,3)\cdot 4 + f(1,3)\cdot 4 + f(-1,3)\cdot 4 + f(-1,5)\cdot 4 + f(1,5)\cdot 4 + f(3,5)\cdot 4$
$= 2\cdot 4 + 4\cdot 4 + 6\cdot 4 + 36\cdot 4 + 30\cdot 4 + 24\cdot 4 + 70\cdot 4 + 80\cdot 4 + 90\cdot 4 = 1368$

6. Let $f(x,y) = xy + 3y^2$. $\Delta A = 2 \times 2 = 4$. $u_i = 1, 3, 5$. $v_i = -1, 1, 3$. $\iint_R (xy + 3y^2)dA \approx$
$4[f(1,-1) + f(1,1) + f(1,3) + f(3,-1) + f(3,1) + f(3,3) + f(5,-1) + f(5,1) + f(5,3)] = 504$ using
4 × [sum seq$(3 - x, x, 1, 5, 2)$ + sum seq$(x + 3, x, 1, 5, 2)$ + sum seq$(3x + 27, x, 1, 5, 2)$]ENTER

7. Let $f(x,y) = x^2y - 2xy^2$. $\iint_R (x^2y - 2xy^2)dA$
$\approx f(-2,-1)\cdot 4 + f(0,-1)\cdot 4 + f(-2,1)\cdot 4 + f(0,1)\cdot 4 + f(-2,3)\cdot 4 + f(0,3)\cdot 4 + f(-2,5)\cdot 4 + f(0,5)\cdot 4$
$= 0\cdot 4 + 0\cdot 4 + 8\cdot 4 + 0\cdot 4 + 48\cdot 4 + 0\cdot 4 + 120\cdot 4 + 0\cdot 4 = 704$

8. $\iint_R (x^2y - 2xy^2)dA$; P(−3,−2); Q(1,6); Δ: $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $y_1 = -1$, $y_2 = 0$, $y_3 = 1$, $y_4 = 2$, $y_5 = 3$, $y_6 = 4$, $y_7 = 5$.



▹ The figure shows the region R which is partitioned into 32 subregions of equal area $\Delta A = 1 \times 1 = 1$ square unit, with midpoints (u_i, v_i). The u_i are −2.5, −1.5, −0.5, and 0.5 and the v_i are −1.5, −0.5, 0.5, 1.5, 2.5, 3.5, 4.5, 5.5. If f is the function defined by $f(x,y) = x^2y - 2xy^2$, we have
$$\iint_R (x^2y - 2xy^2)dA$$
$\approx f(-2.5,-1.5) + f(-2.5,-0.5) + f(-2.5,0.5) + f(-2.5,1.5) + f(-2.5,2.5) + f(-2.5,3.5) + f(-2.5,4.5) + f(-2.5,5.5)$
$+ f(-1.5,-1.5) + f(-1.5,-0.5) + f(-1.5,0.5) + f(-1.5,1.5) + f(-1.5,2.5) + f(-1.5,3.5) + f(-1.5,4.5) + f(-1.5,5.5)$
$+ f(-0.5,-1.5) + f(-0.5,-0.5) + f(-0.5,0.5) + f(-0.5,1.5) + f(-0.5,2.5) + f(-0.5,3.5) + f(-0.5,4.5) + f(-0.5,5.5)$
$+ f(0.5,-1.5) + f(0.5,-0.5) + f(0.5,0.5) + f(0.5,1.5) + f(0.5,2.5) + f(0.5,3.5) + f(0.5,4.5) + f(0.5,5.5) = 736.000$
which is calculated as
sum seq$(6.25y + 5y^2, y, -1.5, 5.5, 1)$
+ sum seq$(2.25y + 3y^2, y, -1.5, 5.5, 1)$
+ sum seq$(.25y + y^2, y, -1.5, 5.5, 1)$
+ sum seq$(.25y - y^2, y, -1.5, 5.5, 1)$ENTER

In Exercises 9–12, find an approximate value of the double integral, where R is the rectangular region having the vertices P and Q, Δ is a partition of R and (u_i, v_i) is an arbitrary point in each subregion.

9. Let $f(x,y) = x^2 + y$. Because $\Delta_i A = 1$, $\int\int_R (x^2+y)dA$

$\approx [f(.25,.5) + f(1.75,0) + f(2.5,.25) + f(4,1) + f(.75,1.75) + f(1.25,1.5) + f(2.5,2) + f(3,1)]1$
$= (.25^2 + .5) + 1.75^2 + (2.45^2 + .25) + (4^2+1) + (.75^2 + 1.75) + (1.25^2 + 1.5) + (2.5^2 + 2) + (3^2+1) = 50.75$

10. Let $f(x,y) = 2 - x - y$. Because $\Delta_i A = 4$ and $(2,2)$ belongs to 4 rectangles, $\int\int_R (2-x-y)dA$
$\approx [f(0.5,1.5) + f(3,1) + f(5.5,0.5) + f(2,2) + f(2,2) + f(5,3)]4$
$= [0-2-4-2-2-6]4 = -64$

11. Let $f(x,y) = xy + 3y^2$. Because $\Delta_i A = 4$, $\int\int_R (xy+3y^2)dA$
$\approx [f(-.5,.5) + f(1,1.5) + f(2.5,2) + f(-1.5,3.5) + f(0,3) + f(4,4) + f(-1,4.5) + f(1,4.5) + f(3,4.5)]4$
$= [0.5 + 8.75 + 17 + 31.5 + 27 + 64 + 56.25 + 165.25 + 74.75]4 = 1376$

12. $\int\int_R (xy+3y^2)dA$; $P(-2,0)$; $Q(4,6)$; Δ: $x_1 = 0$; $x_2 = 2$; $y_1 = 2$; $y_2 = 4$; $(u_1,v_1) = (-2,0)$; $(u_2,v_2) = (0,0)$; $(u_3,v_3) = (2,0)$; $(u_4,v_4) = (-2,2)$; $(u_5,v_5) = (0,2)$; $(u_6,v_6) = (2,2)$; $(u_7,v_7) = (-2,4)$; $(u_8,v_8) = (0,4)$; $(u_9,v_9) = (2,4)$

▹ If ΔA is the number of square units in each region, then $\Delta A = 2 \times 2 = 4$. If $f(x,y) = xy + 3y^2$, then

$$\iint_R (xy+3y^2)dA \approx 4[f(-2,0) + f(0,0) + f(2,0) + f(-2,2) + f(0,2) + f(2,2) + f(-2,4) + f(0,4) + f(2,4)]$$
$$= 4[0+0+0+8+12+16+40+48+56] = 720$$

which is calculated as
$4 \times (\text{sum seq}(0, x, -2, 2, 2) + \text{sum seq}(2x+12, x, -2, 2, 2) + \text{sum seq}(4x+48, x, -2, 2, 2))\boxed{\text{ENTER}}$

13. Let V cubic units be the volume of the solid. $\Delta_i A = 1$. Let $f(x,y) = \sqrt{64 - x^2 - y^2}$. Then

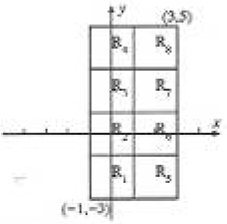
$$V = \iint_R \sqrt{64-x^2-y^2}\,dA \approx f(\tfrac{1}{2},\tfrac{1}{2}) + f(\tfrac{3}{2},\tfrac{1}{2}) + f(\tfrac{5}{2},\tfrac{1}{2}) + f(\tfrac{1}{2},\tfrac{3}{2}) + f(\tfrac{3}{2},\tfrac{3}{2}) + f(\tfrac{5}{2},\tfrac{3}{2}) + f(\tfrac{1}{2},\tfrac{5}{2}) + f(\tfrac{3}{2},\tfrac{5}{2}) + f(\tfrac{5}{2},\tfrac{5}{2})$$
$$= \sqrt{64-\tfrac{1}{4}-\tfrac{1}{4}} + \sqrt{64-\tfrac{9}{4}-\tfrac{1}{4}} + \sqrt{64-\tfrac{25}{4}-\tfrac{1}{4}} + \sqrt{64-\tfrac{1}{4}-\tfrac{9}{4}} + \sqrt{64-\tfrac{9}{4}-\tfrac{9}{4}} + \sqrt{64-\tfrac{25}{4}-\tfrac{9}{4}} + \sqrt{64-\tfrac{1}{4}-\tfrac{25}{4}}$$
$$+ \sqrt{64-\tfrac{9}{4}-\tfrac{25}{4}} + \sqrt{64-\tfrac{25}{4}-\tfrac{25}{4}}$$
$$= \tfrac{1}{2}(\sqrt{254} + \sqrt{246} + \sqrt{230} + \sqrt{246} + \sqrt{238} + \sqrt{222} + \sqrt{230} + \sqrt{222} + \sqrt{206}) \approx 68.6$$

14. $\Delta_i A = 1$. Let $f = 2x + y + 4$. $V = \iint_R (2x+y+4)dA$
$\approx f(.5,.5) + f(.5,1.5) + f(.5,2.5) + f(1.5,.5) + f(1.5,1.5) + f(1.5,2.5) = 5.5 + 6.5 + 7.5 + 7.5 + 8.5 + 9.5 = 45$

15. Let V cubic units be the volume of the solid. $\Delta_i A = 1$. Let $f(x,y) = 10 - \frac{1}{4}x^2 - \frac{1}{9}y^2$. Then

$$V = \iint_R (10 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2)dA \approx f(\tfrac{1}{2},\tfrac{1}{2}) + f(\tfrac{3}{2},\tfrac{1}{2}) + f(\tfrac{1}{2},\tfrac{3}{2}) + f(\tfrac{3}{2},\tfrac{3}{2})$$
$$= (10 - \tfrac{1}{16} - \tfrac{1}{36}) + (10 - \tfrac{9}{16} - \tfrac{1}{36}) + (10 - \tfrac{1}{16} - \tfrac{9}{36}) + (10 - \tfrac{9}{16} - \tfrac{9}{36}) \approx 38.2$$

16. Approximate the volume of the solid bounded by the surface $100z = 300 - 25x^2 - 4y^2$, the planes $x = -1$, $x = 3$, $y = 1$ and $y = 3$, and the xy plane. To find an approximate value of the double integral, take a partition of the region in the xy plane formed by the lines $x = 1$, $y = -1$, and $y = 3$, and take (u_i, v_i) at the center of the ith region.



▹ The figure shows a sketch of the region R in the xy plane which is the base of the solid. R is divided into 8 subrectangles each of area $\Delta A = 2 \times 2 = 4$ square units. If the midpoints are (u_i, v_i), then the u_i are 0 and 2, and the v_i are −2, 0, 2 and 4. Solving the equation of the surface for z and setting $z = f(x,y)$, we get
$f(x,y) = 3 - \frac{1}{4}x^2 - \frac{1}{25}y^2$
The number of cubic units in the volume of the solid is

$$\iint_R (3 - \tfrac{1}{4}x^2 - \tfrac{1}{25}y^2)dA$$
$$\approx 4[f(0,-2) + f(0,0) + f(0,2) + f(0,4) + f(2,-2) + f(2,0) + f(2,2) + f(2,4)] = 72.32$$

which is calculated as
$4 \times (\text{sum seq}(3 - y^2/25, y, -2, 4, 2) + \text{sum seq}(2 - y^2/25, y, -2, 4, 2))\boxed{\text{ENTER}}$
We conclude that the volume is approximately 72.3 cubic units. The exact value is about 69.2 cubic units.

In Exercises 17–20, apply Theorem 13.2.8 to find a closed interval containing the value of the double integral.

17. Let $f(x,y) = 2x + 5y$. Because f is linear, its absolute extrema occur at vertices of the rectangular region R.
$f(0,0) = 0 \quad f(1,0) = 2 \quad f(1,2) = 12 \quad f(0,2) = 10$
Therefore $0 \leq f(x,y) \leq 12$ for all (x,y) in R. Because $\mathrm{A} = 2$, we have from Theorem 13.2.8
$0 \cdot 2 \leq \iint_R (2x+5y)d\mathrm{A} \leq 12 \cdot 2; \;\; 0 \leq \iint_R (2x+5y)d\mathrm{A} \leq 24$

18. $f(x,y) = x^2 + y^2$ is the square of the distance from $(0,0)$. Its least value is 0 at $(0,0)$; its greatest value is 2 at $(1,1)$. $\mathrm{A} = 1$. $0 = 0 \cdot 1 \leq \iint_R (x^2+y^2)d\mathrm{A} \leq 2 \cdot 1 = 2$

19. Let $f(x,y) = e^{xy}$. Because e^t is increasing, $e^{xy} \geq {}^{0 \cdot 0} = 1$ and $e^{xy} \leq e^{1 \cdot 1} = e$ for all (x,y) in R. Because $\mathrm{A} = 1$, we have from Theorem 13.2.8 $1 \cdot 1 \leq \iint_R e^{xy}d\mathrm{A} \leq e \cdot 1$; $1 \leq \iint_R e^{xy}d\mathrm{A} \leq e$

20. $\iint_R (\sin x + \sin y)d\mathrm{A}$ where R is the rectangular region having vertices $(0,0)$, $(\pi,0)$, (π,π), and $(0,\pi)$.

▸ Let $f(x,y) = \sin x + \sin y$. Because f is the sum of two sine functions each of which having a minimum value of 0 and maximum value of 1 on R, we take $m = 0 + 0 = 0$ and $\mathrm{M} = 1 + 1 = 2$ in Theorem 13.2.8. Furthermore, because the area of the region is π^2 square units, we have $\mathrm{A} = \pi^2$ in the theorem. Thus,
$m\mathrm{A} \leq \iint_R f(x,y)d\mathrm{A} \leq \mathrm{MA}$
becomes
$0 \leq \iint_R (\sin x + \sin y)d\mathrm{A} \leq 2\pi^2$

In Exercises 21–30, evaluate the iterated integral.

21. $\int_0^2 \int_0^{2x} xy^3 dy\, dx = \int_0^2 \left[\int_0^{2x} xy^3 dy\right] dx = \int_0^2 \left[\frac{1}{4}xy^4\right]_0^{2x} dx = \int_0^2 4x^5 dx = \frac{2}{3}x^6\Big]_0^2 = \frac{2}{3}(64-1) = 42$

22. $\int_0^4 \int_0^y dx\, dy = \int_0^4 \left[\int_0^y dx\right] dy = \int_0^4 y\, dy = \frac{1}{2}y^2\Big]_0^4 = 8$

23. $\int_0^4 \int_0^y \sqrt{9+y^2}\, dx\, dy = \int_0^4 \left[\int_0^y \sqrt{9+y^2}\, dx\right] dy = \int_0^4 \left[x\sqrt{9+y^2}\right]_0^y = \int_0^4 y(9+y^2)^{1/2} dy = \frac{1}{3}(9+y^2)^{3/2}\Big]_0^4$
$= \frac{1}{3}(125-27) = \frac{98}{3}$

24. $\int_{-1}^1 \int_1^{e^x} \frac{x}{y} dy\, dx$

▸ $\int_{-1}^1 \int_1^{e^x} \frac{x}{y} dy\, dx = \int_{-1}^1 \left[x \ln y\right]_1^{e^x} dx = \int_{-1}^1 x(\ln e^x - \ln 1)dx = \int_{-1}^1 x^2 dx = \frac{1}{3}x^3\Big]_{-1}^1 = \frac{2}{3}$

25. $\int_1^4 \int_{y^2}^y \sqrt{\frac{y}{x}}\, dx\, dy = \int_0^4 \left[\int_{y^2}^y y^{1/2}x^{-1/2} dx\right] dy = \int_1^4 \left[2y^{1/2}x^{1/2}\right]_{y^2}^y dy = 2\int_1^4 (y - y^{3/2})dy = 2\left[\frac{1}{2}y^2 - \frac{2}{5}y^{5/2}\right]_1^4$
$= 2\left[\left(8 - \frac{64}{5}\right) - \left(\frac{1}{2} - \frac{2}{5}\right)\right] = -\frac{49}{5}$

26. $\int_1^4 \int_{x^2}^x \sqrt{\frac{y}{x}} dy\, dx = \int_1^4 x^{-1/2}\left[\int_{x^2}^x y^{1/2} dy\right] dx = \int_1^4 x^{-1/2}\left[\frac{2}{3}y^{3/2}\right]_{x^2}^x dx = \frac{2}{3}\int_1^4 x^{-1/2}(x^{3/2} - x^3)dx$
$= \frac{2}{3}\int_1^4 (x - x^{5/2})dx = \frac{2}{3}\left[\frac{1}{2}x^2 - \frac{2}{7}x^{7/2}\right]_1^4 = -\frac{403}{21}$

27. Because $|x-y|$ is symmetric with respect to the line $y = x$,
$\int_0^1 \int_0^1 |x-y| dy\, dx = 2\int_0^1 \left[\int_0^x |x-y| dy\right] dx = 2\int_0^1 \left[\int_0^x (x-y)dy\right] dx = 2\int_0^1 \left[xy - \frac{1}{2}y^2\right]_0^x dx = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$

28. $\int_0^3 \int_0^x x^2 e^{xy} dy\, dx$

▸ $\int_0^3 \int_0^x x^2 e^{xy} dy\, dx = \int_0^3 \left[xe^{xy}\right]_0^x dx = \int_0^3 (xe^{x^2} - x)dx = \frac{1}{2}e^{x^2} - \frac{1}{2}x^2\Big]_0^3 = \left(\frac{1}{2}e^9 - \frac{9}{2}\right) - \frac{1}{2}e^0 = \frac{1}{2}e^9 - 5$

29. $\int_{\pi/2}^{\pi} \int_0^x \sin(4x-y)dy\, dx = \int_{\pi/2}^{\pi} \left[\int_0^x \sin(4x-y)dy\right] dx = \int_{\pi/2}^{\pi} \left[\cos(4x-y)\right]_0^x dx$
$= \int_{\pi/2}^{\pi} (\cos 3x - \cos 4x)dx = \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x\Big]_{\pi/2}^{\pi} = 0 - \left(\frac{1}{3}\sin\frac{3}{2}\pi - \frac{1}{4}\sin 2\pi\right) = \frac{1}{3}$

30. $\int_{\pi/2}^{\pi} \int_0^{y^2} \sin\frac{x}{y}\, dx\, dy = \int_{\pi/2}^{\pi} \left[-y\cos\frac{x}{y}\right]_{x=0}^{y^2} dy = \int_{\pi/2}^{\pi} (y - y\cos y)dy = \left[\frac{1}{2}y^2 - y\sin y - \cos y\right]_{\pi/2}^{\pi} = \frac{3}{8}\pi^2 + \frac{1}{2}\pi + 1$

In Exercises 31–38, find the exact value of the double integral.

31. $\iint_R (3x-2y+1)dA = \int_0^3\int_{-2}^0 (3x-2y+1)dy\,dx = \int_0^3 \Big[3xy-y^2+y\Big]_{-2}^0 dx = \int_0^3 (6x+6)dx = 3x^2+6x\Big]_0^3 = 45$

32. $\iint_R (y^2-4x)dA$, where R is the rectangular region having opposite vertices $(-1,0)$ and $(1,3)$.

▸ $\iint_R (y^2-4x)dA = \int_{-1}^1\int_0^3 (y^2-4x)dy\,dx = \int_{-1}^1 \Big[\frac{1}{3}y^3-4xy\Big]_0^3 dx = \int_{-1}^1 (9-12x)dx = 9x-6x^2\Big]_{-1}^1 = 18$

33. $\iint_R (x^2+y)dA = \int_0^4\int_0^2 (x^2+y)dy\,dx = \int_0^4 \Big[x^2y+\frac{1}{2}y^2\Big]_0^2 dx = \int_0^4 (2x^2+2)dx = \frac{2}{3}x^3+2x\Big]_0^4 = \frac{128}{3}+8 = \frac{152}{3}$

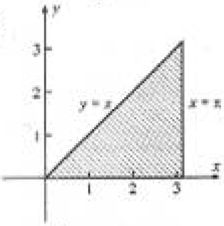
34. $\iint_R (xy+3y^2)dA = \int_0^6\int_{-2}^4 (xy+3y^2)dy\,dx = \int_0^6 \Big[\frac{1}{2}xy^2+y^3\Big]_{-2}^4 dx = \int_0^6 (6x+72)dx = 3x^2+72x\Big]_0^6 = 540$

35. $\iint_R \sin x\, dA = \int_0^\pi\int_{x/2}^{2x} \sin x\, dy\,dx = \int_0^\pi \Big[y\sin x\Big]_{x/2}^{2x} dx = \int_0^\pi \frac{3}{2}x\sin x\, dx = \frac{3}{2}\Big[\sin x - x\cos x\Big]_0^\pi = \frac{3}{2}\pi$

36. $\iint_R \cos(x+y)dA$, where R is bounded by the lines $y=x$ and $x=\pi$, and the x axis.

▸ The figure shows a sketch of the region R, which is bounded by below by the line $y=0$, bounded above by the line $y=x$, bounded on the left by the line $x=0$, and bounded on the right by the line $x=\pi$. Therefore, we have

$$\iint_R \cos(x+y)dA = \int_0^\pi\int_0^x \cos(x+y)dy\,dx = \int_0^\pi \Big[\sin(x+y)\Big]_{y=0}^x$$
$$= \int_0^\pi (\sin 2x - \sin x)dx = -\frac{1}{2}\cos 2x + \cos x\Big]_0^\pi = -2$$



37. Because the integrand is symmetric with respect to the x and y axes,

$$\iint_R x^2\sqrt{9-y^2}\, dA = 4\int_0^3\int_0^{\sqrt{9-y^2}} x^2\sqrt{9-y^2}\, dx\,dy = 4\int_0^3 \Big[\frac{1}{3}x^3\sqrt{9-y^2}\Big]_0^{\sqrt{9-y^2}} dy = \frac{4}{3}\int_0^3 (9-y^2)^2 dy$$
$$= \frac{4}{3}\int_0^3 (81-18y^2+y^4)dy = \frac{4}{3}\Big[81y-6y^3+\frac{1}{5}y^5\Big]_0^3 = \frac{4}{3}(81\cdot 3 - 6\cdot 3^3 + \frac{1}{5}\cdot 3^5) = \frac{864}{5}$$

38. $\iint_R \frac{y^2}{x^2}dA = \int_1^2\int_{1/y}^y y^2x^{-2}dx\,dy = \int_1^2 \Big[-y^2x^{-1}\Big]_{1/y}^y dy = \int_1^2 (y^3-y)dy = \Big[\frac{1}{4}y^4-\frac{1}{2}y^2\Big]_1^2 = \frac{9}{4}$

39. $V = 4\iint_R x\, dA = 8\int_0^4\int_0^{\sqrt{16-x^2}} x\, dy\,dx = 8\int_0^4 \Big[xy\Big]_0^{\sqrt{16-x^2}} dx = 8\int_0^4 x(16-x^2)^{1/2}dx = -\frac{8}{3}(16-x^2)^{3/2}\Big]_0^4 = \frac{512}{3}$

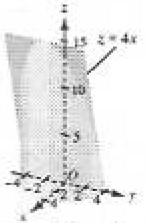
40. Find the volume of the solid bounded by the planes $x = y+2z+1$, $x=0$, $y=0$, and $3y+z=3$.

▸ Refer to the figure below. The projection of the solid onto the xy plane is not contained in the base in the xy plane, and the projection onto the xz plane is not contained in the base in the xz plane, while the projection onto the yz plane is contained in the base in the yz plane. Hence, we let R be the region in the yz plane bounded by the lines $y=0$, $y=1$, $z=0$, and $z=3-3y$ and let $f(y,z)=y+2z+1$. If V cubic units is the volume of the solid, then

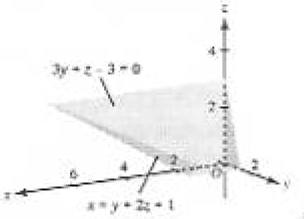
$$V = \iint_R f(y,z)dA = \int_0^1\int_0^{3-3y} (y+2z+1)dz\,dy = \int_0^1 \Big[yz+z^2+z\Big]_{z=0}^{3-3y}$$
$$= \int_0^1 [y(3-3y)+(3-3y)^2+(3-3y)]dy = \int_0^1 (6y^2-18y+12)dy = 2y^3-9y^2+12y\Big]_0^1 = 5$$

Thus, the volume is 5 cubic units.

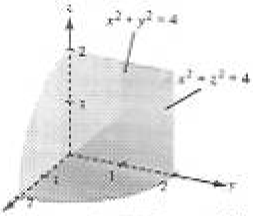
Exercise 39



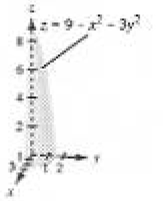
Exercise 40



Exercise 41



Exercise 42



41. $V = \iint_R \sqrt{4-x^2}\, dA = \int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{4-x^2}\, dy\,dx = \int_0^2 \Big[y\sqrt{4-x^2}\Big]_0^{\sqrt{4-x^2}} dx = \int_0^2 (4-x^2)dx$

$= 4x - \frac{1}{3}x^3\Big]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$

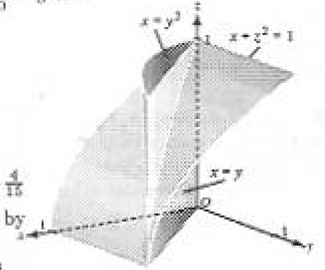
42. $V = \int_0^{\sqrt{3}}\int_0^{\sqrt{9-3y^2}}(9-3y^2-x^2)dx\,dy = \int_0^{\sqrt{3}}\Big[(9-3y^2)x - \frac{1}{3}x^3\Big]_0^{\sqrt{9-3y^2}}dy = \frac{2}{3}\int_0^{\sqrt{3}}(9-3y)^{3/2}dy$

Let $y = \sqrt{3}\sin\theta$. $V = \frac{2}{3}\int_0^{\pi/2}(3\cos\theta)^3\sqrt{3}\cos\theta\,d\theta = 18\sqrt{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$

$= 18\sqrt{3}\Big[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta)\Big]_0^{\pi/2} = \frac{27}{8}\pi$

Alternatively, a z-slice is a quarter of the ellipse $x^2+3y^2 = 9-z$, $\dfrac{x^2}{9-z} + \dfrac{y^2}{(9-z)/3} = 1$ of area

$\frac{1}{4}\sqrt{9-z}\sqrt{\frac{1}{3}(9-z)} = \frac{1}{4}\sqrt{3}(9-z)$. $V = \frac{1}{12}\sqrt{3}\int_0^9(9-z)dz = -\frac{1}{24}\sqrt{3}(9-z)^2\Big]_0^9 = \frac{27}{8}\sqrt{3}$



43. $V = \iint_R\sqrt{1-x}\,dA = \int_0^1\int_{y^2}^{y}(1-x)^{1/2}dx\,dy = \int_0^1\Big[-\frac{2}{3}(1-x)^{3/2}\Big]_{y^2}^{y}dy$

$= -\frac{2}{3}\int_0^1\,[(1-y)^{3/2} - (1-y^2)^{3/2}]dy$. Let $y = \sin\theta$, $dy = \cos\theta\,d\theta$ in the second term. Then $V = \Big[\frac{4}{15}(1-y)^{5/2}\Big]_0^1 + \frac{2}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$

$= -\frac{4}{15} + \frac{2}{3}\Big[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{2}\Big(\frac{1}{2}\cos\theta\sin\theta - \frac{1}{2}\theta\Big)\Big]_0^{\pi/2} = -\frac{4}{15} + \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{8}\pi - \frac{4}{15}$

44. Find by double integration the volume of the portion of the solid bounded by the sphere $x^2+y^2+z^2 = 16$ and lying in the first octant.

▹ Let f be the function defined by $f(x,y) = \sqrt{16-x^2-y^2}$. Then the solid is bounded below by a region R in the xy plane and bounded above by the surface $z = f(x,y)$. R is bounded by the lines $x = 0$, $x = 4$, and the curves $y = 0$ and $y = \sqrt{16-x^2}$. Then

$$V = \iint_R f(x,y)dA = \int_0^4\int_0^{\sqrt{16-x^2}}\sqrt{16-x^2-y^2}\,dy\,dx$$

To evaluate the inner integral, we regard x as a constant. Let $y = \sqrt{16-x^2}\sin\theta$. Then $dy = \sqrt{16-x^2}\cos\theta\,d\theta$, and $\sqrt{16-x^2-y^2} = \sqrt{16-x^2}\cos\theta$. When $y = 0$, then $\theta = 0$, and when $y = \sqrt{16-x^2}$, then $\theta = \frac{1}{2}\pi$, and the inner integral becomes

x² + y² + z² = 16

$$\int_0^{\pi/2}(16-x^2)\cos^2\theta\,d\theta = \frac{1}{2}(16-x^2)\int_0^{\pi/2}(1+\cos 2\theta)d\theta = \frac{1}{2}(16-x^2)\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = \frac{1}{4}\pi(16-x^2)$$

Alternatively, for each value of x the inner integral represents a quarter of the area of a circle of radius $\sqrt{16-x^2}$ which is $\frac{1}{4}\pi(16-x^2)$, just as if we had used slices parallel to the xy plane. Hence,

$$V = \frac{1}{4}\pi\int_0^4(16-x^2)dx = \frac{1}{4}\pi\Big[16x - \frac{1}{3}x^3\Big]_0^4 = \frac{32}{3}\pi$$

Thus, the volume of the solid is $\frac{32}{3}\pi$ cubic units.

In Exercises 45–48, use double integrals to find the area of the region bounded by the curves in the xy plane. Sketch the region.

45. $A = \iint_R dA = \int_0^1\int_{x^3}^{x^2}dy\,dx = \int_0^1\Big[y\Big]_{x^3}^{x^2}dx = \int_0^1(x^2-x^3)dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

46. $y^2 = 4x$, $x^2 = 4y$. $x^4 = 16y^2 = 16\cdot 4x$. $x = 0, 4$. $A = \int_0^4\int_{x^2/4}^{2\sqrt{x}}dy\,dx = \int_0^4(2x^{1/2} - \frac{1}{4}x^2)dx = \Big[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\Big]_0^4 = \frac{16}{3}$

47. Because the region is symmetric with respect to the x and y axes,

$A = \iint_R dA = 4\int_0^3\int_0^{9-x^2}dy\,dx = 4\int_0^3\Big[y\Big]_0^{9-x^2}dx = 4\int_0^3(9-x^2)dx = 4\Big[9x - \frac{1}{3}x^3\Big]_0^3 = 4(27-9) = 72$

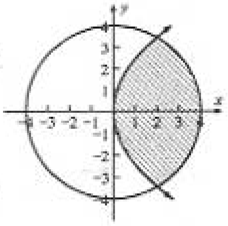
48. $x^2+y^2 = 16$ and $y^2 = 6x$

▹ The figure shows the region, which is bounded by the lines $y = \pm 2\sqrt{3}$, and by the curves $x = \frac{1}{6}y^2$ and $x = \sqrt{16-y^2}$. Let R be the upper half of the region. Thus,

$$A = 2\iint_R dx\,dy = 2\int_0^{2\sqrt{3}}\int_{y^2/6}^{\sqrt{16-y^2}}dx\,dy = 2\int_0^{2\sqrt{3}}\Big[x\Big]_{y^2/16}^{\sqrt{16-y^2}}dy$$

$$= 2\int_0^{2\sqrt{3}}[\sqrt{16-y^2} - \frac{1}{6}y^2]dy = 2\int_0^{2\sqrt{3}}\sqrt{16-y^2}dy - \Big[\frac{1}{9}y^3\Big]_0^{2\sqrt{3}}$$

$$= 2\int_0^{2\sqrt{3}}\sqrt{16-y^2}dy - \frac{8}{3}\sqrt{3} \qquad (1)$$



For the remaining integral in (1), we take $y = 4\sin\theta$. Thus,

$$A = 2\int_0^{\pi/3} 16\cos^2\theta\, d\theta - \tfrac{8}{3}\sqrt{3} = 16\int_0^{\pi/3}(1+\cos 2\theta)d\theta - \tfrac{8}{3}\sqrt{3} = 16\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/3} - \tfrac{8}{3}\sqrt{3} = \tfrac{16}{3}\pi + \tfrac{4}{3}\sqrt{3}$$

Therefore, the area is $\frac{4}{3}(4\pi + \sqrt{3})$ square units.

49. Taking 8 times the volume in the first octant,

$$V = 8c\iint_R \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dA = 8c\int_0^a\int_0^{b\sqrt{1-(x/a)^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

To evaluate this integral, let $x = au$, $dx = a\,du$, $y = bv$, $dy = b\,dv$. Then

$$V = abc\cdot 8\int_0^1\int_0^{\sqrt{1-u^2}}\sqrt{1-u^2-v^2}\,dv\,du = abc\cdot(\text{volume of unit sphere}) = \tfrac{4}{3}\pi abc$$

50. $4x = y^2 = 5 - x^2$, $0 = x^2 + 4x - 5 = (x+5)(x-1)$. $x = 1$ (reject -5), $y = \pm 2$.

(a) $A = \int_0^2\int_{y^2/4}^{\sqrt{5-y^2}} dx\,dy = \int_0^2(\sqrt{5-y^2} - \frac{1}{4}y^2)dy$. Let $y = \sqrt{5}\sin\theta$, $\sin b = \frac{2}{5}\sqrt{5}$, $\cos b = \frac{1}{5}\sqrt{5}$.

$A = \int_0^b \sqrt{5}\cos\theta\cdot\sqrt{5}\cos\theta\, d\theta - \frac{1}{12}y^3\Big]_0^2 = \frac{5}{2}\int_0^b(1+\cos 2\theta)d\theta - \frac{2}{3} = \frac{5}{2}\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^b - \frac{2}{3} = (\frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + 1) - \frac{2}{3}$

$= \frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + \frac{1}{3}$ (b) $A = \int_0^1 2x^{1/2}dx + \int_1^{\sqrt{5}}\sqrt{5-x^2}\,dx$. Let $x = \sqrt{5}\cos\theta$.

$A = \frac{4}{3}x^{3/2}\Big]_0^1 + \int_b^0\sqrt{5}\sin\theta(-\sqrt{5}\sin\theta\, d\theta) = \frac{4}{3} + \frac{5}{2}\int_0^b(1-\cos 2\theta)d\theta = \frac{4}{3} + (\frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} - 1) = \frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + \frac{1}{3}$

51. The parabola $y^2 = 2x$ and the line $2x + 3y = 10$ intersect in the first quadrant at $(2,2)$, the parabola intersects the x axis at $(0,0)$, and the line intersects the x axis at $(5,0)$.

$V = \iint_R\Big(4 - \frac{1}{2}x - \frac{4}{3}y\Big)dA$ (a) $V = \int_0^2\int_{y^2/2}^{(10-3y)/2}\Big(4 - \frac{1}{2}x - \frac{4}{3}y\Big)dx\,dy = \int_0^2\Big[4x - \frac{1}{2}x^2 - \frac{4}{2}xy\Big]_{y^2/2}^{(10-3y)/2}dy$

$= \int_0^2(\frac{55}{4} - \frac{107}{12}y - \frac{9}{16}y^2 + \frac{2}{3}y^3 + \frac{1}{16}y^4)dy = \frac{55}{4}y - \frac{107}{24}y^2 - \frac{3}{16}y^3 + \frac{1}{6}y^4 + \frac{1}{80}y^5\Big]_0^2 = \frac{55}{2} - \frac{107}{6} - \frac{3}{2} + \frac{8}{3} + \frac{2}{5} = \frac{337}{30}$

(b) $V = \int_0^2\int_0^{\sqrt{2x}}\Big(4 - \frac{1}{2}x - \frac{4}{3}y\Big)dy\,dx + \int_2^5\int_0^{(10-2x)/3}\Big(4 - \frac{1}{2}x - \frac{4}{3}y\Big)dy\,dx$

$= \int_0^2\Big[4y - \frac{1}{2}xy - \frac{2}{3}y^2\Big]_0^{\sqrt{2x}}dx + \int_0^5\Big[4y - \frac{1}{2}xy - \frac{2}{2}y^2\Big]_0^{(10-2x)/3}dx$

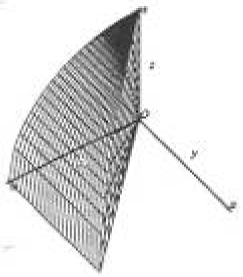
$= \int_0^2(4\sqrt{2}x^{1/2} - \frac{1}{2}\sqrt{2}x^{3/2} - \frac{4}{3}x)dx + \int_2^5\Big(\frac{160}{27} - \frac{37}{27}x + \frac{1}{27}x^2\Big)dx$

$= \Big[\frac{8}{3}\sqrt{2}x^{3/2} - \frac{1}{5}\sqrt{2}x^{5/2} - \frac{2}{3}x^2\Big]_0^2 + \Big[\frac{160}{27}x - \frac{37}{54}x^2 + \frac{1}{81}x^3\Big]_2^5 = \frac{32}{5} + \frac{87}{18} = \frac{337}{30}$

In Exercises 52 and 53, (a) sketch the solid, the measure of whose volume is represented by the given iterated integral; (b) evaluate the iterated integral; (c) write the iterated integral the gives the measure of the volume of the same solid with the order of integration reversed.

52. $\int_0^a\int_0^x\sqrt{a^2-x^2}\,dy\,dx$

▸ (a) The base of the solid is the triangle bounded by the lines $x = 0$ and $x = a$, and by the lines $y = 0$ and $y = x$. And the solid is bounded by the cylindrical surface $z = \sqrt{a^2-x^2}$. The figure shows a sketch of the solid.



(b) $$\int_0^a\int_0^x\sqrt{a^2-x^2}\,dy\,dx = \int_0^a\Big[y\sqrt{a^2-x^2}\Big]_{y=0}^x dx$$
$$= \int_0^a x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2-x^2)^{3/2}\Big]_0^a = \tfrac{1}{3}a^3$$

(c) We may regard the triangular region in the xy plane as being bound by the lines $y = 0$, $y = a$, and the lines $x = y$ and $x = a$. Therefore,

$$\int_0^a\int_0^x\sqrt{a^2-x^2}\,dy\,dx = \int_0^a\int_y^a\sqrt{a^2-x^2}\,dx\,dy$$

53. (b) $\frac{2}{3}\int_0^a\int_0^{\sqrt{a^2-x^2}}(2x+y)dy\,dx = \frac{2}{3}\int_0^a\Big[2xy + \frac{1}{2}y^2\Big]_0^{\sqrt{a^2-x^2}}dx = \frac{2}{3}\int_0^a\Big[2x\sqrt{a^2-x^2} + \frac{1}{2}(a^2-x^2)\Big]dx$

$= \frac{2}{3}\Big[-\frac{2}{3}(a^2-x^2)^{3/2} + \frac{1}{2}(a^2x - \frac{1}{3}x^3)\Big] = \frac{2}{3}(\frac{2}{3}a^3 + \frac{1}{3}a^3) = \frac{2}{3}a^3$ (c) $\frac{2}{3}\iint_R(2x+y)dA = \frac{2}{3}\int_0^a\int_0^{\sqrt{a^2-y^2}}(2x+y)dx\,dy$

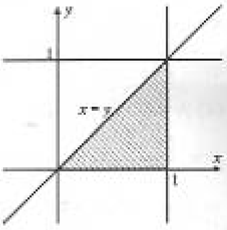
54. Let the axes of the cylinders be the x- and y-axes. The volume is 16 times the region above the part of the first octant where $y \le x$. There $z = \sqrt{r^2 - x^2}$ (the part *not* visible) and so

$$V = 16\int_0^r \int_0^x \sqrt{r^2 - x^2}\,dy\,dx = 16\int_0^r x\sqrt{r^2 - x^2}\,dx = 16\Big[-\tfrac{1}{3}(r^2 - x^2)^{3/2}\Big]_0^r = \tfrac{16}{3}r^3$$

In Exercises 55 and 56, the iterated integral cannot be evaluated exactly in terms of elementary functions by the given order of integration. Reverse the order of integration and perform the computation.

55. $$\int_0^4\int_{\sqrt{x}}^2 \sin \pi y^3\,dy\,dx = \int_0^2\int_0^{y^2} \sin \pi y^3\,dx\,dy = \int_0^2 \Big[x \sin \pi y^3\Big]_0^{y^2} dy = \int_0^2 y^2 \sin \pi y^3\,dy = -\frac{1}{3\pi}\cos \pi y^3\Big]_0^2$$
$$= -\frac{1}{3\pi}(\cos 8\pi - \cos 0) = -\frac{1}{3\pi}(1-1) = 0$$

56. $$\int_0^1\int_y^1 e^{x^2}\,dx\,dy$$



▸ The region R over which the double integral is being taken is bounded by the lines $y = 0$, $y = 1$, and by the "curves" $x = y$ and $x = 1$, as illustrated in the figure. We may also regard R as being bounded by the lines $x = 0$, $x = 1$, and by the "curves" $y = 0$ and $y = x$. Therefore,

$$\int_0^1\int_y^1 e^{x^2}\,dx\,dy = \int_0^1\int_0^x e^{x^2}\,dy\,dx = \int_0^1 \Big[ye^{x^2}\Big]_{y=0}^x dx = \int_0^1 xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2}\Big]_0^1$$
$$= \tfrac{1}{2}(e - 1)$$

13.3 APPLICATIONS OF DOUBLE INTEGRALS

If a lamina has the shape of a closed region R in the xy plane and $\rho(x,y)$ is the measure of the area density of the lamina at any point (x,y) of R, where ρ is continuous on R, then the measure of mass of the lamina is given by

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(u_i, v_i)\Delta_i A = \iint_R \rho(x,y)\,dA$$

The measure M_x of the moment of mass with respect to the x axis of the lamina is given by

$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i \rho(u_i, v_i)\Delta_i A = \iint_R y\rho(x,y)\,dA$$

Analogously, the measure M_y of its moment of mass with respect to the y axis is given by

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i \rho(u_i, v_i)\Delta_i A = \iint_R x\rho(x,y)\,dA$$

The center of mass of the lamina is denoted by the point $(\bar{x}, \bar{y})$ and

$$\bar{x} = \frac{M_y}{M} \quad \bar{y} = \frac{M_x}{M}$$

If the region R and the density function are symmetric in x and y, then $\bar{x} = \bar{y}$. See Exercise 12.

The *moment of inertia* of a particle, whose mass is m kg, about an axis is mr^2 kg-m^2, where r meters is the perpendicular distance from the particle to the axis.

Suppose that we are given a lamina occupying a region R in the xy plane such that the area density at the point (x,y) has measure $\rho(x,y)$, where ρ is continuous on R. Then the measure of the moment of inertia of the lamina about the x axis, denoted by I_x, is defined by

$$I_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i^2 \rho(u_i, v_i)\Delta_i A = \iint_R y^2\rho(x,y)\,dA$$

Similarly, the measure of the moment of inertia of the lamina about the y axis, denoted by I_y, is defined by

$$I_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i^2 \rho(u_i, v_i)\Delta_i A = \iint_R x^2\rho(x,y)\,dA$$

Finally, the measure of the *polar moment of inertia*, denoted by I_0, is defined by

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x,y)\,dA$$

If I is the measure of the moment of inertia about an axis L of a distribution of mass in a plane and M is the measure of the total mass of the distribution, then the *radius of gyration* of the distribution about L has measure r, where

$$r^2 = \frac{I}{M}$$

Surface Area If a surface is defined by the parametric equations over a region R
$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$
then the surface area is given by

$$\sigma = \iint_R \|(x_u, y_u, z_u) \times (x_v, y_v, z_v)\| du\, dv = \iint_R \sqrt{\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}^2 + \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}^2 + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}^2}\, du\, dv$$

In the special case $x = x \quad y = y \quad z = f(x,y)$
we have (Theorem 13.3.4)

$$\sigma = \iint_R \sqrt{f_x{}^2(x,y) + f_y{}^2(x,y) + 1}\; dx\, dy$$

If a surface is given by $g(x,y,z) = 0$ and its projection on the xy plane is R, then

$\sigma = \iint_R [\|\nabla g(x,y,z)\| / |g_z(x,y,z)|]\; dx\, dy$. See Exercise 44.

Cone For the cone $x^2 + y^2 = z^2$, $\sigma = \iint_R \sqrt{2}\; dA = \sqrt{2}$ area(R). See Exercise 32.

Revolution Suppose that the function f is positive on $[a,b]$ and f' is continuous on $[a,b]$. Then the measure of the area of the surface of revolution obtained by revolving the curve $y = f(x)$, with $a \le x \le b$, about the x axis is given by $\sigma = 2\pi \int_a^b f(x)\sqrt{[f'(x)]^2 + 1}\,dx$

Pappus Theorem If an arc of length L units is revolved about the x axis, the surface area is $2\pi\bar{y}L$ square units.

Sphere The surface area of a sphere of radius r units is $4\pi r^2$ square units. The area of a *zone*, the part of the surface cut off by two parallel planes h units apart, is $2\pi rh$ square units. See Ex. 38.

Exercises 13.3

In Exercises 1–12, find the mass and the center of mass of a lamina in the given shape if the area density is as indicated. Mass is measured in kilograms and distance is measured in meters.

- In Exercises 1–12, M kg is the mass of the lamina R, M_x kg-m and M_y kg-m are the moments of mass with respect to the x and y axes, and the center of mass is at $(\bar{x}, \bar{y})$. $\rho(x,y)$ kg/m^2 is the area density at the point (x,y) and k is a constant of proportionality.

1. R is bounded by $x = 3$, $y = 2$, and the coordinate axes. $\rho(x,y) = xy^2$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i v_i{}^2 \Delta_i A = \iint_R xy^2 dA = \int_0^3 \int_0^2 xy^2 dy\, dx = \int_0^3 \frac{1}{3} xy^3 \Big]_0^2 dx = \int_0^3 \frac{8}{3} x\, dx = \frac{4}{3} x^2 \Big]_0^3 = 12$$

$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i(u_i v_i{}^2) \Delta_i A = \iint_R xy^3 dA = \int_0^3 \int_0^2 xy^3 dy\, dx = \int_0^3 \frac{1}{4} xy^4 \Big]_0^2 dx = \int_0^3 4x\, dx = 2x^2 \Big]_0^3 = 18$$

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i(u_i v_i{}^2) \Delta_i A = \iint_R x^2 y^2 dA = \int_0^3 \int_0^2 x^2 y^2 dy\, dx = \int_0^3 \frac{1}{3} x^2 y^3 \Big]_0^2 dx = \int \frac{8}{3} x^2 dx = \frac{8}{9} x^3 \Big]_0^3 = 24$$

$$\bar{x} = \frac{M_y}{M} = \frac{24}{12} = 2 \text{ and } \bar{y} = \frac{M_x}{M} = \frac{18}{12} = \frac{3}{2}$$

2. R is bounded by the coordinates axes, $x = 4$ and $y = 5$. $\rho(x,y) = x^2 + y$.

$$M = \iint_R (x^2 + y) dA = \int_0^4 \int_0^5 (x^2 + y) dy\, dx = \int_0^4 \Big[x^2 y + \tfrac{1}{2} y^2 \Big]_0^5 dx = \int_0^4 (5x^2 + \tfrac{25}{2}) dx = \Big[\tfrac{5}{3} x^3 + \tfrac{25}{2} x \Big]_0^4 = \tfrac{470}{3}$$

$$M_y = \iint_R (x^2 + y)x\, dA = \int_0^4 \int_0^5 (x^2 + y)x\, dy\, dx = \int_0^4 (5x^3 + \tfrac{25}{2} x) dx = \Big[\tfrac{5}{4} x^4 + \tfrac{25}{4} x^2 \Big]_0^4 = 45.\ \bar{x} = \tfrac{3}{470} \cdot 45 = \tfrac{81}{94}$$

$$M_x = \iint_R (x^2 + y)y\, dA = \int_0^4 \int_0^5 (x^2 y + y^2) dy\, dx = \int_0^4 \Big[\tfrac{1}{2} x^2 y^2 + \tfrac{1}{3} y^3 \Big]_0^5 dx = \int_0^4 (\tfrac{25}{2} x^2 + \tfrac{125}{3}) dx = \tfrac{25}{6} x^3 + \tfrac{125}{3} x \Big]_0^4$$

$$= \tfrac{1300}{3}.\ \bar{y} = \tfrac{3}{470} \cdot \tfrac{1300}{3} = \tfrac{130}{47}$$

3. R is bounded by the coordinate axes and $x + 2y = 6$. $\rho(x,y) = y^2$.

$$M = \iint_R y^2 dA = \int_0^3 \int_0^{6-2y} y^2 dx\, dy = \int_0^3 xy^2 \Big]_0^{6-2y} dy = \int_0^3 (6y^2 - 2y^3) dy = 2y^3 - \frac{1}{2} y^4 \Big]_0^3 = 54 - \frac{81}{2} = \frac{27}{2}$$

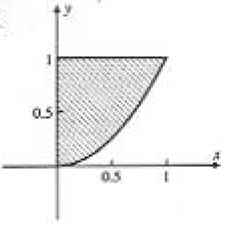
$$M_x = \iint_R y \cdot y^2 dA = \int_0^3 \int_0^{6-2y} y^3 dx\, dy = \int_0^3 xy^3 \Big]_0^{6-2y} dy = \int_0^3 (6y^3 - 2y^4) dy = \frac{3}{2} y^4 - \frac{2}{5} y^5 \Big]_0^3$$

$$= \frac{1}{2}(3^5) - \frac{2}{5}(3^5) = \frac{1}{10}(3^5) = \frac{243}{10}.\ \bar{y} = \frac{1}{M} M_x = \frac{2}{27} \cdot \frac{243}{10} = \frac{9}{5}.$$

$$M_y = \iint_R xy^2 dA = \int_0^3 \int_0^{6-2y} xy^2 dx\, dy = \int_0^3 \frac{1}{2} x^2 y^2 \Big]_0^{6-2y} dy = 2\int_0^3 (3-y)^2 y^2 dy$$

$$= 2\int_0^3 (9y^2 - 6y^3 + y^4) dy = 2\Big[3y^3 - \frac{3}{2} y^4 + \frac{1}{5} y^5 \Big]_0^3 = 2\Big[3^4 - \frac{3}{2}(3^4) + \frac{3}{5}(3^4)\Big] = \frac{81}{5}.\ x = \frac{1}{M} M_y = \frac{2}{27} \cdot \frac{81}{5} = \frac{6}{5}.$$

4. The region in the first quadrant bounded by the parabola $y = x^2$, the line $y = 1$, and the y axis. The area density at any point is $(x + y)\text{kg/m}^2$.



▹ The figure shows the region R. Let $\rho(x,y) = x + y$. Then the number of kilograms in the mass of the lamina is given by

$$\mathrm{M} = \iint_R \rho(x,y)d\mathrm{A} = \int_0^1\int_{x^2}^1 (x+y)dy\,dx = \int_0^1\left[xy + \tfrac12 y^2\right]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac12 + x - x^3 - \tfrac12 x^4)dx = \tfrac12 x + \tfrac12 x^2 - \tfrac14 x^4 - \tfrac1{10}x^5\Big]_0^1 = \tfrac{13}{20} \qquad (1)$$

Thus, the mass is $\frac{13}{20}$ kilograms. Moreover, if $(\bar x, \bar y)$ is the center of mass, we have

$$\mathrm{M}_y = \iint_R \rho(x,y)x\,d\mathrm{A} = \int_0^1\int_{x^2}^1 (x+y)x\,dy\,dx = \int_0^1 x\left[xy + \tfrac12 y^2\right]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac12 x + x^2 - x^4 - \tfrac12 x^5)dx = \tfrac14 x^2 + \tfrac13 x^3 - \tfrac15 x^5 - \tfrac1{12}x^6\Big]_0^1 = \tfrac{3}{10} \qquad (2)$$

and

$$\mathrm{M}_x = \iint_R \rho(x,y)y\,d\mathrm{A} = \int_0^1\int_{x^2}^1 (x+y)y\,dy\,dx = \int_0^1\left[\tfrac12 xy^2 + \tfrac13 y^3\right]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac13 + \tfrac12 x - \tfrac12 x^5 - \tfrac13 x^6)dx = \tfrac13 x + \tfrac14 x^2 - \tfrac1{12}x^6 - \tfrac1{21}x^7\Big]_0^1 = \tfrac{19}{42} \qquad (3)$$

From (1), (2) and (3), we have

$\bar x = \frac{1}{\mathrm{M}}\cdot \mathrm{M}_y = \frac{20}{13}\cdot\frac{3}{10} = \frac{6}{13}$ and $\bar y = \frac{1}{\mathrm{M}}\cdot \mathrm{M}_x = \frac{20}{13}\cdot\frac{19}{42} = \frac{190}{273}$

Therefore, the center of mass is at $(\frac{6}{13}, \frac{190}{273})$.

5. R is in the first quadrant bounded by $x^2 = 8y$, $y = 2$ and the y axis. $\rho(x,y) = k(y+1)$.

$$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n k(v_i+1)\Delta_i\mathrm{A} = k\iint_R (y+1)d\mathrm{A} = k\int_0^2\int_0^{\sqrt{8y}} (y+1)dx\,dy = k\int_0^2 x(y+1)\Big]_0^{\sqrt{8y}} dy$$
$$= k\sqrt8\int_0^2 (y^{3/2} + y^{1/2})dy = k\sqrt8\left[\frac25 y^{5/2} + \frac23 y^{3/2}\right]_0^2 = k\sqrt8\left(\frac85\sqrt2 + \frac43\sqrt2\right) = k\sqrt8\left(\frac{44}{15}\sqrt2\right) = \frac{176}{15}k$$

$$\mathrm{M}_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i[k(v_i+1)]\Delta_i\mathrm{A} = k\iint_R (y^2+y)d\mathrm{A} = k\int_0^2\int_0^{\sqrt{8y}}(y^2+y)dx\,dy = k\int_0^2 x(y^2+y)\Big]_0^{\sqrt{8y}}$$
$$= k\sqrt8\int_0^2 (y^{5/2} + y^{3/2})dy = k\sqrt8\left[\frac27 y^{7/2} + \frac25 y^{5/2}\right]_0^2 = k\sqrt8\left(\frac{16}{7}\sqrt2 + \frac85\sqrt2\right) = k\sqrt8\left(\frac{136}{35}\sqrt2\right) = \frac{544}{35}k$$

$$\mathrm{M}_y = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n u_i[k(v_i+1)]\Delta_i\mathrm{A} = k\iint_R x(y+1)d\mathrm{A} = k\int_0^2\int_0^{\sqrt{8y}} x(y+1)dx\,dy = k\int_0^2 \frac12 x^2(y+1)\Big]_0^{\sqrt{8y}} dy$$
$$= k\int_0^2 4y(y+1)dy = 4k\left[\frac13 y^3 + \frac12 y^2\right]_0^2 = 4k\left(\frac83 + 2\right) = \frac{56}{3}k$$

$$\bar x = \frac{1}{\mathrm{M}}\mathrm{M}_y = \frac{15}{176k}\cdot\frac{56k}{3} = \frac{35}{22} \text{ and } \bar y = \frac{1}{\mathrm{M}}\mathrm{M}_x = \frac{15}{176k}\cdot\frac{544k}{35} = \frac{102}{77}$$

6. R is bounded by the coordinate axes, $y = e^x$, and $x = 1$. $\rho(x,y) = ky$.

$$\mathrm{M} = \int_0^1\int_0^{e^x} ky\,dy\,dx = \int_0^1\left[\tfrac12 ky^2\right]_0^{e^x} dx = \int_0^1 \tfrac12 ke^{2x}dx = \tfrac14 ke^{2x}\Big]_0^1 = \tfrac14 k(e^2 - 1)$$

$$\mathrm{M}_y = \int_0^1\int_0^{e^x} kxy\,dy\,dx = \int_0^1 \tfrac12 kxe^{2x}dx = \left[\tfrac12 k(\tfrac12 xe^{2x} - \tfrac14 e^{2x})\right]_0^1 = \tfrac12 k(\tfrac14 e^2 + \tfrac14).\ \bar x = \frac{\frac18 k(e^2+1)}{\frac14 k(e^2-1)} = \frac{e^2+1}{2(e^2-1)}$$

$$\mathrm{M}_x = \int_0^1\int_0^{e^x} ky^2dy\,dx = \int_0^1\left[\tfrac13 ky^2\right]_0^{e^x} dx = \int_0^1 \tfrac13 kxe^{3x}dx = \tfrac19 ke^{3x}\Big]_0^1 = \tfrac19 k(e^2 - 1).\ \bar y = \frac{\frac19 k(e^3-1)}{\frac14 k(e^2-1)} = \frac{4(e^2-1)}{9(e^2-1)}$$

7. R is in the first quadrant bounded by $x^2 + y^2 = a^2$. $\rho(x,y) = k(x+y)$.

$$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n k(u_i+v_i)\Delta_i\mathrm{A} = \iint_R k(x+y)d\mathrm{A} = \int_0^a\int_0^{\sqrt{a^2-x^2}} k(x+y)dy\,x = k\int_0^a\left[xy + \frac12 y^2\right]_0^{\sqrt{a^2-x^2}} dx$$
$$= k\int_0^a\left[x\sqrt{a^2-x^2} + \frac12(a^2-x^2)\right]dx = k\left[-\frac13(a^2-x^2)^{3/2} + \frac12 a^2x - \frac16 x^3\right]_0^a = k\left[\frac12 a^3 - \frac16 a^3 + \frac13 a^3\right] = \frac23 ka^3$$

$$\mathrm{M}_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i[k(u_i+v_i)]\Delta_i\mathrm{A} = \iint_R k(xy+y^2)d\mathrm{A} = k\int_0^a\int_0^{\sqrt{a^2-x^2}}(xy+y^2)dy\,dx$$
$$= k\int_0^a \frac12 xy^2 + \frac13 y^3\Big]_0^{\sqrt{a^2-x^2}} = k\int_0^a\left[\frac12 x(a^2-x^2) + \frac13(a^2-x^2)^{3/2}\right]dx$$

Let $x = a\sin\theta$, $dx = a\cos\theta$ in the second term. From Exercise 13.2.43,

$$M_x = k\Big[\frac{1}{4}a^2x^2 - \frac{1}{8}x^4\Big]_0^a + \frac{ka^4}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = k\Big(\frac{1}{4}a^4 - \frac{1}{8}a^4\Big) + \frac{ka^4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{16}ka^4(2+\pi)$$

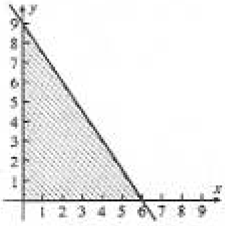
$\bar{y} = \frac{1}{M}M_x = \frac{3}{2ka^3}\cdot\frac{ka^4(2+\pi)}{16} = \frac{3}{32}a(2+\pi)$. $\bar{x} = \bar{y}$ because of symmetry about the line $y = x$.

8. The region bounded by the triangle whose sides are segments of the coordinate axes and the line $3x + 2y = 18$. The area density varies as the product of the distances from the coordinate axes.

▸ The figure shows the region R. Let $\rho(x,y) = kxy$. The measure of the mass is

$$M = \iint_R \rho(x,y)dA = \int_0^6\int_0^{(-3/2)x+9} kxy\,dy\,dx = \tfrac{1}{2}k\int_0^6 xy^2\Big]_{y=0}^{(-3/2)x+9}dx$$

$$= \tfrac{1}{2}k\int_0^6 (\tfrac{9}{4}x^3 - 27x^2 + 81x)dx = \tfrac{1}{2}k\Big[\tfrac{9}{16}x^4 - 9x^3 + \tfrac{81}{2}x^2\Big]_0^6 = \tfrac{243}{2}k \qquad (1)$$



Thus, the mass is $\frac{243}{2}k$ kilograms. If $(\bar{x},\bar{y})$ is the center of mass, we have

$$M_y = \iint_R \rho(x,y)x\,dA = \int_0^6\int_0^{(-3/2)x+9} kx^2y\,dy\,dx$$

$$= \tfrac{1}{2}k\int_0^6 x^2y^2\Big]_{y=0}^{(-3/2)x+9}dx = \tfrac{1}{2}k\int_0^6(\tfrac{9}{4}x^4 - 27x^3 + 81x^2)dx$$

$$= \tfrac{1}{2}k\Big[\tfrac{9}{20}x^5 - \tfrac{27}{4}x^4 + 27x^3\Big]_0^6 = \tfrac{1458}{5}k \qquad (2)$$

and, reversing the order of integration to expand a lower power,

$$M_x = \iint_R kxy^2dA = \tfrac{1}{2}k\int_0^9\int_0^{(-2/3)y+6} kxy^2dx\,dy = \tfrac{1}{2}k\int_0^9 y^2x^2\Big]_{x=0}^{(-2/3)y+6}dy$$

$$= \tfrac{1}{2}k\int_0^9(\tfrac{4}{9}y^4 - 8y^3 + 36y^2)dy = \tfrac{1}{2}k\Big[\tfrac{4}{45}y^5 - 2y^4 + 12y^3\Big]_0^9 = \tfrac{2187}{5}k \qquad (3)$$

Substituting from (1), (2) and (3), we have

$$\bar{x} = \frac{1}{M}\cdot M_y = \frac{2}{243k}\cdot\frac{1458k}{5} = \frac{12}{5} \text{ and } \bar{y} = \frac{1}{M}\cdot M_x = \frac{2}{243k}\cdot\frac{2187k}{5} = \frac{18}{5}$$

Therefore, the center of mass is $(\frac{12}{5},\frac{18}{5})$.

9. R is bounded by $y = \sin x$ and $[0,\pi]$ on the x axis. $\rho(x,y) = ky$.

$$M = \iint_R ky\,dA = k\int_0^\pi\int_0^{\sin x} y\,dy\,dx = k\int_0^\pi \frac{1}{2}y^2\Big]_0^{\sin x}dx = \tfrac{1}{2}k\int_0^\pi \sin^2x\,dx = \tfrac{1}{2}k\Big[-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\Big]_0^\pi = \tfrac{1}{4}k\pi$$

$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i(kv_i)\Delta_iA = \iint_R ky^2dA = k\int_0^\pi\int_0^{\sin x} y^2dy\,dx = k\int_0^\pi \frac{1}{3}y^3\Big]_0^{\sin x}dx = \tfrac{1}{3}k\int_0^\pi \sin^3x\,dx$$

$$= \frac{1}{3}k\int_0^\pi (1-\cos^2x)\sin x\,dx = \frac{1}{3}k\Big[-\cos + \frac{1}{3}\cos^3x\Big]_0^\pi = \frac{1}{3}k\Big[\Big(1-\frac{1}{3}\Big) - \Big(-1+\frac{1}{3}\Big)\Big] = \tfrac{4}{9}k.\ \bar{y} = \frac{1}{M}M_x = \frac{4}{k\pi}\cdot\frac{4k}{9} = \frac{16}{9\pi}.$$

$\bar{x} = \frac{1}{2}\pi$ because $y = \sin x$ is symmetric with respect to $x = \frac{1}{2}\pi$ and ρ is independent of x.

10. R is bounded by $y = \sqrt{x}$ and $y = x$. $\rho(x,y) = kx$.

$$M = \int_0^1\int_{y^2}^y kx\,dx\,dy = \int_0^1 \tfrac{1}{2}kx^2\Big]_{y^2}^y dy = \int_0^1 \tfrac{1}{2}k(y^4 - y^2)dy = \tfrac{1}{2}k\Big[\tfrac{1}{5}y^5 - \tfrac{1}{3}y^3\Big]_0^1 = \tfrac{1}{15}k$$

$$M_y = \int_0^1\int_{y^2}^y kx^2dx\,dy = \int_0^1 \tfrac{1}{3}kkx^3\Big]_{y^2}^y dy = \int_0^1 \tfrac{1}{3}k(y^6 - y^3)dy = \tfrac{1}{3}k\Big[\tfrac{1}{7}y^7 - \tfrac{1}{4}y^4\Big]_0^1 = \tfrac{1}{28}k.\ \bar{x} = \tfrac{15}{28}$$

$$M_x = \int_0^1\int_{y^2}^y kxy\,dx\,dy = \int_0^1 \tfrac{1}{2}k(y^5 - y^3)dy = \tfrac{1}{2}k\Big[\tfrac{1}{6}y^6 - \tfrac{1}{4}y^4\Big]_0^1 = \tfrac{1}{24}k.\ \bar{y} = \tfrac{15}{24} = \tfrac{5}{8}$$

11. R is in the first quadrant bounded by $x^2 + y^2 = 4$ and $x + y = 2$. $\rho(x,y) = xy$.

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n u_iv_i\Delta_iA = \iint_R xy\,dA = \int_0^2\int_{2-x}^{\sqrt{4-x^2}} xy\,dy\,dx = \int_0^2 \frac{1}{2}xy^2\Big]_{2-x}^{\sqrt{4-x^2}}$$

$$= \frac{1}{2}\int_0^2[x(4-x^2) - x(2-x)^2]dx = \frac{1}{2}\int_0^2(4x^2 - 2x^3)dx = \frac{1}{2}\Big[\frac{4}{3}x^3 - \frac{1}{2}x^4\Big]_0^2 = \frac{1}{2}\Big(\frac{32}{3} - 8\Big) = \frac{4}{3}$$

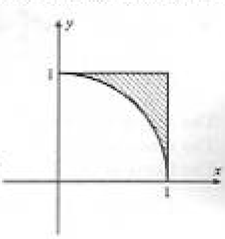
$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i(u_iv_i)\Delta_iA = \iint_R xy^2dA = \int_0^2\int_{2-x}^{\sqrt{4-x^2}} xy^2dy\,dx = \int_0^2 \frac{1}{3}xy^3\Big]_{2-x}^{\sqrt{4-x^2}}dx$$

$$= \frac{1}{3}\int_0^2[x(4-x^2)^{3/2} - x(2-x)^3]dx = \frac{1}{3}\Big[-\frac{1}{5}(4-x^2)^{5/2} + \frac{1}{4}x(2-x)^4 + \frac{1}{20}(2-x)^5\Big]_0^2 = \frac{1}{3}\Big[\frac{32}{5} + 0 + \frac{32}{20}\Big] = \frac{8}{5}$$

$\bar{y} = \frac{1}{M}M_x = \frac{3}{4}\cdot\frac{8}{5} = \frac{6}{5}$. $\bar{x} = \bar{y}$ because of symmetry with respect to the line $y = x$.

12. The region bounded by the circle $x^2+y^2=1$ and lines $x=1$, $y=1$. The area density at (x,y) is xy kg/m^2.

▹ The figure shows the region R. The number of kilograms in the mass is given by



$$\mathrm{M}=\iint_R xy\,d\mathrm{A}=\int_0^1\int_{\sqrt{1-x^2}}^1 xy\,dy\,dx=\tfrac{1}{2}\int_0^1 xy^2\Big]_{y=\sqrt{1-x^2}}^1 dx$$
$$=\tfrac{1}{2}\int_0^1 x[1-(1-x^2)]dx=\tfrac{1}{2}\int_0^1 x^3dx=\tfrac{1}{8}$$

Thus, the mass is $\frac{1}{8}$ kilogram. If $(\bar{x},\bar{y})$ is the center of mass, then $\bar{y}=\bar{x}$ because both the region and the density function are symmetric in x and y. Thus,

$$\mathrm{M}_y=\iint_R x\cdot xy\,d\mathrm{A}=\int_0^1\int_{\sqrt{1-x^2}}^1 x^2y\,dy\,dx=\tfrac{1}{2}\int_0^1 x^2y^2\Big]_{y=\sqrt{1-x^2}}^1 dx$$
$$=\tfrac{1}{2}\int_0^1 x^4dx=\tfrac{1}{10}$$

and

$$\bar{x}=\frac{1}{\mathrm{M}}\cdot\mathrm{M}_y=8\cdot\tfrac{1}{10}=\tfrac{4}{5}$$

Therefore, the center of mass is $(\frac{4}{5},\frac{4}{5})$.

In Exercises 13–18, find the moment of inertia of the homogeneous lamina of the given shape about the indicated axis if the area density is k kilograms per square meter and distance is measured in meters.

▹ In Exercises 13–22, I_x kg-m^2 and I_y kg-m^2 are the moments of inertia of the lamina R about the x and y axes, r m is the radius of gyration about the x axis, and I_0 kg-m^2 is the polar moment of inertia.

13. R is bounded by $4y=3x$, $x=4$, and the x axis; about the x axis.

$$\mathrm{I}_x=\lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i^{\,2}\Delta_i\mathrm{A}=\iint_R ky^2d\mathrm{A}=k\int_0^4\int_0^{3x/4}y^2dy\,dx=k\int_0^4\Big[\tfrac{1}{3}y^3\Big]_0^{3x/4}dx=\frac{9k}{64}\int_0^4 x^3dx=\frac{9k}{64}\Big[\frac{1}{4}x^4\Big]_0^4=9k$$

14. R is bounded by $4y=3x$, $x=4$, and the x axis; about the line $x=4$.

$$\mathrm{I}_{x=4}=k\int_0^4\int_0^{3x/4}(4-x)^2dy\,dx=\tfrac{3}{4}k\int_0^4 x(4-x)^2dx=\tfrac{3}{4}k\int_0^4(16x-8x^2+x^3)dx=\tfrac{3}{4}k\Big[8x^2-\tfrac{8}{3}x^3+\tfrac{1}{4}x^4\Big]_0^4=16k$$

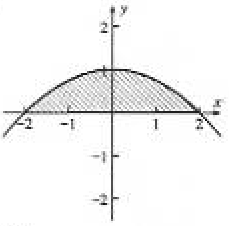
15. R is bounded by $x^2+y^2=a^2$. $\mathrm{I}_0=\iint_R k(x^2+y^2)d\mathrm{A}=\iint_R kx^2d\mathrm{A}+\iint_R ky^2d\mathrm{A}=2\iint_R ky^2d\mathrm{A}$

$$=8k\int_0^a\int_0^{\sqrt{a^2-x^2}}y^2dy\,dx=8k\int_0^a\frac{1}{3}y^3\Big]_0^{\sqrt{a^2-x^2}}dx=\frac{8k}{3}\int_0^a(a^2-x^2)^{3/2}dx.$$ Let $x=a\sin\theta$, $dx=a\cos\theta$.

From Exercise 13.2.43, $\mathrm{I}_0=\dfrac{8ka^4}{3}\displaystyle\int_0^{\pi/2}\cos^4\theta\,d\theta=\frac{8ka^4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{k\pi a^4}{2}$.

16. The region bounded by the parabola $x^2=4-4y$ and the x axis; about the x axis.

▹ The figure shows the region R. The number of kg-m^2 in the moment of inertia about the x axis is given by



$$\mathrm{I}_x=\iint_R ky^2d\mathrm{A}=k\int_{-2}^2\int_0^{1-x^2/4}y^2dy\,dx=\tfrac{1}{3}k\int_{-2}^2 y^3\Big]_0^{1-x^2/4}dx$$
$$=\tfrac{1}{3}k\int_{-2}^2(1-\tfrac{3}{4}x^2+\tfrac{3}{16}x^4-\tfrac{1}{64}x^6)dx=\tfrac{1}{3}k\Big[x-\tfrac{1}{4}x^3+\tfrac{3}{80}x^5-\tfrac{1}{448}x^7\Big]_{-2}^2=\tfrac{64}{105}k$$

Thus, the moment of inertia about the x axis is $\frac{64}{105}k$ kg-m^2.

17. R is bounded by$x^2=4-4y$ and the x axis.

$$\mathrm{I}_0=\lim_{\|\Delta\|\to 0}\sum_{i=1}^n k(u_i^{\,2}+v_i^{\,2})\Delta_i\mathrm{A}=\iint_R k(x^2+y^2)d\mathrm{A}=k\int_{-2}^2\int_0^{(4-x^2)/4}(x^2+y^2)dy\,dx$$
$$=k\int_{-2}^2\Big[x^2y+\frac{1}{3}y^3\Big]_0^{(4-x^2)/4}dx=k\int_{-2}^2\left[\frac{4x^2-x^4}{4}+\frac{(4-x^2)^3}{192}\right]dx=2k\int_0^2\left(x^2-\frac{x^4}{4}+\frac{1}{3}-\frac{x^2}{4}+\frac{x^4}{16}-\frac{x^6}{192}\right)dx$$
$$=2k\int_0^2\left(\frac{1}{3}+\frac{3}{4}x^2-\frac{3}{16}x^4-\frac{1}{192}x^6\right)dx=2k\Big[\frac{1}{3}x+\frac{1}{4}x^3-\frac{3}{5\cdot 16}x^5-\frac{1}{7\cdot 192}x^7\Big]_0^2=2k\left(\frac{2}{3}+2-\frac{6}{5}-\frac{2}{21}\right)=\frac{96}{35}k$$

18. A lamina in the shape of a triangle of sides a m, b m, c m, about side a.

▹ Let h m be the altitude. Place side a on the x axis. A y-section has length $\frac{a}{h}(h-y)$. The area of the triangle is $\frac{1}{2}ah=\sqrt{s(s-a)(s-b)(s-c)}$ where $s=\frac{1}{2}(a+b+c)$. Then

$$\mathrm{I}_a=\int_0^h\int ky^2dx\,dy=\frac{ak}{h}\int_0^h y^2(h-y)dy=\frac{ak}{h}\Big[\tfrac{1}{3}y^3h-\tfrac{1}{4}y^4\Big]_0^h=\frac{kah^3}{12}=\frac{k}{12a^2}\cdot 8(\tfrac{1}{2}ah)^3=\frac{2k}{3a^2}[s(s-a)(s-b)(s-c)]^{3/2}$$

In Exercises 19–22, find for the lamina each of the following: (a) the moment of inertia about the x axis; (b) the moment of inertia about the y axis; (c) the radius of gyration about the x axis; (d) the polar moment of inertia.

19. R is bounded by $x = 3$, $y = 2$, and the coordinate axes. $\rho(x,y) = xy^2$.

(a) $I_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i^2(u_i v_i^2)\Delta_i A = \iint_R xy^4 dA = \int_0^3 \int_0^2 xy^4 dy\, dx = \int_0^3 \frac{1}{5}xy^5\Big]_0^2 dx = \frac{32}{5}\int_0^3 x\, dx = \frac{16}{5}x^2\Big]_0^3 = \frac{144}{5}$

(b) $I_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i^2(u_i v_i^2)\Delta_i A = \iint_R x^3y^2 dy\, dx = \int_0^3 \int_0^2 x^3y^2 dy\, dx = \int_0^3 \frac{1}{3}x^3y^3\Big]_0^2 = \frac{8}{3}\int_0^3 x^3\, dx = \frac{2}{3}x^4\Big]_0^3 = 54$

(c) From Exercise 1, M = 12. Thus $r = \sqrt{\frac{1}{m} \cdot I_x} = \sqrt{\frac{1}{12} \cdot \frac{144}{5}} = \sqrt{\frac{12}{5}} = \frac{2}{5}\sqrt{15}$.

(d) $I_0 = I_x + I_y = \frac{144}{5} + 54 = \frac{414}{5}$

20. The lamina of Exercise 4.

▹ The figure for Exercise 4 shows the region R. In Exercise 4 we are given that $\rho(x,y) = x + y$.

(a) $I_x = \iint_R (x+y)y^2 dA = \int_0^1 \int_0^{\sqrt{y}} (x+y)y^2 dx\, dy = \int_0^1 y^2\Big[\frac{1}{2}x^2 + xy\Big]_{x=0}^{\sqrt{y}} dy$

$= \int_0^1 y^2[\frac{1}{2}y + y^{3/2}]dy = \int_0^1 (\frac{1}{2}y^3 + y^{7/2})dy = \frac{1}{8}y^4 + \frac{2}{9}y^{9/2}\Big]_0^1 = \frac{25}{72}$

Thus, the moment of inertia about the x axis is $\frac{25}{72}$ kg-m^2.

(b) $I_y = \iint_R (x+y)x^2 dA = \int_0^1 \int_{x^2}^1 (x+y)x^2 dy\, dx = \int_0^1 x^2\Big[xy + \frac{1}{2}y^2\Big]_{y=x^2}^1 dx = \int_0^1 x^2[x + \frac{1}{2} - x^3 - \frac{1}{2}x^4]dx$

$= \int_0^1 (\frac{1}{2}x^2 + x^3 - x^5 - \frac{1}{2}x^6)dx = \frac{1}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6 - \frac{1}{14}x^7\Big]_0^1 = \frac{5}{28}$

Thus, the moment of inertia about the y axis is $\frac{5}{28}$kg-m^2.

(c) Substituting from parts (a) and (b), we have

$r^2 = \frac{1}{M}I_x = \frac{20}{13} \cdot \frac{25}{72} = \frac{125}{234}$

$r = \frac{5}{78}\sqrt{130}$

Thus, the radius of gyration about the x axis is $\frac{5}{78}\sqrt{130}$ meters.

(d) $I_0 = I_x + I_y = \frac{25}{72} + \frac{5}{28} = \frac{265}{504}$

Thus, the polar moment of inertia is $\frac{265}{504}$ kg-m^2.

21. R is bounded by $y = \sin x$ and $[0,\pi]$ on the x axis. $\rho(x,y) = ky$.

(a) $I_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i^2(kv_i)\Delta_i A = \iint_R ky^3 dA = k\int_0^\pi \int_0^{\sin x} y^3 dy\, dx = k\int_0^\pi \frac{1}{4}y^4\Big]_0^{\sin x} dx = \frac{1}{2}\int_0^\pi \sin^4 x\, dx$

$= \frac{1}{4}k\Big[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\Big(-\frac{1}{2}\sin x \cos x + \frac{1}{2}x\Big)\Big]_0^\pi = \frac{1}{4}k \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi = \frac{3}{32}\pi k$

(b) $I_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i^2(kv_i)\Delta_i A = \iint_R kx^2y dA = k\int_0^\pi \int_0^{\sin x} x^2y\, dy\, dx = k\int_0^\pi \frac{1}{2}x^2y^2\Big]_0^{\sin x} dx = \frac{1}{2}k\int_0^\pi x^2\sin^2 x\, dx$

$= \frac{1}{4}k\int_0^\pi (x^2 - x^2\cos 2x)dx = \frac{1}{4}k\Big[\frac{1}{3}x^3 - \frac{1}{2}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\Big]_0^\pi = \frac{1}{4}k\Big(\frac{1}{3}\pi^3 - \frac{1}{2}\pi\Big)$

(c) From Exercise 9, $M = \frac{1}{4}\pi k$. Therefore $r = \sqrt{\frac{1}{M} \cdot I_x} = \sqrt{\frac{4}{\pi k} \cdot \frac{3\pi k}{32}} = \frac{1}{4}\sqrt{6}$.

(d) $I_0 = I_x + I_y = \frac{3}{32}\pi k + \frac{1}{12}\pi^3 k - \frac{1}{8}\pi k = (\frac{1}{12}\pi^3 - \frac{1}{32}\pi)k$

22. R is bounded by $y = \sqrt{x}$ and $y = x$. $\rho(x,y) = kx$.

(a) $I_x = \int_0^1 \int_{y^2}^{y} kxy^2 dx\, dy = \frac{1}{2}k\int_0^1 x^2y^2\Big]_{x=y^2}^{y} dy = \frac{1}{2}k\int_0^1 y^2(y^2 - y^4)dy = \frac{1}{2}k\Big[\frac{1}{5}y^5 - \frac{1}{7}y^7\Big]_0^1 = \frac{1}{35}k$

(b) $I_y = \int_0^1 \int_{y^2}^{y} kx^3 dx\, dy = \frac{1}{4}k\int_0^1 x^4\Big]_{y^2}^{y} dy = \frac{1}{4}k\int_0^1 (y^4 - y^8)dy = \frac{1}{4}k\Big[\frac{1}{5}y^5 - \frac{1}{9}y^9\Big]_0^1 = \frac{1}{45}k$

(c) From Exercise 10, $M = \frac{1}{15}k$. Therefore $r = \sqrt{\frac{1}{M} \cdot I_x} = \sqrt{\frac{15}{35}} = \frac{1}{7}\sqrt{21}$

(d) $I_0 = I_x + I_y = \frac{1}{35}k + \frac{1}{45}k = \frac{16}{315}k$

23. R is bounded by $y = 2x - x^2$ and the x axis. $\rho(x,y) = k(4-y)$. If I kg-m^2 is the moment of inertia of the lamina about the line $y = 4$,

$I = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - v_i)^2[k(4-y)]\Delta_i A = k\iint_R (4-y)^3 dA = k\int_0^2 \int_0^{2x-x^2} (4-y)^3 dy\, dx$

$$= k\int_0^2 -\tfrac{1}{4}(4-y)^4\Big]_0^{2x-x^2} dx = -\tfrac{1}{4}k\int_0^2 [(4-2x+x^2)^4-256]dx = -\tfrac{1}{4}k\int_0^2 \{[3+(x-1)^2]^4-256\}dx$$

Let $w = x - 1$, then $dw = dx$. Thus

$$\mathrm{I} = -\tfrac{1}{4}k\int_0^1 [(w^2+3)^4-256]dw = -\tfrac{1}{2}k\int_0^1 (w^8+12w^6+54w^4+108w^2+81-256)dw$$

$$= -\tfrac{1}{2}k\Big[\tfrac{1}{9}w^9+\tfrac{12}{7}w^7+\tfrac{54}{5}w^5+36w^3-175w\Big]_0^1 = -\tfrac{1}{2}k\Big(\tfrac{1}{9}+\tfrac{12}{7}+\tfrac{54}{5}+36-175\Big) = \frac{19{,}904}{315}k$$

24. A homogeneous lamina of area density k slugs/ft^2 is in the shape of the region bounded by the curve $x = \sqrt{y}$, the x axis, and the line $x = a$, where $a > 0$. Find the moment of inertia of the lamina about the line $x = a$.

▹ The figure shows the region R. $r = a - x$ is the distance from any point (x, y) to the axis $x = a$. The measure of the moment of inertia of the lamina about the line $x = a$ is

$$\mathrm{I} = \iint_R k(a-x)^2 d\mathrm{A} = \int_0^a\int_0^{x^2} k(a-x)^2 dy\,dx$$

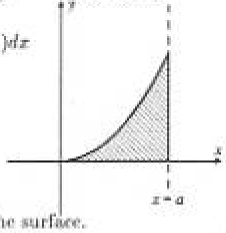
$$= \int_0^a k(a-x)^2x^2dx = k\int_0^a (a^2x^2-2ax^3+x^4)dx$$

$$= k\Big[\tfrac{1}{3}a^2x^3-\tfrac{1}{2}ax^4+\tfrac{1}{5}x^5\Big]_0^a$$

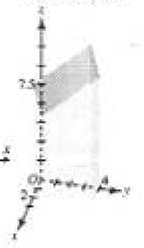
$$= k(\tfrac{1}{3}a^5-\tfrac{1}{2}a^5+\tfrac{1}{5}a^5) = \tfrac{1}{30}ka^5$$

The moment of inertia is $\frac{1}{30}ka^5$ slug-ft^2.

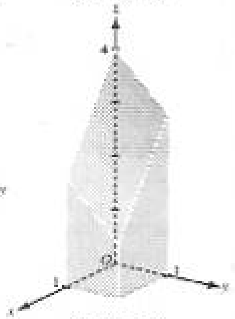
Exercise 24



Exercise 26



Exercise 25



In Exercises 25–42, σ square units is the area of the surface.

25. $z = f(x, y) = 4 - 2x - y$; $f_x(x, y) = -2$; $f_y(x, y) = -1$.

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y)+f_y^{\,2}(x,y)+1}\,dx\,dy$$

$$= \int_0^1\int_0^1 \sqrt{4+1+1}\,dx\,dy = \sqrt{6}\int_0^1 x\Big]_0^1 dy$$

$$= \sqrt{6}\int_0^1 dy = \sqrt{6}$$

26. $z = f(x, y) = 5 + 2x + y$; $f_x(x, y) = 2$; $f_y(x, y) = 1$.

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y)+f_y^{\,2}(x,y)+1}\,dx\,dy = \int_0^4\int_0^2 \sqrt{4+1+1}\,dx\,dy = 8\sqrt{6}$$

27. The region R in the xy plane is bounded by the triangle with sides $x = 0$, $y = 0$, and $y = -\frac{9}{4}x + 9$.

$z = f(x, y) = 16 - 4x - \frac{16}{9}y$; $f_x(x, y) = -4$; $f_y(x, y) = -\frac{16}{9}$.

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y)+f_y^{\,2}(x,y)+1}\,dy\,dx = \int_0^4\int_0^{-9x/4+9} \sqrt{16+\tfrac{256}{81}+1}\,dy\,dx = \tfrac{1}{9}\sqrt{1633}\int_0^4 (-\tfrac{9}{4}x+9)dx$$

$$= \tfrac{1}{9}\sqrt{1633}\Big[-\tfrac{9}{8}x^2+9x\Big]_0^4 = \sqrt{1633}(-2+4) = 2\sqrt{1633}$$

28. Find the area of the surface that is cut from the plane $z = ax + by$ by the planes $x = 0$, $x = a$, $y = 0$, and $y = b$, where $a > 0$ and $b > 0$.

▹ Let $f(x, y) = ax + by$. R is the region in the xy plane bounded by lines $x = 0$, $x = a$, $y = 0$, and $y = b$. Then $f_x(x, y) = a$ and $f_y(x, y) = b$

Therefore the number of square units in the area of the surface is

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y)+f_y^{\,2}(x,y)+1}\,dx\,dy = \int_0^b\int_0^a \sqrt{a^2+b^2+1}\,dx\,dy = \int_0^b a\sqrt{a^2+b^2+1}\,dy$$

$$= ab\sqrt{a^2+b^2+1}$$

The area of the surface is $ab\sqrt{a^2+b^2+1}$ square units.

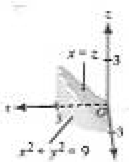
▹ In Exercises 29 and 30, the cylinder is not the graph of a function of x and y.

29. In the first octant $y \geq 0$; $x^2 + y^2 = 9$ gives $y = \sqrt{9-x^2}$; $\dfrac{\partial y}{\partial x} = -\dfrac{x}{\sqrt{9-x^2}}$;

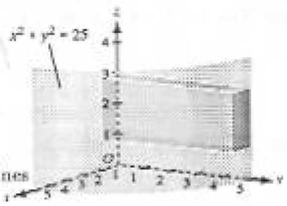
$$\frac{\partial z}{\partial x} = 0.\ \sigma = \iint_R \sqrt{y_x^{\,2}+y_z^{\,2}+1}\,dz\,dx = \int_0^3\int_0^x \sqrt{\frac{x^2}{9-x^2}+0+1}\,dz\,dx$$

$$= \int_0^3\int_0^x 3(9-x^2)^{-1/2}dz\,dx = 3\lim_{b\to 3^-}\int_0^b x(9-x^2)^{-1/2}dx$$

$$= 3\lim_{b\to 3^-}\Big[-(9-x^2)^{1/2}\Big]_0^b = 3\lim_{b\to 3^-}[-(9-b^2)^{1/2}+3] = 9$$



30. In the first octant $y = f(x,z) = \sqrt{25 - x^2}$. $y_x = -\dfrac{x}{\sqrt{25 - x^2}}$; $y_z = 0$



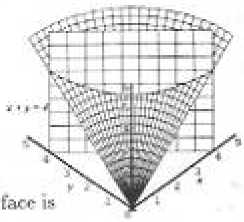
$$\sigma = \iint_R \sqrt{{y_x}^2 + {y_z}^2 + 1}\, dA = \int_0^1 \int_1^3 \sqrt{\frac{x^2}{25 - x^2} + 0 + 1}\, dz\, dx$$
$$= 2\int_0^1 \frac{5dx}{\sqrt{25 - x^2}} = 10 \sin^{-1} \frac{x}{5}\Big]_0^1 = 10 \sin^{-1} \tfrac{1}{5}$$

31. The region R in the xy plane is bounded by the triangle whose sides are the lines $y = 2x$, $x = 0$, and $y = 4$. $z = f(x,y) = 2 + 5x + y^2$; $f_x = 5$; $f_y(x,y) = 2y$.

$$\sigma = \iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\, dx\, dy = \int_0^4 \int_0^{y/2} \sqrt{25 + 4y^2 + 1}\, dx\, dy = \int_0^4 x\sqrt{4y^2 + 26}\Big]_0^{y/2} dy$$
$$= \frac{1}{2}\int_0^4 y(4y^2 + 26)^{1/2} dy = \frac{1}{24}(4y^2 + 26)^{3/2}\Big]_0^4 = \frac{1}{24}(90\sqrt{90} - 26\sqrt{26}) = \frac{1}{12}(135\sqrt{10} - 13\sqrt{26})$$

32. Find the area of the surface in the first octant which is cut from the cone $x^2 + y^2 = z^2$ by the plane $x + y = 4$.

▹ The figure shows the part of the surface that lies over the triangle R in the xy plane bounded by the x axis, the y axis, and the line $x + y = 4$, viewed from behind the plane. We take



$f(x,y) = \sqrt{x^2 + y^2}$

Thus,

$$f_x(x,y) = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } f_y(x,y) = \frac{y}{\sqrt{x^2 + y^2}}$$
$${f_x}^2(x,y) + {f_y}^2(x,y) + 1 = \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1 = 2$$

The area of the triangle is $\frac{1}{2}(4)(4) = 8$ square units. Hence, the measure of the surface is

$$\sigma = \iint_R \sqrt{2}\, dA = 8\sqrt{2}$$

Thus, the area of the surface is $8\sqrt{2}$ square units.

33. The required area is 8 times the part above the first quadrant.

$z = f(x,y) = \sqrt{4 - x^2}$; $f_x(x,y) = \dfrac{-x}{\sqrt{4 - x^2}}$; $f_y(x,y) = 0$. $\sigma = 8\iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\, dA$

$$= 8\iint_R \sqrt{\frac{x^2}{4 - x^2} + 0 + 1}\, dA = 16\int_0^2 \int_0^{\sqrt{4 - x^2}} \frac{1}{\sqrt{4 - x^2}}\, dy\, dx = 16\int_0^2 dx = 32$$

34. $x^2 + y^2 = 2x$ is $(x - 1)^2 + y^2 = 1$. The required area is 4 times the part above the first quadrant.

$f(x,y) = \sqrt{x^2 + y^2}$ so ${f_x}^2 + {f_y}^2 + 1 = 2$. Area of the semicircle is $\frac{1}{2}\pi$. $\sigma = 4\iint_R \sqrt{2}\, dA = 4\sqrt{2} \cdot \frac{1}{2}\pi = 2\sqrt{2}\pi$

35. $z = f(x,y) = \sqrt{x^2 + y^2}$ so ${f_x}^2 + {f_y}^2 + 1 = 2$. $\sigma = 2\iint_R \sqrt{2}\, dA = 2\int_{-1}^2 \int_{y^2}^{2+y} \sqrt{2}\, dx\, dy$

$$= 2\sqrt{2}\int_{-1}^2 (2 + y - y^2) dy = 2\sqrt{2}\Big[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\Big]_{-1}^2 = 2\sqrt{2}\Big[\Big(4 + 2 - \frac{8}{3}\Big) - \Big(-2 + \frac{1}{2} + \frac{1}{3}\Big)\Big] = 9\sqrt{2}$$

36. Find the area of the portion of the plane $x = z$ that lies between the planes $y = 0$ and $y = 6$ and within the hyperboloid $9x^2 - 4y^2 + 16z^2 = 144$.

▹ The projection of the intersection of $x = z$ and $9x^2 - 4y^2 + 16z^2 = 144$ on the xy plane is the hyperbola $25x^2 - 4y^2 = 144$. We take twice the part over the first quadrant where $x = \frac{2}{5}\sqrt{y^2 + 36}$. Let $z = f(x,y) = x$.

Then

$f_x(x,y) = 1 \quad f_y(x,y) = 0$

Therefore,

$$\sigma = \iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\, dA = 2\int_0^6 \int_0^{2\sqrt{y^2 + 36}/5} \sqrt{1^2 + 0 + 1}\, dx\, dy = 2\sqrt{2}\int_0^6 x\Big]_0^{2\sqrt{y^2 + 36}/5} dy$$
$$= \frac{4}{5}\sqrt{2}\int_0^6 \sqrt{y^2 + 36}\, dy = \frac{2}{5}\sqrt{2}\Big[y\sqrt{36 + y^2} + 36 \ln(y + \sqrt{36 + y^2})\Big]_0^6 = \frac{2}{5}\sqrt{2}[36\sqrt{2} + 36 \ln(6 + 6\sqrt{2}) - 36 \ln 6]$$
$$= \frac{72}{5}[2 + \sqrt{2} \ln(1 + \sqrt{2})]$$

Thus, the area of the surface is $\frac{72}{5}[2 + \sqrt{2} \ln(1 + \sqrt{2})]$ square units.

37. The line from (0,0) to (a,b) has equation $y = f(x) = \frac{b}{a}x$; $f'(x) = \frac{b}{a}$. From Theorem 13.3.4,

$$\sigma = 2\pi\int_0^a f(x)\sqrt{[f'(x)]^2+1}\,dx = 2\pi\int_0^a \frac{b}{a}x\sqrt{\frac{b^2}{a^2}+1}\,dx = \frac{2\pi b\sqrt{a^2+b^2}}{a^2}\int_0^a x\,dx = \frac{2\pi b\sqrt{a^2+b^2}}{a^2}\cdot\frac{x^2}{2}\Big]_0^a = \pi b\sqrt{a^2+b^2}$$

Or, by Pappus, segment has length $\sqrt{a^2+b^2}$ and the midpoint is $\frac{1}{2}b$ from the x axis, so $\sigma = 2\pi\cdot\frac{1}{2}b\cdot\sqrt{a^2+b^2}$.

38. We revolve the circle $x^2+y^2=r^2$ about the x axis. $y = f(x) = \sqrt{r^2-x^2}$, $f'(x) = -x/\sqrt{r^2-x^2}$.

$$\sigma = 2\pi\int_a^b f(x)\sqrt{[f'(x)]^2+1}\,dx = 2\pi\int_a^b \sqrt{r^2-x^2}\sqrt{\frac{x^2}{r^2-x^2}+1}\,dx = 2\pi\int_a^b r\,dx = 2\pi r(b-a) = 2\pi rh$$

If $h = 2r$, we find the area of the entire sphere is $4\pi r^2$.

39. Let σ_x and σ_y denote the measure of the surface when a curve is revolved about the x and y axes. Reversing x and y, Theorem 13.3.2 gives $\sigma_x = 2\pi\int_a^b y\text{ ds}$. Also, $\sigma_y = \int_c^d x\text{ ds}$. If $y = a\cosh\frac{x}{a}$,

$$\sigma_y = 2\pi\int_0^a x\sqrt{1+y'^2}\,dx = 2\pi\int_0^a x\sqrt{1+\sinh^2\frac{x}{a}}\,dx = 2\pi\int_0^a x\cosh\frac{x}{a}\,dx = 2\pi\Big[ax\sinh\frac{x}{a} - a^2\cosh\frac{x}{a}\Big]_0^a$$

$$= 2\pi(a^2\sinh 1 - a^2\cos 1 + a^2) = 2\pi a^2(1-e^{-1})$$

40. Find the area of the surface of revolution obtained by revolving the catenary $y = a\cosh(x/a)$ from $x = 0$ to $x = a$ about the x axis.

▹ We take

$f(x) = a\cosh(x/a)$

$f'(x) = \sinh(x/a)$

$\sqrt{[f'(x)]^2+1} = \sqrt{\sinh^2(x/a)+1} = \sqrt{\cosh^2(x/a)} = \cosh(x/a)$

Therefore, the measure of the surface area is

$$\sigma = 2\pi\int_0^a a\cosh(x/a)\cdot\cosh(x/a)\,dx = \pi a\int_0^a [1+\cosh(2x/a)]dx$$

$$= \pi a\Big[x + \tfrac{1}{2}a\sinh(2x/a)\Big]_0^a = \pi a(1+\tfrac{1}{2}a\sinh 2)$$

Hence the surface area is $\pi a(1+\frac{1}{2}a\sinh 2)$ square units.

41. Taking $y \geq 0$, $18y^2 = x(6-x)^2$ gives $y = f(x) = \frac{1}{6}\sqrt{2}(6x^{1/2} - x^{3/2})$; $f'(x) = \frac{1}{6}\sqrt{2}(3x^{-1/2} - \frac{3}{2}x^{1/2})$.

$$\sigma = 2\pi\int_0^6 f(x)\sqrt{[f'(x)]^2+1}\,dx = 2\pi\int_0^6 \tfrac{1}{6}\sqrt{2}(6x^{1/2}-x^{3/2})\sqrt{1+\tfrac{1}{18}(3x^{-1/2}-\tfrac{3}{2}x^{1/2})^2}\,dx$$

$$= \frac{\pi}{3}\sqrt{2}\int_0^6 x^{1/2}(6-x)\sqrt{1+\tfrac{1}{18}(9x^{-1}-9+\tfrac{9}{4}x)}\,dx = \frac{\pi}{3}\sqrt{2}\pi\int_0^6 x^{1/2}(6-x)\sqrt{\tfrac{1}{2}(x^{-1}+1+\tfrac{1}{4}x)}dx$$

$$= \frac{\pi}{3}\sqrt{2}\int_0^6 x^{1/2}(6-x)\tfrac{1}{2}\sqrt{2}\sqrt{(x^{-1/2}+\tfrac{1}{2}x^{1/2})^2}dx = \frac{\pi}{3}\int_0^6 x^{1/2}(x^{-1/2}+\tfrac{1}{2}x^{1/2})dx = \frac{\pi}{3}\int_0^6 (6+2x-\tfrac{1}{2}x^2)dx$$

$$= \frac{\pi}{3}\Big[6x+x^2-\tfrac{1}{6}x^3\Big]_0^6 = \frac{\pi}{3}(36+36-36) = 12\pi$$

42. $y = \ln x$, $x = e^y$. $\sigma = 2\pi\int_0^{\ln 2} x\sqrt{1+(dx/dy)^2}dy = 2\pi\int_0^{\ln 2}\sqrt{1+e^{2y}}e^y dy$. Let $e^y = \tan u$, $e^y dy = \sec^2 u\,du$,

$\tan b = 2$, $\sec b = \sqrt{5}$. $\sigma = 2\pi\int_{\pi/4}^b \sec u(\sec^2 u\,du) = \pi(\sec u\tan u + \ln|\sec u + \tan u|)\Big]_{\pi/4}^b$

$= \pi[2\sqrt{5} - \sqrt{2} + \ln(\sqrt{5}+2) - \ln(\sqrt{2}+1)]$

43. Choose the coordinate axes so that the vertices of the triangle are at $(h, -\frac{1}{2}b)$, $(h, \frac{1}{2}b)$ and (0,0). An equation of the side of the triangle in the first quadrant is $y = bx/2h$.

$$\text{I}_x = k\iint_R y^2 d\text{A} = 2k\int_0^h\int_0^{bx/2h} y^2 dy\,dx = 2k\int_0^h \frac{1}{3}y^3\Big]_0^{bx/2h} dx = \frac{kb^3}{12h^3}\int x^3 dx = \frac{kb^3}{12h^3}\Big[\frac{1}{4}x^4\Big]_0^h = \frac{1}{48}khb^3.$$

$$\text{M} = k\iint_R d\text{A} = 2k\int_0^h\int_0^{bx/2h} dy\,dx = 2k\int_0^h \frac{bx}{2h}dx = 2k\Big[\frac{bx^2}{4h}\Big]_0^h = \frac{1}{2}khb.$$

The x axis is the axis of symmetry and $r = \sqrt{\frac{1}{\text{M}}\cdot\text{I}_x} = \sqrt{\frac{2}{kbh}\cdot\frac{khb^3}{48}} = \sqrt{\frac{6b^2}{144}} = \frac{1}{12}b\sqrt{6}$.

44. Suppose that f and its first partial derivatives are continuous on the closed region R in the xy plane. Show that if σ square units is the area of the portion of of the surface $z = f(x,y)$ that lies over R, then $\sigma = \iint_R \|\nabla g(x,y,z)\| dx\, dy$ where $g(x,y,z) = z - f(x,y)$.

▹ We apply the definition of gradient and norm to the right side of the required equation. Thus,

$$\iint_R \|\nabla g(x,y,z)\| dx\, dy = \iint_R \|\langle f_x(x,y), f_y(x,y), -1\rangle\| dx\, dy = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dx\, dy = \sigma$$

More generally, if a surface is given by $g(x,y,z) = 0$ and its projection on the xy plane is R, then

$$\sigma = \iint_R \sqrt{z_x^2 + z_y^2 + 1} dx\, dy = \iint_R \sqrt{\left(-\frac{g_x}{g_z}\right)^2 + \left(-\frac{g_y}{g_z}\right)^2 + 1} dx\, dy = \iint_R \frac{\sqrt{g_x^2 + g_y^2 + g_z^2}}{|g_z|} dx\, dy$$

$$= \iint_R \frac{\|\nabla g\|}{|g_z|} dx\, dy$$

13.4 THE DOUBLE INTEGRAL IN POLAR COORDINATES

Let f be a function of two variables r and θ. Let R be a region in the polar plane bounded by the rays $\theta = \theta_1$, $\theta = \theta_2$, and by the curves $r = g_1(\theta)$ and $r = g_2(\theta)$ where $0 \le g_1(\theta) \le g_2(\theta)$. Then the double integral of f on R is equivalent to an iterated integral, and

$$\iint_R f(r,\theta) dA = \iint_R f(r,\theta) r\, dr\, d\theta = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\, dr\, d\theta \qquad \text{(I)}$$

Double integrals in polar coordinates are used to calculate areas, mass, volume, moments, and centers of mass, just as were double integrals in Cartesian coordinates. For example, if S is a solid that is bounded below by some region R in the polar plane, bounded above by the surface $z = f(r,\theta)$, where $f(r,\theta) \ge 0$, and bounded on the sides by a cylindrical surface whose rulings are perpendicular to the polar plane, then the measure of the volume of S is given by (I) and the surface area is given by

$$\sigma = \iint_R \sqrt{f_r^2(r,\theta) + r^{-2} f_\theta^2(r,\theta) + 1}\; r\, dr\, d\theta \qquad \text{(II)}$$

Proof Substituting from Exercise 12.5.51 into Theorem 13.3.4, we obtain

$$\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

$$= \iint_R \sqrt{(\cos\theta\, f_r - r^{-1} \sin\theta f_\theta)^2 + (\sin\theta\, f_r + r^{-1} \cos\theta\, f_\theta)^2}\; r\, dr\, d\theta$$

which is equivalent to (II).

Exercises 13.4

In Exercises 1–6, use double integrals to find the area of the region.

▹ A square units is the required area and V cubic units is the required volume. M kg is the mass of the given lamina and M_x kg-m and M_y kg-m are its moments about the x and y axes. I_0 kg-m^2 is the polar moment of inertia, r m is the required radius of gyration, $\rho(r,\theta)$ kg/m^2 is the area density at the point (r,θ), and k is a constant of proportionality.

1. The region is inside the cardioid $r = 2(1 + \sin\theta)$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \bar{r}_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = 2\int_{-\pi/2}^{\pi/2} \int_0^{2(1+\sin\theta)} r\, dr\, d\theta = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big]_0^{2(1+\sin\theta)} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta) d\theta = 4\Big[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2} = 4\cdot\frac{3}{2}\cdot\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 6\pi$$

2. One leaf of the rose $r = a\cos 2\theta$

$$A = 2\int_0^{\pi/4} \int_0^{a\cos 2\theta} r\, dr\, d\theta = \int_0^{\pi/4} a^2 \cos^2 2\theta\, d\theta = \tfrac{1}{2}a^2 \int_0^{\pi/4} (1 + \cos 4\theta) d\theta = \tfrac{1}{2}a^2\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \tfrac{1}{8}a^2\pi$$

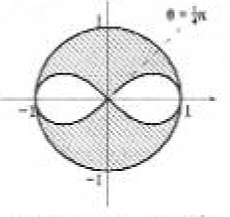
3. The region is inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} r_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big]_a^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (a^2 + 2a^2\cos\theta + a^2\cos^2\theta - a^2) d\theta = \frac{1}{2}a^2 \int_{-\pi/2}^{\pi/2} (2\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{1}{2}a^2\Big[2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2} = \frac{1}{2}a^2\left[\left(2 + 0 + \frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(-2 + 0 - \frac{1}{2}\cdot\frac{\pi}{2}\right)\right] = \frac{1}{2}a^2\left(4 + \frac{1}{2}\pi\right) = \frac{1}{4}a^2(8 + \pi)$$

4. The region inside the circle $r = 1$ and outside the lemniscate $r^2 = \cos 2\theta$.



▹ The figure shows the region. Let R be the first-quadrant part of the region inside the lemniscate. We take the area of the circle minus four times the area of R. Because $\cos 2\theta = 0$ when $\theta = \frac{1}{4}\pi$, the region R is bounded by the rays $\theta = 0$ and $\theta = \frac{1}{4}\pi$. Furthermore, R is bounded by the curves $r = 0$ and $r = \sqrt{\cos 2\theta}$ for $0 \le \theta \le \frac{1}{4}\pi$. Thus, the measure of the area of R is given by

$$A = \iint_R r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4} r^2\Big]_0^{\sqrt{\cos 2\theta}} d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4} \cos 2\theta\,d\theta = \frac{1}{4}\sin 2\theta\Big]_0^{\pi/4} = \frac{1}{4}$$

Hence, $4A = 1$. Because the area of the circle is π square units, then the area of the region outside the lemniscate and inside the circle is $(\pi - 1)$ square units.

5. A_1 and A_2 square units are the areas of the large and small loops of $r = 2 - 4\sin\theta$.

$$A_1 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i\Delta_i r\,\Delta_i\theta = \iint_R r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/6}\int_0^{2-4\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/6}\frac{1}{2}r^2\Big]_0^{2-4\sin\theta}$$
$$= \int_{-\pi/2}^{\pi/6}(2-4\sin\theta)^2 d\theta = 4\int_{-\pi/2}^{\pi/6}(1-4\sin\theta+4\sin^2\theta)d\theta = 4\Big[\theta + 4\cos\theta + 2\theta - \sin 2\theta\ \Big]_{-\pi/2}^{\pi/6}$$
$$= 4\Big[\Big(\frac{\pi}{6} + 2\sqrt{3} + \frac{\pi}{2} - \frac{1}{2}\sqrt{3}\Big) - \Big(-\frac{\pi}{2} + 0 - \pi + 0\Big)\Big] = 4\Big(2\pi + \frac{3}{2}\sqrt{3}\Big) = 8\pi + 6\sqrt{3}$$
$$A_2 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i\Delta_i r\,\Delta_i\theta = \iint_R r\,dr\,d\theta = 2\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta-2} r\,dr\,d\theta = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}r^2\Big]_0^{4\sin\theta-2}$$
$$= \int_{\pi/6}^{\pi/2}(2-4\sin\theta)^2 d\theta = 4\Big[\theta + 4\cos\theta + 2\theta - \sin 2\theta\ \Big]_{\pi/6}^{\pi/2} = 4\Big[\big(\tfrac{1}{2}\pi + 0 + \pi - 0\big) - \big(\tfrac{1}{6}\pi + 2\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{1}{2}\sqrt{3}\big)\Big]$$

$= 4(\pi - \frac{3}{2}\sqrt{3}) = 4\pi - 6\sqrt{3}$. Therefore $A = A_1 - A_2 = (8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.

6. The limaçon $r = 3 - \cos\theta$ meets the circle $r = 5\cos\theta$ when $3 - \cos\theta = 5\cos\theta$, $\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{1}{3}\pi$.

$$A = 2\Big[\int_{\pi/3}^{\pi}\int_0^{3-\cos\theta} r\,dr\,d\theta - \text{segment of circle}\Big] = 2\Big[\frac{1}{2}\int_{\pi/3}^{\pi}(3-\cos\theta)^2 d\theta - \big[\tfrac{1}{6}\pi(\tfrac{5}{2})^2 - \tfrac{1}{4}\sqrt{3}(\tfrac{5}{2})^2\big]\Big]$$
$$= \int_{\pi/3}^{\pi}[9 - 6\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)]d\theta - \tfrac{25}{24}\pi + \tfrac{25}{16}\sqrt{3} = \Big[\tfrac{19}{2}\theta - 6\sin\theta + \tfrac{1}{4}\sin 2\theta\Big]_{\pi/3}^{\pi} - \tfrac{25}{12}\pi + \tfrac{25}{8}\sqrt{3}$$
$$= \tfrac{19}{2}\cdot\tfrac{2}{3}\pi + 6\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{25}{12}\pi + \tfrac{25}{8}\sqrt{3} = \tfrac{17}{4}\pi + 6\sqrt{3}$$

In Exercises 7–12, find the volume of the solid.

7. The solid is bounded by the ellipsoid $z^2 + 9r^2 = 9$.

$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 3(1-\bar{r}_i^{\,2})^{1/2}\bar{r}_i\Delta_i r\,\Delta_i\theta = 3\iint_R r(1-r^2)^{1/2}dr\,d\theta = 24\int_0^{\pi/2}\int_0^1 r(1-r^2)^{1/2}dr\,d\theta$$
$$= 24\int_0^{\pi/2} -\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1 d\theta = 8\int_0^{\pi/2} d\theta = 8\theta\Big]_0^{\pi/2} = 4\pi$$

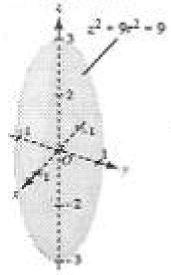
8. The solid cut out of the sphere $z^2 + r^2 = 4$ by the cylinder $r = 1$.

▹ The required volume is twice that of the part above the polar plane. See the figure. R is bounded by the circle $r = 1$. Solving the equation of the sphere for the positive value of z, we obtain $z = f(r,\theta) = \sqrt{4 - r^2}$. If V is the measure of the volume of the entire solid, we have V =

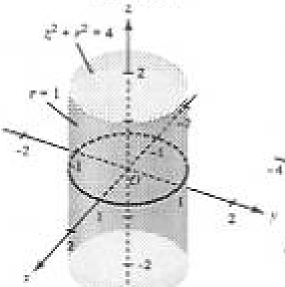
$$2\iint_R f(r,\theta)r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^1\sqrt{4-r^2}\,r\,dr\,d\theta = -\frac{2}{3}\int_0^{2\pi}(4-r^2)^{3/2}\Big]_0^1 d\theta = -\frac{2}{3}\int_0^{2\pi}(3^{3/2}-8)d\theta = \frac{4}{3}\pi(8-3\sqrt{3})$$

Therefore, the volume of the solid is $\frac{4}{3}\pi(8-3\sqrt{3})$ cubic units.

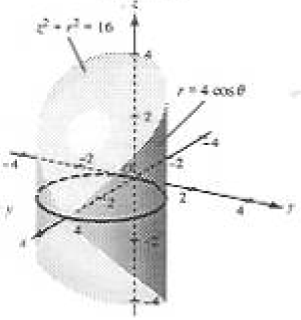
Exercise 7



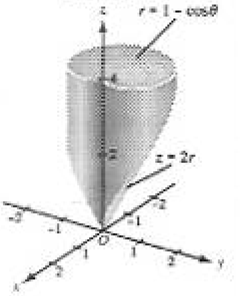
Exercise 8



Exercise 9



Exercise 10



9. The solid is cut out of the sphere $z^2+r^2=16$ by the cylinder $r=4\cos\theta$.

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(16-\bar{r}_i^{\,2})^{1/2}\bar{r}_i\Delta_i r\,\Delta_i\theta=\iint_R r(16-r^2)^{1/2}dr\,d\theta=4\int_0^{\pi/2}\int_0^{4\cos\theta}r(16-r^2)^{1/2}dr\,d\theta$$
$$=4\int\Big[-\tfrac{1}{3}(16-r^2)^{3/2}\Big]_0^{4\cos\theta}d\theta=-\frac{4}{3}\int_0^{\pi/2}[(16-16\cos^2\theta)^{3/2}-64]d\theta=$$
$$=-\frac{256}{3}\int_0^{\pi/2}\sin\theta(1-\cos^2\theta)d\theta+\frac{256}{3}\Big[\theta\Big]_0^{\pi/2}=-\frac{256}{3}\Big[-\cos\theta+\frac{1}{3}\cos^3\theta\Big]_0^{\pi/2}+\frac{128}{2}\pi$$
$$=-\tfrac{256}{3}\Big[(-0+0)-(-1+\tfrac{1}{3})\Big]+\tfrac{128}{3}\pi=\tfrac{128}{3}\pi-\tfrac{512}{9}=\tfrac{128}{9}(3\pi-4)$$

10. $$V=2\int_0^{\pi}\int_0^{1-\cos\theta}2r\cdot r\,dr\,d\theta=\tfrac{4}{3}\int_0^{\pi}(1-\cos\theta)^3d\theta=\tfrac{4}{3}\int_0^{\pi}(1-3\cos\theta+3\cos^2\theta-3\cos^3\theta)d\theta$$
$$=\tfrac{4}{3}\int_0^{\pi}[1+\tfrac{3}{2}(1+\cos 2\theta)]d\theta+\tfrac{4}{3}\int_0^{\pi}(-3\cos\theta-3\cos^3\theta)d\theta=\tfrac{4}{3}\Big[\tfrac{5}{2}\theta+\tfrac{3}{4}\sin 2\theta\Big]_0^{\pi}+0=\tfrac{10}{3}\pi$$

11. The solid is bounded by the paraboloid $z=4-r^2$, the cylinder $r=1$, and the polar plane.

$$V=\iint_R(4-r^2)r\,dr\,d\theta=\int_0^{2\pi}\int_0^{1}(4r-r^3)dr\,d\theta=\int_0^{2\pi}\Big[2r^2-\frac{1}{4}r^4\Big]_0^1d\theta=\frac{7}{4}\int_0^{2\pi}d\theta=\frac{7}{4}\theta\Big]_0^{2\pi}=\tfrac{7}{2}\pi$$

12. The solid above the paraboloid $z=r^2$ and below the plane $z=2r\sin\theta$.

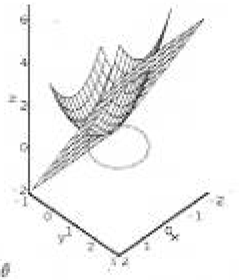
▸ We find the intersection of the paraboloid and the plane. Eliminating z from the two given equations, we have

$$r^2-2r\sin\theta=0$$
$$r(r-2\sin\theta)=0$$

Hence

$r=0$ (the pole) and $r=2\sin\theta$ (which contains the pole)

The given surfaces intersect in a curve whose projection onto the polar plane is the circle $r=2\sin\theta$. See the figure. The volume of the solid is given by



$$V=\iint_R(2r\sin\theta-r^2)dA=\int_0^{\pi}\int_0^{2\sin\theta}(2r\sin\theta-r^2)r\,dr\,d\theta$$
$$=\int_0^{\pi}\Big[\tfrac{2}{3}r^3\sin\theta-\tfrac{1}{4}r^4\Big]_0^{2\sin\theta}d\theta=\tfrac{4}{3}\int_0^{\pi}\sin^4\theta\,d\theta=\tfrac{1}{3}\int_0^{\pi}(1-2\cos 2\theta+\cos^2 2\theta)d\theta$$
$$=\tfrac{1}{3}\int_0^{\pi}(\tfrac{3}{2}-2\cos 2\theta+\tfrac{1}{2}\cos 4\theta)d\theta=\tfrac{1}{3}\Big[\tfrac{3}{2}\theta-\sin 2\theta+\tfrac{1}{8}\sin 4\theta\Big]_0^{\pi}=\tfrac{1}{2}\pi$$

Thus, the volume is $\frac{1}{2}\pi$ cubic units.

In Exercises 13–19, find the mass and center of mass (in Cartesian coordinates) of a lamina having the shape of the given region if the area density is as indicated. Mass is measured in kilograms and distance is measured in meters.

13. The region is inside the cardioid $r=2(1+\sin\theta)$. $\rho(r,\theta)=kr$.

$$M=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta=k\iint_R r^2dr\,d\theta=2k\int_{-\pi/2}^{\pi/2}\int_0^{2(1+\sin\theta)}r^2dr\,d\theta=2k\int_{-\pi/2}^{\pi/2}\frac{1}{3}r^3\Big]_0^{2(1+\sin\theta)}d\theta$$
$$=\tfrac{2}{3}k\int_{-\pi/2}^{\pi/2}8(1+3\sin\theta+3\sin^2\theta+\sin^3\theta)d\theta$$

Because $\sin\theta$ and $\sin^3\theta$ are odd and $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$ is even,

$$M=\frac{16}{3}k\int_0^{\pi/2}[2+3(1-\cos 2\theta)]d\theta=\frac{16}{3}k\Big[5-\frac{3}{2}\sin 2\theta\Big]_0^{\pi/2}=\frac{16}{3}\cdot 5\cdot\frac{\pi}{2}=\frac{40}{3}k\pi$$
$$M_x=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{y}_i(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{r}_i\sin\bar{\theta}_i)(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta=\iint_R kr^3\sin\theta\,dr\,d\theta$$
$$=2k\int_{-\pi/2}^{\pi/2}\int_0^{2(1+\sin\theta)}r^3\sin\theta\,dr\,d\theta=2k\int_{-\pi/2}^{\pi/2}\frac{1}{4}r^4\Big]_0^{2(1+\sin\theta)}\sin\theta\,d\theta=\frac{k}{2}\int_{-\pi/2}^{\pi/2}16(1+\sin\theta)^4\sin\theta\,d\theta$$
$$=8k\int_{-\pi/2}^{\pi/2}(\sin\theta+4\sin^2\theta+6\sin^3\theta+4\sin^4\theta+\sin^5\theta)d\theta=64k\int_0^{\pi/2}(\sin^4\theta+\sin^2\theta)d\theta$$
$$=64k\Big[-\frac{1}{4}\sin^3\theta\cos\theta+\frac{7}{4}\int\sin^2\theta\,d\theta\Big]_0^{\pi/2}=64k\Big[0+\frac{7}{4}\Big(-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta\Big)\Big]_0^{\pi/2}=64k\cdot\frac{7}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=28k\pi$$

Because R is symmetric with respect to the y axis, the center of mass is at $(0,\bar{y})$ and

$$\bar{y}=\frac{1}{M}M_x=\frac{3}{40k\pi}\cdot 28k\pi=\frac{21}{10}$$

14. The region is inside the rose $r = a\cos 2\theta$. $\rho(r,\theta) = kr$. $\mathrm{M} = 2\int_0^{\pi/4}\int_0^{a\cos 2\theta} kr\cdot r\,dr\,d\theta = \frac{2}{3}ka^3\int_0^{\pi/4}\cos^3 2\theta\,d\theta$

$= \frac{2}{3}ka^3\int_0^{\pi/4}(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \frac{1}{3}ka^3\Big[\sin 2\theta - \frac{1}{3}\sin^3 2\theta\Big]_0^{\pi/4} = \frac{1}{3}ka^3\cdot\frac{2}{3} = \frac{2}{9}ka^3$. $\bar{y} = 0$ by symmetry.

$\mathrm{M}_y = 2\int_0^{\pi/4}\int_0^{a\cos\theta} kr(r\cos\theta)r\,dr\,d\theta = \frac{1}{2}ka^4\int_0^{\pi/4}\cos\theta\cos^3 2\theta\,d\theta = \frac{1}{2}ka^4\int_0^{\pi/4}(1-2\sin^2\theta)^3\cos\theta\,d\theta$

$= \frac{1}{2}ka^4\int_0^{\pi/4}(1 - 6\sin^2\theta + 12\sin^4\theta - 8\sin^6\theta)d(\sin\theta) = \frac{1}{2}ka^4\Big[\sin\theta - 2\sin^3\theta + \frac{12}{5}\sin^5\theta - \frac{8}{7}\sin^7\theta\Big]_0^{\pi/4}$

$= \frac{1}{2}ka^4(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} + \frac{3}{5}\sqrt{2} - \frac{1}{7}\sqrt{2}) = \frac{4}{35}ka^4\sqrt{2}$. $\bar{x} = \frac{9}{2}\cdot\frac{4}{35}\sqrt{2}a = \frac{18}{35}\sqrt{2}a$

15. The region is inside the limaçon $r = 2 - \cos\theta$. $\rho(r,\theta) = kr$.

$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(kr_i)r_i\Delta_i r\,\Delta_i\theta = k\iint_R r^2 dr\,d\theta = k\int_0^{2\pi}\int_0^{2-\cos\theta} r^2 dr\,d\theta = k\int_0^{2\pi}\frac{1}{3}r^3\Big]_0^{2-\cos\theta}d\theta$

$= \frac{k}{3}\int_0^{2\pi}(2-\cos\theta)^3 d\theta = \frac{k}{3}\int_0^{2\pi}(8 - 12\cos\theta + 6\cos^2\theta - \cos^3\theta)d\theta = \frac{k}{3}\int_0^{2\pi}(8 + 6\cos^2\theta)d\theta$

$= \frac{k}{3}\int_0^{2\pi}(11 + 3\cos 2\theta)d\theta = \frac{k}{3}\int_0^{2\pi}11\,d\theta = \frac{22}{3}k\pi$

$\mathrm{M}_y = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{x}_i(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{r}_i\cos\bar{\theta}_i)(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \iint_R kr^3\cos\theta\,dr\,d\theta$

$= k\int_0^{2\pi}\int_0^{2-\cos\theta} r^3\cos\theta\,dr\,d\theta = k\int_0^{2\pi}\frac{1}{4}r^4\Big]_0^{2-\cos\theta}\cos\theta\,d\theta = \frac{k}{4}\int_0^{2\pi}(2-\cos\theta)^4\cos\theta\,d\theta$

$= \frac{k}{4}\int_0^{2\pi}(16\cos\theta - 32\cos^2\theta + 24\cos^3\theta - 8\cos^4\theta + \cos^5\theta)d\theta = -2k\int_0^{2\pi}(\cos^4\theta + 4\cos^2\theta)d\theta$

$= -2k\Big[\frac{1}{4}\cos^3\theta\sin\theta + \Big(\frac{3}{4}+4\Big)\int\cos^2\theta\,d\theta\Big]_0^{2\pi} = -2k\cdot\frac{19}{4}\Big[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\Big]_0^{2\pi} = -2k\cdot\frac{19}{4}\cdot\pi = -\frac{19}{2}k\pi$

Because R is symmetric with respect to the polar axis, the centroid is at $(\bar{x},0)$ and

$\bar{x} = \frac{1}{\mathrm{M}}\mathrm{M}_y = \frac{3}{22k\pi}\cdot\frac{-19k\pi}{2} = -\frac{57}{44}$

16. The region bounded by the convex limaçon $r = 2 + \cos\theta$, $0 \le \theta \le \pi$, and the polar axis. The area density at any point is $k\sin\theta$ kg/m^2.

▹ The measure of the mass is given by

$$\mathrm{M} = \iint_R k\sin\theta\,d\mathrm{A} = \int_0^{\pi}\int_0^{2+\cos\theta} k\sin\theta\,r\,dr\,d\theta = \frac{1}{2}k\int_0^{\pi}\sin\theta\,r^2\Big]_0^{2+\cos\theta}d\theta$$

$$= \frac{1}{2}k\int_0^{\pi}(2+\cos\theta)^2\sin\theta\,d\theta = -\frac{1}{6}k(2+\cos\theta)^3\Big]_0^{\pi} = \frac{13}{3}k \tag{1}$$

Thus, the mass is $\frac{13}{3}k$ kilograms.

$$\mathrm{M}_y = \iint_R (k\sin\theta)x\,d\mathrm{A} = \int_0^{\pi}\int_0^{2+\cos\theta}(k\sin\theta)(r\cos\theta)r\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta}\sin\theta\cos\theta\,r^2 dr\,d\theta$$

$$= \frac{1}{3}k\int_0^{\pi}\sin\theta\cos\theta(2+\cos\theta)^3 d\theta = \frac{1}{3}k\int_0^{\pi}[8\cos\theta + 12\cos^2\theta + 6\cos^3\theta + \cos^4\theta](-d\cos\theta)$$

$$= -\frac{1}{3}k\Big[4\cos^2\theta + 4\cos^3\theta + \frac{3}{2}\cos^4\theta + \frac{1}{5}\cos^5\theta\Big]_0^{\pi} = \frac{14}{5}k \tag{2}$$

and

$$\mathrm{M}_x = \iint_R (k\sin\theta)y\,d\mathrm{A} = \int_0^{\pi}\int_0^{2+\cos\theta}(k\sin\theta)(r\sin\theta)r\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^2\sin^2\theta\,dr\,d\theta$$

$$= \frac{1}{3}k\int_0^{\pi}\sin^2\theta(2+\cos\theta)^3 d\theta = \frac{1}{3}k\int_0^{\pi}\sin^2\theta(8 + 12\cos\theta + 6\cos^2\theta + \cos^3\theta)d\theta$$

$$= \frac{1}{3}k\Big[8\int_0^{\pi}\sin^2\theta\,d\theta + 12\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta + 6\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta + \int_0^{\pi}\sin^2\theta\cos^3\theta\,d\theta\Big]$$

$$= \frac{1}{3}k\Big[4\int_0^{\pi}(1-\cos 2\theta)d\theta + 12\cdot 0 + \frac{3}{4}\int_0^{\pi}(1-\cos 4\theta)d\theta + 0\Big]$$

$$= \frac{1}{3}k\Big[4(\theta - \frac{1}{2}\sin 2\theta) + \frac{3}{4}(\theta - \frac{1}{4}\sin\theta)\Big]_0^{\pi} = \frac{19}{12}\pi k \tag{3}$$

The integrals involving odd powers of cosine over $[0,\pi]$ vanish. Substituting from (1), (2) and (3), we have

$\bar{x} = \frac{3}{13}\cdot\frac{14}{5} = \frac{42}{65}$ and $\bar{y} = \frac{3}{13}\cdot\frac{19}{12}\pi = \frac{19}{52}\pi$

Therefore, the center of mass is $(\frac{42}{65}, \frac{19}{52}\pi)$.

17. The region is the top half of the limaçon $r = 2r\cos\theta$. $\rho(r,\theta) = kr\sin\theta$.

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(k\bar r_i\sin\bar\theta_i)\bar r_i\Delta_i r\,\Delta_i\theta = \iint_R kr^2\sin\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= k\int_0^{\pi}\frac13 r^3\Big]_0^{2+\cos\theta}\sin\theta\,d\theta = \frac{k}{3}\int_0^{\pi}(2+\cos\theta)^3\sin\theta\,d\theta = \frac{k}{3}\Big[-\frac14(2+\cos\theta)^4\Big]_0^{\pi} = -\frac{k}{12}(1^4-3^4) = \frac{20}{3}k$$

$$M_y = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar x_i(k\bar r_i\sin\bar\theta_i)\bar r_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(r_i\cos\theta_i)(kr_i\sin\theta_i)r_i\Delta_i r\,\Delta_i\theta$$

$$= k\iint_R r^3\sin\theta\cos\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^3\sin\theta\cos\theta\,dr\,d\theta = k\int_0^{\pi}\frac14 r^4\Big]_0^{2+\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(2+\cos\theta)^4\sin\theta\cos\theta\,d\theta = \frac{k}{4}\int_0^{\pi}(16\cos\theta + 32\cos^2\theta + 24\cos^3\theta + 8\cos^4\theta + \cos^5\theta)\sin\theta\,d\theta$$

$$= \frac{k}{4}\Big[-8\sin^2\theta - \frac{32}{3}\cos^3\theta - 6\cos^4\theta - \frac85\cos^5\theta - \frac16\cos^6\theta\Big]_0^{\pi} = \frac{k}{4}\Big(\frac{32}{3}+\frac85+\frac{32}{3}+\frac85\Big) = \frac{92}{15}k$$

$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar y_i(k\bar r_i\sin\bar\theta_i)\bar r_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(r_i\sin\theta_i)(kr_i\sin\theta_i)r_i\Delta_i r\,\Delta_i\theta$$

$$= k\iint_R r^3\sin^2\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^3\sin^2\theta\,dr\,d\theta = k\int_0^{\pi}\frac14 r^4\Big]_0^{2+\cos\theta}\sin^2\theta\,d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(2+\cos\theta)^4\sin^2\theta\,d\theta = \frac{k}{4}\int_0^{\pi}(16 + 32\cos\theta + 24\cos^2\theta + 8\cos^3\theta + \cos^4\theta)(1-\cos^2\theta)d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(-\cos^6\theta - 8\cos^5\theta - 23\cos^4\theta - 24\cos^3\theta + 8\cos^2\theta + 32\cos\theta + 16)d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(-\cos^6\theta - 23\cos^4\theta + 8\cos^2\theta + 16)d\theta = \frac{k}{4}\Big[-\frac16\cos^5\theta\sin\theta + \int\Big(\Big(-\frac56 - 23\Big)\cos^4\theta + 8\cos^2\theta + 16\Big)d\theta\Big]_0^{\pi}$$

$$= \frac{k}{4}\Big[0 - \frac{143}{6}\cdot\frac14\cos^3\theta\sin\theta + \int\Big(\Big(-\frac{143}{6}\cdot\frac34 + 8\Big)\cos^2\theta + 16\Big)d\theta\Big]_0^{\pi}$$

$$= \frac{k}{4}\Big[0 - \frac{79}{8}\cdot\frac12\cos\theta\sin\theta + \Big(-\frac12\cdot\frac{79}{8} + 16\Big)\theta\Big]_0^{\pi} = \frac{k}{4}\Big(0 + \frac{177}{16}\pi\Big) = \frac{177}{64}k\pi$$

The center of mass is at $(\bar x,\bar y)$ and $x = \frac{1}{M}M_y = \frac{3}{20k}\cdot\frac{92k}{15} = \frac{23}{25}$, $\bar y = \frac{1}{M}\cdot M_x = \frac{3}{20k}\cdot\frac{177k\pi}{64} = \frac{531}{1280}\pi$.

18. $M = 2k\Big[\int_{\pi/3}^{\pi/2}\int_{5\cos\theta}^{3-\cos\theta} r\cdot r\,dr\,d\theta + \int_{\pi/2}^{\pi} r\cdot r\,dr\,d\theta\Big] = \frac23 k\Big[\int_{\pi/3}^{\pi/2}[(3-\cos\theta)^3 - (5\cos\theta)^3]d\theta + \int_{\pi/2}^{\pi}(3-\cos\theta)^3 d\theta\Big]$

$= k(14\pi - \frac{500}{9} + \frac{159}{4}\sqrt3)$. $\bar y = 0$ by symmetry. $M_y = 2k\Big[\int_{\pi/3}^{\pi/2}\int_{5\cos\theta}^{3-\cos\theta} r(r\cos\theta)r\,dr\,d\theta + \int_{\pi/2}^{\pi} r(r\cos\theta)r\,dr\,d\theta\Big]$

$= \frac12 k\Big[\int_{\pi/3}^{\pi/2}[(3-\cos\theta)^4 - (5\cos\theta)^4]d\theta + \int_{\pi/2}^{\pi}(3-\cos\theta)^4 d\theta\Big] = k(\frac{11613}{160}\sqrt3 - \frac{500}{3} - \frac{39}{2}\pi)$. $\bar x = M_y/M$

19. The region is the small loop of the limaçon $r = 2 - 4\sin\theta$. $\rho(r,\theta) = kr$.

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(kr_i)r_i\Delta_i r\,\Delta_i\theta = k\iint_R r^2\,dr\,d\theta = k\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta-2} r^2\,dr\,d\theta = 2k\int_{\pi/4}^{\pi/2}\frac13 r^3\Big]_0^{4\sin\theta-2}d\theta$$

$$= \frac23 k\int_{\pi/6}^{\pi/2}(4\sin\theta-2)^3 d\theta = \frac{16}{3}\int_{\pi/6}^{\pi/2}(2\sin\theta-1)^3 d\theta = \frac{16}{3}k\int_{\pi/6}^{\pi/2}(8\sin^3\theta - 12\sin^2\theta + 6\sin\theta - 1)d\theta$$

$$= \frac{16}{3}k\Big\{-\frac13\cdot 8\sin^2\theta\cos\theta\Big]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2}\Big[-12\sin^2\theta + \Big(\frac23\cdot 8 + 6\Big)\sin\theta - 1\Big]d\theta\Big\}$$

$$= \frac{16}{3}k\Big\{\frac83\cdot\frac14\cdot\frac12\sqrt3 + 6\sin\theta\cos\theta\Big]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2}\Big[\frac{34}{3}\sin\theta - (6+1)\Big]d\theta\Big\}$$

$$= \frac{16}{3}k\Big\{\frac13\sqrt3 - 6\cdot\frac12\cdot\frac12\sqrt3 + \Big[-\frac{34}{3}\cos\theta - 7\theta\Big]_{\pi/6}^{\pi/2} = \frac{16}{3}k\Big(\frac13\sqrt3 - \frac32\sqrt3 + \frac{34}{3}\cdot\frac12\sqrt3 - 7\cdot\frac13\pi\Big) = \frac89(27\sqrt3 - 14\pi)k$$

$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar y_i(k\bar r_i)\bar r_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar r_i\sin\bar\theta_i)(k\bar r_i)\bar r_i\Delta_i r\,\Delta_i\theta = \iint_R kr^3\sin\theta\,dr\,d\theta$$

$$= 2k\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta-2} r^3\sin\theta\,dr\,d\theta = 2k\int_{\pi/6}^{\pi/2}\frac14 r^4\Big]_0^{4\sin\theta-2}\sin\theta\,d\theta = \frac{k}{2}\int_{\pi/6}^{\pi/2}(2-4\sin\theta)^4\sin\theta\,d\theta$$

$= 8k\int_{\pi/6}^{\pi/2}(2\sin\theta-1)^4\sin\theta\,d\theta = 8k1$. Using recursion for powers of sine gives

$$\int(2\sin\theta-1)^4\sin\theta\,d\theta = \int(16\sin^5\theta - 32\sin^4\theta + 24\sin^3\theta - 8\sin^2\theta + \sin\theta)d\theta$$

$$= -16\cdot\frac15\sin^4\theta\cos\theta + 32\cdot\frac14\sin^3\theta\cos\theta + \int\Big[\Big(16\cdot\frac45 + 24\Big)\sin^3\theta - \Big(32\cdot\frac34 + 8\Big)\sin^2\theta + \sin\theta\Big]d\theta$$

$$= -\tfrac{16}{5}\sin^4\theta\cos\theta + 8\sin^3\theta\cos\theta - \tfrac{184}{5}\cdot\tfrac{1}{3}\sin^2\theta\cos\theta + 32\cdot\frac{1}{3}\sin\theta\cos\theta + \int\left[\left(\tfrac{184}{5}\cdot\tfrac{2}{3}+1\right)\sin\theta - 32\cdot\tfrac{1}{2}\right]d\theta$$

$$= -\tfrac{16}{5}\sin^4\theta\cos\theta + 8\sin^3\theta\cos\theta - \tfrac{184}{15}\sin^2\theta\cos\theta + 16\sin\theta\cos\theta - \tfrac{383}{15}\cos\theta - 16\theta$$

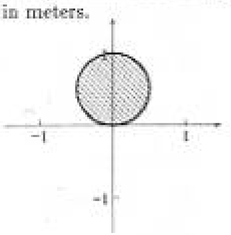
$$8kI = 8k\left[\tfrac{16}{5}(\tfrac{1}{2})^4\tfrac{1}{2}\sqrt{3} - 8(\tfrac{1}{2})^3\tfrac{1}{2}\sqrt{3} + \tfrac{184}{15}(\tfrac{1}{2})^2\tfrac{1}{2}\sqrt{3} - 16(\tfrac{1}{2})\sqrt{3} + \tfrac{383}{15}\cdot\tfrac{1}{2}\sqrt{3} - 16\cdot\tfrac{1}{3}\pi\right] = \tfrac{4}{15}(297\sqrt{3} - 160\pi)$$

Because R is symmetric with respect to the y axis the center of mass is at $(0,\bar{y})$ and

$$\bar{y} = \tfrac{1}{M}M_x = \frac{3(297\sqrt{3} - 160\pi)}{10(27\sqrt{3} - 14\pi)} \approx 1.268$$

In Exercises 20–24, find the moment of inertia of a lamina in the shape of the given region and area density k kg/m^2 about the indicated axis or point. Mass is measured in kilograms and distance in meters.

20. The region enclosed by the circle $r = \sin\theta$, about the $\frac{1}{2}\pi$ axis.



▹ The figure shows the region. The distance from any point (r,θ) to the $\frac{1}{2}\pi$ axis is $x = r\cos\theta$ meters. Thus, the measure of the moment of inertia about the $\frac{1}{2}\pi$ axis is given by

$$I = \iint_R k(r\cos\theta)^2 dA = k\int_0^\pi\int_0^{\sin\theta} r^3\cos^2\theta\,dr\,d\theta = \tfrac{1}{4}k\int_0^\pi \sin^4\theta\cos^2\theta\,d\theta$$

$$= \tfrac{1}{4}k\int_0^\pi \sin^2\theta(\sin\theta\cos\theta)^2 d\theta = \tfrac{1}{32}k\int_0^\pi (1-\cos 2\theta)\sin^2 2\theta\,d\theta$$

$$= \tfrac{1}{32}k\int_0^\pi (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta - \sin^2 2\theta\cos 2\theta)d\theta = \tfrac{1}{32}k\left[\tfrac{1}{2}\theta\right]_0^\pi - 0 - 0 = \tfrac{1}{64}k\pi$$

- The moment of inertia is $\frac{1}{64}\pi k$ kg·m^2.

21. $I_x = \lim_{\|\Delta\|\to 0}\sum \bar{y}_i^2(k)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n(\bar{r}_i\sin\bar{\theta}_i)(k)\bar{r}_i\Delta_i r\Delta_i\theta = k\iint_R r^3\sin^2\theta\,dr\,d\theta$

$$= k\int_0^\pi\int_0^{\sin\theta} r^3\sin^2\theta\,dr\,d\theta = k\int_0^\pi \frac{1}{4}r^4\Big]_0^{\sin\theta}\sin^2\theta = \frac{k}{4}\int_0^\pi \sin^6\theta\,d\theta = \frac{k}{4}\left[-\frac{1}{6}\sin^5\theta\cos\theta + \frac{5}{6}\int\sin^4\theta\,d\theta\right]_0^\pi$$

$$= \frac{k}{4}\left[0 - \frac{5}{6}\cdot\frac{1}{4}\sin^4\theta\cos\theta + \frac{5}{6}\cdot\frac{3}{4}\int\sin^2\theta\,d\theta\right]_0^\pi = \frac{k}{4}\left[0 - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta + \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\theta\right]_0^\pi = \frac{k}{4}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi = \frac{5}{64}k\pi$$

22. $I_0 = 2k\int_0^\pi\int_0^{a(1-\cos\theta)} r^2\,r\,dr\,d\theta = \frac{1}{2}ka^4\int_0^\pi(1-\cos\theta)^4 d\theta$. Let $u = \frac{1}{2}\theta$. $I_0 = 16ka^4\int_0^{\pi/2}\sin^8 u\,du$

$$= 16ka^4\left[-\tfrac{1}{8}\sin^7 u\cos u + \tfrac{7}{8}\int_0^{\pi/2}\sin^6 u\,du\right] = 16ka^4\left[0 - \tfrac{7}{8}\cdot\tfrac{1}{6}\sin^5 u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\int_0^{\pi/2}\sin^4 u\,du\right]$$

$$= 16ka^4\left[0 - \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{1}{4}\sin^3 u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\int_0^{\pi/2}\sin^2 u\,du\right] = 16ka^4\left[0 - \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\sin u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\int_0^{\pi/2}du\right]$$

$$= 16ka^4\cdot\tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi = \tfrac{35}{16}ka^4\pi$$

23. The region is bounded by a cardioid $r = a(1+\cos\theta)$ and a circle $r = 2a\cos\theta$. $\rho(r,\theta) = k$.

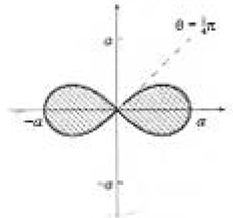
$$I_0 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n \bar{r}_i^2(k)\bar{r}_i\Delta_i r\Delta_i\theta = k\iint_R r^3 dr\,d\theta = 2k\int_0^\pi\int_0^{a(1+\cos\theta)} r^3 dr\,d\theta - 2k\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3 dr\,d\theta$$

$$= 2k\int_0^\pi \frac{1}{4}r^4\Big]_0^{a(1+\cos\theta)}d\theta - 2k\int_0^{\pi/2}\frac{1}{4}r^4\Big]_0^{2a\cos\theta}d\theta = \frac{k}{2}a^4\int_0^\pi[(1+\cos\theta)^4 - 8\cos^4\theta]d\theta$$

$$= \frac{k}{2}a^4\int_0^\pi(-7\cos^4\theta + 4\cos^3\theta + 6\cos^2\theta + 4\cos\theta + 1)d\theta = \frac{k}{2}a^4\int_0^\pi(-\cos^4\theta + 6\cos^2\theta + 1)d\theta$$

$$= \frac{k}{2}a^4\left[-\frac{7}{4}\cos^3\theta\sin\theta + \int\left(\left(-\frac{21}{4}+6\right)\cos^2\theta + 1\right)d\theta\right]_0^\pi = \frac{k}{2}a^4\left[0 + \frac{3}{4}\cdot\frac{1}{2}\cos\theta\sin\theta + \left(\frac{3}{4}\cdot\frac{1}{2}+1\right)\theta\right]_0^\pi = \frac{11}{2}k\pi a^4$$

24. The region enclosed by the lemniscate $r^2 = a^2\cos 2\theta$; about the polar axis.



▹ The figure shows the region. By symmetry, the moment is 4 times the moment of the part in the first quadrant. The distance from a point (r,θ) in the first quadrant to the polar axis is $y = r\sin\theta$. Thus the moment of inertia is

$$I = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} k(r\sin\theta)^2 r\,dr\,d\theta = 4k\int_0^{\pi/4}\left[\tfrac{1}{4}r^4\sin^2\theta\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= ka^4\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\,d\theta = \tfrac{1}{2}ka^4\int_0^{\pi/4}\cos^2 2\theta(1-\cos 2\theta)d\theta$$

$$= \tfrac{1}{2}ka^4\int_0^{\pi/4}[\tfrac{1}{2}(1+\cos 4\theta) - (1-\sin^2 2\theta)\cos 2\theta]d\theta$$

$$= \tfrac{1}{2}ka^4\left[\tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}ka^4(\tfrac{1}{8}\pi + \tfrac{1}{8}\sin\pi - \tfrac{1}{2}\sin\tfrac{1}{2}\pi + \tfrac{1}{6}\sin^3\tfrac{1}{2}\pi$$

$$= \tfrac{1}{2}ka^4(\tfrac{1}{8}\pi - \tfrac{1}{3}) = \tfrac{1}{48}ka^4(3\pi - 8)$$

- The moment of inertia is $\frac{1}{48}ka^4(3\pi - 8)$ kg·m^2.

25. The region is bounded by one loop of the lemniscate $r^2 = \cos 2\theta$. $\rho(r,\theta) = k$.

$$\mathrm{M} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k\bar{r}_i \Delta_i r\, \Delta_i\theta = k\iint_R r\,dr\,d\theta = 2k\int_0^{\pi}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = 2k\int_0^{\pi} \tfrac{1}{2}r^2\Big]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= k\int_0^{\pi/4} \cos 2\theta\,d\theta = \tfrac{k}{2}\sin 2\theta\Big]_0^{\pi/4} = \tfrac{k}{2}$$

$$\mathrm{I}_0 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \bar{r}_i^{\,2}(k)\bar{r}_i\Delta_i r\,\Delta_i\theta = k\iint_R r^3 dr\,d\theta = 2k\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r^3 dr\,d\theta = 2k\int_0^{\pi/4} \tfrac{1}{4}r^4\Big]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \tfrac{k}{2}\int_0^{\pi/4}\cos^2 2\theta\,d\theta = \tfrac{k}{4}\int_0^{\pi/4}(1 + \cos\theta)d\theta = \tfrac{k}{4}\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \tfrac{k}{4}\cdot\tfrac{\pi}{4} = \tfrac{k\pi}{16}.\ r = \sqrt{\tfrac{1}{\mathrm{M}}\mathrm{I}_0} = \sqrt{\tfrac{2}{k}\cdot\tfrac{k\pi}{16}} = \tfrac{1}{4}\sqrt{2\pi}.$$

26. $\mathrm{M} = \int_0^{2\pi}\int_0^4 r\cdot r\,dr\,d\theta = \frac{64}{3}\int_0^{2\pi} d\theta = \frac{128}{3}\pi$. $\mathrm{I}_0 = \int_0^{2\pi}\int_0^4 r^2(r)r\,dr\,d\theta = \frac{1024}{5}\int_0^{2\pi} d\theta = \frac{2048}{5}\pi$.

$r = \sqrt{\frac{3}{128}\cdot\frac{2048}{5}} = \frac{4}{5}\sqrt{15}$

27. $\iint e^{x^2+y^2}\,d\mathrm{A} = \int_0^{2\pi}\int_1^3 e^{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}e^{r^2}\Big]_1^3 d\theta = \frac{1}{2}\int_0^{2\pi}(e^9 - e)d\theta = (e^9 - e)\pi$

28. Evaluate by polar coordinates the double integral $\iint_R \frac{x}{\sqrt{x^2+y^2}}d\mathrm{A}$
where R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 1$ and the coordinate axes.

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}d\mathrm{A} = \int_0^{\pi/2}\int_0^1 \frac{r\cos\theta}{r} r\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\cos\theta\,d\theta = \tfrac{1}{2}\sin\theta\Big]_0^{\pi/2} = \tfrac{1}{2}$$

Because $\left|x/\sqrt{x^2+y^2}\right| \leq |x/x| = 1$ and the integrand is continuous except at the origin, the integral exists.

29. $x^2 + y^2 + z^2 = 4x$ and $y^2 + z^2 = x^2$ intersect in the circle $x = 2$, $y^2 + z^2 = 4$ which bisects the sphere $(x-2)^2 + y^2 + z^2 = 4$. The sphere has radius 2. Thus from Example 7 with $a = 2$, $\sigma = 2\pi(2)^2 = 8\pi$.

30. $x^2 + y^2 + z^2 = 36, x^2 + y^2 \leq 9. z = \sqrt{36 - x^2 - y^2}. z_x{}^2 + z_y{}^2 + 1 = \dfrac{x^2}{36 - x^2 - y^2} + \dfrac{y^2}{36 - x^2 - y^2} + 1 = \dfrac{36}{36 - x^2 - y^2}$

$$\sigma = 2\iint_R \frac{6}{\sqrt{36 - x^2 - y^2}}d\mathrm{A} = 12\int_0^{2\pi}\int_0^3 \frac{1}{\sqrt{36 - r^2}}r\,dr\,d\theta = 12\int_0^{2\pi} -\sqrt{36 - r^2}\Big]_0^3 d\theta = 12\int_0^{2\pi}(6 - 3\sqrt{3})d\theta$$

$= 72\pi(2 - \sqrt{3})$. Or, $z = f(r,\theta) = \sqrt{36 - r^2}$, $\sqrt{f_r{}^2 + r^{-2}f_\theta{}^2 + 1} = \sqrt{\left(\dfrac{-r}{\sqrt{36 - r^2}}\right)^2 + 0 + 1} = \dfrac{6}{\sqrt{36 - r^2}}$

31. $x^2 + y^2 + z^2 = 4z$ and $x^2 + y^2 = 3z$ intersect in the origin and the circle $z = 1$, $x^2 + y^2 = 3$. Because the larger part of the sphere is within the paraboloid, we find the area σ of the smaller part, which lies below the intersection, and subtract it from the area of of the whole sphere of radius 2 units. From Example 7 with $a = 2$, this area is $2\cdot 2\pi(2)^2 = 16\pi$ square units.

$z = f(x,y) = 2 - \sqrt{4 - x^2 - y^2}$; $f_x(x,y) = \dfrac{-x}{\sqrt{4 - x^2 - y^2}}$; $f_y(x,y) = \dfrac{-y}{\sqrt{4 - x^2 - y^2}}$

$$\sigma = \iint_R \sqrt{f_x{}^2(x,y) + f_y{}^2(x,y) + 1}\,d\mathrm{A} = \iint_R \sqrt{\frac{x^2}{4 - x^4 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1}\,d\mathrm{A} = \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}}\,d\mathrm{A}$$

We use polar coordinates. If $x^2 + y^2 = r^2$ and $d\mathrm{A} = r\,dr\,d\theta$, then

$$\sigma = \int_0^{2\pi}\int_0^{\sqrt{3}} 2(4 - r^2)^{-1/2} r\,dr\,d\theta = \int_0^{2\pi} -2(4 - r^2)^{1/2}\Big]_0^{\sqrt{3}} d\theta = \int_0^{2\pi}(-2 + 4)d\theta = 2\int_0^{2\pi} d\theta = 4\pi$$

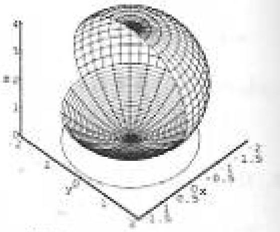
Alternatively, $z = f(r,\theta) = 2 - \sqrt{4 - r^2}$, $\sqrt{f_r{}^2 + r^{-2}f_\theta{}^2 + 1} = \sqrt{\left(\dfrac{r}{\sqrt{4 - r^2}}\right)^2 + 0 + 1} = \dfrac{2}{\sqrt{4 - r^2}}$

Hence the number of square units in the required area is $16\pi - 4\pi = 12\pi$.

32. Find the area of the portion of the surface of the paraboloid $x^2+y^2=3z$ which lies within the sphere $x^2+y^2+z^2=4z$.

▹ We find the intersection of the paraboloid and the sphere. Eliminating x^2 and y^2 from the equations, we get $z^2=z$. Therefore, the surfaces intersect in the planes $z=0$ and $z=1$. The curve of intersection in the surface $z=1$ is the circle $x^2+y^2=3$, $z=1$. Completing the square on z in the equation of the sphere, we have

$x^2+y^2+(z-2)^2=4$



Thus, the sphere has center at $(0,0,2)$ and radius 2. The figure shows the back half of the sphere, and the surface, which lies over R, the circle $x^2+y^2=3$ in the xy plane. We take

$f(x,y)=\frac{1}{3}(x^2+y^2)$

Thus,

$f_x(x,y)=\frac{2}{3}x$ and $f_y(x,y)=\frac{2}{3}y$

We calculate the measure of the area and switch to polar coordinates.

$$\sigma=\iint_R\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)+1}\,dA=\iint_R\sqrt{\tfrac{4}{9}(x^2+y^2)+1}\,dA$$

$$=\int_0^{\sqrt{3}}\int_0^{2\pi}\sqrt{\tfrac{4}{9}r^2+1}\;r\,d\theta\,dr=2\pi\int_0^{\sqrt{3}}\sqrt{\tfrac{4}{9}r^2+1}\;r\,dr=2\pi(\tfrac{9}{8})\tfrac{2}{3}(\tfrac{4}{9}r^2+1)^{3/2}\Big]_0^{\sqrt{3}}=\tfrac{3}{2}\pi(\tfrac{7}{9}\sqrt{21}-1)$$

Therefore, the area of the surface is $\frac{3}{2}\pi(\frac{7}{9}\sqrt{21}-1)\approx 3.85\pi$ square units, while the area of the zone of the sphere from $z=0$ to $z=1$ is $2\pi rh=2\pi(2)1=4\pi$ square units.

33. $az=xy=\frac{1}{4}(x+y)^2-\frac{1}{4}(x-y)^2$ is a hyperbolic paraboloid. $z=a^{-1}xy$. Inside $x^2+y^2=a^2$ in the first octant

$$\sigma=\iint_R\sqrt{z_x{}^2+z_y{}^2+1}\,dA=\iint_R\sqrt{(a^{-1}y)^2+(a^{-1}x)^2+1}\,dA=\iint_R a^{-1}\sqrt{x^2+y^2+a^2}\,dA$$

$$=\int_0^a\int_0^{\pi/2}a^{-1}\sqrt{r^2+a^2}\,r\,dr\,d\theta=\tfrac{1}{2}\pi a^{-1}\int_0^a(r^2+a^2)^{1/2}r\,dr=\tfrac{1}{3}\cdot\tfrac{1}{2}\pi a^{-1}(r^2+a^2)^{3/2}\Big]_0^a$$

$$=\tfrac{1}{6}\pi a^{-1}[(2a^2)^{3/2}-(a^2)^{3/2}]=\tfrac{1}{6}\pi(2\sqrt{2}-1)a^2$$

34. Find the area of the surface cut from the hyperbolic paraboloid $y^2-x^2=6z$ by the cylinder $x^2+y^2=36$.

▹ We take $f(x,y)=\frac{1}{6}(y^2-x^2)$. Then $f_x(x,y)=-\frac{1}{3}x$ and $f_y(x,y)=\frac{1}{3}y$ and

$\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)+1}=\sqrt{\frac{1}{9}(x^2+y^2)+1}$

The surface lies over (and under) the circle R in the xy plane that is the graph of $x^2+y^2=36$. Therefore,

$$\sigma=\iint_R\sqrt{\tfrac{1}{9}(x^2+y^2)+1}\,dA=\int_0^6\int_0^{2\pi}\sqrt{\tfrac{1}{9}r^2+1}\;r\,dr\,d\theta=2\pi\int_0^6\sqrt{\tfrac{1}{9}r^2+1}\;r\,dr$$

$$=2\pi(\tfrac{9}{2})(\tfrac{2}{3})(\tfrac{1}{9}r^2+1)^{3/2}\Big]_0^6=6\pi(5\sqrt{5}-1)$$

13.5 THE TRIPLE INTEGRAL

If f is a continuous function of x, y, and z on some region S in R^3, then the *triple integral of f on S* is given by

$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}f(u_i,v_i,w_i)\Delta_i V=\iiint_S f(x,y,z)dV$$

where S is divided into boxes by planes parallel to the coordinate planes and $\|\Delta\|$ is the length of the longest diagonal of a box.

We use the triple integral to find the total mass of a solid with variable density. If $\rho(x,y,z)$ is the measure of the volume density of a solid that occupies the region S, then the measure of the mass of the solid is given by

$$M=\iiint_S\rho(x,y,z)dV$$

When $\rho(x,y,z)=1$, the triple integral for mass gives the measure of the volume of the solid; earlier method are usually easier. We may use a thrice-iterated integral to calculate the value of a triple integral. If S is bounded by the planes $x=x_1$ and $x=x_2$, bounded by the cylinders $y=g_1(x)$ and $y=g_2(x)$ that are perpendicular to the xy plane, and bounded by the surfaces $z=h_1(x,y)$ and $z=h_2(x,y)$, where $x_1<x_2$, $g_1(x)\le g_2(x)$, and $h_1(x,y)\le h_2(x,y)$ and the functions g_1, g_2, h_1, and h_2 are smooth (i.e. they have continuous derivatives or partial derivatives), then

$$\iiint_S f(x,y,z)dV=\int_{x_1}^{x_2}\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)}f(x,y,z)dz\,dy\,dx$$

The roles of x, y, and z may be interchanged, so that there are six possible orders of integration that may be used to evaluate a triple integral.

If the limits of integration are constants, so that R is a rectangular parallelpiped and f can be expressed as the product $f(x,y,z) = g(x)h(y)k(z)$, then the iterated integral can be expressed as a product

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\,dy\,dx = \int_{x_1}^{x_2} g(x)dx\int_{y_1}^{y_2} h(y)dy\int_{z_1}^{z_2} k(z)dz$$

Simpson's Rule In §7.6 we showed that Simpson's rule was exact for polynomials of degree ≤ 3. The following analog in 3-space can shorten many calculations. Let S be a tetrahedron with vertices A, B, C, D, centroid G and volume V and let $f(x,y,z)$ be a polynomial of degree ≤ 2 then

$$\iiint_S f(x,y,z)dV = \frac{V}{20}[f(A) + f(B) + f(C) + f(D) + 16f(G)].$$

Exercises 13.5

In Exercises 1–8, evaluate the iterated integral.

1. $\int_0^1\int_0^{1-x}\int_{2y}^{1+y^2} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\Big[xz\Big]_0^1 dy\,dx = \int_0^1\int_0^{1-x} x(1+y^2-2y)dy\,dx$

$= \int_0^1\int_0^{1-x} x(y-1)^2dy\,dx = \frac13\int_0^1 x(y-1)^3\Big]_0^{1-x} dx = \frac13\int_0^1 (x-x^4)dx = \frac13\Big[\frac12 x^2 - \frac15 x^5\Big]_0^1 = \frac1{10}$

2. $\int_1^2\int_0^x\int_1^{x+xy} xy\,dz\,dy\,dx = \int_1^2\int_0^x xy(x+xy-1)dy\,dx = \int_1^2\int_0^x(x^2y+x^2y^2-xy)dy\,dx$

$= \int_1^2\Big[\frac12x^2y^2+\frac13x^2y^3-\frac12xy^2\Big]_0^x dx = \int_1^2(\frac12x^4+\frac13x^5-\frac12x^3)dx = \Big[\frac1{10}x^5+\frac1{18}x^6-\frac18x^4\Big]_1^2 = \frac{189}{40}$

3. $\int_0^1\int_0^x\int_0^{x+y}(x+y+z)dz\,dy\,dx = \int_0^1\int_0^x\Big[xz-yz+\frac12 z^2\Big]_0^{x+y} dy\,dx = \int_0^1\int_0^x\left(\frac32 x^2+3xy+\frac32 y^2\right)dy\,dx$

$= \int_0^1\Big[\frac32 x^2y+\frac32 xy^2+\frac12\Big]_0^x dx = \int_0^1\frac72 x^3dx = \Big[\frac78 x^4\Big]_0^1 = \frac78$

4. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{2-y} z\,dx\,dz\,dy$

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} zx\Big]_{x=0}^{2-y}dz\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}(2-y)z\,dz\,dy$$

$$= \int_0^2(2-y)\frac12z^2\Big]_{y=0}^{\sqrt{4-y^2}} = \frac12\int_0^2(2-y)(4-y^2)dy$$

$$= \frac12\int_0^2(8-4y-2y^2+y^3)dy = \frac12\Big[8y-2y^2-\frac23y^3+\frac14y^4\Big]_0^2 = \frac{10}3$$

5. $\int_{-1}^0\int_e^{2e}\int_0^{\pi/3} y\ln z\tan x\,dx\,dz\,dy = \int_{-1}^0\int_e^{2e}\Big[y\ln z\ln|\sec x|\Big]_0^{\pi/3}dz\,dy = \int_{-1}^0\int_e^{2e} y\ln z\ln 2\,dz\,dy$

$= \int_{-1}^0\Big[y(\ln 2)z(\ln z-1)\Big]_e^{2e} dy = \int_{-1}^0 2e(\ln 2)^2y\,dy = \Big[e(\ln 2)^2y^2\Big]_{-1}^0 = -e(\ln 2)^2$

6. $\int_1^2\int_y^{y^2}\int_0^{\ln x} ye^zdz\,dx\,dy = \int_1^2\int_y^{y^2} ye^z\Big]_0^{\ln x}dx\,dy = \int_1^2\int_y^{y^2} y(x-1)dx\,dy = \int_1^2 y\Big[\frac12x^2-x\Big]_y^{y^2}dy$

$= \int_1^2 y(\frac12y^4-\frac12y^2-y^2+y)dy = \frac1{12}y^6-\frac38y^4+\frac13y^3\Big]_1^2 = \frac{47}{24}$

7. $\int_0^{\pi/2}\int_z^{\pi/2}\int_0^{xz}\cos\frac yz\,dy\,dx\,dz = \int_0^{\pi/2}\int_z^{\pi/2}(z\sin x)dx\,dz = \int_0^{\pi/2} z\cos z\,dz = \Big[z\sin z+\cos z\Big]_0^{\pi/2} = \frac12\pi-1$

8. $\int_0^2\int_0^y\int_0^{\sqrt3 z}\frac{z}{x^2+z^2}dx\,dz\,dy$

▸ $$\int_0^2\int_0^y\int_0^{\sqrt3 z}\frac{z}{x^2+z^2}dx\,dz\,dy = \int_0^2\int_0^y\Big[\tan^{-1}\left(\frac xz\right)\Big]_{x=0}^{\sqrt3 z} = \int_0^2\int_0^y(\tan^{-1}\sqrt3-\tan^{-1}0)dz\,dy$$

$$= \int_0^2\int_0^y\frac13\pi\,dz\,dy = \frac13\pi\int_0^2 z\Big]_0^y dy = \frac13\pi\int_0^2 y\,dy = \frac16\pi y^2\Big]_0^2 = \frac23\pi$$

In Exercises 9–18, evaluate the triple integral.

In Exercises 9 and 10, S is bounded by the plane $12x+20y+15z=60$ and the coordinate planes.

9. $\iiint_S y\,dV = \int_0^5\int_0^{4(1-x/5)}\int_0^{3(1-x/5-z/4)} y\,dy\,dz\,dx = \int_0^5\int_0^{4(1-x/5)}\frac12y^2\Big]_0^{3(1-x/5-z/4)}dz\,dx$

$= \frac92\int_0^5\int_0^{4(1-x/5)}\left(1-\frac x5-\frac z4\right)dz\,dx = \frac92\int_0^5 -\frac43\left(1-\frac x5-\frac z4\right)^3\Big]_0^{4(1-x/5)}dx = 6\int_0^5\left(1-\frac x5\right)^3dx = 6\Big[-\frac52\left(1-\frac x2\right)^4\Big]_0^5 = \frac{15}2$

10. $\iiint_S (x^2+z^2)dV = \int_{y=0}^{3}\left[\int_{z=0}^{4(1-y/3)}\int_{x=0}^{5(1-y/3-z/4)} x^2 dx\, dz + \int_{x=0}^{5(1-y/3)}\int_{z=0}^{4(1-x/5-y/3)} z^2 dz\, dx\right]$

$= \int_0^3\left[\int_0^{4(1-y/3)} \frac{1}{3}x^3\Big]_0^{5(1-y/3-z/4)} dz + \int_0^{5(1-y/3)} \frac{1}{3}z^3\Big]_0^{4(1-x/5-y/3)} dx\right]dy$

$= \int_0^3\left[\int_0^{4(1-y/3)} \frac{125}{3}\left(1-\frac{y}{3}-\frac{z}{4}\right)^3 dz + \int_0^{5(1-y/3)} \frac{64}{3}\left(1-\frac{x}{5}-\frac{y}{3}\right)^3 dx\right]dy$

$= \int_0^3\left[-\frac{125}{3}\left(1-\frac{y}{3}\right)^4 - \frac{80}{3}\left(1-\frac{y}{3}\right)^4\right]dy = \frac{205}{3}\cdot\frac{3}{5}\left(1-\frac{y}{3}\right)^5\Big]_0^3 = 41$. Alternatively, because $f(x,y,z)$ is of degree 2, we may use Simpson's rule. The vertices are $(5,0,0), (0,3,0), (0,0,4)$, and $(0,0,0)$. $V = \frac{1}{6}\cdot 5\cdot 3\cdot 4 = 20$. $G = (\frac{5}{4}, \frac{3}{4}, 1)$

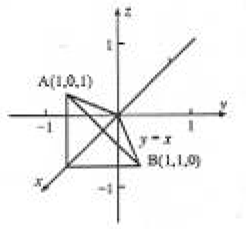
$I = \frac{V}{20}[f(5,0,0)+f(0,3,0)+f(0,0,4)+f(0,0,0)+16f(\frac{5}{4},\frac{3}{4},1)] = \frac{1}{2}(25+0+16+0+16\cdot\frac{41}{16}) = 41$

11. The vertices $(1,1,0)$, $(1,0,0)$, $(1,0,1)$ lie in the plane $z = x - y$; the 4th vertex is O.

$\iiint_S z\, dV = \int_0^1\int_0^x\int_0^{x-y} z\, dz\, dy\, dx = \int_0^1\int_0^x \frac{1}{2}(x-y)^2 dy\, dx = \int_0^1 -\frac{1}{6}(x-y)^3\Big]_0^x dx = \frac{1}{6}\int_0^1 x^3\, dx = \frac{1}{6}\left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{24}$

12. $\iiint_S yz\, dV$ is S is the region bounded by the tetrahedron having vertices $(0,0,0)$, $(1,1,0)$, $(1,0,0)$ and $(1,0,1)$.

▹ The figure shows the region S, which is bounded by the planes $x = 0$ and $x = 1$, the cylinders $y = 0$ and $y = x$, and the surfaces $z = 0$ and the plane of $(0,0,0)$, $(1,0,1)$ and $(1,1,0)$ which has equation $x - y - z = 0$ or, equivalently, $z = x - y$. Therefore, the given triple integral may be expressed as an iterated integral as follows:



$$\iiint_S yz\, dV = \int_0^1\int_0^x\int_0^{x-y} yz\, dz\, dy\, dx = \frac{1}{2}\int_0^1\int_0^x y(x-y)^2 dy\, dx$$
$$= \frac{1}{2}\int_0^1\int_0^x (x^2y - 2xy^2 + y^3)dy\, dx = \frac{1}{2}\int_0^1\left[\frac{1}{2}x^2y^2 - \frac{2}{3}xy^3 + \frac{1}{4}y^4\right]_{y=0}^x$$
$$= \frac{1}{2}\int_0^1 (\frac{1}{2}x^2 - \frac{2}{3}x^4 + \frac{1}{4}x^4)dx = \frac{1}{24}\int_0^1 x^4 dx = \frac{1}{120}$$

Also, because $f(x,y,z) = yz$ is of degree 2, we may use Simpson's rule. $G = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4})$ and $V = \frac{1}{6}\cdot 1\cdot 1\cdot 1 = \frac{1}{6}$.

$\iiint_S yz dV = \frac{V}{20}[f(0,0,0)+f(1,1,0)+f(1,0,0)+f(1,0,1)+16f(\frac{3}{4},\frac{1}{4},\frac{1}{4})] = \frac{1}{120}[0+0+0+0+16\cdot\frac{1}{16}] = \frac{1}{120}$

13. S is bounded by the coordinate planes and the planes $x = 2$, $y = 3$, and $z = 4$.

$\iiint_S xy\, dV = \int_0^2\int_0^3\int_0^4 xy\, dz\, dy\, dx = \int_0^2\int_0^3 4xy\, dy\, dx = \int_0^2 [2xy^2]_0^3\, dx = \int_0^2 18x\, dx = 9x^2\Big]_0^2 = 36$

14. S is the tetrahedron bounded by the planes $x + 2y + 3z = 6$, $x = 0$, $y = 0$, $z = 0$.

$\iiint_S x\, dV$. Method 1. Use the method of Exercise 9, integrating first with respect to z. Method 2. $f(x,y,z) = x$ is of degree 1. The vertices are $(6,0,0)$, $(0,3,0)$, $(0,0,2)$, $(0,0,0)$. $V = \frac{1}{6}\cdot 6\cdot 3\cdot 2 = 6$. $G = (\frac{3}{2}, \frac{3}{4}, \frac{1}{2})$.

$\iiint_S x\, dV = \frac{V}{20}[f(6,0,0)+f(0,3,0)+f(0,0,2)+f(0,0,0)+16f(\frac{3}{2},\frac{3}{4},\frac{1}{2})] = \frac{6}{20}(6+0+0+0+16\cdot\frac{3}{2}) = 9$

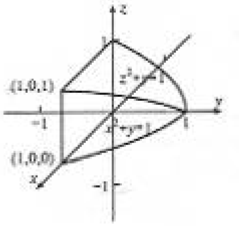
15. S is bounded by $z = x^2 + y^2$, $z = 27 - 2x^2 - 2y^2$ intersecting in the circle R: $x^2 + y^2 = 9$, $z = 9$.

$\iiint_S dV = \iint_R \int_{x^2+y^2}^{27-2x^2-2y^2} dz\, dA = \iint_R (27 - 3x^2 - 3y^2)dA = \int_0^{2\pi}\int_0^3 (27 - 3r^2)r\, dr\, d\theta$

$= \int_0^{2\pi}\left[\frac{27}{2}r^2 - \frac{3}{4}r^4\right]_0^3 d\theta = \int_0^{2\pi}\frac{243}{4}\, d\theta = \frac{243}{2}\pi$

16. $\iiint_S y^2 dV$ if S is the region bounded by the cylinders $x^2 + y = 1$ and $z^2 + y = 1$ and the plane $y = 0$.

▹ The region S and integrand y^2 are symmetric with respect to the xy and yz planes. The figure shows S_1, the part of S in the first octant. We calculate the value of the triple integral over S_1 and multiply by 4. The region S_1 is bounded above by the cylinder $z = \sqrt{1-y}$ and below by the xy plane. Furthermore, the region S_1 is bounded by the cylinder $x = \sqrt{1-y}$ and the xz plane. The value of the given interval is thus



$$\iiint_S y^2 dV = 4\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} y^2 dz\, dx\, dy = 4\int_0^1\int_0^{\sqrt{1-y}} y^2\sqrt{1-y}\,dx\, dy$$
$$= 4\int_0^1 y^2\sqrt{1-y}\sqrt{1-y}\,dy = 4\int_0^1 (y^2 - y^3)dy = 4\left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{3}$$

17. S is bounded by the cylinder$x^2 + z^2 = 9$ and the planes $x + y = 3$, $z = 0$, and $y = 0$.

$$\iiint_S (xz + 3z)dV = \int_{-3}^{3}\int_0^{3-x}\int_0^{\sqrt{9-x^2}} z(x+3)dz\,dy\,dx = \tfrac{1}{2}\int_{-3}^{3}\int_0^{3-x}(9-x^2)(x+3)dy\,dx$$

$$= \frac{1}{2}\int_{-3}^{3}(9-x^2)(9-x^2)dx = \int_0^3 (81 - 18x^2 + x^4)dx = \left[81x - 6x^3 + \frac{1}{5}x^5\right]_0^3 = \frac{648}{5}$$

18. Take 8 times the part in the first octant. $V = 8\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}}\int_{z=0}^{\sqrt{4-x^2}} xyz\,dz\,dy\,dx$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} xyz^2\Big]_{z=0}^{\sqrt{4-x^2}} dy\,dx = 44\int_0^2\int_0^{\sqrt{4-x^2}} x(4-x^2)y\,dy\,dx = 2\int_0^2 x(4-x^2)^2dx = -\tfrac{1}{3}(4-x^2)^3\Big]_0^2 = \tfrac{64}{3}$$

In Exercises 19–32, use triple integration.

▸ V cubic units is the volume of the region S and M kg is its mass.

19. S is bounded by the xy plane, the plane $z = y$, the cylinder $y^2 = x$ and the plane $x = 1$.

$$V = \iiint_S dV = \int_0^1\int_0^{\sqrt{x}}\int_0^y dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}} y\,dy\,dx = \int_0^1 \frac{1}{2}y^2\Big]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}x^2\Big]_0^1 = \frac{1}{4}$$

20. Find the volume of the solid in the first octant bounded by the cylinder $x^2 + z^2 = 16$, the plane $x + y = 2$, and the three coordinate planes.

▸ The figure shows the solid S, which is bounded by the planes $x = 0$ and $x = 2$, the cylinders $y = 0$ and $y = 2 - x$, and the surfaces $z = 0$ and $z = \sqrt{16 - x^2}$. Therefore, the measure of the volume of S is

$$V = \iiint_S dV = \int_0^2\int_0^{2-x}\int_0^{\sqrt{16-x^2}} dz\,dy\,dx = \int_0^2\int_0^{2-x}\sqrt{16-x^2}\,dy\,dx$$

$$= \int_0^2 (2-x)\sqrt{16-x^2}dx = 2\int_0^2\sqrt{16-x^2}dx - \int_0^2 x\sqrt{16-x^2}dx \qquad (1)$$

For the first integral in (1), let $x = 4\sin\theta$. Then

$$\int_0^2\sqrt{16-x^2}dx = \int_0^{\pi/6} 16\cos^2\theta\,d\theta = 8\int_0^{\pi/6}(1+\cos 2\theta)d\theta$$

$$= 8\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/6} = 8(\tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3}) = \tfrac{4}{3}\pi + 2\sqrt{3} \qquad (2)$$

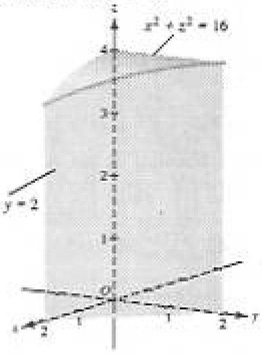
For the second integral in (1) we have

$$\int_0^2 x\sqrt{16-x^2}dx = -\tfrac{1}{3}(16-x^2)^{3/2}\Big]_0^2 = -\tfrac{1}{3}(24\sqrt{3} - 64) = \tfrac{64}{3} - 8\sqrt{3} \qquad (3)$$

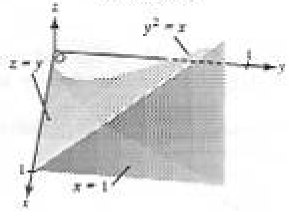
Substituting from (2) and (3) into (1), we have

$$V = 2(\tfrac{4}{3}\pi + 2\sqrt{3}) - (\tfrac{64}{3} - 8\sqrt{3}) = \tfrac{8}{3}\pi + 12\sqrt{3} - \tfrac{64}{3}$$

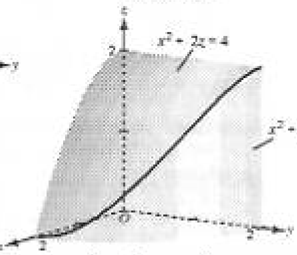
Therefore, the volume is $\frac{4}{3}(2\pi + 9\sqrt{3} - 16)$ cubic units.



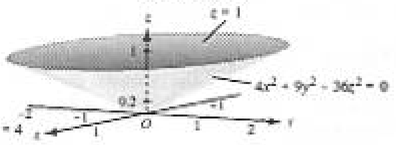
Exercise 19



Exercise 21



Exercise 22



21. S is in the first octant bounded by $x^2 + y^2 = 4$, $x^2 + 2z = 4$ and the coordinate planes.

$$V = \iiint_S dV = \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{(4-x^2)/2} dz\,dy\,dx = \frac{1}{2}\int_0^2\int_0^{\sqrt{4-x^2}}(4-x^2)dy\,dx = \frac{1}{2}\int_0^2 (4-x^2)^{3/2}dx$$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$. Then

$$V = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = 8\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int\cos^2\theta\,d\theta\right]_0^{\pi/2} = 8\left[0 + \frac{3}{4}\cdot\frac{1}{2}\cos\theta\sin\theta + \frac{3}{4}\cdot\frac{1}{2}\theta\right]_0^{\pi/2} = 8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{2}\pi$$

22. The volume of a cone is $\frac{1}{3}$(base area)height. The base is the ellipse $4x^2 + 9y^2 = 36$, $x^2/9 + y^2/4 = 1$ of area $\pi(3)2 = 6\pi$. $V = \frac{1}{3}(6\pi)1 = 2\pi$

23. The projection of the intersection of $z = 3x^2 + y^2$ and $z = 4 - x^2$ is $4x^2 + y^2 = 4$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta_i V = \iiint_S dV = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} \int_{3x^2+y^2}^{4-x^2} dz\, dy\, dx = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} (4 - 4x^2 - y^2)dy\, dx$$

Let $y = 2u$, $dy = 2\, du$. Then use $x^2 + u^2 = r^2$, $du\, dx = dA = r\, dr\, d\theta$.

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} (4 - 4x^2 - 4u^2)du\, dx = 32\int_0^{\pi/2} \int_0^1 (1 - r^2)r\, dr\, d\theta = 32\int_0^{\pi/2} \Big[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\Big]_0^1 d\theta = 8\int_0^{\pi/2} d\theta = 4\pi$$

24. Find the volume of the solid enclosed by the sphere $x^2 + y^2 + z^2 = a^2$.

▹ If V cubic units is the volume of the solid,

$$V = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, dy\right]dx$$

The integral in brackets represents the area of a circle of radius $\sqrt{a^2 - x^2}$ and has the value $\pi(a^2 - x^2)$. Thus,

$$V = \int_{-a}^{a} \pi(a^2 - x^2)dx \tag{1}$$

$$= \pi\Big[a^2x - \tfrac{1}{3}x^3\Big]_{-a}^{a} = \tfrac{4}{3}\pi a^3$$

Therefore, the volume of the solid is $\frac{4}{3}\pi a^3$ cubic units. Note that (1) is equivalent to the method of slicing.

25. S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

$$V = \iiint_S dV = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\, dy\, dz = 8c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

Let $x = au$, $dx = a\, du$, $y = bv$, $dy = b\, dV$. Then let $u^2 + v^2 = r^2$, $du\, dv = dA = r\, dr\, d\theta$.

$$V = 8abc\int_0^1 \int_0^{\sqrt{1-u^2}} \sqrt{1 - u^2 - v^2}\, dV\, du = 8abc\int_0^{\pi/2} \int_0^1 (1 - r^2)^{1/2} r\, dr\, d\theta = 8abc\int_0^{\pi/2} -\tfrac{1}{3}(1 - r^2)^{3/2}\Big]_0^1 d\theta$$

$$= 8abc\int_0^{\pi/2} \frac{1}{3}\, d\theta = \frac{4}{3}\pi abc$$

26. S is bounded by the cylinders $z = 5x^2$ and $z = 3 - x^2$ and the planes $y + z = 4$, $y = 0$. $5x^2 = 3 - x^2$, $x^2 = \frac{1}{2}$

$$V = \iiint_S dV = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{5x^2}^{3-x^2} \int_0^{4-z} dy\, dz\, dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{5x^2}^{3-x^2} (4 - z)dz\, dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (4z - \tfrac{1}{2}z^2)\Big]_{5x^2}^{3-x^2} dx$$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} [12 - 4x^2 - \tfrac{1}{2}(9 - 6x^2 + x^4) - 20x^2 + \tfrac{25}{2}x^4]dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\tfrac{15}{2} - 21x^2 + 12x^4)dx$$

$$= 2\Big[\tfrac{15}{2}x - 7x^3 + \tfrac{12}{5}x^5\Big]_0^{\sqrt{2}/2} = \tfrac{23}{5}\sqrt{2}$$

27. S is bounded by the cylinder $z = 4 - x^2$, $y = 5$ and the coordinate planes; $\rho(x, y, z) = k$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k\, \Delta_i V = k\iiint_S dV = k\int_0^2 \int_0^5 \int_0^{4-x^2} dz\, dy\, dx = k\int_0^2 \int_0^5 z\Big]_0^{4-x^2} dy\, dx = k\int_0^2 \int_0^5 (4 - x^2)dy\, dx$$

$$= k\int_0^2 4y - x^2y\Big]_0^5 dx = k\int_0^2 (20 - 5x^2)dx = k\Big[20x - \frac{5}{3}x^3\Big]_0^2 = k\Big(40 - \frac{40}{3}\Big) = \frac{80}{3}k$$

28. Find the mass of the solid enclosed by the tetrahedron formed by the plane $100x + 25y + 16z = 400$ and the coordinate planes if the volume density varies as the distance from the yz plane. The volume density is measured in kg/m^3.

▹ Let $\rho(x, y, z) = kx$ be the measure of the volume density at (x, y, z). Then the measure of the mass of the solid S is given by

$$M = \iiint_S kx\, dV = \int_0^{25} \int_0^{(400-16z)/25} \int_0^{(400-25y-16z)/100} kx\, dx\, dy\, dz$$

$$= \int_0^{25} \int_0^{(400-16z)/25} \frac{k}{2}x^2\Big]_0^{(400-25y-16z)/100} dy\, dz = \int_0^{25} \int_0^{(400-16z)/25} \frac{k}{20{,}000}(400 - 25y - 16z)^2\, dy\, dz$$

$$= \int_0^{25} \frac{k}{20{,}000} \cdot \frac{1}{3(-25)}(400 - 25y - 16z)^3\Big]_0^{(400-16z)/25} dz = \int_0^{25} \frac{k}{1{,}500{,}000}(400 - 16z)^3\, dz$$

$$= \frac{k}{1{,}500{,}000} \cdot \frac{1}{4(-16)}(400 - 16z)^4\Big]_0^{25} = \frac{(400)^4 k}{1{,}500{,}000(64)} = \frac{800k}{3}$$

Thus, the mass is $\frac{800}{3}k$ kilograms.

Also, because $f(x, y, z) = kx$ is of degree 1, we may use Simpson's rule. The vertices are $(4, 0, 0)$, $(0, 16, 0)$, $(0, 0, 25)$, and $(0, 0, 0)$. The centroid is $G = (1, 4, \frac{25}{4})$ and $V = \frac{1}{6} \cdot 4 \cdot 16 \cdot 25 = \frac{800}{3}$. Therefore,

$$\iiint_S kx dV = \frac{V}{20}[f(4, 0, 0) + f(0, 16, 0) + f(0, 0, 25) + f(0, 0, 0) + 16f(1, 4, \tfrac{25}{4})] = \tfrac{40}{3}[4k + 0 + 0 + 0 + 16k] = \tfrac{800}{3}k$$

29. S is bounded by $x = z^2$, $y = x^2$ and the planes $y = 0$ and $z = 0$. $\rho(x,y,z) = kxyz$ kg/m^3.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k u_i v_i w_i \Delta_i V = k \iiint_S xyz\, dV = k \int_0^1 \int_0^{x^2} \int_0^{\sqrt{x}} xyz\, dz\, dy\, dx = k \int_0^1 \int_0^{x^2} \frac{1}{2} xyz^2 \Big]_0^{\sqrt{x}} dy\, dx$$
$$= \frac{k}{2} \int_0^1 \int_0^{x^2} x^2 y\, dy\, dx = \frac{k}{2} \int_0^1 \frac{1}{2} x^2 y^2 \Big]_0^{x^2} dx = \frac{k}{4} \int_0^1 x^6 dx = \frac{1}{28} kx^7 \Big]_0^1 = \frac{1}{28} k$$

30. Take 4 times the part in the first quadrant. $m = 4 \int_{y=0}^{2} \int_{x=0}^{\sqrt{4-y^2}/2} \int_{z=0}^{4-4x^2-y^2} 3xz\, dz\, dx$

$$= 6 \int_0^2 \int_0^{\sqrt{4-y^2}/2} x(4 - 4x^2 - y^2)^2 dx\, dy = 6(-\tfrac{1}{24}) \int_0^2 (4 - 4x^2 - y^2)^3 \Big]_0^{\sqrt{4-y^2}/2} dy = \tfrac{1}{4} \int_0^2 (4 - y^2)^3 dy$$
$$= \tfrac{1}{4} \int_0^2 (64 - 48y^2 + 12y^4 - y^6) dy = \tfrac{1}{4}(64 \cdot 2 - 16 \cdot 8 + \tfrac{12}{5} \cdot 32 - \tfrac{1}{8} \cdot 128) = \tfrac{512}{35}$$

31. S is bounded by $z = xy$ and the planes $x = 1$, $y = 1$ and $z = 0$. $\rho(x,y,z) = 3\sqrt{x^2 + y^2}$ kg/m^3.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 3\sqrt{u_i^2 + v_i^2}\, \Delta_i V = 3 \iiint_S \sqrt{x^2 + y^2}\, dV = 3 \int_0^1 \int_0^1 \int_0^{xy} \sqrt{x^2 + y^2}\, dz\, dy\, dx$$
$$= 3 \int_0^1 \int_0^1 xy(x^2 + y^2)^{1/2} dy\, dx = 3 \int_0^1 \tfrac{1}{3} x(x^2 + y^2)^{3/2} \Big]_0^1 dx = \int_0^1 [x(x^2 + 1)^{3/2} - x^4] dx = \tfrac{1}{5}(x^2 + 1)^{5/2} - \tfrac{1}{5} x^5 \Big]_0^1$$
$$= \tfrac{4}{5}\sqrt{2} - \tfrac{1}{5} - \tfrac{1}{5} = \tfrac{2}{5}(2\sqrt{2} - 1)$$

32. A solid has the shape of a right-circular cylinder of base radius r meters and height h meters. find the mass of the solid if the volume density varies as the distance from one of the bases. The volume density is measured in kg/m^3.

▸ Let one base of the solid be the region R enclosed by a circle in the xy plane with center at the origin and radius r meters. An equation of the circle is $x^2 + y^2 = r^2$. Let $\rho(x,y,z) = kz$. Then the number of units in the mass of the solid is

$$M = \iiint_S \rho(x,y,z) dV = \iint_R \int_0^h kz\, dz\, dA = \iint_R \tfrac{1}{2} kz^2 \Big]_0^h dA = \tfrac{1}{2} kh^2 \iint_R dA = \tfrac{1}{2} kh^2 (\pi r^2) = \tfrac{1}{2} \pi k h^2 r^2$$

The mass of the solid is $\frac{1}{2}\pi k h^2 r^2$ kilograms.

13.6 THE TRIPLE INTEGRAL IN CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical Coordinates Let f be a continuous function of r, θ, and z. Let S be a region in R^3 bounded by the planes $\theta = \theta_1$ and $\theta = \theta_2$, the cylinders $r = g_1(\theta)$ and $r = g_2(\theta)$ that are perpendicular to the polar plane, and by the surfaces $z = h_1(r,\theta)$ and $z = h_2(r,\theta)$, where $\theta_1 < \theta_2$, $g_1(\theta) \le g_2(\theta)$, and $h_1(r,\theta) \le h_2(r,\theta)$, and the functions g_1, g_2, h_1, and h_2 are smooth. Then the triple integral of f on S is equivalent to an iterated integral with

$$\iiint_S f(r,\theta,z) dV = \iiint_S f(r,\theta,z) r\, dz\, dr\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r,\theta,dz) r\, dz\, dr\, d\theta$$

Other orders of integration are also possible. Appropriate when there is symmetry with respect to an axis or a cylinder which contains an axis.

Spherical Coordinates Let f be a continuous function of ρ, θ, and ϕ. Let S be a region in R^3 bounded by the planes $\theta = \theta_1$ and $\theta = \theta_2$, the cones $\phi = g_1(\theta)$ and $\phi = g_2(\theta)$ and the surfaces $\rho = h_1(\phi,\theta)$ and $\rho = h_2(\phi,\theta)$ where $\theta_1 < \theta_2$, $g_1(\theta) \le g_2(\theta)$ and $h_1(\phi,\theta) \le h_2(\phi,\theta)$ and the functions g_1, g_2, h_1, and h_2 are smooth. Then the triple integral of f on S is equivalent to an iterated integral with

$$\iiint_S f(\rho,\theta,\phi) dV = \iiint_S f(\rho,\theta,\phi) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho,\theta,\phi) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Other orders of integration are also possible. Appropriate when a boundary is a cone, or a sphere centered at the origin or containing the origin.

Because the factors r and ρ^2, some integrals which are improper in rectangular coordinates become proper when we switch to cylindrical or spherical coordinates. See Ex. 32.

Exercises 13.6

In Exercises 1–6, evaluate the iterated integral.

1. $\int_0^{\pi/4}\int_0^{a}\int_0^{r\cos\theta} r\sec^3\theta\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a} z\Big]_0^{r\cos\theta} r\sec^3\theta\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a} r^2\sec^2\theta\,dr\,d\theta$
$= \int_0^{\pi/4}\frac{1}{3}r^3\Big]_0^{a}\sec^2\theta\,d\theta = \frac{1}{3}a^3\tan\theta\Big]_0^{\pi/4} = \frac{1}{3}a^3$

2. $\int_0^{\pi/4}\int_{2\sin\theta}^{2\cos\theta}\int_0^{r\sin\theta} r^2\cos\theta\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_{2\sin\theta}^{2\cos\theta} r^3\sin\theta\cos\theta\,dr\,d\theta$
$= \int_0^{\pi/4}\frac{1}{4}[(2\cos\theta)^4-(2\sin\theta)^4]\sin\theta\cos\theta\,d\theta = 4\int_0^{\pi/4}(1-2\sin^2\theta)\sin\theta\cos\theta\,d\theta = 4\Big[\frac{1}{2}\sin^2\theta-\frac{1}{2}\sin^4\theta\Big]_0^{\pi/4} = \frac{1}{2}$

3. $\int_0^{\pi}\int_0^{4}\int_0^{1} re^z\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{4} re^z\Big]_0^{1}dr\,d\theta = (e-1)\int_0^{\pi}\int_0^{4} r\,dr\,d\theta = (e-1)\int_0^{\pi}\frac{1}{2}r^2\Big]_0^{4}d\theta = (e-1)\int_0^{\pi}6\,d\theta = 6\pi(e-1)$

4. $\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta$

▸ $\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\Big[\frac{1}{4}\rho^4\sin\phi\Big]_0^{2}d\phi\,d\theta = 4\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = 4\int_0^{2\pi}-\cos\phi\Big]_0^{\pi}d\theta$
$= 4\int_0^{2\pi}2\,d\theta = 16\pi$

5. $\int_0^{\pi/4}\int_0^{2a\cos\phi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\rho\,d\phi = \int_0^{\pi/4}\int_0^{2a\cos\phi}\theta\Big]_0^{2\pi}\rho^2\sin\phi\,d\rho\,d\phi = 2\pi\int_0^{\pi/4}\int_0^{2a\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi$
$= 2\pi\int_0^{\pi/4}\frac{1}{3}\rho^3\Big]_0^{2a\cos\phi}\sin\phi\,d\phi = \frac{16}{3}\pi a^3\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi = -\frac{16}{3}\pi a^3\Big[\frac{1}{4}\cos^4\phi\Big]_0^{\pi/4} = -\frac{4}{3}\pi a^3(\frac{1}{4}-1) = \pi a^3$

6. $\int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\int_0^{a\csc\theta}\rho^3\sin^2\theta\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\frac{1}{4}(a^4\csc^4\theta)\sin^2\theta\sin\phi\,d\theta\,d\phi = \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\csc^2\theta\sin\phi\,d\phi$
$= \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}(1-\cot\phi)\sin\phi\,d\phi = \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}(\sin\phi-\cos\phi)d\phi = \frac{1}{4}a^4\Big[-\cos\phi-\sin\phi\Big]_{\pi/4}^{\pi/2} = \frac{1}{4}a^4(\sqrt{2}-1)$

7. (a) $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{a} rz\Big]_0^{\sqrt{a^2-r^2}}dr\,d\theta$
$= 2\int_0^{2\pi}\int_0^{a} r(a^2-r^2)^{1/2}dr\,d\theta = -\frac{2}{3}\int_0^{2\pi}(a^2-r^2)^{3/2}\Big]_0^{a}d\theta = \frac{2}{3}a^3\int_0^{2}d\theta = \frac{4}{3}\pi a^3$

(b) $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}\rho^3\Big]_0^{a}\sin\phi\,d\theta\,d\phi$
$= \frac{1}{3}a^3\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = \frac{1}{3}a^3\int_0^{\pi}\theta\Big]_0^{2\pi}\sin\phi\,d\phi = \frac{2}{3}\pi a^3\int_0^{\pi}\sin\phi\,d\phi = -\frac{2}{3}\pi a^3\cos\phi\Big]_0^{\pi} = \frac{4}{3}\pi a^3$

8. If S is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=16$ and the coordinate planes, evaluate the triple integral $\iiint_S xyz\,dV$ by three methods: (a) using spherical coordinates; (b) using rectangular coordinates; (c) using cylindrical coordinates.

▸ (a) In spherical coordinates we have
$x=\rho\sin\phi\cos\theta \quad y=\rho\sin\phi\sin\theta \quad z=\rho\cos\phi \quad dV=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$
Thus,

$$\iiint_S xyz\,dV = \iiint_S \rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$

Because s is bounded by the point sphere $\rho=0$ and the sphere $\rho=4$, bounded by the cylinders $\theta=0$ and $\theta=\frac{1}{2}\pi$, and by the cones $\phi=0$ and $\phi=\frac{1}{2}\pi$, the triple integral may be replaced by an iterated integral. Thus,

$$\iiint_S xyz\,dV = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{4}\rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{4}\rho^5d\rho\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi = \Big[\frac{1}{6}\rho\Big]_0^{4}\cdot\Big[\frac{1}{2}\sin^2\theta\Big]_0^{\pi/2}\cdot\Big[\frac{1}{4}\sin^4\phi\Big]_0^{\pi/2}$$
$$= \frac{1}{6}\cdot 4096\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{256}{3}$$

(b) In rectangular coordinates, s is bounded by the surface $z=0$ and $z=\sqrt{16-x^2-y^2}$, bounded by the cylinders $y=0$ and $y=\sqrt{16-x^2}$, and bounded by the planes $x=0$ and $x=4$. Thus,

$$\iiint_S xyz\,dV = \int_0^{4}\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{1}{2}\int_0^{4}\int_0^{\sqrt{16-x^2}} xy(16-x^2-y^2)dy\,dx$$
$$= -\frac{1}{8}\int_0^{4} x(16-x^2-y^2)^2\Big]_{y=0}^{\sqrt{16-x^2}}dx = \frac{1}{8}\int_0^{4} x(16-x^2)^2dx = -\frac{1}{48}(16-x^2)^3\Big]_0^{4} = \frac{256}{3}$$

(c) In cylindrical coordinates, we have
$x = r\cos\theta \quad y = r\sin\theta \quad dV = r\,dz\,dr\,d\theta$
Thus,

$$\iiint_S xyz\,dV = \iiint_S r^3\sin\theta\cos\theta\,z\,dz\,dr\,d\theta$$

Because S is bounded by the surfaces $z = 0$ and $z = \sqrt{16-r^2}$, by the cylinders $r = 0$ and $r = 4$, and by the planes $\theta = 0$ and $\theta = \frac{1}{2}\pi$, the triple integral may be replaced by an iterated integral. Thus,

$$\iiint_S xyz\,dV = \int_0^{\pi/2}\int_0^4\int_0^{\sqrt{16-r^2}} r^3\sin\theta\cos\theta\,z\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^4 \sin\theta\cos\theta\,r^3(16-r^2)dr\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^4(16r^3-r^5)dr = \frac{1}{2}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}\cdot\left[4r^4-\frac{1}{6}r^6\right]_0^4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1024}{3} = \frac{256}{3}$$

In Exercises 9–16, use cylindrical coordinates.

▸ V cubic units is the required volume, M kg is the required mass, p is the volume density.

9. $x = r\cos\theta$. $\displaystyle V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{\pi/2}\int_0^1\int_0^{r\cos\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^2\cos\theta\,dr\,d\theta$

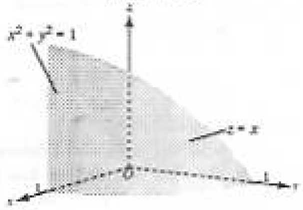
$\displaystyle = \frac{1}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}\sin\theta\Big]_0^{\pi/2} = \frac{1}{3}$

10. $z = 1-(x^2+y^2) = 1-r^2$. $\displaystyle V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(1-r^2)dr\,d\theta = \int_0^{2\pi}d\theta\left[\frac{1}{2}r^2-\frac{1}{4}r^4\right]_0^1 = \frac{1}{2}\pi$

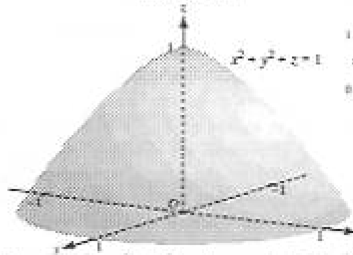
11. $x^2+y^2 = r^2$. $\displaystyle V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{2\pi}\int_0^2\int_8^{12-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi} 2r^2-\frac{1}{4}r^4\Big]_0^2\,d\theta = \int_0^{2\pi}4\,d\theta = 8\pi$

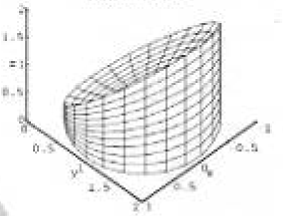
Exercise 9



Exercise 10



Exercise 12



12. Find the volume of the solid bounded by cylinder $x^2+y^2 = 2y$, paraboloid $x^2+y^2 = 2z$, and the xy plane.

▸ The figure shows the solid S, which is bounded above by the paraboloid whose cylindrical equation is $2z = r^2$, and below by the xy plane, and bounded by the cylinder whose cylindrical equation is $r = 2\sin\theta$, $0 \le \theta \le \pi$. The volume of the solid is given by

$$V = \iiint_S dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2/2} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta}\frac{1}{2}r^3\,dr\,d\theta = \int_0^{\pi}2\sin^4\theta\,d\theta$$
$$= 2\left[-\frac{1}{4}\sin^3\theta\cos\theta\Big]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\theta\,d\theta\right] = 2\left[0-\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta+\frac{3}{4}\cdot\frac{1}{2}\theta\right]_0^{\pi} = \frac{3}{4}\pi$$

13. The solid is bounded by the sphere $r^2+z^2 = a^2$. $p(r,\theta,z) = k(r^2+z^2)\text{kg/m}^3$.

$\displaystyle M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}k(r_i{}^2+z_i{}^2)\Delta_i V = k\iiint_S(r^2+z^2)dV = 2k\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}}(r^2+z^2)r\,dz\,dr\,d\theta$

$\displaystyle = 2k\int_0^{2\pi}\int_0^a r^2z+\frac{1}{3}z^3\Big]_0^{\sqrt{a^2-r^2}} r\,dr\,d\theta = 2k\int_0^{2\pi}\int_0^a\left[r^2\sqrt{a^2-r^2}+\frac{1}{3}(a^2-r^2)^{3/2}\right]r\,dr\,d\theta$

Let $u^2 = a^2-r^2$, $2u\,du = -2r\,dr$. Then $\displaystyle M = 4\pi k\int_0^a\left[(a^2-u^2)u+\frac{1}{3}u^3\right]u\,du = 4\pi k\left[\frac{1}{3}a^2u^3-\frac{2}{15}u^5\right]_0^a = \frac{4}{5}a^5\pi k$

14. $x^2+y^2 = 4x$, $r^2 = 4r\cos\theta$, $r = 4\cos\theta$, $0 \le \theta \le \frac{1}{2}\pi$. $x^2+y^2+z^2 = r^2+z^2 = 16$, $p = kz$.

$\displaystyle M = \int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}}kz\,r\,dz\,dr\,d\theta = k\int_0^{\pi/2}\int_0^{4\cos\theta}\frac{1}{2}(16-r^2)r\,dr\,d\theta = -\frac{1}{8}k\int_0^{\pi/2}(16-r^2)^2\Big]_0^{4\cos\theta}d\theta$

$\displaystyle = 32k\int_0^{\pi/2}(1-\sin^4\theta)d\theta = 16k\pi - 32k\left[-\frac{1}{4}\sin^3\theta\cos\theta\Big]_0^{\pi/2}+\frac{3}{4}\int_0^{\pi/2}\sin^2\theta\right]$

$\displaystyle = 16k\pi - 32k\left[0-\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta\Big]_0^{\pi/2}+\frac{3}{4}\cdot\frac{1}{2}\theta\Big]_0^{\pi/2}\right] = 16k\pi - 6k\pi = 10k\pi$

15. S is bounded by $r = 5$, $z = r$, and the xy plane. $p = k$ slug/ft^3. I_z is in slug-ft^2.

$$I_z = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \bar{r}_i^{\,2} k\, \Delta_i V = k \iiint_S r^2 dV = k \int_0^{2\pi} \int_0^5 \int_0^r r^2 (r\, dz\, dr\, d\theta) = k \int_0^{2\pi} \int_0^5 r^3 z \Big]_0^r dr\, d\theta = k \int_0^{2\pi} \int_0^5 r^4 dr\, d\theta$$

$$= k \int_0^{2\pi} \frac{1}{5} r^5 \Big]_0^5 d\theta = 625k \int_0^{2\pi} d\theta = 1250\pi k$$

16. Find the moment of inertia of the solid bounded by a right-circular cylinder of altitude h meters and radius a meters, with respect ot the axis of the cylinder. The volume density varies as the distance from the axis of the cylinder and is measured in kg/m^3.

▸ Take the z axis as the axis of the cylinder with the base of the cylinder in the polar plane. We are given that the measure of the volume density is $\rho(r, \theta, z) = kr$. If S is the region occupied by the solid, then the measure of its moment of inertia with respect to the axis of the cylinder is given by

$$I_z = \iiint_S r^2 (kr)(r\, dr\, d\theta\, dz) = k\left[\int_0^h dz\right] \cdot \left[\int_0^{2\pi} d\theta\right] \cdot \left[\int_0^a r^4 dr\right] = k \cdot h \cdot 2\pi \cdot \tfrac{1}{5} a^5 = \tfrac{2}{5}\pi k h a^5$$

Thus, the moment of inertia is $\frac{2}{5}\pi k h a^5$ kg-m^3.

17. $$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k \bar{\rho}^2 \Delta_i = \iiint_S k\rho^2 dV = 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= 8k \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{5} a^5 \sin\phi\, d\theta\, d\phi = \frac{8}{5} a^5 k \int_0^{\pi/2} \frac{\pi}{2} \cdot \sin\phi\, d\phi = \frac{4}{5} a^5 \pi k \Big[-\cos\phi\Big]_0^{\pi/2} = \frac{4}{5} a^5 \pi k$$

18. $p = kr = k\rho \sin\phi$. $$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a k\rho \sin\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = k \int_0^{2\pi} d\theta \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2\phi) d\phi \int_0^a \rho^3 d\rho$$

$$= k \cdot 2\pi \cdot \tfrac{1}{2}\Big[\phi - \tfrac{1}{2}\sin 2\phi\Big]_0^{\pi/2} \cdot \tfrac{1}{4} a^4 = \tfrac{1}{8} k a^4 \pi^2$$

In Exercises 19–22, use spherical coordinates.

19. $x^2 + y^2 + z^2 = 4z$; $\rho^2 = 4\rho\cos\phi$; $\rho = 4\cos\phi$ and $x^2 + y^2 + z^2 = 2z^2$; $\rho^2 = 2\rho^2\cos^2\phi$; $\cos^2\phi = \frac{1}{2}$; $\phi = \frac{1}{4}\pi$

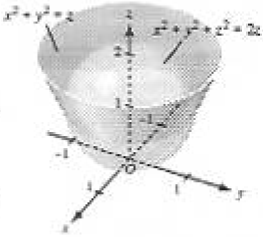
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta_i V = \iiint_S dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{4\cos\phi} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \rho^3 \Big]_0^{4\cos\phi} \sin\phi\, d\phi\, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3\phi \sin\phi\, d\phi\, d\theta = \frac{64}{3} \cdot 2\pi\Big[-\frac{1}{4}\cos^4\phi\Big]_0^{\pi/4} = \frac{32}{3}\pi\Big(-\frac{1}{4} + 1\Big) = 8\pi$$

20. Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 2z$ and above the paraboloid $x^2 + y^2 = z$.

▸ Eliminating x and y from the given equations, we obtain $z^2 = z$. Thus, the sphere and the paraboloid intersect in the plane $z = 1$, where $\phi = \frac{1}{4}\pi$, and in the origin, corresponding to $\phi = \frac{1}{2}\pi$. Because the sphere has the equation $x^2 + y^2 + (z - 1)^2 = 1$, its center lies in the plane $z = 1$ and its radius is 1. The figure shows the solid. Because $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$, a spherical equation of the sphere is

$$\rho^2 = 2\rho\cos\phi$$
$$\rho = 2\cos\phi$$



Because $x^2 + y^2 = r^2 = (\rho\sin\phi)^2$, a spherical equation of the paraboloid is

$$\rho\cos\phi = \rho^2\sin^2\phi$$
$$\rho = \cot\phi \csc\phi$$

The measure of the volume of the entire sphere is $V_1 = \frac{4}{3}\pi$. The measure of the volume of the solid is $V_1 - V_2$, where V_2 is the measure of the volume of the sphere outside the paraboloid. Then

$$V_2 = \int_{\pi/4}^{\pi/2} \int_{\cot\phi\csc\phi}^{2\cos\phi} \int_0^{2\pi} \rho^2 \sin\phi\, d\theta\, d\rho\, d\phi = 2\pi \int_{\pi/4}^{\pi/2} \int_{\cot\phi\csc\phi}^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \Big[\frac{1}{3}\rho^3 \sin\phi\Big]_{\cot\phi\csc\phi}^{2\cos\phi} d\phi = \frac{2}{3}\pi \int_{\pi/4}^{\pi/2} (\cos^3\phi \sin\phi - \cot^3\phi \csc^2\phi) d\phi$$

$$= \frac{2}{3}\pi\Big[-\frac{1}{4}\cos^4\phi + \frac{1}{4}\cot^4\phi\Big]_{\pi/4}^{\pi/2} = \frac{2}{3}\pi \cdot \frac{1}{4} = \frac{1}{6}\pi$$

Because $V_1 - V_2 = \frac{4}{3}\pi - \frac{1}{6}\pi = \frac{7}{6}\pi$, the volume of the solid is $\frac{7}{6}\pi$ cubic units.

21. $x^2 + y^2 = 2x$; $r^2 = 2r\cos\theta$; $r = 2\cos\theta$; $\rho = 2\sin\theta\csc\phi$ and $x^2 + y^2 + z^2 = 2z^2$; $\rho^2 = 2\rho^2\cos^2\phi$; $\phi = \frac{1}{4}\pi$. If I_z kg-m^2 is the moment of inertia with respect to the z axis, then

$$\mathrm{I}_z = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar\rho_i\sin\bar\phi_i)^2 k\,\Delta_i\mathrm{V} = \iiint_S k\rho^2\sin^2\phi\,d\mathrm{V} = k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta\csc\phi}(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\frac{1}{5}\rho^5\Big]_0^{2\cos\theta\csc\phi}\sin^3\phi\,d\phi\,d\theta = \frac{32}{5}k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\cos^5\theta\csc^2\phi\,d\phi\,d\theta$$

$$= \frac{32}{5}k\int_{-\pi/2}^{\pi/2}-\cos^5\theta\cot\phi\Big]_{\pi/4}^{\pi/2}d\theta = \frac{32}{5}k\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta \overset{\text{even}}{=} \frac{64}{5}k\int_0^{\pi/2}(1 - 2\sin^2\theta + \sin^4\theta)\cos\theta\,d\theta$$

$$= \frac{64}{5}k\Big[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\Big]_0^{\pi/2} = \frac{64}{5}k\Big(1 - \frac{2}{3} + \frac{1}{5}\Big) = \frac{512}{75}k$$

22. $r^2 = (\rho\sin\phi)^2$. $\mathrm{I}_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^2\sin^2\phi\cdot k\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = k\int_0^{2\pi}d\theta\int_0^{\pi}(1 - \cos^2\phi)\sin\phi\,d\phi\int_0^{2}\rho^4 d\rho$

$$= 2k\pi\Big[\tfrac{1}{3}\cos^3\phi - \cos\phi\Big]_0^{\pi}\Big[\tfrac{1}{5}\rho^5\Big]_0^{2} = 2k\pi\cdot\tfrac{4}{3}\cdot\tfrac{32}{5} = \tfrac{256}{15}k\pi$$

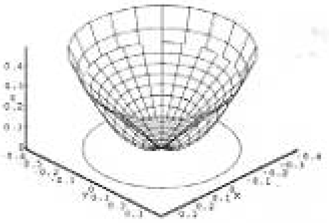
In Exercises 23–28, use the coordinate system that you decide is best for the problem.

23. The solid is the upper half of the sphere $\rho = 2$. $\mathrm{p}(\rho,\theta,\phi) = k\rho$ kg/m^3.

$$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}k\bar\rho_i\Delta_i\mathrm{V} = k\iiint_S\rho\,d\mathrm{V} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2}\rho(r^2\sin\phi\,d\rho\,d\phi\,d\theta) = 4k\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\rho^4\Big]_0^{2}\sin\phi\,d\phi\,d\theta$$

$$= 16k\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta = 16k\int_0^{\pi/2}(-\cos\phi)\Big]_0^{\pi/2}d\theta = 16k\int_0^{\pi/2}d\theta = 8k\pi$$

24. Find the mass of the homogeneous solid inside the paraboloid $3x^2 + 3y^2 = z$ and outside the cone $x^2 + y^2 = z^2$ if the constant volume density is k kg/m^3.



▹ Cylindrical coordinates are best because a cylindrical equation of the paraboloid is $z = 3r^2$ and a cylindrical equation of the cone is $z = r$. Eliminating z from the cylindrical equations, we obtain $r = 3r^2$. Thus, $r = 0$ or $r = \frac{1}{3}$. The figure shows the solid, which is bounded below by the paraboloid (shown cut open) above by the cone, and bounded by the cylinder $r = \frac{1}{3}$. If M is the measure of the mass of the solid, then

$$\mathrm{M} = \int_0^{1/3}\int_{3r^2}^{r}\int_0^{2\pi}kr\,d\theta\,dz\,dr = 2\pi k\int_0^{1/3}\int_{3r^2}^{r}r\,dz\,dr$$

$$= 2\pi k\int_0^{1/3}r(3 - 3r^2)dr = 2\pi k\Big[\tfrac{1}{3}r^3 - \tfrac{3}{4}r^4\Big]_0^{1/3} = \tfrac{1}{162}\pi k$$

The mass is $\frac{1}{162}\pi k$ kg.

In Exercises 25 and 26, the solid is bounded by the spheres $\rho = a$ and $\rho = 2a$. $\mathrm{p}(\rho,\theta,\phi) = k\rho^{-2}$ slugs/ft^3.

25. If I_z slug-ft^2 is the moment of inertia of the solid about the z axis, then

$$\mathrm{I}_z = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar\rho_i\sin\bar\phi_i)^2(k\bar\rho_i^{\,-2})\Delta_i\mathrm{V} = \iiint_S\rho^2\sin^2\phi(k\rho^{-2})d\mathrm{V} = k\int_0^{\pi}\int_0^{2\pi}\int_a^{2a}\sin^2\phi(\rho^2\sin\phi\,d\rho\,d\theta\,d\phi)$$

$$= k\int_0^{\pi}\int_0^{2\pi}\frac{1}{3}\rho^3\Big]_a^{2a}\sin^3\phi\,d\theta\,d\phi = \frac{7}{3}a^3k\int_0^{\pi}\int_0^{2\pi}\sin^3\phi\,d\theta\,d\phi = \frac{7}{3}a^3k\int_0^{\pi}\theta\Big]_0^{2\pi}\sin^3\phi\,d\phi = \frac{14}{3}\pi a^3k\int_0^{\pi}\sin^3\phi\,d\phi$$

$$= \frac{14}{3}\pi a^3k\int_0^{\pi}(1 - \cos^2\phi)\sin\phi\,d\phi = \frac{14}{3}\pi a^3k\Big[-\cos\phi + \frac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{14}{3}\pi a^3k\Big(\frac{4}{3}\Big) = \frac{56}{9}\pi a^3k$$

26. $\mathrm{M} = k\int_0^{\pi}\int_0^{2\pi}\int_a^{2a}k\rho^{-2}(\rho^2\sin\phi\,d\rho\,d\theta\,d\phi) = k\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_a^{2a}d\rho = k\cdot 2\cdot 2\pi\cdot a = 4\pi ka$

In Exercises 27 and 28, the solid is inside the paraboloid $x^2 + y^2 = z$ and outside the cone $x^2 + y^2 = z^2$. The constant volume density is k kg/m^3.

27. $x^2 + y^2 = z$; $z = r^2$ and $x^2 + y^2 = z^2$; $z = r$

$$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}k\,\Delta_i\mathrm{V} = k\iiint_S d\mathrm{V} = k\int_0^{2\pi}\int_0^{1}\int_{r^2}^{r}r\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^{1}rz\Big]_{r^2}^{r}dr\,d\theta = k\int_0^{2\pi}\int_0^{1}(r^2 - r^3)dr\,d\theta$$

$$= k\int_0^{2\pi}\frac{1}{3}r^3 - \frac{1}{4}r^4\Big]_0^{1}d\theta = \frac{1}{12}k\int_0^{2\pi}d\theta = \frac{1}{6}\pi k$$

If M_{xy} kg-m is the moment of mass with respect to the xy plane, then

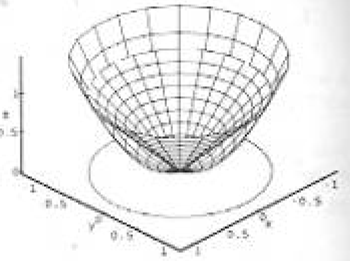
$$\mathrm{M}_{xy} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}z_i(k)\Delta_i\mathrm{V} = k\iiint_S z\,d\mathrm{V} = \int_0^{2\pi}\int_0^{1}\int_{r^2}^{r}zr\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^{1}\frac{1}{2}rz^2\Big]_{r^2}^{r}dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^{1}(r^3 - r^5)dr\,d\theta = \frac{k}{2}\int_0^{2\pi}\frac{1}{4}r^4 - \frac{1}{6}r^6\Big]_0^{1}d\theta = \frac{1}{24}k\int_0^{2\pi}d\theta = \frac{1}{12}\pi k$$

Because of symmetry with respect to the z axis the center of mass is at $(0,0,\bar{z})$ where $\bar{z} = \frac{1}{M}M_{xy} = \frac{6}{\pi k}\cdot\frac{\pi k}{12} = \frac{1}{2}$

28. Find the moment of inertia with respect to the z axis of the solid.

▸ Eliminating x and y from the given equations, we have $z^2 = z$. Thus, the given surfaces intersect in the planes $z = 0$ and $z = 1$. The figure shows the solid and its projection on the xy plane. We use a cylindrical coordinate system. Thus, the solid is bounded below by the paraboloid $z = r^2$ (shown cut open) and bounded above by the cone $z = r$, bounded by the cylinders $r = 0$ and $r = 1$, $0 \le \theta \le 2\pi$. Therefore, the measure of the moment of inertia with respect to the z axis is given by



$$I_z = \iiint_S kr^2\,dV = k\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^2(r\,dz\,dr\,d\theta)$$
$$= k\left[\int_0^{2\pi} d\theta\right]\cdot\int_0^1 r^3(r-r^2)\,dr = 2\pi k\left[\tfrac{1}{5}r^5 - \tfrac{1}{6}r^6\right]_0^1 = 2\pi k\cdot\tfrac{1}{30} = \tfrac{1}{15}k\pi$$

The moment of inertia is $\frac{1}{15}\pi k$ kg-m^2.

In Exercises 29–32, evaluate the iterated integral by using either cylindrical or spherical coordinates.

29. $\int_0^4\int_0^3\int_0^{\sqrt{9-x^2}}\sqrt{x^2+y^2}\,dy\,dx\,dz = \int_0^{\pi/2}\int_0^3\int_0^4 r(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^3 r^2 z\Big]_0^4 dr\,d\theta$
$= 4\int_0^{\pi/2}\int_0^3 r^2\,dr\,d\theta = 4\int_0^{\pi/2}\frac{1}{3}r^3\Big]_0^3 d\theta = 36\int_0^{\pi/2} d\theta = 18\pi$

30. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{z}{\sqrt{x^2+y^2}} = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}}\frac{z}{r}(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^1\frac{1}{2}(1-r^2)\,dr\,d\theta = \frac{1}{2}\pi\left[\frac{1}{2}r - \frac{1}{6}r^3\right]_0^1 = \frac{1}{6}\pi$

31. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^1\int_r^{\sqrt{2-r^2}} z^2(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^1 \frac{1}{3}rz^3\Big]_r^{\sqrt{2-r^2}}$
$= \frac{1}{3}\int_0^{\pi/2}\int_0^1 [r(2-r^2)^{3/2} - r^4]\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/2}\left[-\frac{1}{5}(2-r^2)^{5/2} - \frac{1}{5}r^5\right]_0^1 = -\frac{1}{15}\int_0^{\pi/2}(2-4\sqrt{2})\,d\theta$
$= \frac{1}{15}(4\sqrt{2}-2)\theta\Big]_0^{\pi/2} = \frac{1}{15}\pi(2\sqrt{2}-1)$

32. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}}\frac{1}{x^2+y^2+z^2}\,dz\,dx\,dy$

▸ The given iterated integral is equivalent to a triple integral over a region S which is bounded by the surfaces $z = 0$ and $z = \sqrt{4-x^2-y^2}$, bounded by the cylinders $x = 0$ and $x = \sqrt{4-y^2}$, and bounded by the planes $y = 0$ and $y = 2$. Thus, S is the part of the spherical solid enclosed by $x^2+y^2+z^2$ that is in the first octant. We use spherical coordinates. We have
$x^2+y^2+z^2 = \rho^2$ and $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$
Therefore

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}}\frac{1}{x^2+y^2+z^2}\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\frac{1}{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \left[\int_0^{\pi/2}\sin\phi\,d\phi\right]\cdot\left[\int_0^{\pi/2}d\theta\right]\cdot\left[\int_0^2 d\rho\right] = 1\cdot\tfrac{1}{2}\pi\cdot 2 = \pi$$

Note that while the original integral is improper, the spherical integral is proper.

Miscellaneous Exercises for Chapter 13

1. $P_{\text{spherical}} = (3,\pi,\frac{1}{3}\pi)$. $r = \rho\sin\phi = 3\sin\frac{1}{3}\pi = 3(\frac{1}{2}\sqrt{3}) = \frac{3}{2}\sqrt{3}$. $\theta = \pi$, $z = 3\cos\frac{1}{3}\pi = 3(\frac{1}{2}) = \frac{3}{2}$.
Hence $P_{\text{cylindrical}} = (\frac{3}{2}\sqrt{3},\pi,\frac{3}{2})$.

2. $P_{\text{cartesian}} = (-3,\sqrt{3},2)$. $\rho = \sqrt{9+3+4} = 4$, $\theta = \tan^{-1}\frac{\sqrt{3}}{-3} + \pi = \frac{5}{6}\pi$. $\phi = \cos^{-1}\frac{2}{4} = \frac{1}{3}\pi$. $P_{\text{spherical}} = (4,\frac{5}{6}\pi,\frac{1}{3}\pi)$

3. (a) $S_{\text{cartesian}}$ is $(x+y)^2+1 = z$. Because $x = r\cos\theta$ and $y = r\sin\theta$, $S_{\text{cylindrical}}$ is
$z = (r\cos\theta + r\sin\theta)^2 + 1$; $z = r^2\cos^2\theta + 2r^2\sin\theta\cos\theta + r^2\sin^2\theta + 1$; $z = r^2 + r^2\sin 2\theta + 1$
(b) $S_{\text{cartesian}}$ is $25x^2+4y^2 = 100$. Because $x = r\cos\theta$ and $y = r\sin\theta$, $S_{\text{cylindrical}}$ is
$25r^2\sin^2\theta + 4r^2\sin^2\theta = 100$; $r^2(25\cos^2\theta + 4\sin^2\theta) = 100$

4. Find an equation in spherical coordinates of the graph of each of the following equations:
(a) $x^2 + y^2 + 4z^2 = 4$ (b) $4x^2 - 4y^2 + 9z^2 = 36$

▷ (a) We have
$(x^2 + y^2 + z^2) + 3z^2 = 4$
Because $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, we obtain

$$\rho^2 + 3\rho^2\cos^2\phi = 4$$
$$\rho^2(1 + 3\cos^2\phi) = 4$$

(b) Substituting from Eqs. (III) of Section 15.9 into the given equation, we obtain

$$4\rho^2\sin^2\phi\cos^2\theta - 4\rho^2\sin^2\phi\sin^2\theta + 9\rho^2\cos^2\phi = 36$$
$$4\rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta) + 9\rho^2\cos^2\phi = 36$$
$$4\rho^2\sin^2\phi\cos 2\theta + 9\rho^2\cos^2\phi = 36$$
$$\rho^2(4\sin^2\phi\cos 2\theta + 9\cos^2\phi) = 36$$

In Exercises 5–12, evaluate the iterated integral.

5. $\int_0^1\int_x^{\sqrt{x}} x^2y\,dy\,dx = \int_0^1 \frac{1}{2}y^2\Big]_x^{\sqrt{x}} x^2\,dx = \frac{1}{2}\int_0^1 (x^3 - x^4)dx = \frac{1}{2}\Big[\frac{1}{4}x^4 - \frac{1}{5}x^5\Big]_0^1 = \frac{1}{2}\Big(\frac{1}{4} - \frac{1}{5}\Big) = \frac{1}{40}$

6. $\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy\,dx\,dy = 0$ because R is symmetrical with respect to the axes and the integrand is odd.

7. $\int_0^{\pi/2}\int_0^{2\sin\theta} r\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}r^2\Big]_0^{2\sin\theta}\cos^2\theta\,d\theta = 2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$
$= \frac{1}{4}\int_0^{\pi/2}(1 - \cos 4\theta)d\theta = \frac{1}{4}\Big[\theta - \frac{1}{4}\sin 4\theta\Big]_0^{\pi/2} = \frac{1}{8}\pi$

8. $\int_0^{\pi}\int_0^{3(1+\cos\theta)} r^2\sin\theta\,dr\,d\theta$

▷ $\int_0^{\pi}\int_0^{3(1+\cos\theta)} r^2\sin\theta\,dr\,d\theta = \int_0^{\pi}\sin\theta\Big[\frac{1}{3}r^3\Big]_0^{3(1+\cos\theta)}d\theta = 9\int_0^{\pi}(1 + \cos\theta)^3\sin\theta\,d\theta = 9\Big[-\frac{1}{4}(1 + \cos\theta)^4\Big]_0^{\pi} = 36$

9. $\int_0^1\int_0^z\int_0^{y+z} e^xe^ye^z\,dx\,dy\,dz = \int_0^1\int_0^z e^x\Big]_0^{y+z} e^ye^z\,dy\,dz = \int_0^1\int_0^z (e^{y+z} - 1)e^ye^z\,dy\,dz$
$= \int_0^1\int_0^z(e^{2y}e^{2z} - e^ye^z)dy\,dz = \int_0^1\Big[\frac{1}{2}e^{2y}e^{2z} - e^ye^z\Big]_0^z dz = \int_0^1\Big(\frac{1}{2}e^{4z} - e^{2z} - \frac{1}{2}e^{2z} + e^z\Big)dz$
$= \int_0^1\Big(\frac{1}{2}e^{4z} - \frac{3}{2}e^{2z} + e^z\Big)dz = \Big[\frac{1}{8}e^{4z} - \frac{3}{4}e^{2z} + e^z\Big]_0^1 = \frac{1}{8}e^4 - \frac{3}{4}e^2 + e - \frac{3}{8}$

10. $\int_1^2\int_3^x\int_0^{\sqrt{3}y}\frac{y}{y^2 + z^2}dz\,dy\,dx = \int_1^2\int_3^x\Big[\tan^{-1}\frac{z}{y}\Big]_0^{\sqrt{3}y}dy\,dx = \frac{1}{3}\pi\int_1^2\int_3^x dy\,dx = \frac{1}{3}\pi\int_1^2(x - 3)dx = \frac{1}{6}\pi(x - 3)^2\Big]_1^2 = -\frac{1}{2}\pi$

11. $\int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^2 r^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\frac{1}{4}\rho^4\Big]_0^2\sin\phi\cos\phi\,d\phi\,d\theta = 4\int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\sin\phi(\cos\phi)d\phi\,d\theta$
$= 4\int_0^{\pi/2}\frac{1}{2}\sin^2\phi\Big]_{\pi/6}^{\pi/2}d\theta = 2\int_0^{\pi/2}\Big(1 - \frac{1}{4}\Big)d\theta = \frac{3}{2}\int_0^{\pi/2}d\theta = \frac{3}{4}\pi$

12. $\int_0^a\int_0^{\pi/2}\int_0^{\sqrt{a^2-z^2}} zre^{-r^2}dr\,d\theta\,dz$

▷ $\int_0^a\int_0^{\pi/2}\int_0^{\sqrt{a^2-z^2}} zre^{-r^2}dr\,d\theta\,dz = \int_0^a\int_0^{\pi/2} z\Big[-\frac{1}{2}e^{-r^2}\Big]_0^{\sqrt{a^2-z^2}}d\theta\,dz = -\frac{1}{2}\int_0^a\int_0^{\pi/2}(ze^{z^2-a^2} - z)d\theta\,dz$
$= -\frac{1}{4}\pi\int_0^a(ze^{z^2-a^2} - z)dz = -\frac{1}{8}\Big[e^{z^2-a^2} - z^2\Big]_0^a = \frac{1}{8}\pi(e^{-a^2} + a^2 - 1)$

In Exercises 13–16, evaluate the multiple integral.

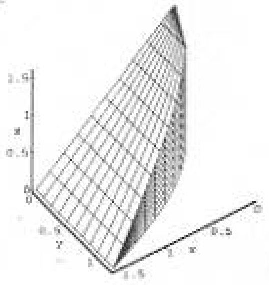
13. $\iint_R xy\,dA = \int_0^{\pi/2}\int_0^1(r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\sin\theta\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}r^4\Big]_0^1\sin\theta\cos\theta\,d\theta$
$= \frac{1}{4}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{1}{8}\sin^2\theta\Big]_0^{\pi/2} = \frac{1}{8}$

14. $\iint_R(x + y)d\mathrm{A} \overset{x\text{ is odd}}{=} \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y\,dy\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2x\,dx = \frac{1}{4}\int_{-\pi/2}^{\pi/2}(1 + \cos 2x)dx = \frac{1}{4}\pi$

15. $\iiint_S z^2d\mathrm{V} = 4\int_0^1\int_0^{\sqrt{1-z}}\int_0^{\sqrt{1-z}} z^2\,dy\,dx\,dz = 4\int_0^1\int_0^{\sqrt{1-z}} y\Big]_0^{\sqrt{1-z}} z^2\,dx\,dz$
$= 4\int_0^1\int_0^{\sqrt{1-z}} z^2\sqrt{1-z}\,dx = 4\int_0^1 x\Big]_0^{\sqrt{1-z}} z^2\sqrt{1-z}\,dz = 4\int_0^1(z^2 - z^3)dz = 4\Big[\frac{1}{3}z^3 - \frac{1}{4}z^4\Big]_0^1 = 4\Big(\frac{1}{3} - \frac{1}{4}\Big) = \frac{1}{3}$

16. $\iiint_S y\cos(x+z)dV$; S is the region bounded by the cylinder $x = y^2$ and the planes $x + z = \frac{1}{2}\pi$, $y = 0$, $z = 0$.

▹ The figure shows the solid S, which is bounded below by the plane $z = 0$, above by the plane $z = \frac{1}{2}\pi - x$, and bounded on the sides by the cylinders $y = 0$ and $y = \sqrt{x}$, $0 \le x \le \frac{1}{2}\pi$. Therefore



$$\iiint_S y\cos(x+z)dV = \int_0^{\pi/2}\int_0^{\sqrt{x}}\int_0^{\pi/2-x} y\cos(x+z)dz\,dy\,dx$$

$$= \int_0^{\pi/2}\int_0^{\sqrt{x}} y\Big[\sin(x+z)\Big]_0^{\pi/2-x} dy\,dx = \int_0^{\pi/2}\int_0^{\sqrt{x}} y(1-\sin x)dy\,dx$$

$$= \tfrac{1}{2}\int_0^{\pi/2} x(1-\sin x)dx = \tfrac{1}{2}\int_0^{\pi/2} x\,dx - \tfrac{1}{2}\int_0^{\pi/2} x\{\sin x\,dx\}$$

$$= \tfrac{1}{4}x^2\Big]_0^{\pi/2} - \tfrac{1}{2}\Big[-x\cos x + \sin x\Big]_0^{\pi/2} = \tfrac{1}{16}\pi^2 - \tfrac{1}{2}(1) = \tfrac{1}{16}(\pi^2 - 8)$$

17. $\iint_R \frac{1}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_1^2 \frac{1}{r^2}(r\,dr\,d\theta) = \int_0^{\pi/2}\int_1^2 \frac{1}{r}\,dr\,d\theta$

$$= \int_0^{\pi/2} \ln|r|\Big]_1^2 d\theta = \ln 2\int_0^{\pi/2} d\theta = \tfrac{1}{2}\pi\ln 2$$

18. $\int_0^1\int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln(x^2+y^2)dx\,dy = \int_0^{\pi/6}\int_0^2 \ln r\{2r\,dr\}d\theta = \frac{1}{6}\pi\Big[\lim_{a\to 0^+} \ln r\cdot r^2\Big]_a^2 - \int_0^2 r\,dr\Big] = \frac{1}{6}\pi(4\ln 2 - 2)$

$$= \tfrac{1}{3}\pi(2\ln 2 - 1)$$

In Exercises 19 and 20, evaluate the iterated integral by reversing the order of integration.

19. $\int_0^1\int_x^1 \sin y^2 dy\,dx = \int_0^1\int_0^y \sin y^2 dx\,dy = \int_0^1 x\Big]_0^y \sin y^2 dy = \int_0^1 y\sin y^2 dy = -\frac{1}{2}\cos y^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

20. $\int_0^1\int_0^{\cos^{-1} y} e^{\sin x}dx\,dy$

▹ The region R in the xy plane on which the double integral is taken is bounded by the curves $x = 0$ and $x = \cos x$ and the lines $x = 0$ and $x = \frac{1}{2}\pi$. Thus,

$$\int_0^1\int_0^{\cos^{-1} y} e^{\sin x}dx\,dy = \int_0^{\pi/2}\int_0^{\cos x} e^{\sin x}dy\,dx = \int_0^{\pi/2}\cos x\,e^{\sin x}dx = e^{\sin x}\Big]_0^{\pi/2} = e - 1$$

In Exercises 21 and 22, used double integrals to find the area of the region bounded by the curves. Sketch.

21. $A = \iint_R dA = \int_{-1}^1\int_{x^4}^{x^2} dy\,dx = \int_0^1 y\Big]_{x^4}^{x^2} dx = \int_{-1}^1 (x^2 - x^4)dx = \frac{1}{3}x^3 - \frac{1}{5}x^5\Big]_{-1}^1 = (\frac{1}{3} - \frac{1}{5}) - (-\frac{1}{3} + \frac{1}{5}) = \frac{4}{15}$

22. $A = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \int_0^1 (x^{1/2} - x^3)dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^4\Big]_0^1 = \frac{5}{12}$

In Exercises 23 and 24, evaluate the iterated integral by changing to either cylindrical or spherical coordinates.

23. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^2 \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^3\int_0^2 r(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^3 z\Big]_0^2 r^2 dr\,d\theta = 2\int_0^{\pi/2}\int_0^3 r^2 dr\,d\theta$

$$= 2\int_0^{\pi/2} \tfrac{1}{3}r^3\Big]_0^3 d\theta = 18\int_0^{\pi/2} d\theta = 9\pi$$

24. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} z\sqrt{4-x^2-y^2}\,dz\,dx\,dy$

▹ The given iterated integral is equivalent to a triple integral over a region S which is bounded by the surfaces $z = 0$ and $z = \sqrt{4-x^2-y^2}$, bounded by the cylinders $x = 0$ and $x = \sqrt{4-y^2}$, and bounded by the planes $y = 0$ and $y = 2$. Thus, S is the part of the spherical solid enclosed by $x^2 + y^2 + z^2$ that is in the first octant. Because the integrand $z\sqrt{4-x^2-y^2}$ is more easily expressed in terms of cylindrical coordinates, we use cylindrical coordinates to evaluate the given integral. A cylindrical equation of the part of the sphere in the the first octant is $z = \sqrt{4-r^2}$, and an equation of its projection onto the xy plane is the circle $r = 2$, $0 \le \theta \le \frac{1}{2}\pi$. Therefore

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} z\sqrt{4-x^2-y^2}\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 \Big[\tfrac{1}{2}z^2 r\sqrt{4-r^2}\Big]_0^{\sqrt{4-r^2}} dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\int_0^2 r(4-r^2)^{3/2}dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\Big[-\tfrac{1}{5}(4-r^2)^{5/2}\Big]_0^2 d\theta$$

$$= -\tfrac{1}{10}\int_0^{\pi/2}(-4^{5/2})d\theta = \tfrac{8}{5}\pi$$

In Exercises 25 and 27, R is in the first quadrant bounded by the parabolas $x^2 = 4y$ and $x^2 = 8 - 4y$.

25. $A = \iint_R dx\, dy = \int_0^1 \int_0^{2\sqrt{y}} dx\, dy + \int_1^2 \int_0^{2\sqrt{2-y}} dx\, dy = \int_0^1 x\Big]_0^{2\sqrt{y}} dy + \int_1^2 x\Big]_0^{2\sqrt{2-y}} dy$

$= 2\int_0^1 y^{1/2} dy + 2\int_1^2 (2-y)^{1/2} dy = \frac{4}{3} y^{3/2}\Big]_0^1 - \frac{4}{3}(2-y)^{3/2}\Big]_1^2 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

27. $A = \iint_R dy\, dx = \int_0^2 \int_{x^2/4}^{(8-x^2)/4} dy\, dx = \int_0^2 y\Big]_{x^2/4}^{(8-x^2)/4} dx = \frac{1}{4}\int_0^2 (8 - 2x^2) dx = \frac{1}{4}\Big[8x - \frac{2}{3}x^3\Big]_0^2 = \frac{1}{4}(16 - \frac{16}{3}) = \frac{8}{3}$

In Exercises 26 and 28, use double integration to find the area of the region in the xy plane bounded by the parabolas $y = 9 - x^2$ and $y = x^2 + 1$.

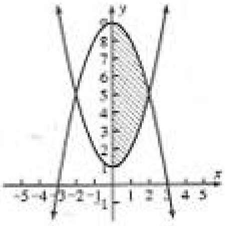
26. Integrate first with respect to y.

$A = 2\int_0^2 \int_{x^2+1}^{9-x^2} dy\, dx = 2\int_0^2 (8 - 2x^2) dx = 2\Big[8x - \frac{2}{3}x^3\Big]_0^2 = \frac{64}{3}$

28. Integrate first with respect to x.

▸ The figure shows the region of which half is shaded. The parabolas intersect when

$$\begin{aligned} 9 - x^2 &= x^2 + 1 \\ 2x^2 &= 8 \\ x &= \pm 2 \\ y &= 5 \end{aligned}$$



Let

$$f(y) = \begin{cases} \sqrt{y-1} & \text{if } 1 \le y \le 5 \\ \sqrt{9-y} & \text{if } 5 \le y \le 9 \end{cases}$$

Then

$$A = \iint_R dx\, dy = 2\int_1^9 \int_0^{f(y)} dx\, dy = 2\Big[\int_1^5 \sqrt{y-1} dy + \int_5^9 \sqrt{9-y} dy\Big]$$

$$= 2\cdot\frac{2}{3}(y-1)^{3/2}\Big]_1^5 + 2\Big(-\frac{2}{3}\Big)(9-y)^{3/2}\Big]_5^9 = \frac{4}{3}\cdot 8 + \frac{4}{3}\cdot 8 = \frac{64}{3}$$

Therefore, the area is $\frac{64}{3}$ square units.

In Exercises 29 and 31, use double integration to find the the volume of the solid bounded by the planes $x = y$, $y = 0$, $z = 0$, $x = 1$, and $z = 1$.

29. $V = \iint_R 1\, dA = \int_0^1 \int_y^1 dx\, dy = \int_0^1 (1 - y) dy = -\frac{1}{2}(1-y)^2\Big]_0^1 = \frac{1}{2}$

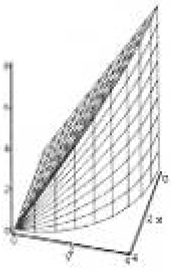
31. $V = \iint_R 1\, dA = \int_0^1 \int_0^x dy\, dx = \int_0^1 x\, dx = \frac{1}{2} x^2\Big]_0^1 = \frac{1}{2}$

In Exercises 30 and 32, use double integration to find the volume of the solid above the xy plane bounded by the cylinder $x^2 + y^2 = 16$ and the plane $z = 2y$.

30. $V = 2\int_0^4 \int_0^{\sqrt{16-y^2}} 2y\, dx\, dy = 4\int_0^4 \sqrt{16 - y^2}\, y\, dy = -\frac{4}{3}(16 - y^2)^{3/2}\Big]_0^4 = \frac{256}{3}$

32. Integrate first with respect to y.

▸ The figure shows the half of the solid that is in the first octant. We take R to be the first quadrant part of the circle $x^2 + y^2 = 16$. We regard R as being bounded by the lines $x = 0$ and $x = 4$ and the curves $y = 0$ and $y = \sqrt{16 - x^2}$. Thus, the measure of the volume is given by



$$V = 2\iint_R 2y\, dA = 2\int_0^4 \int_0^{\sqrt{16-x^2}} 2y\, dy\, dx = 2\int_0^4 (16 - x^2) dx$$

$$= 2\Big[16x - \frac{1}{3}x^3\Big]_0^4 = \frac{256}{3}$$

Therefore, the volume is $\frac{256}{3}$ cubic units.

33. $V = \iint_R (x^2 + y) dA = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} (x^2 + y) dy\, dx = \int_0^4 x^2 y + \frac{1}{2} y^2\Big]_{x^2/4}^{2\sqrt{x}} dx = \int_0^4 \Big(x^{5/2} + 2x - \frac{9}{32} x^4\Big) dx$

$= \frac{4}{7} x^{7/2} + x^2 - \frac{9}{160} x^5\Big]_0^4 = \frac{512}{7} + 16 - \frac{288}{5} = \frac{1104}{35}$

34. $x^2 = x + 1$, $x = -1, 2$. $M = \int_{-1}^2 \int_{x^2}^{x+2} x^2 y^2 dy\, dx = \frac{1}{3}\int_{-1}^2 x^2[(x+2)^3 - x^6] dx$

$= \frac{1}{3}\int_{-1}^2 [x^5 + 6x^4 + 12x^3 + 8x^2 - x^7] dx = \frac{1}{3}\Big[\frac{1}{6}x^6 + \frac{6}{5}x^5 + 3x^4 + \frac{8}{3}x^3 - \frac{1}{8}x^8\Big]_{-1}^2 = \frac{207}{10}$

35. In the first octant $y \geq 0$ so $x^2 + y^2$ gives $y = \sqrt{9 - x^2}$. Then $\dfrac{\partial y}{\partial x} = -\dfrac{x}{\sqrt{9 - x^2}}$ and $\dfrac{\partial y}{\partial z} = 0$.

$$\sigma = \int_0^3 \int_x^{3x} \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dz\, dx = \int_0^3 \int_0^{3x} \sqrt{\frac{x^2}{9 - x^2} + 0 + 1}\, dz\, dx = \int_0^3 \frac{3z}{\sqrt{9 - x^2}}\Big]_x^{3x} dx = 6 \int_0^3 \frac{x}{\sqrt{9 - x^2}}\, dx$$

$$= 6 \lim_{b \to 3^-} \int_0^b x(9 - x^2)^{-1/2} dx = -6 \lim_{b \to 3^-} (9 - x^2)^{1/2}\Big]_0^b = -6 \lim_{b \to 3^-} [(9 - b^2)^{1/2} - 3] = 18$$

36. Find the area of the surface of the part of the cylinder $x^2 + y^2 = a^2$ that lies inside the cylinder $y^2 + z^2 = a^2$.

▹ The part of the surface that lies in the first octant is shown shaded in the figure. This is one-eighth of the entire surface. We take the yz plane as the horizontal plane. Solving the equation of the cylinder $x^2 + y^2 = a^2$ for x, we obtain $x = \pm \sqrt{a^2 - y^2}$. We take

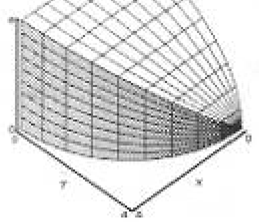
$$f(y, z) = \sqrt{a^2 - y^2}$$

Thus,

$$f_y(y, z) = \frac{-y}{\sqrt{a^2 - y^2}} \quad \text{and } f_z(y, z) = 0$$

Hence,

$$\sqrt{f_y^{\,2}(y, z) + f_z^{\,2}(y, z) + 1} = \sqrt{\frac{y^2}{a^2 - y^2} + 1} = \frac{a}{\sqrt{a^2 - y^2}}$$



Let R be the region in the first quadrant of the yz plane bounded by the circle $y^2 + z^2 = a^2$. The measure of the surface area is given by

$$\sigma = 8 \iint_R \frac{a}{\sqrt{a^2 - y^2}} dA = \lim_{b \to a^-} 8a \int_0^b \int_0^{\sqrt{a^2 - y^2}} \frac{1}{\sqrt{a^2 - y^2}} dz\, dy = \lim_{b \to a^-} 8a \int_0^b dy = 8a^2$$

▪ The surface area is $8a^2$ square units. Because the integrand is unbounded near $(0, a, 0)$, the integral is improper.

37. $$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} r_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_{1/(1 + \cos\theta)}^{1} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2\Big]_{1/2\cos^2(\theta/2)}^{1} d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(1 - \frac{1}{4} \sec^4 \frac{\theta}{2}\right) d\theta = \int_0^{\pi/2} \left[1 - \frac{1}{4}\left(\tan^2 \frac{\theta}{2} + 1\right) \sec^2 \frac{\theta}{2}\right] d\theta = \left[\theta - \frac{1}{2}\left(\frac{1}{3} \tan^3 \frac{\theta}{2} + \tan \frac{\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3}$$

38. The limaçon $r = 3 - \cos\theta$ meets the circle $r = 5\cos\theta$ when $3 - \cos\theta = 5\cos\theta$, $\cos\theta = \frac{1}{2}$, $\theta = \pm \frac{1}{3}\pi$.

$$M = 2 \int_0^{\pi/3} \int_{3 - \cos\theta}^{5\cos\theta} 2 \sin\theta\, r\, dr\, d\theta = 2 \int_0^{\pi/3} [25\cos^2\theta - (3 - \cos\theta)^2] \sin\theta\, d\theta = -\frac{2}{3}[25\cos^3\theta + (3 - \cos\theta)^3]_0^{\pi/3} = \frac{19}{2}$$

39. $$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (u_i v_i^{\,2}) \Delta_i A = \iint_R x y^2 dA = \int_0^3 \int_0^2 x y^2 dy\, dx = \int_0^3 \frac{1}{3} x y^3\Big]_0^2 dx = \frac{8}{3} \int_0^3 x\, dx = \frac{4}{3} x^2\Big]_0^3 = 12$$

If M_x kg-m and M_y kg-m are the moments of mass with respect to the x and y axes,

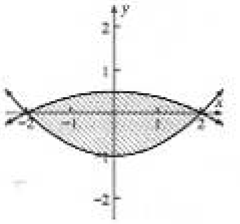
$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i (u_i v_i^{\,2}) \Delta_i A = \iint_R x y^3 dA = \int_0^3 \int_0^2 x y^3 dy\, dx = \int_0^3 \frac{1}{4} x y^4\Big]_0^2 dx = 4 \int_0^3 x\, dx = 2x^2\Big]_0^3 = 18$$

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i (u_i v_i^{\,2}) \Delta_i A = \iint_R x^2 y^2 dA = \int_0^3 \int_0^2 x^2 y^2 dy\, dx = \int_0^3 \frac{1}{3} x^2 y^3\Big]_0^2 dx = \frac{8}{3} \int_0^3 x^2\, dx = \frac{8}{9} x^3\Big]_0^3 = 24$$

If $(\bar{x}, \bar{y})$ is the center of the mass, $\bar{x} = \dfrac{M_y}{M} = \dfrac{24}{12} = 2$ and $\bar{y} = \dfrac{M_x}{M} = \dfrac{18}{12} = \dfrac{3}{2}$.

40. Find the center of mass of the lamina in the shape of the region bounded by the parabolas $x^2 = 4 + 4y$ and $x^2 = 4 - 8y$ if the area density at any point is kx^2 kg/m^2.

▹ The figure shows the region R, which is bounded by the lines $x = -2$ and $x = 2$, bounded below by the curve $y = \frac{1}{4}x^2 - 1$, and above by the curve $y = -\frac{1}{8}x^2 + \frac{1}{2}$. Because the region is symmetrical with respect to the y axis, $\bar{x} = 0$. The measure of the total mass is given by



$$M = \iint_R kx^2 dA = k \int_{-2}^{2} \int_{x^2/4 - 1}^{-x^2/8 + 1/2} x^2 dy\, dx$$

$$= k \int_{-2}^{2} x^2[(-\tfrac{1}{8}x^2 + \tfrac{1}{2}) - (\tfrac{1}{4}x^2 - 1)] dx = 2k \int_0^2 (-\tfrac{3}{8}x^4 + \tfrac{3}{2}x^2) dx$$

$$= 2k\Big[-\tfrac{3}{40}x^5 + \tfrac{1}{2}x^3\Big]_0^2 = \tfrac{16}{5}k$$

Furthermore,

$$M_x = \iint_R kx^2 y\, dA = k \int_{-2}^{2} \int_{x^2/4 - 1}^{-x^2/8 + 1/2} x^2 y\, dy\, dx = \tfrac{1}{2}k \int_{-2}^{2} x^2[(-\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - (\tfrac{1}{4}x^2 - 1)^2] dx$$

$$= k\int_0^2 x^2[(-\tfrac{1}{8})^2(x^2-4)^2-(\tfrac{1}{4})^2(x^2-4)^2]dx = -\tfrac{3}{64}\int_0^2 x^2(x^2-4)^2dx = -\tfrac{3}{64}\int_0^2(x^6-8x^4+16x^2)dx$$

$$= -\tfrac{3}{64}\Big[\tfrac{1}{7}x^7-\tfrac{8}{5}x^5+\tfrac{16}{3}x^3\Big]_0^2 = -\tfrac{16}{35}k$$

Thus,

$$\bar{y} = \frac{1}{M}\cdot M_x = \frac{5}{16k}\Big(-\frac{16k}{35}\Big) = -\frac{1}{7}$$

Therefore, the center of mass is $(0, -\frac{1}{7})$.

41. $M = \lim\limits_{\|\Delta\|\to 0}\sum\limits_{i=1}^{n}(\bar{r}_i\theta_i)(\bar{r}_i\Delta_i r\,\Delta_i\theta) = \iint_R r^2\theta dr\,d\theta = \int_0^{\pi/4}\int_0^{\cos 2\theta} r^2\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{3}r^3\theta\Big]_0^{\cos 2\theta}d\theta$

$= \frac{1}{3}\int_0^{\pi/4}\theta\cos^3 2\theta\,d\theta = \frac{1}{3}\int_0^{\pi/4}\theta\{(1-\sin^2 2\theta)\cos 2\theta\,d\theta\} = \frac{1}{6}\Big[\theta\Big(\sin 2\theta - \frac{1}{3}\sin^3 2\theta\Big)\Big]_0^{\pi/4} - \frac{1}{6}\int_0^{\pi/4}\Big(\sin 2\theta - \frac{1}{3}\sin^3 2\theta\Big)d\theta$

$= \frac{1}{6}\cdot\frac{\pi}{4}\cdot\frac{2}{3} - \frac{1}{6}\int_0^{\pi/4}\Big(\frac{2}{3}+\frac{1}{3}\cos^2 2\theta\Big)\sin 2\theta\,d\theta = \frac{\pi}{36} + \frac{1}{12}\Big[\frac{2}{3}\cos 2\theta + \frac{1}{9}\cos^3 2\theta\Big]_0^{\pi/4} = \frac{\pi}{36} - \frac{1}{12}\cdot\frac{7}{9} = \frac{3\pi-7}{108}$

42. $I_x = \iint_R y^2k\sqrt{x^2+y^2}dA = k\int_0^a\int_0^{2\pi}r^2\sin^2\theta\cdot r\cdot r\,dr\,d\theta = k\int_0^a r^4dr\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)d\theta = k\cdot\frac{1}{5}a^5\cdot\pi$

43. $x^2+y^2 = 4z$; $r^2 = 4z$ and $x^2+y^2 = 4ay$; $r^2 = 4ar\sin\theta$; $r = 4a\sin\theta$.

$V = \int_0^{\pi}\int_0^{4a\sin\theta}\int_0^{r^2/4} r\,dz\,d\theta = \frac{1}{4}\int_0^{\pi}\int_0^{4a\sin\theta} r^3dr\,d\theta = \frac{1}{16}\int_0^{\pi} r^4\Big]_0^{4a\sin\theta}d\theta = 16a^4\int_0^{\pi}\sin^4\theta\,d\theta$

$= 16a^4\Big[-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\Big(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\Big)\Big]_0^{\pi} = 16a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi = 6\pi a^4$

44. Use spherical coordinates to find the mass of a spherical solid of radius a meters if the volume density at each point is proportional to the distance of the point from the center of the sphere. The volume density is measured in kg/m^3.

▸ With the center of the sphere at the origin, a spherical equation of the sphere is $\rho = a$. We are given that the volume density is $k\rho$ kg/m^3. The measure of the mass of the solid is given by

$$M = \int_0^{\pi}\int_0^{2\pi}\int_0^{a}(k\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = k\Big[\int_0^{\pi}\sin\phi\,d\phi\Big]\cdot\Big[\int_0^{2\pi}d\theta\Big]\cdot\Big[\int_0^{a}\rho^3\Big] = -k\cos\phi\Big]_0^{\pi}\cdot 2\pi\cdot\tfrac{1}{4}a^4 = \pi ka^4$$

The mass is πka^4 kg.

45. $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}[r(4-r^2)^{1/2}-r]dr\,d\theta = \int_0^{2}\Big[-\frac{1}{3}(4-r^2)^{3/2}-\frac{1}{2}r^2\Big]_0^{\sqrt{3}}d\theta$

$= 2\pi\Big[(-\frac{1}{3}-\frac{3}{2})-(-\frac{8}{3})\Big] = \frac{5\pi}{3}$

46. $V = \int_0^2\int_0^{2x^2}\int_0^{8-y}dz\,dy\,dx = \int_0^2\int_0^{2x^2}(8-y)dy\,dx = \int_0^2\Big[8y-\frac{1}{2}y^2\Big]_0^{2x^2}dx = \int_0^2(16x^2-2x^4)dx$

$= \Big[\frac{16}{3}x^3-\frac{2}{5}x^5\Big]_0^2 = \frac{448}{15}$

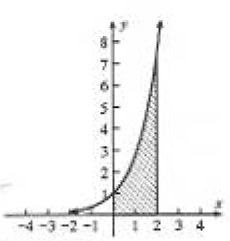
In Exercises 47 and 48, a lamina is the shape of the region bounded by the curve $y = e^x$, the line $x = 2$, and the coordinate axes. The area density at any point is xy kg/m^3.

47. If I_x kg-m^2 is the moment of inertia of the lamina about the x axis

$I_x = \lim\limits_{\|\Delta\|\to 0}\sum\limits_{i=1}^{n}v_i(u_iv_i)\Delta_iA = \iint_R xy^3dA = \int_0^2\int_0^{e^x}xy^3dy\,dx = \int_0^2\frac{1}{4}xy^4\Big]_0^{e^x}dx = \frac{1}{4}\int_0^2 xe^{4x}dx$

$= \frac{1}{4}\Big[\frac{1}{4}xe^{4x}-\frac{1}{16}e^{4x}\Big]_0^2 = \frac{1}{64}[(8e^8-e^8)+1] = \frac{1}{64}(7e^8+1)$

48. Find the moment of inertia about the y axis of the lamina.



▸ The figure shows the region. The measure of the moment of inertia about the y axis is given by

$$I_y = \iint_R x^3y\,dA = \int_0^2\int_0^{e^x}x^3y\,dy\,dx = \int_0^2 x^3\{\tfrac{1}{2}e^{2x}\,dx\}$$

We use integration by parts repeatedly with dv indicated with braces. Thus,

$$I_y = \Big[\tfrac{1}{4}x^3e^{2x}-\tfrac{3}{4}\int x^2\{e^{2x}dx\}\Big]_0^2 = \Big[\tfrac{1}{4}x^3e^{2x}-\tfrac{3}{8}x^2e^{2x}+\tfrac{3}{4}\int x\{e^{2x}dx\}\Big]_0^2$$

$$= \Big[\tfrac{1}{4}x^3e^{2x}-\tfrac{3}{8}x^2e^{2x}+\tfrac{3}{8}xe^{2x}-\tfrac{3}{8}\int e^{2x}dx\Big]_0^2 = \tfrac{1}{16}e^{2x}(4x^3-6x^2+6x-3)\Big]_0^2 = \tfrac{1}{16}(17e^4+3)$$

Thus, the moment of inertia is $\frac{1}{16}(17e^4+3)$ kg-m^2.

In Exercises 49–51, lamina is the region bounded by the lemniscate $r^2 = 4\cos 2\theta$ and the area density is k kg/m^2.

49. If I_y kg-m^2 is the moment of inertia of the lamina about the y axis,

$$I_y = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (\bar{r}_i \cos \bar{\theta}_i)^2 (k)\Delta_i A = k \iint_R r^2\cos^2\theta \, r\, dr\, d\theta = 2k\int_{-\pi/4}^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r^3\cos^2\theta \, dr\, d\theta$$

$$= 2k\int_{-\pi/4}^{\pi/4} \frac{1}{4} r^4\Big]_0^{2\sqrt{\cos 2\theta}} \cos^2\theta\, d\theta = 8k\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \cos^2\theta\, d\theta \overset{\text{even}}{=} 16k\int_0^{\pi/4}\cos^2 2\theta\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 8k\int_0^{\pi/4}(\cos^2 2\theta + \cos^3 2\theta)d\theta = 8k\int_0^{\pi/4}\left[\frac{1}{2}+\frac{1}{2}\cos 4\theta + (1-\sin^2 2\theta)\cos 2\theta\right]d\theta$$

$$= 8k\left[\frac{1}{2}\theta + \frac{1}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta - \frac{1}{6}\sin^3 2\theta\right]_0^{\pi/4} = 8k\left(\frac{1}{8}\pi + \frac{1}{2} - \frac{1}{6}\right) = k\left(\pi + \frac{8}{3}\right)$$

50. $M = 4k\int_0^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\, dr\, d\theta = 8k\int_0^{\pi/4}\cos 2\theta\, d\theta = 4k\sin 2\theta\Big]_0^{\pi/4} = 4k$. The mass is $4k$ kg.

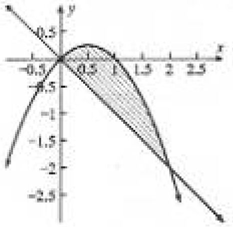
51. If I_0 kg-m^2 is the polar moment of inertia of the lamina,

$$I_0 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i^2(k)\Delta_i A = k\iint_R r^2(r\, dr\, d\theta) = 2k\int_{-\pi/4}^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r^3 dr\, d\theta = 2k\int_{-\pi/4}^{\pi/4}\frac{1}{4}r^4\Big]_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$= 8k\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\, d\theta = 8k\int_0^{\pi/4}(1+\cos 4\theta)d\theta = 8k\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = 2\pi k$$

From Exercises 50, we have $M = 4k$. If r meters is the radius of gyration, $r = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{2\pi k}{4k}} = \frac{1}{2}\sqrt{2\pi}$.

52. Find the moment of inertia about the y axis of the lamina in the shape of the region bounded by the parabola $y = x - x^2$ and the line $x + y = 0$, if the area density at any point is $(x+y)$kg/m^2.



▹ The figure shows the region R. Let $\rho(x,y) = x + y$. The number of units in the moment of inertia about the y axis is

$$I_y = \iint_R \rho(x,y)x^2 dA = \int_0^2\int_{-x}^{x-x^2} x^2(x+y)dy\, dx = \int_0^2 x^2\left[\frac{1}{2}(x+y)^2\right]_{-x}^{x-x^2} dx$$

$$= \frac{1}{2}\int_0^2 x^2(2x-x^2)^2 dx = \frac{1}{2}\int_0^2 (x^6 - 4x^5 + 4x^4)dx = \frac{1}{2}\left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + \frac{4}{5}x^5\right]_0^2$$

$$= \frac{2^7}{14} - \frac{2^6}{3} + \frac{2^6}{5} = 2^6\left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}\right) = \frac{64}{105}$$

The moment of inertia is $\frac{64}{105}$ kg-m^2.

In Exercises 53 and 54, the solid is bounded by the spheres $\rho = 2$ and $\rho = 3$. $p = k\rho$ kg/m^2.

53. $M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(k\rho_i)\Delta_i V = \iiint_S k\rho\, dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 k\rho\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = k\int_0^{\pi}\int_0^{2\pi}\frac{1}{4}\rho^4\Big]_2^3$

$$= \frac{65}{4}k\int_0^{\pi}\int_0^{2\pi}\sin\phi\, d\theta\, d\phi = \frac{65}{4}k\int_0^{\pi}\theta\Big]_0^{2\pi}\sin\phi\, d\phi = \frac{65}{2}\pi k\int_0^{\pi}\sin\phi\, d\phi = \frac{65}{2}\pi k\Big[-\cos\phi\Big]_0^{\pi} = 65\pi k$$

54. $M_z = \int_0^{\pi}\int_0^{2\pi}\int_2^3(\rho\sin\phi)^2\cdot k\rho\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = 2\pi k\int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi\int_2^3\rho^5 d\rho$

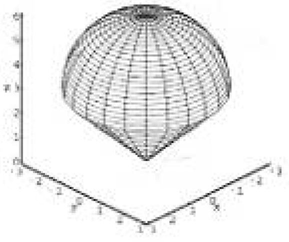
$$= 2\pi k\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi}\cdot\frac{665}{6} = \frac{2660}{9}\pi k$$

55. The solid is bounded by $z^2 = 4x^2 + 4y^2$; $z^2 = 4r^2$, $z = 0$ and $z = 4$. If I_z is in kg-m^2,

$$I_z = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i^2 k\,\Delta_i V = \iiint_S kr^2 dV = k\int_0^{2\pi}\int_0^2\int_{2r}^4 r^2(r\, dz\, dr\, d\theta) = k\int_0^{2\pi}\int_0^2 r^3 z\Big]_{2r}^4 dr\, d\theta$$

$$= 2k\int_0^{2\pi}\int_0^2(4r^3 - 2r^4)dr\, d\theta = 2k\int_0^{2\pi}\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 d\theta = 2k\int_0^{2\pi}\frac{8}{5}d\theta = 2k\cdot\frac{8}{5}\cdot 2\pi = \frac{32}{5}\pi k$$

56. Find the center of mass of the solid bounded by the sphere $x^2 + y^2 + z^2 - 6z = 0$ and the cone $x^2 + y^2 = z^2$ and above the cone, if the volume density at any point is kz kg/m^3.



▹ Eliminating $x^2 + y^2$ from the given equations, we have $6z - z^2 = z^2$, $z(z-3) = 0$. Thus, the surfaces intersect in the planes $z = 0$ and $z = 3$. Furthermore, by completing the square we have for the equation of the sphere

$x^2 + y^2 + (z-3)^2 = 9$

The sphere has center $(0,0,3)$ and radius 3. The figure shows the solid S. Because the sphere and cone contain the origin, we use spherical coordinates. Because

$x^2+y^2+z^2=\rho^2$ and $z=\rho\cos\phi$
the given equation of the sphere becomes

$$\rho^2=6\rho\cos\phi$$
$$\rho=6\cos\phi$$

in spherical coordinates. Because on the cone $z=r$, the equation of the cone is

$$\phi=\tfrac{1}{4}\pi$$

Thus, S is bounded by the surfaces $\rho=0$ and $\rho=6\cos\phi$, bounded by the cones $\phi=0$ and $\phi=\frac{1}{4}\pi$, for $0\le\theta\le 2\pi$. Hence, the measure of the mass of the solid is

$$\mathrm{M}=\iiint_S kz\,d\mathrm{V}=\int_0^{\pi/4}\int_0^{2\pi}\int_0^{6\cos\phi}(k\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\tfrac{1}{4}k\int_0^{\pi/4}\int_0^{2\pi}\sin\phi\cos\phi(6\cos\phi)^4d\theta\,d\phi$$
$$=648\pi k\int_0^{\pi/4}\sin\phi\cos^5\phi\,d\phi=648\pi k\Big[-\tfrac{1}{6}\cos^6\phi\Big]_0^{\pi/4}=648\pi k(1-\tfrac{1}{8})=\tfrac{189}{2}\pi k$$

Furthermore,

$$\mathrm{M}_{xy}=\iiint_S kz^2d\mathrm{V}=\int_0^{\pi/4}\int_0^{2\pi}\int_0^{6\cos\phi}k(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$=\tfrac{1}{5}k\int_0^{\pi/4}\int_0^{2\pi}\sin\phi\cos^2\phi(6\cos\phi)^5d\theta\,d\phi=\tfrac{15552}{5}\pi k\int_0^{\pi/4}\sin\phi\cos^7\phi\,d\phi=\tfrac{1944}{5}\pi k\Big[-\cos^8\phi\Big]_0^{\pi/4}$$
$$=\tfrac{1944}{5}\pi k(1-\tfrac{1}{16})=\tfrac{729}{2}\pi k$$

Then

$$\bar{z}=\frac{1}{\mathrm{M}}\mathrm{M}_{xy}=\frac{2}{189\pi k}\cdot\frac{729\pi k}{2}=\frac{27}{2}$$

Because S and the density function are symmetric with respect to the z axis, the center of mass lies on the z axis. Thus, the center of mass is $(0,0,\frac{27}{7})$.

FOURTEEN

INTRODUCTION TO THE CALCULUS OF VECTOR FIELDS

14.1 VECTOR FIELDS

14.1.1 Theorem Suppose that M and N are function of two variables x and y defined on an open disk $B((x_0, y_0; r)$ in R^2, and M_y and N_x are continuous on B. Then the vector
$M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
is a gradient on B if and only if
$M_y(x, y) = N_x(x, y)$
at all points in B.

14.1.2 Theorem Let M, N, and R be functions of three variables x, y, and z defined on an open ball $B((x_0, y_0, z_0); r)$ in R^3, and M_y, M_z, N_x, N_z, R_x, and R_y are continuous on B. Then the vector
$M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$
is a gradient on B if and only if
$M_y(x, y, z) = N_x(x, y, z) \quad M_z(x, y, z) = R_x(x, y, z) \quad N_z(x, y, z) = R_y(x, y, z)$
at all points in B.

A *vector field* is a function that associates a vector with a point in space. A *scalar field* is a function that associates a scalar with a point in space.

Conservative If $\mathbf{F}$ is a vector field that is the gradient of some scalar field ϕ, that is, $\mathbf{F} = \nabla\phi$, then $\mathbf{F}$ is called a *conservative vector field* and ϕ is a *potential function* for $\mathbf{F}$. Also $d\phi = M\,dx + N\,dy + R\,dz$ is said to be an *exact differential.*

Let $\mathbf{F}$ be a vector field on some open ball in R^3 such that
$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

Curl The *curl* of $\mathbf{F}$ is defined by

$$\text{curl } \mathbf{F}(x, y, z) = \left(\frac{\partial R}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

if these partial derivatives exist.
In terms of vectors, Theorem 14.1.2 becomes

14.1.6 Theorem Let $\mathbf{F}$ be differentiable on an open ball B in R^3. Then $\mathbf{F} = \nabla f$ if and only if curl $\mathbf{F} \equiv \mathbf{0}$.
If curl $\mathbf{F} \equiv \mathbf{0}$, then $\mathbf{F}$ is said to be *irrotational.* In other words, $\mathbf{F}$ is irrotational if and only if it is conservative.

Divergence The divergence of $\mathbf{F}$, denoted by div $\mathbf{F}$, is defined by

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$

if these partial derivatives exist.
If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$ and $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$

14.1.5 Theorem Let $\mathbf{G}$ be differentiable on an open ball B. Then $\mathbf{G} = \text{curl } \mathbf{F}$ if and only if div $\mathbf{G} \equiv 0$. See Ex. 47.
If div $\mathbf{G} \equiv 0$, then $\mathbf{G}$ is said to be *solenoidal.*

If the del operator in three dimensions is defined by $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ then

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & R \end{vmatrix} \quad \text{and} \quad \text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

The Laplacian of f is given by

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

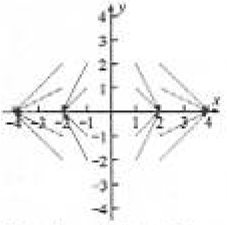
Two Dimensions If $\mathbf{F}$ is a vector field on some open disk in R^2, such that
$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
then $\text{curl } \mathbf{F}(x, y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$ and $\text{div } \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

Exercises 14.1

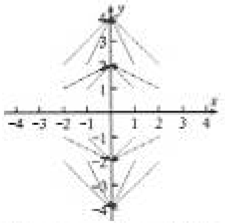
In Exercises 1–6, show on a figure the representations having initial point at (x, y) of the vectors in the vector field, where x is ± 1 or ± 2 and y is ± 1 or ± 2.

1. $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$

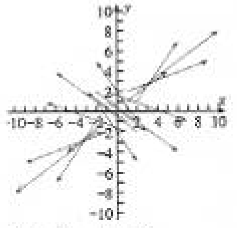
▹



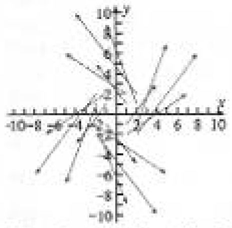
2. $\mathbf{F}(x,y) = -x\mathbf{i} + y\mathbf{j}$



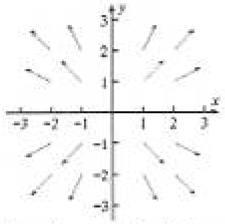
3. $\mathbf{F}(x,y) = 4y\mathbf{i} + 3x\mathbf{j}$



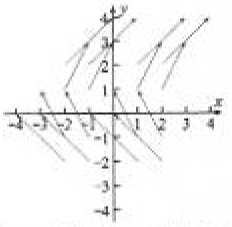
4. $\mathbf{F}(x,y) = -3y\mathbf{i} + 4x\mathbf{j}$



5. $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$



6. $\mathbf{F}(x,y) = y\mathbf{i} + 2\mathbf{j}$



▹ (Exercise 4) The table below lists the function value for $\mathbf{F}$ at each of the given points. For each point (x, y) in the table, we use (x, y) as the initial point, and sketch the geometric representation of the vector $\mathbf{F}(x, y)$. See the figure.

		y			
		-2	-1	1	2
	-2	$6\mathbf{i} - 8\mathbf{j}$	$3\mathbf{i} - 8\mathbf{j}$	$-3\mathbf{i} - 8\mathbf{j}$	$-6\mathbf{i} - 8\mathbf{j}$
x	-1	$6\mathbf{i} - 4\mathbf{j}$	$3\mathbf{i} - 4\mathbf{j}$	$-3\mathbf{i} - 4\mathbf{j}$	$-6\mathbf{i} - 4\mathbf{j}$
	1	$6\mathbf{i} + 4\mathbf{j}$	$3\mathbf{i} + 4\mathbf{j}$	$-3\mathbf{i} + 4\mathbf{j}$	$-6\mathbf{i} + 4\mathbf{j}$
	2	$6\mathbf{i} + 8\mathbf{j}$	$3\mathbf{i} + 8\mathbf{j}$	$-3\mathbf{i} + 8\mathbf{j}$	$-6\mathbf{i} + 8\mathbf{j}$

In Exercises 7–14, find a conservative vector field having the given potential function.

▹ The desired conservative vector field is $\mathbf{F} = \nabla f$.

7. $f(x,y) = 3x^2 + 2y^3$. $\mathbf{F}(x,y) = \nabla f(x,y) = 6x\mathbf{i} + 6y\mathbf{j}$.

8. $f(x,y) = 2x^4 - 5x^2y^2 + 4y^4$

▹ The required conservative vector field $\mathbf{F}$ is the gradient of f. Thus,
$\mathbf{F}(x,y) = \nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = (8x^3 - 10xy^2)\mathbf{i} + (-10x^2y + 16y^3)\mathbf{j}$

9. $f(x,y) = \tan^{-1}x^2y$. $\mathbf{F}(x,y) = \nabla f(x,y) = \dfrac{2xy}{1+x^4y^2}\,\mathbf{i} + \dfrac{x^2}{1+x^4y^2}\,\mathbf{j}$.

10. $f(x,y) = ye^x - xe^y$. $\mathbf{F}(x,y) = \nabla f(x,y) = (ye^x - e^y)\mathbf{i} + (e^x - xe^y)\mathbf{j}$

11. $f(x,y,z) = 2x^3 - 3x^2y + xy^2 - 4y^3$. $\mathbf{F}(x,y,z) = \nabla f(x,y,z) = (6x^2 - 6xy + y^2)\mathbf{i} + (-3x^2 + 2xy - 12y^2)\mathbf{j}$.

12. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

▹ If $\mathbf{F}$ is the required vector field, then

$$\mathbf{F}(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$$

13. $f(x,y,z) = x^2ye^{-4z}$. $\mathbf{F}(x,y,z) = \nabla f(x,y,z) = 2xye^{-4z}\mathbf{i} + x^2e^{-4z}\mathbf{j} - 4x^2ye^{-4z}\mathbf{k}$.

14. $f(x,y,z) = z\sin(x^2 - y)$. $\mathbf{F}(x,y,z) = \nabla f(x,y,z) = 2xz\cos(x^2 - y)\mathbf{i} - z\cos(x^2 - y)\mathbf{j} + \sin(x^2 - y)\mathbf{k}$

In Exercises 15–20, determine if the vector field is conservative.

15. $\mathbf{F}(x,y) = (3x^2 - 2y^2)\mathbf{i} + (3 - 4xy)\mathbf{j} = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$. $\mathrm{M}_y(x,y) = -4y$, $\mathrm{N}_x(x,y) = -4y$.
Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathbf{F}$ is conservative.

16. $\mathbf{F}(x,y) = (e^x e^y + 6e^{2x})\mathbf{i} + (e^x e^y - 2e^y)\mathbf{j}$
▹ Let
$\mathrm{M}(x,y) = e^x e^y + 6e^{2x}$ $\quad$ $\mathrm{N}(x,y) = e^x e^y - 2e^y$
$\mathrm{M}_y(x,y) = e^x e^y$ $\quad$ $\mathrm{N}_x(x,y) = e^x e^y$
Because $\mathrm{M}_y(x,y) = \mathrm{N}_x(x,y)$, it follows that $\mathbf{F}$ is a gradient, and hence a conservative vector field.

17. $\mathbf{F}(x,y) = y\cos(x+y)\mathbf{i} - x\sin(x+y)\mathbf{j} = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$.
$\mathrm{M}_y(x,y) = \cos(x+y) - y\sin(x+y)$ $\quad$ $\mathrm{N}_x(x,y) = -\sin(x+y) - x\cos(x+y)$
Because $\mathrm{M}_y \neq \mathrm{N}_x$, $\mathbf{F}$ is not conservative.

18. $\mathbf{F}(x,y,z) = (3x^2 + 2yz)\mathbf{i} + (2xz + 6yz)\mathbf{j} + (2xy + 3y^2 - 2z)\mathbf{k} = \mathrm{M}(x,y,z)\mathbf{i} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$
$\mathrm{M}_y(x,y,z) = 2z$ $\quad$ $\mathrm{N}_x(x,y,z) = 2z$ $\quad$ $\mathrm{R}_x(x,y,z) = 2y$
$\mathrm{M}_z(x,y,z) = 2y$ $\quad$ $\mathrm{N}_z(x,y,z) = 2x + 6y$ $\quad$ $\mathrm{R}_y(x,y,z) = 2x + 6y$
Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, $\mathbf{F}$ is conservative.

19. $\mathbf{F}(x,y,z) = (2ye^{2x} + e^z)\mathbf{i} + (3ze^{3y} + e^{2x})\mathbf{j} + (xe^z + e^{3y})\mathbf{k} = \mathrm{M}(x,y,z)\mathbf{i} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$
$\mathrm{M}_y(x,y,z) = 2e^{2x}$ $\quad$ $\mathrm{N}_x(x,y,z) = 2e^{2x}$ $\quad$ $\mathrm{R}_x(x,y,z) = e^z$
$\mathrm{M}_z(x,y,z) = e^z$ $\quad$ $\mathrm{N}_z(x,y,z) = 3e^{3y}$ $\quad$ $\mathrm{R}_y(x,y,z) = 3e^{3y}$
Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, $\mathbf{F}$ is conservative.

20. $\mathbf{F}(x,y,z) = y\sec^2 x\mathbf{i} + (\tan x - z\sec^2 y)\mathbf{j} + x\sec z\tan z\mathbf{k}$
▹ We apply Theorem 14.1.2 to determine if $\mathbf{F}$ is a gradient. Let
$\mathrm{M}(x,y,z) = y\sec^2 x$ $\quad$ $\mathrm{N}(x,y,z) = \tan x - z\sec^2 y$ $\quad$ $\mathrm{R}(x,y,z) = x\sec z\tan z$
$\mathrm{M}_y(x,y,z) = \sec^2 x$ $\quad$ $\mathrm{N}_x(x,y,z) = \sec^2 x$ $\quad$ $\mathrm{R}_x(x,y,z) = \sec z\tan z$
$\mathrm{M}_z(x,y,z) = 0$ $\quad$ $\mathrm{N}_z(x,y,z) = -\sec^2 y$ $\quad$ $\mathrm{R}_y(x,y,z) = 0$
We have $\mathrm{M}_y(x,y,z) = \mathrm{N}_x(x,y,z)$. However, because $\mathrm{M}_z(x,y,z) \neq \mathrm{R}_x(x,y,z)$, then $\mathbf{F}$ is not a gradient, and thus not conservative.

In Exercises 21–32, prove that the vector field is conservative and find a potential function.
▹ We may prove that $\mathbf{F}$ is conservative by exhibiting the potential function ϕ such that $\mathbf{F} = \nabla\phi$.

21. $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j} = \nabla\phi$ where $\phi(x,y) = xy + \mathrm{C}$.

22. $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j} = \nabla\phi$ where $\phi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \mathrm{C}$.

23. $\mathbf{F}(x,y) = e^x\sin y\mathbf{i} + e^x\cos y\mathbf{j} = \nabla\phi$ where $\phi(x,y) = e^x\sin y + \mathrm{C}$.

24. $\mathbf{F}(x,y) = (\sin y\sinh x + \cos y\cosh x)\mathbf{i} + (\cos y\cosh x - \sin y\sinh x)\mathbf{j}$
▹ $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is the gradient of some function ϕ. We let
$\mathrm{M}(x,y) = \sin y\sinh x + \cos y\cosh x$
$\mathrm{N}(x,y) = \cos y\cosh x - \sin y\sinh x$
Then
$\mathrm{M}_y(x,y) = \cos y\sinh x - \sin y\cosh x = \mathrm{N}_x(x,y)$
Hence $\mathbf{F}$ is the gradient of some function ϕ. Furthermore, we have
$\phi_x(x,y) = \sin y\sinh x + \cos y\cosh x \quad (1)$
$\phi_y(x,y) = \cos y\cosh x - \sin y\sinh x \quad (2)$
Integrating with respect to x on both sides of (1), we have
$\phi(x,y) = \sin y\cosh x + \cos y\sinh x + c(y) \quad (3)$
Partial differentiating with respect to y on both sides of (3), we have
$\phi_x(x,y) = \cos y\cosh x - \sin y\sinh x + c'(y) \quad (4)$
Comparing (2) and (4) we find
$c'(y) = 0$
$c(y) = \mathrm{C}$
Substituting for $c(y)$ into (3), we obtain the required potential function
$\phi(x,y) = \sin y\cosh x + \cos y\sinh x + \mathrm{C}$

25. $\mathbf{F}(x,y) = (2xy^2 - y^3)\mathbf{i} + (2x^2y - 3xy^2 + 2)\mathbf{j} = \nabla\phi$ where $\phi(x,y) = x^2y^2 - xy^3 + 2y + \mathrm{C}$.

26. $\mathbf{F}(x,y) = (3x^2 + 2y - y^2e^x)\mathbf{i} + (2x^2y - 3xy^2 + 2)\mathbf{j} = \nabla\phi$ where $\phi(x,y) = x^3 + 2xy - y^2e^x + \mathrm{C}$.

27. $\mathbf{F}(x,y,z) = (x^2 - y)\mathbf{i} - (x - 3z)\mathbf{j} + (z + 3y)\mathbf{k} = \nabla\phi$ where $\phi(x,y,z) = \frac{1}{3}x^3 - xy + 3yz + \frac{1}{2}z^2 + \mathrm{C}$.

28. $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

▹ Because **F** is conservative if and only if **F** is a gradient, we let

$\mathrm{M}(x,y,z) = yz \qquad \mathrm{N}(x,y,z) = xz \qquad \mathrm{R}(x,y,z) = xy$

$\mathrm{M}_y(x,y,z) = z \qquad \mathrm{N}_x(x,y,z) = z \qquad \mathrm{R}_x(x,y,z) = y$

$\mathrm{M}_z(x,y,z) = y \qquad \mathrm{N}_z(x,y,z) = x \qquad \mathrm{R}_y(x,y,z) = x$

Because

$\mathrm{M}_y(x,y,z) = \mathrm{N}_x(x,y,z) \quad \mathrm{M}_z(x,y,z) = \mathrm{R}_x(x,y,z) \quad \mathrm{N}_z(x,y,z) = \mathrm{R}_y(x,y,z)$

then **F** is the gradient of a potential function ϕ, such that

$\phi_x(x,y,z) = yz \quad \phi_y(x,y,z) = xz \quad \phi_z(x,y,z) = xy \qquad (1)$

Because $\phi_x(x,y,z) = yz$, then by integrating with respect to x, we have

$\phi(x,y,z) = xyz + g(y,z)$

Partial-differentiating with respect to y on both sides, we get

$\phi_y(x,y,z) = xz + g_y(x,y)$

Because $\phi_y(x,y,z) = xz$ in (1), we conclude that $g_y(y,z) = 0$, and thus $g(y,z) = \mathrm{h}(z)$, and

$\phi(x,y,z) = xyz + h(z)$

Partial-differentiating with respect to z, we have

$\phi_z(x,y,z) = xy + h'(z)$.

And because $\phi_z(x,y,z) = xy$ in (1), then $h'(z) = 0$, and $h(z) = \mathrm{C}$. Thus, the potential function is given by

$\phi(x,y,z) = xyz + \mathrm{C}$

It is perfectly legitimate to *guess* that $\phi(x,y,z) = xyz$ and merely verify Equations (1).

29. $\mathbf{F}(x,y,z) = (ze^x + e^y)\mathbf{i} + (xe^y - e^z)\mathbf{j} + (-ye^z + e^x)\mathbf{k} = \nabla\phi$ where $\phi(x,y,z) = ze^x + xe^y - ye^z + \mathrm{C}$.

30. $\mathbf{F}(x,y,z) = (\tan y + 2xy \sec z)\mathbf{i} + (x\sec^2 y + x^2\sec z)\mathbf{j} + \sec z(x^2y\tan z - \sec z)\mathbf{k}$

$\phi(x,y,z) = x\tan y + x^2y\sec z - \tan z + \mathrm{C}$

31. $\mathbf{F}(x,y,z) = (2x\cos y - 3)\mathbf{i} - (x^2\sin y + z^2)\mathbf{j} - (2yz - 2)\mathbf{k} = \nabla\phi$ where $\phi = x^2\cos y - 3x - yz^2 + 2z + \mathrm{C}$.

32. $\mathbf{F}(x,y,z) = (2y^3 - 8xz^2)\mathbf{i} + (6xy^2 + 1)\mathbf{j} - (8x^2z + 3z^2)\mathbf{k}$

▹ We have

$\mathrm{M}(x,y,z) = 2y^3 - 8xz^2 \qquad \mathrm{N}(x,y,z) = 6xy^2 + 1 \qquad \mathrm{R}(x,y,z) = -(8x^2z + 3z^2)$

$\mathrm{M}_y(x,y,z) = 6y^2 \qquad \mathrm{N}_x(x,y,z) = 6y^2 \qquad \mathrm{R}_x(x,y,z) = -16xz$

$\mathrm{M}_z(x,y,z) = -16xz \qquad \mathrm{N}_z(x,y,z) = 0 \qquad \mathrm{R}_y(x,y,z) = 0$

Because

$\mathrm{M}_y(x,y,z) = \mathrm{N}_x(x,y,z) \quad \mathrm{M}_z(x,y,z) = \mathrm{R}_x(x,y,z) \quad \mathrm{N}_z(x,y,z) = \mathrm{R}_y(x,y,z)$

then **F** is the gradient of a potential function ϕ, such that

$\phi_x(x,y,z) = 2xy^3 - 8xz^2 \quad \phi_y(x,y,z) = 6xy^2 + 1 \quad \phi_z(x,y,z) = -8x^2z - 3z^2 \qquad (1)$

Integrating the first equation of (1) with respect to x, we get

$\phi(x,y,z) = 2xy^3 - 4x^2 + g(y,z) \qquad (2)$

Partial-differentiating with respect to y on both sides, we get

$\phi_y(x,y,z) = 6xy^2 + g_y(y,z)$

Comparing with the second equation in (1), we get

$g_y(y,z) = 1$

$g(y,z) = y + h(z)$

Substituting the value for $g(y,z)$ into (2), we have

$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y + h(z) \qquad (3)$

Partial-differentiating with respect to z, we have

$\phi_z(x,y,z) = \; = -8x^2z + h'(z)$

Comparing with the third equation in (1), we have

$h'(z) = -3z^2$

$h(z) = -z^3 + \mathrm{C}$

Substituting into (3), we obtain the potential function

$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y - z^3 + \mathrm{C}$

In Exercises 33–42, find curl $\mathbf{F}$ and div $\mathbf{F}$ for the vector field.

33. $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y\mathbf{j}$. $M(x, y) = 2x$, $N(x, y) = 3y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} = \mathbf{0}$; div $\mathbf{F} = M_x + N_y = 2 + 3 = 5$.

34. $\mathbf{F}(x, y) = \cos x\mathbf{i} - \sin y\mathbf{j}$. $M = \cos x$, $N = -\sin y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} = \mathbf{0}$; div $\mathbf{F} = M_x + N_y = -\sin x - \cos y$

35. $\mathbf{F}(x, y) = e^x\cos y\mathbf{i} + e^x\sin y\mathbf{j}$. $M(x, y) = e^x\cos y$, $N(x, y) = e^x\sin y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} =$ $(e^x\sin y + e^x\sin y)\mathbf{k} = 2e^x\sin y\mathbf{k}$. div $\mathbf{F} = M_x + N_y = e^x\cos y + e^x\cos y = 2e^x\cos y$.

36. $\mathbf{F}(x, y) = -\frac{y}{x}\mathbf{i} + \frac{1}{x}\mathbf{j}$

▹ Let $M(x, y) = -\dfrac{y}{x}$ $\quad N(x, y) = \dfrac{1}{x}$

Then

$$\text{curl } \mathbf{F}(x, y) = [N_x(x, y) - M_y(x, y)]\mathbf{k} = \left(-\frac{1}{x^2} + \frac{1}{x}\right)\mathbf{k} = \frac{x - 1}{x^2}\mathbf{k}$$

and

$$\text{div } \mathbf{F}(x, y) = M_x(x, y) + N_y(x, y) = \frac{y}{x^2}$$

37. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$. $M(x, y, z) = x^2$, $N(x, y, z) = y^2$, $R(x, y, z) = z^2$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = M_x + N_y + R_z = 2x + 2y + 2z$.

38. $\mathbf{F}(x, y, z) = xz^2\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}$. $M(x, y, z) = xz^2$, $N(x, y, z) = y^2$, $R(x, y, z) = x^2z$
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = M_x + N_y + R_z = z^2 + 2y + x^2$.

39. $\mathbf{F}(x, y, z) = \cos y\mathbf{i} + \cos z\mathbf{j} + \cos x\mathbf{k}$. $M(x, y, z) = \cos y$, $N(x, y, z) = \cos z$, $R(x, y, z) = \cos x$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \sin z\mathbf{i} + \sin x\mathbf{j} + \sin y\mathbf{k}$. div $\mathbf{F} = M_x + N_y + R_z = 0$.

40. $\mathbf{F}(x, y, z) = (y^2 + z^2)\mathbf{i} + xe^y\cos z\mathbf{j} - xe^y\cos z\mathbf{k}$

▹ curl $\mathbf{F}(x, y, z)$

$$= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 + z^2 & xe^y\cos z & -xe^y\cos z \end{vmatrix}$$

$$= [D_y(-xe^y\cos z) - D_z(xe^y\cos z)]\mathbf{i} - [D_x(-xe^y\cos z) - D_z(y^2 + z^2)]\mathbf{j} + [D_x(xe^y\cos z) - D_y(y^2 + z^2)]\mathbf{k}$$

$$= (-xe^y\cos z + xe^y\sin z)\mathbf{i} + (e^y\cos z)\mathbf{j} + (e^y\cos z - 2y)\mathbf{k}$$

$$\text{div } \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F} = D_x(y^2 + z^2) + D_y(xe^y\cos z) + D_z(-xe^y\cos z)$$

$$= 0 + xe^y\cos z + xe^y\sin z = xe^y(\cos z + \sin z)$$

41. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + 1}\mathbf{i} + \sqrt{x^2 + y^2 + 1}\mathbf{j} + z^2\mathbf{k}$. $M = \sqrt{x^2 + y^2 + 1}$, $N = \sqrt{x^2 + y^2 + 1}$, $R = z^2$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \dfrac{x - y}{\sqrt{x^2 + y^2 + 1}}\mathbf{k}$. div $\mathbf{F} = \dfrac{x + y}{\sqrt{x^2 + y^2 + 1}} + 2z$.

42. $\mathbf{F}(x, y, z) = \dfrac{x}{(x^2 + y^2)^{3/2}}\mathbf{i} + \dfrac{y}{(x^2 + y^2)^{3/2}}\mathbf{j} + \mathbf{k}$. $M = \dfrac{x}{(x^2 + y^2)^{3/2}}$, $N = \dfrac{y}{(x^2 + y^2)^{3/2}}$, $R = 1$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = -\dfrac{1}{(x^2 + y^2)^{3/2}}$

In Exercises 43–46, prove that the scalar function is harmonic by showing that its Laplacian is zero.

43. $f(x, y) = e^y\sin x + e^x\cos y$. $\dfrac{\partial f}{\partial x} = e^y\cos x + e^x\cos y$, $\dfrac{\partial f}{\partial y} = e^y\sin x - e^x\sin y$.
$\dfrac{\partial^2 f}{\partial x^2} = -e^y\sin x + e^x\cos y$, $\dfrac{\partial^2 f}{\partial y^2} = e^y\sin x - e^x\cos y$. Because $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$, f is harmonic.

44. $f(x,y) = \ln\sqrt{x^2+y^2}$

▹ We have

$f(x,y) = \frac{1}{2}\ln(x^2+y^2)$

Then,

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \text{ and } \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Therefore, the Laplacian of f is

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial f^2}{\partial y^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

Because the Laplacian of f is zero, then f is harmonic.

45. $f(x,y,z) = 2x^2 + 3y^2 - 5z^2$. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = 6y$, $\frac{\partial f}{\partial z} = -10z$.

$\frac{\partial^2 f}{\partial x^2} = 4$, $\frac{\partial^2 f}{\partial y^2} = 6$, $\frac{\partial f^2}{\partial z^2} = -10$. Because $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$, f is harmonic.

46. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$. $f_x = -\frac{1}{2}x(x^2+y^2+z^2)^{-3/2}$.

$f_{xx} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} + \frac{3}{2}x^2(x^2+y^2+z^2)^{-5/2}$

$= \frac{1}{2}(x^2+y^2+z^2)^{-3/2}[-(x^2+y^2+z^2) + 3x^2] = \frac{1}{2}(x^2+y^2+z^2)(-2x^2+y^2+z^2)$. The sum of 3 such terms is 0.

47. Th 14.1.5. $\text{div}(\text{curl } \mathbf{F}) = \text{div}[(R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k}] = (R_{yx} - N_{zx}) + (M_{zy} - R_{xy}) + (N_{xz} - M_{yz})$
The mixed partial derivatives are equal if they are continuous (Th. 12.3.3) and so the sum is zero. Conversely, let $\mathbf{G} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ with $\text{div}\,\mathbf{G} = P_x + Q_y + R_z = 0$. We shall find $\mathbf{F} = S\mathbf{i} + T\mathbf{j}$ such that $\mathbf{G} = \text{curl } \mathbf{F}$, i.e. $P = -T_z$, $Q = S_z$, and $R = T_x - S_y$. For fixed x, y and any function $c(x,y)$, the first two are satisfied by
$T = -\int_{t=0}^{z} P(x,y,t)dt$ and $S = \int_{t=0}^{z} Q(x,y,t)dt + c(x,y)$. To satisfy the third we need
$0 = T_x - S_y - R = -\int_{t=0}^{z} (P_x(x,y,t) + Q_y(x,y,t))dt - c_y(x,y) - R(x,y,z)$
$= \int_{t=0}^{z} R_z(x,y,t)dt - c_y(x,y) - R(x,y,z)$ {hypothesis}
$= R(x,y,z) - R(x,y,0) - c_y(x,y) - R(x,y,z) = -R(x,y,0) - c_y(x,y)$
which is satisfied by $c(x,y) = -\int_{t=0}^{y} R(x,t,0)dt$. If $\mathbf{H}$ is any other solution then $\text{curl}(\mathbf{G} - \mathbf{H}) = \mathbf{G} - \mathbf{G} = 0$ so the difference is the gradient of a scalar function.

48. Prove Theorem 14.1.6.

▹ If f is a scalar field on an open ball in R^3, then the gradient of f is given by

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

The curl of the gradient of f is

$$\text{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

Because the second partial derivatives of f are continuous on B, then by Theorem 12.3.3,

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Therefore, $\text{curl}(\nabla f) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

49. Because $M\mathbf{i} + N\mathbf{j} + R\mathbf{k}$ is a gradient, there exists a function f such that $f_x = M$, $f_y = N$, $f_z = R$. Because M_y, M_z, N_x, N_z, R_x, and R_y exist and are continuous on B, it follows from Theorem 12.3.3 that
$M_y = f_{xy} = f_{yx} = N_x$, $M_z = f_{xz} = f_{zx} = R_x$, and $N_z = f_{yz} = f_{zy} = R_y$.

14.2 LINE INTEGRALS

14.2.2 Definition Let C be a smooth curve lying in an open disk B in R^2 and having the vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad a \le t \le b$

Let **F** be a vector field on B defined by

$\mathbf{F}(x,y) = \mathrm{M}(x,y)\mathbf{j} + \mathrm{N}(x,y)\mathbf{j}$

where M and N are continuous on B. Then, by using the differential form notation, the line integral of $\mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$ over C is given by

$$\int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy = \int_a^b [\mathrm{M}(f(t),g(t))f'(t) + \mathrm{N}(f(t),g(t))g'(t)]dt$$

or, equivalently, by using vector notation, the line integral of **F** over C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt \tag{I}$$

The line integral has the same value for any parametric representation of C. Furthermore, if C is closed curve, the value is the same for any initial point. The differential form notation seems to be easier to apply. See Exercise 4.

14.2.3 Definition Let the curve C consist of smooth arcs $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n$. Then the *line integral* of $\mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$ over C is defined by the following equation:

$$\int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy = \sum_{i=1}^{n}\left(\int_{C_i} \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy\right)$$

or, equivalently, by using vector notation, the line integral of **F** over C is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \sum_{i=1}^{n}\left(\int_{a_i}^{b_i} \mathbf{F}(\mathbf{R}(t) \cdot \mathbf{R}'(t)dt\right)$$

Line integrals are used to calculate the total work done by a variable force that causes a particle to move along a curve.

14.2.1 Definition Let C be a smooth curve lying in the open disk B in R^2 for which a vector equation is $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, $a \le t \le b$. Furthermore, let a force field on B be defined by $\mathbf{F}(x,y) = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$, where M and N are continuous on B. If W is the measure of the work done by a force **F** moving a particle along C from $(f(a), g(a))$ to $(f(b), g(b))$, then

$$\mathrm{W} = \int_a^b [\mathrm{M}(f(t),g(t))f'(t) + \mathrm{N}(f(t),g(t))g'(t)]dt$$

or, equivalently, by using vector notation,

$$\mathrm{W} = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt$$

or, equivalently, using a line integral in differential form notation,

$$\mathrm{W} = \int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$$

The following definitions extend the concept of a line integral to functions of three variables.

Smooth, Closed The curve C having the vector equation $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$, is *smooth* on $[a,b]$ if f', g', and h' are continuous and never all 0 in (a,b). C is *closed* if $\mathbf{R}(a) = \mathbf{R}(b)$.

14.2.4 Definition Let C be a smooth curve lying in an open ball in R^3 and having the vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \le t \le b$

Let **F** be a vector field on B defined by

$\mathbf{F}(x,y) = \mathrm{M}(x,y,z)\mathbf{j} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$

where M, N, and R are continuous on B. Then the line integral of $\mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz$ over C is given by

$$\int_C \mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz =$$
$$\int_a^b [\mathrm{M}(f(t),g(t),h(t))f'(t) + \mathrm{N}(f(t),g(t),h(t))g'(t) + \mathrm{R}(f(t),g(t),h(t)]dt$$

or, equivalently, by using vector notation, the line integral of **F** over C is given by Eq. (I).

Exercises 14.2

In Exercises 1–22, evaluate the line integral over the curve.

1. $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$. $C{:}\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt = \int_0^1 \langle t^2, t\rangle \cdot \langle 1, 2t\rangle dt = \int_0^1 3t^2 dt = t^3\Big]_0^1 = 1$$

2. $\mathbf{F}(x,y) = 2xy\mathbf{i} - 3x\mathbf{j}$; C: $\mathbf{R}(t) = 3t^2\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt = \int_0^1 \langle 2(3t^2)(-t), -3(3t^2)\rangle \cdot \langle 6t, -1\rangle dt = \int_0^1 (-36t^4 + 9t^2)dt = -\tfrac{36}{5}t^5 + 3t^3\Big]_0^1 = -\tfrac{21}{5}$$

3. $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x-2y)\mathbf{j}$. C:$\mathbf{R}(t) = \sin t\mathbf{i} - 2\cos t\mathbf{j}$, $0 \le t \le \pi$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^\pi \langle 2\sin t(-2\cos t), \sin t + 4\cos t\rangle \cdot \langle \cos t, 2\sin t\rangle dt$$

$$= \int_0^\pi (-4\sin t\cos^2 t + 2\sin^2 t + 8\sin t\cos t)dt = \Big[\tfrac{4}{3}\cos^3 t + \Big(t - \tfrac{1}{2}\cos 2t\Big) + 4\sin^2 t\Big]_0^\pi = \pi - \tfrac{8}{3}$$

4. $\int_C \mathbf{F} \cdot d\mathbf{R}$; $\mathbf{F}(x,y) = xy\mathbf{i} - y^2\mathbf{j}$; C: $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, from the point $(1,1)$ to the point $(4,-8)$.

▹ We apply Definition 14.2.2. Because

$$\mathbf{F}(x,y) = \langle xy, -y^2\rangle \quad \text{and} \quad \mathbf{R}(t) = \langle t^2, t^3\rangle$$

then

$$\mathbf{F}(\mathbf{R}(t)) = \mathbf{F}(t^2, t^3) = \langle t^2t^3, -(t^3)^2\rangle = \langle t^5, -t^6\rangle \quad \text{and} \quad \mathbf{R}'(t) = \langle 2t, 3t^2\rangle$$

Furthermore, because $\mathbf{R}(1) = \langle 1,1\rangle$ and $\mathbf{R}(-2) = \langle 4,-8\rangle$, then $t = 1$ at the point $(1,1)$ and $t = -2$ at the point $(4,-8)$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt = \int_1^{-2} \langle t^5, -t^6\rangle \cdot \langle 2t, 3t^2\rangle dt = \int_1^{-2} (2t^6 - 3t^8)dt = \tfrac{2}{7}t^7 - \tfrac{1}{3}t^9\Big]_1^{-2} = \tfrac{939}{7}$$

We may also use the differential form notation. We have

$$x = t^2 \qquad y = t^3$$
$$dx = 2t\,dt \qquad dy = 3t^2\,dt$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C xy\,dx - y^2dy = \int_1^{-2} t^2(t^3)(2t\,dt) - (t^3)^2(3t^2dt) = \int_1^{-2} (2t^6 - 3t^8)dt = \tfrac{2}{7}t^7 - \tfrac{1}{3}t^9\Big]_1^{-2} = \tfrac{939}{7}$$

5. $\mathbf{F}(x,y) = (x-y)\mathbf{i} + (x+y)\mathbf{j}$. C: $\mathbf{R}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le 2\pi$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} \langle 2\cos t - 2\sin t, 2\cos t + 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle dt$$

$$= \int_0^{2\pi} (-4\sin t\cos t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t)dt = \int_0^{2\pi} 4\,dt = 8\pi$$

6. $x = 3\cos t$, $y = 2\sin t$. $\displaystyle\int_C (x-2y)dx + xy\,dy = \int_0^{\pi/2} (3\cos t - 4\sin t)(-3\sin t\,dt) + 6\sin t\cos t(2\cos t\,dt)$

$$= \int_0^{\pi/2} [-9\cos t\sin t + 6(1 - \cos 2t) + 12\cos^2 t\sin t]dt = -\tfrac{9}{2}\sin^2 t + 6t - 3\sin 2t - 4\cos^3 t\Big]_0^{\pi/2} = 3\pi - \tfrac{1}{2}$$

7. $\mathbf{F}(x,y) = y\sin x\mathbf{i} - \cos x\mathbf{j}$. C: $\mathbf{R}(t) = \frac{1}{2}(t+1)\pi\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \langle t\sin(t+1)\tfrac{\pi}{2}, -\cos(t+1)\tfrac{\pi}{2}\rangle \cdot \langle \tfrac{\pi}{2}, 1\rangle dt = \int_0^1 \Big[\tfrac{\pi}{2}t\sin(t+1)\tfrac{\pi}{2} - \cos(t+1)\tfrac{\pi}{2}\Big]dt = -t\cos(t+1)\tfrac{\pi}{2}\Big]_0^1 = 1$$

8. $\int_C \mathbf{F} \cdot d\mathbf{R}$; $\mathbf{F}(x,y) = 9x^2y\mathbf{i} + (5x^2 - y)\mathbf{j}$; C: the curve $y = x^3 + 1$ from $(1,2)$ to $(3,28)$.

▹ Because the equation of C is expressed in terms of x, we apply Definition 14.2.2 with parameter x. We replace y by $x^3 + 1$ and dy by $3x^2dx$ with $1 \le x \le 3$. We have

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C 9x^2y\,dx + (5x^2 - y)dy = \int_1^3 9x^2(x^3+1)dx + [5x^2 - (x^3+1)](3x^2\,dx)$$

$$= \int_1^3 (9x^5 + 9x^2 + 15x^4 - 3x^5 - 3x^2)dx = \int_1^3 (6x^5 + 15x^4 + 6x^2)dx = x^6 + 3x^5 + 2x^3\Big]_1^3 = 1{,}506$$

9. C: $y = x$ from $(0,0)$ to $(2,2)$. With x as parameter,

$$\int_C (x^2 + xy)dx + (y^2 - xy)dy = \int_0^2 (x^2 + x^2)dx + (x^2 - x^2)dx = \int_0^2 2x^2dx = \frac{2}{3}x^3\Big]_0^2 = \frac{16}{3}$$

10. C: $y = \frac{1}{2}x^2$ from $(0,0)$ to $(2,2)$. With x as parameter, $\displaystyle\int_C (x^2 + xy)dx + (y^2 - xy)dy$

$$= \int_0^2 (x^2 + \tfrac{1}{2}x^3)dx + (\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3)(x\,dx) = \int_0^2 (x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^5 - \tfrac{1}{2}x^4)dx = \tfrac{1}{3}x^3 + \tfrac{1}{8}x^4 + \tfrac{1}{24}x^6 - \tfrac{1}{10}x^5\Big]_0^2 = \tfrac{62}{15}$$

11. On C_1, x is parameter, $0 \le x \le 2$, $y = 0$. On C_2, y is parameter, $0 \le y \le 2$, $x = 2$.

$$\int_C (x^2 + xy)dx + (y^2 - xy)dy = \int_0^2 x^2dx + \int_0^2 (y^2 - 2y)dy = \frac{1}{3}x^3\Big]_0^2 + \Big[\frac{1}{3}y^3 - y^2\Big]_0^2 = \frac{8}{3} + \Big(\frac{8}{3} - 4\Big) = \frac{4}{3}$$

12. $\int_C yx^2dx + (x+y)dy$, where C is the line $y = -x$ from the origin to the point $(1,-1)$.

▹ We let x be the parameter, and replace y by $-x$ and dy by $-dx$ with $0 \le x \le 1$. Thus,

$$\int_C yx^2dx + (x+y)dy = \int_0^1 -x^3dx + (x-x)(-dx) = -\int_0^1 x^3dx = -\tfrac{1}{4}x^4\Big]_0^1 = \tfrac{1}{4}$$

13. C: $y = -x^3$, $0 \le x \le 1$. With x as parameter

$$\int_C yx^2dx + (x+y)dy = \int_0^1 (-x^3)x^2dx + (x - x^3)(-3x^2)dx = \int_0^1 (2x^5 - 3x^3)dx = \Big[\tfrac{1}{3}x^6 - \tfrac{3}{4}x^4\Big]_0^1 = -\frac{5}{12}$$

14. C_1: $x = 0$, $y = 0$ to -1 and C_2: $y = -1$, $x = 0$ to 1.

$$\int_C yx^2dx + (x+y)dy = \int_0^{-1} y\,dy + \int_0^1 -x^2dx = \tfrac{1}{2}y^2\Big]_0^{-1} - \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

15. With x as parameter, C_1: $y = 2x + 3$, $0 \le x \le 3$ and C_2: $y = x^2$, $3 \le x \le 5$.

$$\int_C 3xy\,dx + (4x^2 - 3y)dy = \int_0^3 3x(2x+3)dx + [4x^2 - 3(2x+3)]2\,dx + \int_0^5 3x(x^2)dx + (4x^2 - 3x^2)2x\,dx$$

$$= \int_0^3 (14x^2 - 3x - 18)dx + \int_0^5 5x^3dx = \Big[\frac{14}{3}x^3 - \frac{3}{2}x^2 - 18x\Big]_0^3 + \Big[\frac{5}{2}x^4\Big]_0^5 = 14\cdot 9 - \frac{27}{2} - 54 + \frac{5}{4}(5^4 - 3^4) = \frac{1477}{2}$$

16. $\int_C (xy - z)dx + e^x dy + y dz$, where C is the line segment from $(1,0,0)$ to $(3,4,8)$.

▹ The segment has the direction of $\langle 1,2,4\rangle$. Thus, parametric equations of C are

$x = 1 + t \quad y = 2t \quad z = 4t \quad \text{for} \quad 0 \le t \le 2$

Because $dx = dt$, $dy = 2dt$, and $dz = 4dt$, we have

$$\int_C (xy - z)dx + e^x dy + y\,dz = \int_0^2 [(1+t)(2t) - 4t]dt + e^{1+t}(2dt) + (2t)(4dt)$$

$$= \int_0^2 (6t + 2t^2 + 2e^{t+1})dt = 3t^2 + \tfrac{2}{3}t^3 + 2e^{t+1}\Big]_0^2 = \tfrac{52}{3} + 2(e^3 - e)$$

17. C: $x = t$, $y = 2t$, $z = 4t$, $0 \le t \le 1$.

$$\int_C (x+y)dx + (y+z)dy + (x+z)dz = \int_0^1 (t + 2t)dt + (2t + 4t)2\,dt + (t + 4t)4\,dt = \int_0^1 35t\,dt = \frac{35}{2}$$

18. C: $x = t + 1$, $y = t^2$, $z = t^3$, $0 \le t \le 2$

$$\int_C (xy - z)dx + e^x dy + ydz = \int_0^2 (t^2 + 2te^{t+1} + 3t^4)dt = \tfrac{1}{3}t^3 + 2(t-1)e^{t+1} + \tfrac{3}{5}t^5\Big]_0^2 = 2e^3 + 2e + \tfrac{328}{15}$$

19. $\mathbf{F}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$; C: $\mathbf{R}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C \langle t, a\cos t, a\sin t\rangle \cdot \langle -a\sin t, a\cos t, 1\rangle dt = \int_0^{2\pi} (-at\sin t + a^2\cos^2 t + a\sin t)dt$$

$$= a(t\cos t - \sin t) + \tfrac{1}{2}a^2t + \tfrac{1}{4}a^2\sin 2t - a\cos t\Big]_0^{2\pi} = (a\cdot 2\pi + a^2\pi - a) - (-a) = \pi(a^2 + 2a)$$

20. $\int_C \mathbf{F}\cdot d\mathbf{R}$; $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (6y^2 - xz)\mathbf{j} + 10z\mathbf{k}$; C: the twisted cubic $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$.

▹ We apply Definition 14.2.4. We have

$$\begin{array}{lll} x = t & y = t^2 & z = t^3 \\ dx = dt & dy = 2t\,dt & dz = 3t^2dt \end{array}$$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C 2xy\,dx + (6y^2 - xz)dy + 10z\,dz$$

$$= \int_0^1 \{2t(t^2)dt + [6(t^2)^2 - t(t^3)](2t\,dt) + 10t^3(3t^2dt)\} = \int_0^1 (2t^3 + 40t^5)dt = \tfrac{1}{2}t^4 + \tfrac{20}{3}t^6\Big]_0^1 = \tfrac{43}{6}$$

21. $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (6y^2 - xz)\mathbf{j} + 10z\mathbf{k}$

C_1: $\mathbf{R}_1(t) = t\mathbf{k}$, $0 \le t \le 1$; C_2: $\mathbf{R}(t) = t\mathbf{j} + \mathbf{k}$, $0 \le t \le 1$; C_3: $\mathbf{R}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}$, $0 \le t \le 1$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_{C_1} \mathbf{F}\cdot d\mathbf{R}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{R}_2 + \int_{C_3} \mathbf{F}\cdot d\mathbf{R}_3 = \int_0^1 10t\,dt + \int_0^1 6t^3\,dt + \int_0^1 2t\,dt = 5t^2\Big]_0^1 + 2t^3\Big]_0^1 + t^2\Big]_0^1 = 8$$

22. $x = y = z = t$, $0 \le t \le 1$. $\displaystyle\int_C 2xy\,dx + (6y^2 - xz)dy + 10z\,dz = \int_0^1 (2t^2 + 5t^2 + 10t)dt = \tfrac{7}{3}t^3 + 5t^2\Big]_0^1 = \tfrac{22}{3}$

In Exercises 23–36, find the total work done in moving an object along the given curve C if the motion is caused by the given force field. Assume that length is measured in meters and force is measured in newtons.

▹ W joules of work is done.

23. $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$; C: $\mathbf{R}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$.

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 2t^2, 2t^2\rangle \cdot \langle 1,1\rangle dt = \int_0^1 4t^2dt = \frac{4}{3}t^3\Big]_0^1 = \frac{4}{3}$$

24. $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$; C is the arc of the parabola $y^2 = x$ from the origin to the point $(1,1)$.

▹ We use Definition 14.2.1 in differential form notation with y as a parameter. We replace x by y^2, $0 \le y \le 1$, and dx by $2y\,dy$. Thus,

$$W = \int_C 2xy\,dx + (x^2 + y^2)dy = \int_0^1 2(y^2)y(2y\,dy) + [(y^2)^2 + y^2]dy = \int_0^1 (5y^4 + y^2)dy = \Big[y^5 + \tfrac{1}{3}y^3\Big]_0^1 = \tfrac{4}{3}$$

Thus, the total work done is $\frac{4}{3}$ joules.

25. $\mathbf{F}(x,y) = (y-x)\mathbf{i} + x^2y\mathbf{j}$; C: $\mathbf{R}(t)t\mathbf{i} + (3t-2)\mathbf{j}$, $1 \le t \le 2$.
$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 \langle 3t-2-t, t^2(3t-2)\rangle \cdot \langle 1,3\rangle dt = \int_0^2 (9t^3 - 6t^2 + 2t - 2)dt = \frac{9}{4}t^4 - 2t^3 + t^2 - 2t\Big]_0^2 = 20\frac{3}{4}$

26. C: $y = x^2$, $1 \le x \le 2$. $W = \int_C (y-x)dx + x^2ydy = \int_1^2 (x^2 - x)dx + x^4(2x\,dx) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x^6\Big]_1^2 = \frac{131}{6}$

27. $\mathbf{F}(x,y) = (y-x)\mathbf{i} + x^2y\mathbf{j}$; C_1: $\mathbf{R}_1(t) = t\mathbf{i} + t\mathbf{j}$, $1 \le t \le 2$; C_2: $\mathbf{R}_2(t) = 2\mathbf{i} + t\mathbf{j}$, $2 \le t \le 4$.
$W = \int_0^2 \langle t-t, t^2\cdot t\rangle \cdot \langle 1,1\rangle dt + \int_0^4 \langle t-2, 4t\rangle \cdot \langle 0,1\rangle dt = \int_0^2 t^3dt + \int_0^4 4t\,dt = \frac{1}{4}t^4\Big]_0^2 + 2t^2\Big]_0^4 = 27\frac{3}{4}$

28. $\mathbf{F}(x,y) = -x^2y\mathbf{i} + 2y\mathbf{j}$; C is the line segment from $(a,0)$ to $(0,a)$.
▹ The segment has the direction of $\langle -1,1\rangle$. Thus, parametric equations of C are $x = a-t$ and $y = t$, $0 \le t \le a$.

$$W = \int_C -x^2ydx + 2y\,dy = \int_0^a -(a-t)^2t(-dt) + 2t\,dt = \int_0^a (t^3 - 2at^2 + a^2t + 2t)dt$$
$$= \frac{1}{4}t^4 - \frac{2}{3}at^3 + \frac{1}{2}a^2t^2 + t^2\Big]_0^a = \frac{1}{12}a^4 + a^2$$

Therefore, the total work done is $\frac{1}{12}a^4 + a^2$ joules.

29. $\mathbf{F}(x,y) = -x^2y\mathbf{i} + 2y\mathbf{j}$; C: $\mathbf{R}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le \frac{1}{2}\pi$.
$W = \int_0^{\pi/2} \langle -a^2\cos^2t\cdot a\sin t, 2\sin t\rangle \cdot \langle -a\sin t, a\cos t\rangle dt = \int_0^{\pi/2} (a^4\sin^2t\cos^2t + 2a^2\sin t\cos t)dt$
$= \int_0^{\pi/2} \left(\frac{1}{4}a^4\sin^2 2t + 2a^2\sin t\cos t\right)dt = \frac{1}{8}a^4\left(t - \frac{1}{4}\cos 4t\right) + a^2\sin^2t\Big]_0^{\pi/2} = \frac{1}{16}a^4\pi + a^2$

30. C_1: $(a,0)$ to (a,a); C_2: (a,a) to $(0,a)$. $W = \int_{C_1} + \int_{C_2} -x^2y\,dx + 2y\,dy = \int_0^a 2y\,dy + \int_a^0 -ax^2dx = a^2 + \frac{1}{3}a^4$

31. $\mathbf{F}(x,y,z) = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$; C: $t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.
$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle t+t, t+t, t+t\rangle \cdot \langle 1,1,1\rangle dt = \int_0^1 6t\,dt = 3t^2\Big]_0^1 = 3$

32. $\mathbf{F}(x,y,z) = z^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$; C is the line segment from the origin to the point $(4,0,3)$.
▹ Parametric equation of C are
$x = 4t \quad y = 0 \quad z = 3t \quad 0 \le t \le 1$
Thus,

$$W = \int_C z^2dx + y^2dy + xz\,dz = \int_0^1 (3t)^2(4\,dt) + 0 + (4t)(3t)(3\,dt) = \int_0^1 72t^2dt = 24t^3\Big]_0^1 = 24$$

The total work done is 24 joules.

33. $\mathbf{F}(x,y,z) = e^x\mathbf{i} + e^y\mathbf{j} + e^z$; C: $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 2$.
$W = \int_0^2 \langle e^t, e^t, e^t\rangle \cdot \langle 1, t^2, t^3\rangle dt = \int_0^2 (e^t + 2te^t + 3t^2e^t)dt = e^t + e^t + e^t\Big]_0^2 = e^2 + e^4 + e^8 - 3$

34. $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 2$. $W = \int_C (xyz + x)dx + (x^2z + y)dy + (x^2y + z)dz =$
$\int_0^2 [(t^6 + t) + (t^5 + t^2)(2t) + (t^4 + t^3)(3t^2)]dt = \int_0^2 (t^6 + t + 2t^6 + 2t^3 + 3t^6 + 3t^5)dt = \frac{6}{7}t^7 + \frac{1}{2}t^6 + \frac{1}{2}t^4 + \frac{1}{2}t^2\Big]_0^2 = \frac{1062}{7}$

35. $\mathbf{F}(x,y,z) = (xyz + x)\mathbf{i} + (x^2z + y)\mathbf{j} + (x^2y + z)\mathbf{k}$
C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.
$W = \int_{C_1} \mathbf{F}\cdot d\mathbf{R}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{R}_2 + \int_{C_3} \mathbf{F}\cdot d\mathbf{R}_3 = \int_0^1 t\,dt + \int_0^1 t\,dt + \int_0^1 (1+t)dt = \frac{1}{2}t^2\Big]_0^1 + \frac{1}{2}t^2\Big]_0^1 + t + \frac{1}{2}t^2\Big]_0^1 = 2\frac{1}{2}$

36. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + (yz - x)\mathbf{k}$; C is $\mathbf{R}(t) = 2t\mathbf{i} + t^2\mathbf{j} + 4t^3\mathbf{k}$, for $0 \le t \le 1$.
We have

$$\begin{array}{lll} x = 2t & y = t^2 & z = 4t^3 \\ dx = 2\,dt & dy = 2t^2dt & dz = 12t^2dt \end{array}$$

Thus,

$$W = \int_C x\,dx + y\,dy + (yz - x)dz = \int_0^1 (2t)(2\,dt) + (t^2)(2t^2dt) + [t^2(4t^3) - 2t](12t^2dt)$$
$$= \int_0^1 (48t^7 - 22t^3 + 4t)dt = 6t^8 - \frac{11}{2}t^4 + 2t^2\Big]_0^1 = \frac{5}{2}$$

The work is 2.5 joules.

14.3 LINE INTEGRALS INDEPENDENT OF THE PATH

14.3.1 Theorem Let C be any sectionally smooth curve lying in a open disk B in R^2 from the point (x_1, y_1) to the point (x_2, y_2). If **F** is a conservative vector field continuous on B and ϕ is a potential function for **F**, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

is independent of the path C and

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(x_2, y_2) - \phi(x_1, y_1)$$

14.3.2 Theorem If C is any sectionally smooth closed curve lying in some open disk B in R^2 and **F** is a conservative vector field on B, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = 0$$

14.3.3 Theorem Let C be any sectionally smooth curve lying in a open ball B in R^3 from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2). If **F** is a conservative vector field continuous on B and ϕ is a potential function for **F**, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

is independent of the path C and

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Exercises 14.3

In Exercises 1–12, use the result of the indicated Exercise in Exercises 14.1 to prove that the line integral is independent of the path. The evaluate the line integral by applying either Theorem 14.3.1 or 14.3.3 and using the potential function found in the indicated exercise. In each exercise C is any sectionally smooth curve from the point A to the point B.

1. $\mathbf{F} = \nabla\phi$ with $\phi(x, y) = xy$. $\int_C y\,dx + x\,dy = \phi(3, 2) - \phi(1, 4) = 6 - 4 = 2.$

2. $\mathbf{F} = \nabla\phi$ with $\phi(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. $\int_C x\,dx + y\,dy = \phi(1, 3) - \phi(-5, 2) = 5 - \frac{29}{2} = -\frac{19}{2}$

3. $\mathbf{F} = \nabla\phi$ with $\phi(x, y) = e^x \sin y$. $\int_C e^x \sin y\,dx + e^x \cos y\,dy = \phi(2, \frac{1}{2}\pi) - \phi(0, 0) = e^2 - 0 = e^2.$

4. $\int_C (\sin y \sinh x + \cos y \cosh x)dx + (\cos y \cosh x - \sin y \sinh x)dy$; A is $(1, 0)$ and B is $(2, \pi)$; Exercises 24.

▸ In Exercise 14.1.24 we proved that the vector field defined by

$\mathbf{F}(x, y) = (\sin y \sinh x + \cos y \cosh x)dx + (\cos y \cosh x - \sin y \sinh x)dy$

is conservative with a potential function given by

$\phi(x, y) = \sin y \cosh x + \cos y \sinh x$

Therefore, by Theorem 14.3.1, the given line integral is independent of the path C, and the value of the integral is

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(2, \pi) - \phi(1, 0) = (\sin \pi \cosh 2 + \cos \pi \sinh 2) - (\sin 0 \cosh 1 + \cos 0 \sinh 1) = -\sinh 2 - \sinh 1$$

5. $\mathbf{F} = \nabla\phi$ with $\phi(x, y) = x^2y^2 - x^y 3 + 2y$. $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, 2) - \phi(-3, -1) = 0 - 4 = -4.$

6. $\mathbf{F} = \nabla\phi$ with $\phi(x, y) = x^3 + 2xy - y^2e^x$. $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, -3) - \phi(0, 2) = (-5 - 9e) - (-4) = -1 - 9e$

7. $\mathbf{F} = \nabla\phi$ with $\phi(x, y, z) = \frac{1}{3}x^3 - xy + 3yz + \frac{1}{2}z^2$. $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(3, 0, 4) - \phi(-3, 1, 2) = 17 - 2 = 15.$

8. $\int_C yz\,dx + xz\,dy + xy\,dz$; A is $(0, -2, 5)$ and B is $(4, 1, -3)$; Exercise 28.

▸ In Exercise 14.1.28, we proved that the force field defined by

$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

is conservative with a potential function given by

$\phi(x, y, z) = xyz$

Therefore, by Theorem 14.3.3, the given line integral is independent of the path C, and

$$\int_C yz\,dx + xz\,dy + xy\,dz = \phi(4, 1, -3) - \phi(0, -2, 5) = (4)(1)(-3) - (0)(-2)(5) = -12$$

9. $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = ze^x + xe^y - ye^z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,2,1) - \phi(1,0,2) = 1 - 2e - 2e - 1 = -4e$.

10. $\phi = x\tan y + x^2\sec z - \tan z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,\frac{1}{4}\pi,\frac{1}{3}\pi) - \phi(2,\frac{1}{6}\pi,0) = (3\cdot 1 + 9\cdot 2 - \sqrt{3}) - (\sqrt{3}+4) = 17 - 2\sqrt{3}$

11. $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = x^2\cos y - 3x - yz^2 + 2z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,\pi,0) - \phi(-1,0,3) = -4 - 10 = -14$.

12. $\int_C (2y^3 - 8xz^2)dx + (6xy^2+1)dy - (8x^2z + 3z^2)dz$; A is $(2,0,0)$ and B is $(3,2,1)$; Exercise 32.

▸ In Exercise 14.1.32, we proved that the vector field defined by
$\mathbf{F}(x,y,z) = (2y^3 - 8xz^2)\mathbf{i} + (6xy^2+1)\mathbf{j} - (8x^2 + 3z^2)\mathbf{k}$
is conservative with a potential function given by
$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y - z^3$
Therefore, by Theorem 14.3.3, the given line integral is independent of the path C, and

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,2,1) - \phi(2,0,0) = 13$$

In Exercises 13–20, show that the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ for the given function F and C is independent of the path, and evaluate the integral.

▸ In Exercises 13–19 we show the line integral is path-independent by exhibiting a potential.

13. $\mathbf{F}(x,y) = 2(x-y)\mathbf{i} + 2(3y-x)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^2 - 2xy + 3y^2$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,3) - \phi(3,0) = 27 - 9 = 18$

14. $\mathbf{F}(x,y) = (3x^2 + 6xy - 2y^2)\mathbf{i} + (3x^2 - 4xy + 3y^2)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^3 + 3x^2y - 2xy^2 + y^3$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,2) - \phi(3,0) = 8 - 27 = -19$$

15. $\mathbf{F}(x,y) = (4xe^{2x} - 3e^xe^y)\mathbf{i} + (2e^{2y} - 3e^xe^y)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = 2e^x - 3e^{xy} + e^{2y}$. The focus of the parabola $y^2 = 4x$ is at $(1,0)$ so the endpoint of the latus rectum is at $(1,2)$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,2) - \phi(0,0) = (2e^2 - 3e^1e^2 + e^4) - (2 - 3 + 1) = 2e^2 - 3e^3 + e^4$$

16. $\mathbf{F}(x,y) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$; C is the segment of the line $3x + 4y = 12$ from the point where it intersects the y axis to the point where it intersects the x axis.

▸ By inspection, we see that
$\phi(x,y) = e^x\cos y$
is a potential function. In fact
$\nabla\phi(x,y) = D_x(e^x\cos y)\mathbf{i} + D_y(e^x\cos y)\mathbf{j} = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$
Thus, we may apply Theorem 14.3.1 to evaluate the integral.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(4,0) - \phi(0,3) = e^4 - \cos 3$$

17. $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 3y^2\mathbf{j} + \mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = x^2 + y^3 + z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,0,3) - \phi(\frac{3}{2},0,0) = 3 - \frac{9}{4} = \frac{3}{4}$

18. $\mathbf{F}(x,y,z) = (2xy + z^2)\mathbf{i} + (x^2 - 2yz)\mathbf{j} + (2xz - y^2)\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = x^2y + xz^2 - y^2z$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,0,1) - \phi(0,1,0) = 0 - 0 = 0$$

19. $\mathbf{F}(x,y,z) = 2ye^{2x}\mathbf{i} + 2e^{x}\mathbf{j} + 3z^2\mathbf{k} = \mathbf{F}\phi$ with $\phi(x,y,z) = ye^{2x} + z^3$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(\ln 2,2,2) - \phi(\ln 2,1,1) = 8 + 8 - 4 - 1 = 11$$

20. $\mathbf{F}(x,y,z) = \left(\frac{1}{z} - \frac{y}{x^2}\right)\mathbf{i} + \left(\frac{1}{x} + \frac{z}{y^2}\right)\mathbf{j} - \left(\frac{1}{y} + \frac{x}{z^2}\right)\mathbf{k}$;

C is any sectionally smooth curve from the point $(1,2,-1)$ to the point $(2,4,-2)$.

▸ Let

$$M(x,y,z) = \frac{1}{z} - \frac{y}{x^2} \qquad N(x,y,z) = \frac{1}{x} + \frac{z}{y^2} \qquad R(x,y,z) = \frac{-1}{y} - \frac{x}{z^2}$$

Then

$$M_y(x,y,z) = \frac{-1}{x^2} \qquad N_x(x,y,z) = \frac{-1}{x^2} \qquad R_x(x,y,z) = \frac{-1}{z^2}$$

$$M_z(x,y,z) = \frac{-1}{z^2} \qquad N_z(x,y,z) = \frac{1}{y^2} \qquad R_y(x,y,z) = \frac{1}{y^2}$$

Because $M_y(x,y,z) = N_x(x,y,z)$, $M_z(x,y,z) = R_x(x,y,z)$, and $N_z(x,y,z) = R_y(x,y,z)$, then the given vector field **F** is conservative and the line integral is independent of the path C. We let C be a line segment with direction $\langle 1,2,-1\rangle$. Parametric equation for C are

$$x = 1 + t \qquad y = 2 + 2t \qquad z = -(1 + t) \quad 0 \le t \le 1$$

Then

$$dx = dt \qquad dy = 2\,dt \qquad dz = -dt$$

and

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_C \left(\frac{1}{z} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} + \frac{z}{y^2}\right)dy - \left(\frac{1}{y} + \frac{x}{z^2}\right)dz \\ &= \int_0^1 \left(\frac{-1}{1+t} - \frac{2+2t}{(1+t)^2}\right)dt + \left(\frac{1}{1+t} - \frac{1+t}{(2+2t)^2}\right)(2\,dt) - \left(\frac{1}{2+2t} + \frac{1+t}{(1+t)^2}\right)(-dt) \\ &= \int_0^1 (-3 + \tfrac{3}{2} + \tfrac{3}{2})\frac{dt}{1+t} = 0\end{aligned}$$

In Exercises 21–30, show that the line integral is independent of the path, and compute the value in any convenient manner. In Each exercise, C is any sectionally smooth curve from point A to point B.

▸ We may show the line integral is path-independent by exhibiting a potential for the vector function.

21. $\mathbf{F}(x, y) = (2y - x)\mathbf{i} + (y^2 + 2x)\mathbf{j} = \nabla\phi$ with $\phi(x, y) = 2xy - \frac{1}{2}x^2 + \frac{1}{3}y^3$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1, 2) - \phi(0, -1) = (4 - \tfrac{1}{2} + \tfrac{8}{3}) - (-\tfrac{1}{3}) = \tfrac{13}{2}$$

22. $\displaystyle\int_C (\ln x + 2y)dx + (e^y + 2x)dy = \int_C (\ln x\,dx + (2y\,dx + 2x\,dy) + e^y dy = x \ln x - x + 2xy + e^y\Big]_{(3,1)}^{(1,3)}$
$= -3 \ln 3 + 2 + e^3 - e$

23. $\mathbf{F}(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j} = \nabla\phi$ with $\phi(x, y) = x \tan y$. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(4, \tfrac{1}{4}\pi) - \phi(-2, 0) = 4 - 0 = 4.$

24. $\int_C \sin y\,dx + (\sin y + x \cos y)dy$; A is $(-2, 0)$ and B is $(2, \frac{1}{6}\pi)$.

▸ Integrating $\sin y + x \cos y$ with respect to y, we conjecture that
$\phi = -\cos y + x \sin y$
is a potential for the integrand. Checking, we find
$\nabla(-\cos y + x \sin y) = \sin y\mathbf{i} + (\sin y + x \cos y)\mathbf{j}$
Thus, the integral is independent of the path and its value is

$$\phi(2, \tfrac{1}{6}\pi) - \phi(-2, 0) = (-\tfrac{1}{2}\sqrt{3} + 2\cdot\tfrac{1}{2}) - (-1 - 2\cdot 0) = 2 - \tfrac{1}{2}\sqrt{3}$$

25. $\mathbf{F}(x, y) = \dfrac{2y}{(xy+1)^2}\mathbf{i} + \dfrac{2x}{(xy+1)^2} = \nabla\phi$ with $\phi(x, y) = \dfrac{-2}{xy+1}$. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1, 0) - \phi(0, 2) = -2 + 2 = 0$

26. $\displaystyle\int_C \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2} = \int_{(1,0,0)}^{(1,2,3)} d[\tfrac{1}{2}\ln(x^2 + y^2 + z^2)] = \tfrac{1}{2}\ln(x^2 + y^2 + z^2)\Big]_{(1,0,0)}^{(1,2,3)} = \tfrac{1}{2}\ln 14$

27. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k} = \nabla\phi$ with $\phi(x, y, z) = xy + xy + yz$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1, 1, 1) - \phi(0, 0, 0) = 3 - 0 = 3$$

28. $\int_C (yz + x)dx + (xz + y)dy + (xy + z)dz$; A is $(0, 0, 0)$, and B is $(1, 1, 1)$.

▸ Let

$M(x, y, z) = yz + x \qquad N(x, y, z) = xz + y \qquad R(x, y, z) = xy + z$

Then

$M_y(x, y, z) = z \qquad N_x(x, y, z) = z \qquad R_x(x, y, z) = y$
$M_z(x, y, z) = y \qquad N_z(x, y, z) = x \qquad R_y(x, y, z) = x$

Because $M_y(x, y, z) = N_x(x, y, z)$, $M_z(x, y, z) = R_x(x, y, z)$, and $N_z(x, y, z) = R_y(x, y, z)$, then the given vector field $\mathbf{F}$ is conservative and the line integral is independent of the path C. We let C be a line segment from $(0, 0, 0)$ to $(1, 1, 1)$. with direction $\langle 1, 2, -1\rangle$. Parametric equations for C are
$x = t \quad y = t \quad z = t \quad 0 \le t \le 1$
Therefore,

$$\begin{aligned}\int_C (yz + x)dx + (xz + y)dy + (xy + z)dz &= \int_0^1 (t^2 + t)dt + (t^2 + t)dt + (t^2 + t)dt = \int_0^1 (3t^2 + 3t)dt \\ &= t^3 + \tfrac{3}{2}t^2\Big]_0^1 = \tfrac{5}{2}\end{aligned}$$

29. $\mathbf{F}(x, y, z) = (e^x \sin y + yz)\mathbf{i} + (e^x \cos y + z \sin y + xz)\mathbf{j} + (xy - \cos y)\mathbf{k} = \nabla\phi$ with

$\phi(x, y, z) = e^x \sin y + xyz - z \cos y$. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0, \pi, 3) - \phi(2, 0, 1) = 3 + 1 = 4.$

30. $\displaystyle\int_C (2x \ln yz - 5ye^x)dx - (5e^x - x^2 y^{-1})dy + (x^2 z^{-1} + 2z)dz = x^2 \ln y + x^2 \ln z - 5ye^x + z^2\Big]_{(2,1,1)}^{(3,1,e)} = 8 + 6e^2 - 5e^3$

31. $\mathbf{F}(x,y,z) = (4x+2y-z)\mathbf{i} + (2x-2y+z)\mathbf{j} + (-x+y+2z)\mathbf{k} = \nabla\phi$ with
$\phi(x,y,z) = 2x^2 + 2xy - xz - y^2 + yz + z^2$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(-1,2,0) - \phi(4,-2,1) = -6-7 = -13$.

32. Prove Theorem 14.3.3.

▸ If C is a smooth curve, we may let parametric equations of C be
$x = f(t) \quad y = g(t) \quad z = h(t) \quad t_1 \le t \le t_2$
where
$(x_1,y_1,z_1) = (f(t_1), g(t_1), h(t_1))$ and $(x_2,y_2,z_2) = (f(t_2), g(t_2), h(t_2))$
Let
$\mathbf{F}(x,y,z) = \mathrm{M}(x,y,z)\mathbf{i} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$
Because ϕ is a potential function for $\mathbf{F}$
$\phi_x(x,y,z) = \mathrm{M}(x,y,z) \quad \phi_y(x,y,z) = \mathrm{N}(x,y,z) \quad \phi_z(x,y,z) = \mathrm{R}(x,y,z)$
Then, with the help of the chain rule and the second fundamental theorem of the calculus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_C \mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz \\
&= \int_C \phi_x(x,y,z)dx + \phi_y(x,y,z)dy + \phi_z(x,y,z)dz \\
&= \int_{t_1}^{t_2} \phi_x(f(t),g(t),h(t))f'(t)dt + \phi_y(f(t),g(t),h(t))g'(t)dt + \phi_z(f(t),g(t),h(t))h'(t)dt \\
&= \int_{t_1}^{t_2} d\phi(f(t),g(t),h(t)) \\
&= \phi(f(t),g(t),h(t)\Big]_{t_1}^{t_2} \\
&= \phi(f(t_2),g(t_2),h(t_2)) - \phi(f(t_1),g(t_1),h(t_1)) \\
&= \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)
\end{aligned}$$

If C is only sectionally smooth, then apply this result to each smooth piece; the values of ϕ at each intermediate point will cancel.

In Exercises 33–36, find the total work done in moving a particle along arc C if the motion is caused by the force field $\mathbf{F}$. Assume length is measured in meters and force is measured in newtons. (Hint: First show $\mathbf{F}$ is conservative.)

▸ W joules of work is done.

33. $\mathbf{F}(x,y) = 3(x+y)^2\mathbf{i} + 3(x+y)^2\mathbf{j} = \nabla\phi$ with $\phi(x,y) = (x+y)^3$. $\mathrm{W} = \phi(2,4) - \phi(0,0) = 216 - 0 = 216$.

34. $\mathrm{W} = \int_C (2xy - 5y + 2y^2)dx + (x^2 - 5x + 4xy)dy = x^2y - 5xy + 2xy^2\Big]_{(2,0)}^{(0,2)} = 0 - 0 = 0$

35. $\mathbf{F}(x,y,z) = 2y^2z^3\mathbf{i} + 4xyz^3\mathbf{j} + 6xy^2z^2\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = 2xy^2z^3$.
$\mathrm{W} = \phi(2,4,8) - \phi(1,1,1) = 32{,}768 - 2 = 32{,}766$

36. $\mathbf{F}(x,y,z) = 4y^2z\mathbf{i} + 8xyz\mathbf{j} + 4(3z^3 + xy^2)\mathbf{k}$; C: the arc of the circular helix $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = \frac{1}{3}\pi$.

▸ Integrating the z component $12z^3 + 4xy^2$ with respect to z, we conjecture that
$\phi(x,y,z) = 3z^4 + 4xy^2z$
is a potential for the integrand. Checking, we find
$\nabla(3z^4 + 4xy^2z) = 4y^2z\mathbf{i} + 8xyz\mathbf{j} + 4(3z^3 + xy^2)\mathbf{k}$
Thus, the integral $\int_C$ is independent of the path and its value is
$\phi(\frac{3}{2}, \frac{3}{2}\sqrt{3}, \frac{1}{3}\pi) - \phi(3,0,0) = (\frac{27}{2}\pi + \frac{1}{27}\pi^4) - 0 = \frac{27}{2}\pi + \frac{1}{27}\pi^4$
The total work done is $(\frac{27}{2}\pi + \frac{1}{27}\pi^4)$ joules.

37. $\mathbf{F}(x,y,z) = \dfrac{k(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$.

(a) $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = \dfrac{-k}{(x^2+y^2+z^2)^{1/2}}$. $\mathrm{W} = \phi(3,0,4) - \phi(3,0,0) = -\frac{k}{5} + \frac{k}{3} = \dfrac{2}{15}k$.

(b) With z as parameter, C: $x = 3$, $y = 4$, $0 \le z \le 4$.
$\mathrm{W} = k\int_0^4 (\dfrac{3}{(9+z^2)^{3/2}}, 0, \dfrac{z}{(9+z^2)^{3/2}})\cdot(0,0,1)dz = k\int_0^4 z(9+z^2)^{-3/2}dz = -k(9+z^2)^{-1/2}\Big]_0^4 = \dfrac{2}{15}k$

14.4 GREEN'S THEOREM

The work done in moving an object a round a sectionally smooth simple closed curve can be found by evaluating a line integral. Green's theorem gives us the option of using a double integral to evaluate the line integral. The line integral around C in the counterclockwise direction is denoted by $\oint_C$.

In the following three theorems, M and N are functions of two variables x and y which have continuous first partial derivatives on an open disk B in R^2, C is a sectionally smooth simple closed curve lying entirely in B and R is the region bounded by C.

14.4.1 Theorem (Green's Theorem)

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

14.4.3 Theorem (Gauss's Divergence Theorem in the Plane) If $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ and $\mathbf{n}(s)$ is the unit outward normal vector of C at P where s units is the length of arc measured from a particular point on C to P, then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA = \iint_R \nabla\cdot\mathbf{F}\, dA$$

The line integral $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ is called the *flux* of **F** across C.

14.4.4 Theorem (Stokes's Theorem in the Plane) If $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ and $\mathbf{T}(s)$ is the unit tangent vector of C at P, where s units is the length of arc measured from a particular point on C to P, then

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_R \operatorname{curl} \mathbf{F}\cdot\mathbf{k}\, dA$$

The line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds$ is called the *circulation* of **F** around C.

When applying Theorems 14.4.3 and 14.4.4 we may use the equations

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint_C -N(x,y)dx + M(x,y)dy$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M(x,y)dx + N(x,y)dy$$

The following theorem gives a method for using a line integral to find the area of a region bounded by a sectionally smooth simple curve.

14.4.2 Theorem If R is a region having as its boundary a sectionally smooth simple closed curve C, and A square units is the area of R, then

$$A = \tfrac{1}{2}\oint_C x\, dy - y\, dx$$

The area of a hypocycloid of m cusps is $\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)a^2\pi$. The area of an astroid is $\frac{3}{8}a^2\pi$. See Exercise 25.

Exercises 14.4

In Exercises 1–8, evaluate the line integral by Green's theorem. Verify the result by the method of Section 14.2.

1. $\oint_C 4y\, dx + 3x\, dy = \iint_R \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(4y)\right]dA = \iint_R -dA = -1$. Also
C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = (1-t)\mathbf{i} + \mathbf{j}$, $0 \le t \le 1$;
C_4: $\mathbf{R}_4(t) = (1-t)\mathbf{j}$, $0 \le t \le 1$. $\oint_C 4y\, dx + 3x\, dy = \int_0^1 0\, dt + \int_0^1 3\, dt + \int_0^1 (-4)dt + \int_0^1 0\, dt = 3 - 4 = -1$

2. $\oint_C y^2dx + x^2dy = \iint_R [D_x(x^2) - D_y(y^2)]dA = \int_0^1\int_0^1 (2x-2y)dx\, dy = \int_0^1 x^2 - 2xy\Big]_0^1 dy = \int_0^1 (1-2y)dy$
$= y - y^2\Big]_0^1 = 0.$

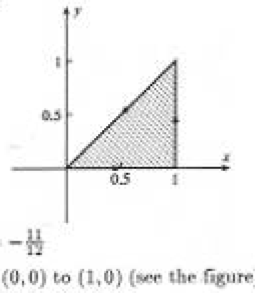
3. $\oint_C 2xy\,dx - x^2y\,dy = \iint_R \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(2xy)\right] dA = \int_0^1 \int_0^{1-x} (-2xy - 2x)dy\,dx$
$= \int_0^1 [-xy^2 - 2xy]_0^{1-x}\,dx = \int_0^1 (-x^3 + 4x^2 - 3x)dx = -\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{3}{2}x^2\Big]_0^1 = -\frac{1}{4} + 143 - \frac{3}{2} = -\frac{5}{12}$. Also

C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = (1-t)_{\mathbf{i}} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = (1-t)\mathbf{j}$, $0 \le t \le 1$.

$\oint_C 2xy\,dx - x^2y\,dy = \int_0^1 0\,dt + \int_0^1 [2(1-t)t(-1) - (1-t)^2t]dt + \int_0^1 0\,dt = \int_0^1 (-t^3 + t^2 - 3t)dt = -\frac{5}{12}$

4. $\oint_C 2xy\,dx - x^2dy$, where C is the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$.

▷ Let
$M(x,y) = 2xy$ and $N(x,y) = -x^2y$
Then
$N_x(x,y) - M_y(x,y) = -2xy - 2x$
Because the region R that is enclosed by the curve C is bounded by the x axis, the line $x = 1$, and the line $y = x$ (see the figure), we have



$$\oint_C 2xy\,dx - x^2y\,dy = \iint_R [N_x(x,y) - M_y(x,y)]dA = \int_0^1 \int_0^x (-2xy - 2x)dy\,dx$$
$$= \int_0^1 -xy^2 - 2xy\Big]_0^x dx = \int_0^1 (-x^3 - 2x^2)dx = -\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\Big]_0^1 = -\tfrac{11}{12}$$

We verify the result by the method of Section 14.2. Let C_1 be the segment from $(0,0)$ to $(1,0)$ (see the figure); let C_2 be the segment from $(1,0)$ to $(1,1)$; and let C_3 be the segment from $(1,1)$ to $(0,0)$. Then

$$\oint_C 2xy\,dx - x^2y\,dy = \oint_{C_1} 2xy\,dx - x^2y\,dy + \oint_{C_2} 2xy\,dx - x^2y\,dy + \oint_{C_3} 2xy\,dx - x^2y\,dy \tag{1}$$

Parametric equations for C_1 are $x = t$, $y = 0$, $0 \le t \le 1$. Thus,

$$\oint_{C_1} 2xy\,dx - x^2y\,dy = \int_0^1 (2t)(0)dt - t^2(0)(0) = 0 \tag{2}$$

Parametric equations for C_2 are $x = 1$, $y = t$, $0 \le t \le 1$. Therefore,

$$\oint_{C_2} 2xy\,dx - x^2y\,dy = \int_0^1 (2t)(0) - t\,dt = -\tfrac{1}{2}t^2\Big]_0^1 = -\tfrac{1}{2} \tag{3}$$

Parametric equations for C_3 are $x = t$, $y = t$, from $t = 1$ to $t = 0$. Hence,

$$\oint_{C_3} 2xy\,dx - x^2y\,dy = \int_1^0 2t^2dt - t^3dt = \tfrac{2}{3}t^3 - \tfrac{1}{4}t^4\Big]_1^0 = 0 - (\tfrac{2}{3} - \tfrac{1}{4}) = -\tfrac{5}{12} \tag{4}$$

Substituting from Eqs. (2), (3), and (4) into Eq. (1), we obtain

$$\oint_C 2xy\,dx - x^2y\,dy = 0 - \tfrac{1}{2} - \tfrac{5}{12} = -\tfrac{11}{12}$$

which agrees with the result obtained by using Green's theorem.

5. $\oint_C x^2y\,dx - y^2x\,dy = \iint_R \left[\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right] dA = -\iint_R (x^2 + y^2)dA = -\int_0^{2\pi}\int_0^1 r^2(r\,dr\,d\theta)$
$= -\int_0^{2\pi} \frac{1}{4}\,d\theta = -\frac{\pi}{2}$. Also, C: $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$.

$\oint_C x^2y\,dx - y^2x\,dy = \int_0^{2\pi} [\cos^2 t \sin t(-\sin t) - \sin^2 t \cos t(\cos t)]dt = -2\int_0^{2\pi} \cos^2 t \sin^2 t\,dt$
$= -\frac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt = -\frac{1}{2}\left[\frac{1}{2}t - \frac{1}{4}\cos 4t\right]_0^{2\pi} = -\frac{\pi}{2}$

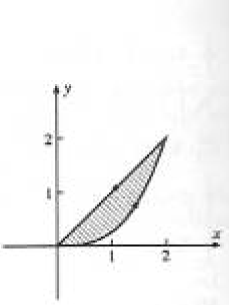
6. $\oint_C (x^2 - y^2)dx + 2xy\,dy = \iint_R [D_x(2xy) - D_y(x^2 - y^2)]dA = \iint_R (2y + 2y)dA = 0$ because y is odd. Also, let $x = \cos\theta$, $y = \sin\theta$. $\oint_C (x^2 - y^2)dx + 2xy\,dy = \int_0^{2\pi} (\cos^2\theta - \sin^2\theta)(-\sin\theta\,d\theta) + 2\sin\theta\cos\theta(\cos\theta\,d\theta) = 0$

7. $\oint_C x^2y\,dx - y^2x\,dy = \iint_R \left[\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right] dA = -\int_0^2 \int_{x^3/4}^x (y^2 + x^2)dy\,dx$
$= -\int_0^2 \left[\frac{1}{3}y^3 + x^2y\right]_{x^3/4}^x dy = -\int_0^2 \left(\frac{4}{3} - \frac{1}{192}x^9 - \frac{1}{4}x^5\right)dx = -\left[\frac{1}{3}x^4 + \frac{1}{1920}x^{10} - \frac{1}{24}x^6\right]_0^2 = -\frac{32}{15}$. Also

C_1: $y = \frac{1}{4}x^3$, $0 \le x \le 2$; $-C_2$: $y = x$, $0 \le x \le 2$. $\oint_C x^2y\,dx - y^2x\,dy$
$= \int_0^2 \left[x^2\left(\frac{1}{4}x^3\right) - \left(\frac{1}{4}x^3\right)^2\left(\frac{3}{4}x^2\right)\right]dx - \int_0^2 (x^2x - x^2x)dx = \int_0^2 \left(\frac{1}{4}x^5 - \frac{3}{64}x^9\right)dx = \frac{1}{24}t^6 - \frac{3}{640}t^{10}\Big]_0^2 = -\frac{32}{15}$

8. $\oint (x^2 - y^2)dx + 2xy\ dy$, where C is the closed curve consisting of the arc of $4y = x^3$ from $(0,0)$ to $(2,2)$ and the line segment from $(2,2)$ to $(0,0)$.

▷ The figure shows the curve C and the region R that is enclosed by C. Applying Green's theorem, we have



$$\oint_C (x^2 - y^2)dx + 2xy\ dy = \iint_R [D_x(2xy) - D_y(x^2 - y^2)]dA = \iint_R (2y + 2y)dA$$
$$= \int_0^2 \int_{x^3/4}^{x} 4y\ dy\ dx = \int_0^2 2y^2\Big]_{x^3/4}^{x} dx$$
$$= 2\int_0^2 (x^2 - \tfrac{1}{16}x^6)dx = 2\Big[\tfrac{1}{3}x^3 - \tfrac{1}{112}x^7\Big]_0^2 = \tfrac{64}{21}$$

We verify the result by the method of Section 14.2. Let C_1 be the arc of the curve $4y = x^3$ from $(0,0)$ to $(2,2)$. Parametric equations of C_1 are

$x = t \quad y = \frac{1}{4}t^3 \quad 0 \le t \le 2$

Thus,

$$\oint_{C_1} (x^2 - y^2)dx + 2xy\ dy = \int_0^2 (t^2 - \tfrac{1}{16}t^6)dt + (\tfrac{1}{2}t^4)(\tfrac{3}{4}t^2 dt) = \int_0^2 (t^2 + \tfrac{5}{16}t^6)dt = \tfrac{1}{3}t^3 + \tfrac{5}{112}t^7\Big]_0^2 = \tfrac{176}{21}$$

Let C_2 be the line segment from $(2,2)$ to $(0,0)$. Parametric equations of C_2 are:

$x = t \quad y = t$ from $t = 2$ to $t = 0$

Hence,

$$\oint_{C_2} (x^2 - y^2)dx + 2xy\ dy = \int_2^0 2t^2 dt = \tfrac{2}{3}t^3\Big]_2^0 = -\tfrac{16}{3}$$

Therefore,

$$\oint_C (x^2 - y^2)dx + 2xy\ dy = \oint_{C_1} (x^2 - y^2)dx + 2xy\ dy + \oint_{C_2} (x^2 - y^2)dx + 2xy\ dy = \tfrac{176}{21} - \tfrac{16}{3} = \tfrac{64}{21}$$

which agrees with the result obtained by using Green's theorem.

In Exercises 9–20, use Green's theorem to evaluate the line integral.

9. $\oint_C (x + y)dx + xy\ dy = \iint_R \Big[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x + y)\Big]dA = \int_0^2 \int_0^{x^3/4} (y - 1)dy\ x = \int_0^2 \Big[\frac{1}{2}y^2 - y\Big]_0^{x^3/4} dx$

$= \int_0^2 \Big(\frac{1}{32}x^6 - \frac{1}{4}x^3\Big)dx = \frac{1}{224}x^7 - \frac{1}{16}x^4\Big]_0^2 = \frac{4}{7} - 1 = -\frac{3}{7}$

10. $\oint_C y^2 dx + x^2 dy = \iint_R [D_x(x^2) - D_y(y^2)]dA = \int_0^1 \int_0^{x^2} (2x - 2y)dy\ dx = \int_0^1 (2xy - y^2)\Big]_0^{x^2} dx = \int_0^1 (2x^3 - x^4)dx$

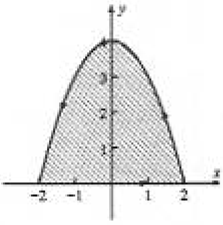
$= \frac{1}{2}x^4 - \frac{1}{5}x^5 = \frac{3}{10}$

11. $\oint_C (-x^2 + x)dy = \iint_R \Big[\frac{\partial}{\partial x}(-x^2 + x) - \frac{\partial}{\partial y}(0)\Big]dA = \int_0^1 \int_{2y}^{2y^2} (-2x + 1)dx\ dy = \int_0^1 -x^2 + x\Big]_{2y}^{2y^2} dy$

$= \int_0^1 (-4y^4 + 6y^2 - 2y)dy = -\frac{4}{5}y^5 + 2y^3 - y^2\Big]_0^1 = \frac{1}{5}$

12. $\oint_C (x^2 + y)dx$, where C is the closed curve determined by the x axis and the parabola $y = 4 - x^2$.

▷ The figure shows the curve C and the region R enclosed by C. We apply Green's theorem with $M(x, y) = x^2 + y$ and $N(x, y) = 0$



$$\oint_C (x^2 + y)dx = \iint_R (N_x - M_y)dA$$
$$= \iint_R (-1)dA$$
$$= -\int_{-2}^{2} \int_0^{4-x^2} dy\ dx$$
$$= -2\int_0^2 (4 - x^2)dx = -2\Big[4x - \tfrac{1}{3}x^3\Big]_0^2 = -\tfrac{32}{3}$$

13. $\oint_C \cos y\ dx + \cos x\ dy = \iint_R \Big[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(\cos y)\Big]dA = \int_0^{\pi/3} \int_0^{\pi/4} (-\sin x + \sin y)dy\ dx$

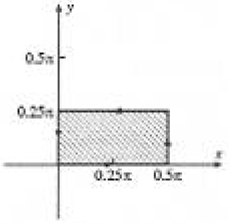
$= \int_0^{\pi/3} \Big[-y\sin x - \cos y\Big]_0^{\pi/4} dx = \int_0^{\pi/3} \Big(-\frac{\pi}{4}\sin x - \frac{1}{2}\sqrt{2} + 1\Big)dx = \Big[\frac{\pi}{4}\cos x + \Big(1 - \frac{1}{2}\sqrt{2}\Big)x\Big]_0^{\pi/3} = \frac{\pi}{24}(5 - 4\sqrt{2})$

14. $\oint_C e^{x+y}dx + e^{x+y}dy = \iint_R [D_x e^{x+y} - D_y e^{x+y}]dA = 0$

15. $\oint_C (\sin^4 x + e^{2x})dx + (\cos^3 y - e^y)dy = \iint_R \left[\frac{\partial}{\partial x}(\cos^3 y - e^y) - \frac{\partial}{\partial y}(\sin^4 x + e^{2x})\right]dA = \iint_R 0\, dA = 0$

16. $\oint_C x \sin y - y \cos x\, dy$, where C is the rectangle with vertices at $(0,0)$, $(\frac{1}{2}\pi, 0)$, $(\frac{1}{2}\pi, \frac{1}{4}\pi)$, and $(0, \frac{1}{4}\pi)$.

▸ The figure shows the rectangle C and the region R enclosed by C.
Applying Green's theorem, we have



$$\begin{aligned}\oint_C x \sin y - y \cos x\, dy &= \iint_R [\mathrm{D}_x(-y \cos x) - \mathrm{D}_y(x \sin y)]dA \\ &= \int_0^{\pi/2}\int_0^{\pi/4} (y \sin x - x \cos y)dy\, dx \\ &= \int_0^{\pi/2}\left[\tfrac{1}{2}y^2 \sin x - x \sin y\right]_0^{\pi/4} \\ &= \int_0^{\pi/2}(\tfrac{1}{32}\pi^2 \sin x - \tfrac{1}{2}\sqrt{2}x)dx \\ &= -\tfrac{1}{32}\pi^2 \cos x - \tfrac{1}{4}\sqrt{2}x^2\Big]_0^{\pi/2} = -\tfrac{1}{4}\sqrt{2}(\tfrac{1}{4}\pi^2) - (-\tfrac{1}{32}\pi^2 = \tfrac{1}{32}(1 - 2\sqrt{2})\pi^2\end{aligned}$$

17. The area of the ellipse $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$ is $\pi(5)(2) = 10\pi$.

$$\oint_C \frac{x^2 y}{x^2+1}dx - \tan^{-1}x\, dy = \iint_R \left[\frac{\partial}{\partial x}(-\tan^{-1}x) + \frac{\partial}{\partial y}\left(\frac{x^2 y}{x^2+1}\right)\right]dA = \iint_R \left(\frac{-1}{1+x^2} - \frac{x^2}{x^2+1}\right)dA = -\iint_R dA = -10\pi$$

18. $\oint_C e^y \cos x\, dx + e^y \sin x\, dy = \iint_R [\mathrm{D}_x(e^y \sin x) - \mathrm{D}_y(e^y \cos x)]dA = \iint_R (e^y \cos x - e^y \cos x)dA = 0$

19. $\oint_C (e^x - x^2 y)dx + 3x^2 y\, dy = \iint_R \left[\frac{\partial}{\partial x}(3x^2 y) - \frac{\partial}{\partial y}(e^x - x^2 y)\right]dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (6xy + x^2)dy\, dx =$

$= \int_0^1 \left[3xy^2 + x^2 y\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 (3x^3 + x^{5/2} - 3x^5 - x^4)dx = x^3 + \frac{2}{7}x^{7/2} - \frac{1}{2}x^6 - \frac{1}{5}x^5\Big]_0^1 = \frac{41}{70}$

20. $\oint_C \tan y + x \tan^2 y\, dy$, where C is the ellipse $x^2 + 4y^2 = 1$.

▸ Applying Green's theorem, we obtain

$$\begin{aligned}\oint_C \tan y + x \tan^2 y\, dy &= \iint_R [\mathrm{D}_x(x \tan^2 y) - \mathrm{D}_y(\tan y)]dA = \iint_R (\tan^2 y - \sec^2 y)dA \\ &= \iint_R (-1)dA = -(\text{area of R}) = -\tfrac{1}{2}\pi\end{aligned}$$

because the area of an ellipse of semiaxes 1 and $\frac{1}{2}$ is $\pi(1)(\frac{1}{2}) = \frac{1}{2}\pi$.

In Exercises 21–26, use Theorem 14.4.2 to find the area of the region.

▸ $\mathrm{A} = \frac{1}{2}\oint_C x\, dy - y\, dx$

21. C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 4$; C_2: $\mathbf{R}_2(t) = (4-t)\mathbf{i} + 2t\mathbf{j}$, $0 \le t \le 1$;
C_3: $\mathbf{R}_3(t) = (3-2t)\mathbf{i} + (2-t)\mathbf{j}$, $0 \le t \le 1$; C_4: $\mathbf{R}_4(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$.

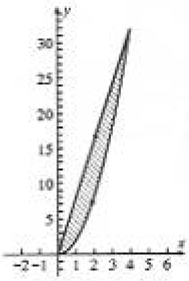
$\mathrm{A} = \frac{1}{2}\int_0^4 0\, dt + \frac{1}{2}\int_0^1 [2(4-t) + 2t]dt + \frac{1}{2}\int_0^1 [-(3-2t) + 2(2-t)]dt + \frac{1}{2}\int_0^1 [-(1-t) + (1-t)]dt$

$= 0 + \frac{1}{2}\int_0^1 8\, dt + \frac{1}{2}\int_0^1 1\, dt + 0 = \frac{9}{2}$

22. $x = a \cos\theta$, $y = a \sin\theta$. $\mathrm{A} = \frac{1}{2}\int_0^{2\pi}(a \cos\theta)(a \cos\theta\, d\theta) - (a \sin\theta)(-a \sin\theta\, d\theta) = \frac{1}{2}a^2 \int_0^{2\pi} d\theta = \pi a^2$

23. C_1: $\mathbf{R}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$; $-C_2$: $\mathbf{R}_2(t) = t\mathbf{i} + t^{1/2}\mathbf{j}$, $0 \le t \le 1$.

$\mathrm{A} = \frac{1}{2}\int_0^1 (2t^2 - t^2)dt - \frac{1}{2}\int_0^1 \left(\frac{1}{2}t^{1/2} - t^{1/2}\right)dt = \frac{1}{2}\int_0^1 \left(t^2 + \frac{1}{2}t^{1/2}\right)dt = \frac{1}{2}\left[\frac{1}{3}t^3 + \frac{1}{3}t^{3/2}\right]_0^1 = \frac{1}{3}$



24. The region bounded by the parabola $y = 2x^2$ and the line $y = 8x$.

▸ The figure shows the region R, bounded by the curve C. Let C_1 be the arc of the parabola $y = 2x^2$ from (0,0) to (4,32), and let C_2 be the line segment $y = 8x$ from (4,32) to (0,0). By Theorem 14.4.2, the area of R is

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_{C_1} x\,dy - y\,dx + \frac{1}{2}\oint_{C_2} x\,dy - y\,dx \qquad (1)$$

Parametric equations of C_1 are $x = t$ and $y = 2t^2$ for $0 \le t \le 4$. Thus,

$$\oint_{C_1} x\,dy - y\,dx = \int_0^4 t(4t\,dt) - 2t^2dt = \int_0^4 2t^2dt = \tfrac{2}{3}t^3\Big]_0^4 = \tfrac{128}{3} \qquad (2)$$

Parametric equations of C_2 are $x = t$ and $y = 8t$ from $t = 4$ to $t = 0$. Thus,

$$\oint_{C_2} x\,dy - y\,dx = \int_4^0 t(8\,dt) - 8t\,dt = 0 \qquad (3)$$

Substituting from Eqs. (2) and (3) into Eq. (1), we obtain

$A = \frac{1}{2}\cdot\frac{128}{3} + \frac{1}{2}\cdot 0 = \frac{64}{3}$

The area is $\frac{64}{3}$ square units.

25. Area of astroid $x = a\cos^3 t$, $y = a\sin^3 t$. $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} [a\cos^3(3a\sin^2 t\cos t) + a\sin^3 t(3a\cos^2 t\sin t)]dt$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)dt = \frac{3a^2}{2}\int_0^{2\pi} \frac{1}{4}\sin^2 2t(1)dt = \frac{3a^2}{2}\cdot\frac{1}{8}\cdot 2\pi = \frac{3a^2\pi}{8}$$

For the hypocyloid of $a/b = m$ cusps, let $k = m - 1$. Then $x = kb\cos t + b\cos kt$, $y = kb\sin t - b\sin kt$.

$$A = \tfrac{1}{2}\int_0^{2\pi}[(kb\cos t + b\cos kt)(kb\cos t - kb\cos kt) - (kb\sin t - b\sin kt)(-kb\sin t - kb\sin kt)]dt$$

$$= \tfrac{1}{2}k(k-1)b^2\int_0^{2\pi}[1 + \cos(k-1)t]dt = k(k-1)b^2\pi = \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)a^2\pi.$$

26. $x = t - \sin t$, $y = 1 - \cos t$. $A = \displaystyle\int_0^{2\pi} 0\,dt + \tfrac{1}{2}\int_{2\pi}^0 (t - \sin t)(\sin t\,dt) - (1 - \cos t)(1 - \cos t\,dt)$

$$= \tfrac{1}{2}\int_{2\pi}^0 (t\sin t + 2\cos t - 2)dt = \tfrac{1}{2}\Big[-t\cos t + \sin t + 2\sin t - 2t\Big]_{2\pi}^0 = 3\pi$$

In Exercises 27–30, verify Gauss's divergence theorem in the plane and Stokes's theorem in the plane for **F** and **R**.

▸ We must show (a) $\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint \text{div}\,\mathbf{F}\,dA$ and (b) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dA$.

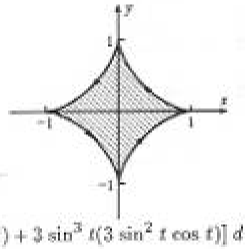
27. $\mathbf{F}(x,y) = 3x\mathbf{i} + 2y\mathbf{j}$, $\mathbf{R}(s) = \cos s\mathbf{i} + \sin s\mathbf{j}$, $0 \le s \le 2\pi$

(a) $\mathbf{N}(s) = \cos s\mathbf{i} + \sin s\mathbf{j}$. $\oint_C \mathbf{F}\cdot\mathbf{N}ds = \displaystyle\int_0^{2\pi}(3\cos s\mathbf{i} + 2\sin s\mathbf{j})\cdot(\cos s\mathbf{i} + \sin s\mathbf{j})ds$

$$= \int_0^{2\pi}(3\cos^2 s + 2\sin^2 s)ds = \frac{1}{2}\int_0^{2\pi}(\cos 2s + 5)ds = \tfrac{1}{2}(5)2\pi = 5\pi \text{ and } \iint_R \text{div}\,\mathbf{F}\,dA = \iint_R (3+2)dA = 5\pi.$$

(b) $\mathbf{T}(s) = -\sin s\mathbf{i} + \cos s\mathbf{j}$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \displaystyle\int_0^{2\pi}(3\cos s\mathbf{i} + 2\sin s\mathbf{j})\cdot(-\sin s\mathbf{i} + \cos s\mathbf{j})ds$

$$= \int_0^{2\pi}(-3\cos s\sin s + 2\sin s\cos s)ds = -\frac{1}{2}\sin^2 s\Big]_0^{2\pi} = 0 \text{ and } \iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}dA = \iint_R \mathbf{0}\cdot\mathbf{k}\,dA = 0.$$



28. $\mathbf{F}(x,y) = 3y\mathbf{i} - 2x\mathbf{j}$ and **R** is the region bounded by $x^{2/3} + y^{2/3} = 1$.

▸ The figure shows the region R. The boundary curve C is an astroid.

Parametric equations of C are

$x = \cos^3 t \quad y = \sin^3 t \quad 0 \le t \le 2\pi$

Thus

$dx = -3\cos^2 t\sin t\,dt \qquad dy = 3\sin^2 t\cos t\,dt$

Because $\mathbf{F}(x,y) = 3y\mathbf{i} - 2x\mathbf{j}$, then

$M(x,y) = 3y = 3\sin^3 t \qquad N(x,y) = -2x = -2\cos^3 t$

The left side of Gauss's divergence theorem in the plane is

$$\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \oint_C -N(x,y)dx + M(x,y)dy = \int_0^{2\pi}[2\cos^3 t(-3\cos^2 t\sin t) + 3\sin^3 t(3\sin^2 t\cos t)]\,dt$$

$$= \int_0^{2\pi}(9\sin^5 t\cos t - 6\cos^5 t\sin t)\,dt = \tfrac{3}{2}\sin^6 t + \cos^6 t\Big]_0^{2\pi} = 0$$

The right side of Gauss's divergence theorem in the plane is

$$\iint_R \text{div } \mathbf{F}\, dA = \iint_R [M_x(x,y) + N_y(x,y)]dA = \iint_R [\text{D}_x(3y) + \text{D}_y(-2x)]dA = \iint_R 0\, dA = 0$$

and we have verified Gauss's divergence theorem in the plane for the given function and region.

The left side of Stokes's theorem in the plane is

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M(x,y)\, dx + N(x,y)\, dy = \int_0^{2\pi} [3\sin^3 t)(-3\cos^2 t \sin t) + (-2\cos^3 t)(3\sin^2 t\cos t)]\, dt$$

$$= \int_0^{2\pi} -3\sin^2 t\cos^2 t(3\sin^2 t + 2\cos^2 t)\, dt = \int_0^{2\pi} -3\left(\frac{\sin 2t}{2}\right)^2\left[3\,\frac{1-\cos 2t}{2} + 2\,\frac{1+\cos 2t}{2}\right] dt$$

$$= -\tfrac{3}{8}\int_0^{2\pi} \sin^2 2t(5-\cos 2t)\, dt = -\tfrac{3}{8}\int_0^{2\pi} [\tfrac{5}{2}(1-\cos 4t) - \sin^2 2t\cos 2t]\, dt$$

$$= -\tfrac{3}{8}\Big[\tfrac{5}{2}t - \tfrac{5}{8}\sin 4t - \tfrac{1}{6}\sin^3 2t\Big]_0^{2\pi} = -\tfrac{3}{8}(5\pi) = -\tfrac{15}{8}\pi$$

The right side of Stokes's theorem in the plane is

$$\iint_R \text{curl } \mathbf{F}\cdot\mathbf{k}\, dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left[\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y)\right] dA = \iint_R -5\, dA$$

$$= -5\int_{-1}^{1}\int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dy\, dx = -10\int_{-1}^{1} (1-x^{2/3})^{3/2}\, dx$$

Let $x = \cos^3\theta$ and $dx = -3\cos^2\theta\sin\theta\, d\theta$. Thus

$$\iint_R \text{curl } \mathbf{F}\cdot\mathbf{k}\, dA = -10\int_{-\pi}^{0} (1-\cos^2\theta)^{3/2}(-3\cos^2\theta\sin\theta)\, d\theta = -30\int_0^{\pi} (\sin^2\theta)^{3/2}\cos^2\theta\sin\theta\, d\theta$$

$$= -30\int_0^{\pi} (\sin^2\theta\cos^2\theta)\sin^2\theta\, d\theta = -30\int_0^{\pi}\left(\frac{\sin 2\theta}{2}\right)^2 \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\tfrac{15}{4}\int_0^{\pi} \sin^2 2\theta(1-\cos 2\theta)\, d\theta = -\tfrac{15}{4}\int_0^{\pi} [\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\cos 2\theta]\, d\theta$$

$$= -\tfrac{15}{4}\Big[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\Big]_0^{\pi} = -\tfrac{15}{4}(\tfrac{1}{2}\pi) = -\tfrac{15}{8}\pi$$

and we have verified Stokes's theorem in the plane for the given vector field and region.

29. $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$. $\mathbf{R}(t) = 5\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le 2\pi$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\, ds = \oint_C -y^2dx + x^2dy = \int_0^{2\pi}[-(4\sin^2 t)(-5\sin t) + (25\cos^2 t)(2\cos t)]dt$

$= \int_0^{2\pi}(20\sin^3 t + 50\cos^3 t)dt = 0$ and $\iint_R \text{div}(x^2\mathbf{i} + y^2\mathbf{j})dA = \iint_R (2x+2y)dA = 0$ (odd powers).

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}ds = \oint_C x^2dx + y^2dy = \int_0^{2\pi}[25\cos^2 t\frac{d}{dt}(\cos t) + 4\sin^2 t\frac{d}{dt}(\sin t)]dt = \frac{25}{3}\cos^3 t + \frac{4}{3}\sin^3 t\Big]_0^{2\pi} = 0$

and $\iint_R \text{curl } \mathbf{F}\cdot\mathbf{k}\, dA = \iint_R \mathbf{0}\cdot\mathbf{k}\, dA = 0$

30. $\mathbf{F}(x,y) = y^2\mathbf{i} + x^2\mathbf{j}$. R: $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\, ds = \oint_C -x^2dx + y^2dx = \oint_C d(-\frac{1}{3}x^3 + \frac{1}{3}y^3) = 0$ and $\iint_R \text{div}(y^2\mathbf{i} + x^2\mathbf{j})dA = \iint_R 0\, dA = 0$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C y^2dx + x^2dy = \int_0^{2\pi} 4\sin^2 t(-5\sin t\, dt) + 25\cos^2 t(2\cos t\, dt) = 0$ and

$\iint_R \text{curl } \mathbf{F}\cdot\mathbf{k}\, dA = \iint_R (2x-2y)dA = 0$ because each term is odd

In Exercises 31–34, use Green's theorem to find the total work done in moving an object in the counterclockwise direction once around curve C if the motion is caused by the force field $\mathbf{F}(x,y)$. Length is measured in meters and force is measured in newtons.

▸ In Exercises 31–33 W joules of work is done.

31. The ellipse $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ has area $\pi(4)(2) = 8\pi$ cm^2.

$\text{W} = \oint_C (3x+y)dx + (4x-5y)dy = \iint_R \left[\frac{\partial}{\partial x}(4x-5y) - \frac{\partial}{\partial y}(3x+y)\right]dA = \iint_R 3\, dA = 3\cdot 8\pi = 24\pi$

32. C is the circle $x^2 + y^2 = 25$: $\mathbf{F}(x,y) = (e^x + y^2)\mathbf{i} + (x^2y + \cos y)\mathbf{j}$

▸ Let R be the region enclosed by the circle. The number of joules in the work is given by

$$W = \oint_C (e^x + y^2)dx + (x^2y + \cos y)dy = \iint_R [D_x(x^2y + \cos y) - D_y(e^x + y^2)]dA = \iint_R (2xy - 2y)dA = 0$$

because the integrand is an odd function of y and R is symmetric with respect to the x axis.

33. $W = \oint_C (e^x + y^2)dx + (e^y + x^2)dy = \iint_R \left[\frac{\partial}{\partial x}(e^y + x^2) - \frac{\partial}{\partial y}(e^x + y^2)\right]dA = \int_0^2 \int_0^{2-x} (2x - 2y)dy\,dx$

$= \int_0^2 \left[2xy - y^2\right]_0^{2-x} dx = \int_0^2 [2x(2-x) - (2-x)^2]dx = \int_0^2 (-3x^2 + 8x - 4)dx = -x^3 + 4x^2 - 4x\Big]_0^2 = 0$

34. $W = \oint_C (xy + y^2)dx + xy\,dy = \iint_R [D_x(xy) - D_y(xy + y^2)]dA = \iint_R (-y - x)dA$

$= -\int_{-2}^{2} \int_0^{3\sqrt{4-x^2}/2} y\,dy\,dx = -\int_0^2 \frac{9}{4}(4 - x^2)dx = -\frac{9}{4}\left[4x - \frac{1}{3}x^3\right]_0^2 = -12$

In Exercises 35–38, find the rate of flow of the fluid out of the region R bounded by the curve C if **F** is the velocity field of the fluid. Velocity is measured in centimeters per second; area is measured in square centimeters.

35. $\mathbf{F}(x,y) = (y^2 + 6x)\mathbf{i} + (2y - x^2)\mathbf{j}$. The ellipse $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ has area $\pi(2)(1) = 2\pi$ cm^2.

From Gauss' divergence theorem the number of cm^2/sec in the outflow of the fluid is

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_R \text{div}\,\mathbf{F}\,dA = \iint_R (6 + 2)dA = 8 \cdot 2\pi = 16\pi$$

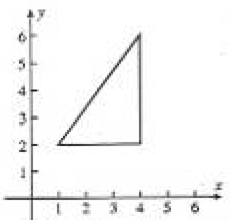
36. $\mathbf{F}(x,y) = (5x - y^2)\mathbf{i} + (3x - 2y)\mathbf{j}$; C is the right triangle having vertices at (1, 2), (4, 2), and (4, 6).

▸ The rate of flow of the fluid is the flux. We apply Gauss's theorem in the plane to evaluate the line integral. The figure shows the the region R. The area of the right triangle is 6 cm^2. Thus,

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \text{div}\,\mathbf{F}\,dA$$

$$= \iint_R (D_x(5x - y^2) + D_y(3x - 2y)]dA = \iint_R 3\,dA = 3(6) = 18$$

The rate of flow of the fluid out of the region R is 18 cm^2/sec.



37. $\mathbf{F}(x,y) = x^3\mathbf{i} + y^3\mathbf{j}$. The number of cm^2/sec in the outflow of the fluid is

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_R \text{div}\,\mathbf{F}\,dA = \iint_R (3x^2 + 3y^2)dA = \int_0^{2\pi}\int_0^1 3r^2(r\,dr\,d\theta) = \int_0^{2\pi} \frac{3}{4}r^4\Big]_0^1 d\theta = \frac{3}{4}\cdot 2\pi = \frac{3}{2}\pi$$

38. $\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \text{div}(xy^2\mathbf{i} + yx^2\mathbf{j})dA = \iint_R (y^2 + x^2)dA = \int_0^{2\pi}\int_0^3 r^2(r\,dr\,d\theta) = 2\pi \cdot \frac{1}{4}r^4\Big]_0^3 = \frac{81}{2}\pi$

In Exercises 39–42, **F** is the velocity field of a fluid around a closed curve C. Use Stokes's theorem in the plane to compute $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$, and from the result determine which of the following applies: (i) the circulation of the fluid is counterclockwise; (ii) the circulation of the fluid is clockwise; (iii) **F** is irrotational; (iv) the circulation is 0 but **F** is not irrotational.

39. $\mathbf{F}(x,y) = 4y\mathbf{i} + 6x\mathbf{j}$. $\text{curl}\,\mathbf{F} = (6 - 4)\mathbf{k} = 2\mathbf{k}$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dA = \iint_R 2\,dA > 0$. Hence (i).

40. $\mathbf{F}(x,y) = 8y\mathbf{i} + 3x\mathbf{j}$; C is the ellipse $4x^2 + 9y^2 = 1$.

▸ The curl of **F** is given by

$$\text{curl}\,\mathbf{F} = [D_x(3x) - D_y(8y)]\mathbf{k} = -5\mathbf{k}$$

The area of the region R bounded by the ellipse is $\pi ab = \pi(\frac{1}{2})(\frac{1}{3}) = \frac{1}{6}\pi$. Therefore, applying Stokes's theorem in the plane, we have

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dA = \iint_R (-5\mathbf{k})\cdot\mathbf{k}\,dA = -5\iint_R dA = -5(\tfrac{1}{6}\pi) = -\tfrac{5}{6}\pi$$

Because $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds < 0$, the circulation of the fluid is clockwise.

41. $\mathbf{F}(x,y) = \sin^2 x\mathbf{i} + \cos^2 y\mathbf{j}$. $\text{curl}\,\mathbf{F} = \mathbf{0}$. Hence (iii).

42. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \text{curl}(y^3\mathbf{i} + x^3\mathbf{j})dA = \iint_R (3x^2 - 3y^2)dA = 0$ Hence (iv).

43. $\oint_C N(x,y)dy = \iint_R \frac{\partial N}{\partial x}\,dA$ is Green's theorem with $M(x,y) = 0$.

14.5 SURFACE INTEGRALS

Let S be the surface that is the graph of the equation $z = f(x,y)$ were f and its first partial derivatives are continuous on a region D in the xy plane, and let G be a function of x, y, z which is continuous on S. Then the surface integral of G over S can be expressed as a double integral over the region D and is given by

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)+1}\,dA$$

In polar coordinates, if the surface is given by $z = f(r,\theta)$, then

$$\iint_S G(r,\theta,z)d\sigma = \iint_R G(r,\theta,f(r,\theta))\sqrt{f_r{}^2(r,\theta)+r^{-2}f_\theta{}^2(r,\theta)+1}\;r\,dr\,d\theta$$

If the measure of the area density at the point (x,y,z) on a surface S is $\rho(x,y,z)$, and if M is the measure of the mass of S, then

$$\mathrm{M} = \iint_S \rho(x,y,z)d\sigma$$

Let $\mathbf{F}(x,y,z) = \mathrm{M}(x,y,z)\mathbf{i} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$ be the velocity field for a fluid that flows through a surface S. If s has the equation $z = f(x,y)$, and S lies over a region D in the xy plane, where f and its first partial derivatives are continuous on D, and $\mathbf{n}$ is a unit upper normal vector for the surface, then the flux of $\mathbf{F}$ across S is the surface integral of $\mathbf{F}\cdot\mathbf{n}$ over S, and this surface integral may be evaluated by a double integral over D. The flux of $\mathbf{F}$ across S is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_D (-\mathrm{M}f_x - \mathrm{N}f_y + \mathrm{R})dA \tag{9}$$

where z is replaced by $f(x,y)$.

Exercises 14.5

In Exercises 1–14, evaluate the surface integral $\iint_S G(x,y,z)d\sigma$ for G and S.

1. $f(x,y) = z = \sqrt{4-x^2-y^2}$, $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y(x,y) = \dfrac{-y}{\sqrt{4-x^2-y^2}}$. Circle D has area 4π. $\iint_S z\,d\sigma$

$$= \iint_D \sqrt{4-x^2-y^2}\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\,dA = \iint_D \sqrt{4-x^2-y^2}\sqrt{\frac{4}{4-x^2-y^2}}\,dA = 2\iint_D dA = 8\pi$$

2. $f(x,y) = z = 1-x-y$, $f_x(x,y) = -1$, $f_y(x,y) = -1$.

$$\iint_S x\,d\sigma = \iint_D x\sqrt{1+1+1}dA = \int_0^1\int_0^{1-x}\sqrt{3}x\,dy\,dx = \sqrt{3}\int_0^1(1-x)x\,dx = \sqrt{3}\Big[\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{6}\sqrt{3}$$

3. $f(x,y) = z = 2-x-y$, $f_x(x,y) = -1$, $f_y(x,y) = -1$.

$$\iint_S (x+2y-z)d\sigma = \iint_D (2x+3y-2)\sqrt{(-1)^2+(-1)^2+1}\,dA = \sqrt{3}\int_0^2\int_0^{2-x}(2x+3y-2)dy\,dx$$

$$= \sqrt{3}\int_0^2\Big[(2x-2)y+\frac{3}{2}y^2\Big]_0^{2-x}dx = \sqrt{3}\int_0^2\Big[(2x-2)(2-x)+\tfrac{3}{2}(2-x)^2\Big]dx$$

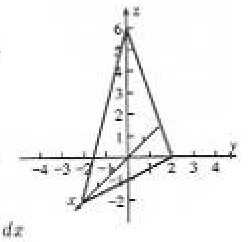
$$= \sqrt{3}\int_0^2\Big[-2x^2+6x-4+\tfrac{3}{2}(2-x)^2\Big]dx = \sqrt{3}\Big[-\tfrac{2}{3}x^3+3x^2-4x-\tfrac{1}{2}(2-x)^3\Big]_0^2 = \tfrac{8}{3}\sqrt{3}$$

4. $G(x,y,z) = z$; S is the portion of the plane $2x+3y+z=6$ in the first octant.

▹ The figure shows the surface S. The projection of S onto the xy plane is the triangular region D, bounded by the x and y axes and the line $2x+3y=6$. Because

$$z = f(x,y) = 6-2x-3y$$

then the surface integral of G over S is given by



$$\iint_S z\,d\sigma = \iint_D (6-2x-3y)\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)}\,dA$$

$$= \int_0^3\int_0^{(6-2x)/3}(6-2x-3y)\sqrt{(-2)^2+(-3)^2+1}\,dy\,dx$$

$$= \sqrt{14}\int_0^3\int_0^{(6-2x)/3}(6-2x-3y)dy\,dx = \int_0^3 (6-2x)y-\tfrac{3}{2}y^2\Big]_0^{(6-2x)/3}dx$$

$$= \sqrt{14}\int_0^3 \tfrac{1}{6}(6-2x)^2\,dx = \sqrt{14}(-\tfrac{1}{36})(6-2x)^3\Big]_0^3 = 6\sqrt{14}$$

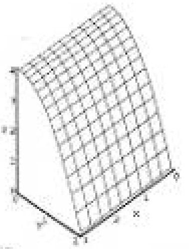
5. $f(x,y) = z = 6 - 2x - 3y$. $f_x(x,y) = -2$, $f_y(x,y) = -3$. $\iint_S xyz\, d\sigma = \iint_D xy(6-2x-3y)\sqrt{4+9+1}\, dA$
$= \sqrt{14}\int_0^3\int_0^{(6-2x)/3} x[(6-2x)y - 3y^2]dy\, dx = \sqrt{14}\int_0^3 x\left[\tfrac{1}{2}(6-2x)y^2 - y^3\right]_0^{(6-2x)/3} dx$
$= \frac{1}{54}\sqrt{14}\int_0^3 x(6-2x)^3 dx = \frac{1}{54}\sqrt{14}\left[-\tfrac{1}{8}x(6-2x)^4 - \frac{1}{80}(6-2x)^5\right]_0^3 = \frac{1}{54}\sqrt{14}\frac{6^5}{80} = \frac{9}{2}\sqrt{14}$

6. $f(x,z) = y = \sqrt{1-x^2}$. $f_x(x,z) = \dfrac{-x}{\sqrt{1-x^2}}$, $f_z(x,z) = 0$. $\iint_S x^2\sqrt{\dfrac{x^2}{1-x^2} + 0 + 1}dA = \int_0^1\int_0^1 \dfrac{x^2}{\sqrt{1-x^2}}dz\, dx$
$= \int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \int_0^{\pi/2}\frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)d\theta = \tfrac{1}{4}\pi$

7. $f(x,y) = z = x^2$. $f_x(x,y) = 2x$, $f_y(x,y) = 0$. $\iint_S x\, d\sigma = \iint_D x\sqrt{4x^2+1}\, dA = \int_0^1\int_0^2 x\sqrt{4x^2+2}\, dy\, dx$
$= \frac{1}{4}\int_0^1 8x\sqrt{4x^2+1}\, dx = \frac{1}{4}\cdot\frac{2}{3}(4x^2+1)^{3/2}\Big]_0^1 = \tfrac{1}{2}(5\sqrt{5}-1)$

8. $G(x,y,z) = y$; S is the portion of the cylinder $z = 4 - y^2$ in the first octant bounded by the coordinate planes and the plane $x = 3$.

▹ The figure shows the surface S. The projection of S onto the xy plane is the rectangular region D, bounded by the x and y axes and the lines $x = 3$ and $y = 2$. Because
$z = f(x,y) = 4 - y^2$
then the surface integral of G over S is given by



$$\iint_S y\, d\sigma = \iint_D y\sqrt{f_x{}^2(x,y) + f_y{}^2(x,y) + 1}dA = \int_0^2\int_0^3 y\sqrt{(-2y)^2+1}dx\, dy$$
$$= \int_0^2 3y\sqrt{4y^2+1}dy = 3(\tfrac{1}{8})(\tfrac{2}{3})(4y^2+1)^{3/2}\Big]_0^2 = \tfrac{1}{4}(17\sqrt{17}-1)$$

In Exercises 9 and 10, $f(x,y) = z = \sqrt{x^2+y^2}$. $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$

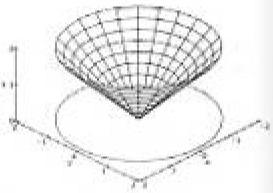
9. $\iint_S z^2 d\sigma = \iint_D (x^2+y^2)\sqrt{\dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} + 1}\, dA = \sqrt{2}\iint_D (x^2+y^2)dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r^2(r\, dr\, d\theta)$
$= \sqrt{2}\int_0^{2\pi}\frac{1}{4}r^4\Big]_0^2 = \frac{1}{4}\sqrt{2}\int_0^{2\pi} 15\, d\theta = \frac{15}{2}\sqrt{2}\pi$

10. $\iint_S xyz\, d\sigma = \sqrt{2}\iint_D xy\sqrt{x^2+y^2}dA = 0$ because integrand is odd

11. $f(x,y) = z = 2 - \frac{2}{3}x - \frac{1}{2}y$. $f_x(x,y) = -\frac{2}{3}$, $f_y(x,y) = -\frac{1}{2}$.
$\iint_S (x+y)d\sigma = \iint_D (x+y)\sqrt{\tfrac{4}{9} + \tfrac{1}{4} + 1}\, dA = \tfrac{1}{6}\sqrt{61}\int_0^3\int_0^{4-4x/3}(x+y)dy\, dx = \tfrac{1}{6}\sqrt{61}\int_0^3 \tfrac{1}{2}(x+y)^2\Big]_0^{4-4x/3}$
$= \frac{1}{12}\sqrt{61}\int_0^3\left[\left(4 - \frac{x}{3}\right)^2 - x^2\right]dx = \frac{1}{12}\sqrt{61}\left[-\left(4 - \frac{x}{3}\right)^3 - \frac{1}{3}x^3\right]_0^3 = \frac{1}{12}\sqrt{61}(-27 - 9 + 64) = \frac{7}{3}\sqrt{61}$

12. $G(x,y,z) = \sqrt{x^2+y^2+z^2}$; S is the portion of the cone $x^2 + y^2 = z^2$ between the xy plane and the plane $z = 2$.

▹ The figure shows the surface S. The projection of S onto the xy plane is the region D bounded by the circle $x^2 + y^2 = 4$. Solving the equation of the surface for z, we obtain



$z = f(x,y) = \sqrt{x^2+y^2}$
Then

$f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$

We evaluate the surface integral of G over S with the help of polar coordinates

$$\iint_S \sqrt{x^2+y^2+z^2}d\sigma = \iint_D \sqrt{x^2+y^2+(x^2+y^2)}\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}dA$$
$$= \iint_D \sqrt{2(x^2+y^2)}\sqrt{2}dA = 2\iint_D \sqrt{x^2+y^2}dA = 2\int_0^{2\pi}\int_0^2 r(r\, dr\, d\theta) = 2(2\pi)\tfrac{1}{3}r^3\Big]_0^2 = \tfrac{32}{3}\pi$$

13. $f(x,y)=z=\sqrt{4-x^2}$. $f_x(x,y)=\dfrac{-x}{\sqrt{4-x^2}}$, $f_y(x,y)=0$

$$\iint_S xyz\,d\sigma=\iint_D xy\sqrt{4-x^2}\sqrt{\frac{x^2}{4-x^2}+1}\,dA=\int_1^3\int_{-1}^2 2xy\,dx\,dy=\int_1^3 x^2y\Big]_{-1}^2\,dy=0$$

14. Let T be the entire sphere. $\iint_S x^2d\sigma=\frac{1}{2}\iint_T x^2d\sigma=\frac{1}{2}\iint_T y^2d\sigma=\frac{1}{2}\iint_T z^2d\sigma=\frac{1}{6}\iint_T(x^2+y^2+z^2)d\sigma$

$$=\tfrac{1}{6}\cdot 81\iint_T d\sigma=\tfrac{1}{6}\cdot 81\cdot 4\pi=54\pi$$

In Exercises 15–20, find the mass of the surface S if the area density at any point (x,y,z) on S is $\rho(x,y,z)$ kg/m^2.

15. $f(x,y)=z=\sqrt{4-x^2-y^2}$. $f_x(x,y)=\dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y(x,y)=\dfrac{-y}{\sqrt{4-x^2-y^2}}$. If M kg is the mass

$$\mathrm{M}=\iint_S \mathrm{k}\sqrt{x^2+y^2+z^2}\,d\sigma=\iint_D 2\mathrm{k}\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\,dA=4\mathrm{k}\iint_D\frac{1}{\sqrt{4-x^2-y^2}}\,dA$$

$$=4\mathrm{k}\int_0^{2\pi}\int_0^1\tfrac{1}{r}(r\,dr\,d\theta)=4\mathrm{k}\int_0^{2\pi}1\,d\theta=4\mathrm{k}\cdot 2\pi=8\pi\mathrm{k}$$

16. S is the portion of the plane $3x+2y+z=6$ in the first octant; $\rho(x,y,z)=y+2z$.

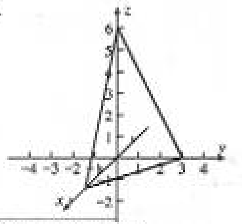
▹ The figure shows the surface S. The mass of the surface is M kg, where M is given by the surface integral

$$\mathrm{M}=\iint_S(y+2z)d\sigma$$

The projection of S onto the xy plane is the region bounded by the axes and the line $3x+2y=6$. solving the equation of the surface for z, we obtain

$z=f(x,y)=6-3x-2y$

Then, substituting for z,



$$\mathrm{M}=\iint_D(y+2z)\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)+1}\,dA=\iint_D(12-6x-3y)\sqrt{(-3)^2+(-2)^2+1}\,dA$$

$$=3\sqrt{14}\int_0^2\int_0^{3(2-x)/2}(4-2x-y)dy\,dx=3\sqrt{14}\int_0^2\Big[2(2-x)y-\tfrac{1}{2}y^2\Big]_0^{3(2-x)/2}dx$$

$$=3\sqrt{14}\int_0^2[3(2-x)^2-\tfrac{9}{8}(2-x)^2]dx=\tfrac{45}{8}\sqrt{14}\int_0^2(2-x)^2dx=-\tfrac{15}{8}\sqrt{14}(2-x)^3\Big]_0^2=15\sqrt{14}$$

17. $f(x,y)=z=9-x^2-y^2$. $f_x(x,y)=-2x$, $f_y(x,y)=-2y$. $d\sigma=\sqrt{4x^2+4y^2+1}\,dA$. D is a circle of radius 3.
If M kg is the mass, $\mathrm{M}=\iint_S d\sigma/\sqrt{4x^2+4y^2+1}=\iint_D dA=9\pi$.

18. Let T be the entire sphere.

$$\mathrm{M}=\iint_S(x^2+y^2)d\sigma=\tfrac{1}{2}\iint_T(x^2+y^2)d\sigma=\tfrac{1}{2}\cdot\tfrac{2}{3}\iint_T(x^2+y^2+z^2)d\sigma=\tfrac{1}{3}\iint_T 1d\sigma=\tfrac{1}{3}\cdot 4\pi=\tfrac{4}{3}\pi$$

19. $f(x,y)=z=\sqrt{x^2+y^2}$. $f_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$. If M kg is the mass

$$\mathrm{M}=\iint_S y^2z^2d\sigma=\iint_D y^2(x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA=\sqrt{2}\iint_D y^2(x^2+y^2)dA$$

$$=\sqrt{2}\int_0^{2\pi}\int_0^3 r^2\sin^2\theta(r^2)r\,dr\,d\theta=\sqrt{2}\int_0^{2\pi}\frac{1}{6}r^6\Big]_0^3\sin^2\theta d\theta=\frac{665}{6}\sqrt{2}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)d\theta=\frac{665}{6}\sqrt{2}\pi$$

20. S is the portion of the sphere $x^2+y^2+z^2=16$ in the first octant; $\rho(x,y,z)=kz^2$, where k is a constant.

▹ The mass of the surface is M kg, where M is given by the surface integral

$$\mathrm{M}=\iint_S kz^2\,d\sigma \qquad (1)$$

By symmetry we also have

$$\mathrm{M}=\iint_S kx^2\,d\sigma\quad(2)\quad\text{and}\qquad\mathrm{M}=\iint_S ky^2\,d\sigma\quad(3)$$

Adding (1), (2) and (3) and dividing by 3, we get

$$\mathrm{M}=\tfrac{1}{3}\iint_S k(x^2+y^2+z^2)\,d\sigma=\tfrac{16}{3}k\iint_S d\sigma=\tfrac{16}{3}k(\tfrac{1}{8}\text{ area of sphere of radius }4)=\tfrac{16}{3}\cdot\tfrac{1}{8}\cdot 4\pi(4)^2=\tfrac{128}{3}k\pi$$

In Exercises 21–24, find the flux of **F** across the surface S where $\mathbf{F}(x, y, z)$ gives the velocity field of the fluid.

21. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $f(x,y) = z = 6 - 3x - 2y$. $f_x = -3$, $f_y = -2$. D is a triangle of area 3.

By equation (9), flux $= \iint_S \mathbf{F}\cdot\mathbf{N}\,ds = \iint_R [-x(-3) - y(-2) + (6 - 3x - 2y)]dA = \iint_D 6\,dA = 18.$

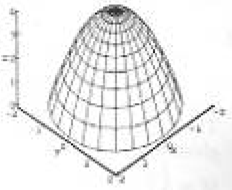
22. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has direction of **n** and so $\mathbf{F}\cdot\mathbf{n} = \sqrt{x^2 + y^2 + z^2} = 1$ and $\iint_S \mathbf{F}\cdot\mathbf{n}ds = \iint_S ds = 2\pi$

23. $\mathbf{F} = -2y\mathbf{i} + 2x\mathbf{j} + 5\mathbf{k}$. $f(x,y) = z = \sqrt{16 - x^2 - y^2}$. $f_x = \dfrac{-x}{\sqrt{16 - x^2 - y^2}} = -\frac{x}{z}$, $f_y = \dfrac{-y}{\sqrt{16 - x^2 - y^2}} = -\frac{y}{z}$

By equation (9), flux $= \iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_D \left[(2y)\left(-\frac{x}{z}\right) + (-2x)\left(-\frac{y}{z}\right) + 5\right]dA = 5\iint_D dA = 5\pi(3)^2 = 45\pi.$

24. $\mathbf{F}(x,y,z) = 3x\mathbf{i} + 3y\mathbf{j} + 6z\mathbf{k}$; S is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy plane.

▸ The figure shows the surface S, which lies above the region D in the xy plane.
D is bounded by the circle $x^2 + y^2 = 4$, whose area is 4π. Let
$z = f(x,y) = 4 - x^2 - y^2$
and
$M(x,y,z) = 3x \quad N(x,y,z) = 3y \quad R(x,y,z) = 6z$
If **n** is a unit upper normal vector for S, then the flux of **F** across S is given by



$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_D (-Mf_x - Nf_y + R)dA$$
$$= \iint_D [(-3x)(-2x) - (3y)(-2y) + 6(4 - x^2 - y^2)]dA = 24\iint_D dA$$
$$= 24(4\pi) = 96\pi$$

25. $\mathbf{F}(x,y,z) = x^2\mathbf{i} + xy\mathbf{j} + 2z\mathbf{k}$. Take the sides of the cube in the order $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \int_0^1\int_0^1 (2z\mathbf{k})\cdot(-\mathbf{i})dy\,dz + \int_0^1\int_0^1 (\mathbf{i} + y\mathbf{j} + 2z\mathbf{k})\cdot\mathbf{i}\,dy\,dz + \int_0^1\int_0^1 (x^2\mathbf{i} + 2z\mathbf{k})\cdot(-\mathbf{j})dx\,dz$$
$$+ \int_0^1\int_0^1 (x^2\mathbf{i} + x\mathbf{j} + 2z\mathbf{k})\cdot\mathbf{j}\,dx\,dz + \int_0^1\int_0^1 (x^2\mathbf{i} + xy\mathbf{j})\cdot(-\mathbf{k})dx\,dy + \int_0^1\int_0^1 (x^2\mathbf{i} + xy\mathbf{j} + 2\mathbf{k})\cdot\mathbf{k}\,dx\,dy$$
$$= 0 + \int_0^1\int_0^1 dy\,dz + 0 + \int_0^1\int_0^1 x\,dx\,dz + 0 + \int_0^1\int_0^1 2\,dx\,dy = 1 + \tfrac{1}{2} + 2 = \tfrac{7}{2}$$

26. $\mathbf{F}(x,y,z) = 3x\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}$. Take the sides of the cube in the order $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \int_0^1\int_0^1 (y^2\mathbf{j} + yz\mathbf{k})\cdot(-\mathbf{i})dy\,dz + \int_0^1\int_0^1 (3\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k})\cdot\mathbf{i}\,dy\,dz + \int_0^1\int_0^1 (3x\mathbf{i})\cdot(-\mathbf{j})dx\,dz$$
$$+ \int_0^1\int_0^1 (3x\mathbf{i} + \mathbf{j} + z\mathbf{k})\cdot\mathbf{j}\,dx\,dz + \int_0^1\int_0^1 (3x\mathbf{i} + y^2\mathbf{j})\cdot(-\mathbf{k})dx\,dy + \int_0^1\int_0^1 (3x\mathbf{i} + y^2\mathbf{j} + y\mathbf{k})\cdot\mathbf{k}\,dx\,dy$$
$$= 0 + \int_0^1\int_0^1 3dy\,dz + 0 + \int_0^1\int_0^1 dx\,dz + 0 + \int_0^1\int_0^1 y\,dx\,dy = 3 + 1 + \tfrac{1}{2} = \tfrac{9}{2}$$

14.6 GAUSS'S DIVERGENCE THEOREM AND STOKES'S THEOREM

In the following two theorems, M, N and R are functions of three variables x, y, and z, having continuous first partial derivatives on an open ball in R^3 and we let
$\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$.

14.6.1 Theorem (Gauss's Divergence Theorem) Let S be a sectionally smooth closed surface lying in B, let E be the region bounded by S, and let **n** be a unit outward normal vector of S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_E \operatorname{div}\mathbf{F}\,dV$$

14.6.2 Theorem (Stokes's Theorem) Let S be a sectionally smooth two-sided surface with one side called "up" lying in B and let C be a sectionally smooth simple curve that is the boundary of S. **n** is a unit upward normal and **T** is a unit tangent vector to C, where s units is the length of arc measured from a particular point on C to P in the counterclockwise direction when viewed from above. Then

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma$$

Stokes's theorem does not apply to a Mobius band: it is one-sided.

Exercises 14.6

In Exercises 1–4, verify Gauss's divergence theorem for **F** and S.

1. $\mathbf{F} = 2z\mathbf{k}$. div $\mathbf{F} = 2$. E is the sphere $x^2+y^2+z^2=1$. $\iiint_E \text{div } \mathbf{F}\, dV = 2\iiint_E dV = 2\left(\frac{4}{3}\pi\right) = \frac{8}{3}\pi$.

   On the upper hemisphere, by equation 14.5.9, $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_D (0+0+2z)dA$
   $= 2\iint_D (1-x^2-y^2)^{1/2} dA = 2\int_0^{2\pi}\int_0^1 (1-r^2)^{1/2} r\, dr\, d\theta = 2\int_0^{2\pi} -\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1 d\theta = \frac{2}{3}\int_0^{2\pi} d\theta = \frac{4}{3}\pi$
   On the lower hemisphere the signs of z and **N** are reversed so the integral is the same.
   Hence the total flux is $\frac{8}{3}\pi$ and Gauss' divergence theorem is verified.

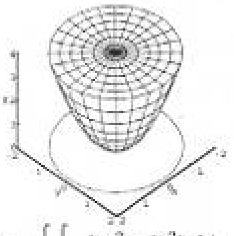
2. $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. E is the sphere $x^2+y^2+z^2=9$. $\iiint_E \text{div } \mathbf{F}\, dV = 3\iiint_E dV = 3\left(\frac{4}{3}\pi 27\right) = 108\pi$

   $\mathbf{F}\cdot\mathbf{n} = \sqrt{x^2+y^2+z^2} = 3$ so $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = 3\cdot 4\pi(3^2) = 108\pi$

3. $\mathbf{F} = xy\mathbf{i}+yz\mathbf{j}$. div $\mathbf{F} = y+0+y = 2y$. E is the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.
   $\iiint_E \text{div } \mathbf{F}\, dV = \int_0^1\int_0^1\int_0^1 2y\, dy\, dx\, dz = \int_0^1 2y\, dy = y^2\Big]_0^1 = 1$. Alternatively,
   $$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \int_0^1\int_0^1 (yz\mathbf{j})\cdot(-\mathbf{i})dy\, dz + \int_0^1\int_0^1 (y\mathbf{i}+yz\mathbf{j})\cdot\mathbf{i}\, dy\, dz + 0 + \int_0^1\int_0^1 (x\mathbf{i}+z\mathbf{j})\cdot\mathbf{j}\, dx\, dz$$
   $$+ \int_0^1\int_0^1 (xy\mathbf{i})\cdot(-\mathbf{k})dx\, dy + \int_0^1\int_0^1 (xy\mathbf{i}+y\mathbf{j})\cdot\mathbf{k}\, dx\, dy$$
   $$= 0 + \int_0^1\int_0^1 y\, dy + 0 + \int_0^1\int_0^1 z\, dx\, dz + 0 + 0 = \frac{1}{2}y^2\Big]_0^1 + \frac{1}{2}z^2\Big]_0^1 = \frac{1}{2}+\frac{1}{2} = 1$$

4. $\mathbf{F}(x,y,z) = 4x\mathbf{i}-2y\mathbf{j}+z\mathbf{k}$; S is the boundary of the region enclosed by the paraboloid $z = x^2+y^2$ and the plane $z=4$.

▸ The figure shows the surface S. S_1 is the part of the paraboloid below the plane and S_2 is the part of S in the plane. The projection of S on the xy plane is the region D bounded by the circle $x^2+y^2=4$. We have
$f(x,y) = x^2+y^2 \quad M(x,y,z) = 4x \quad N(x,y,z) = -2y \quad R(x,y,z) = z = x^2+y^2$



The left side of Gauss's divergence theorem is
$$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}\, d\sigma \qquad (1)$$
Because the outward unit normal to S_1 is downward, we have
$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_D (Mf_x+Nf_y-R)dA = \iint_D [4x(2x)-2y(2y)-(x^2+y^2)]dA = \iint_D (7x^2-5y^2)\, dA$$
We switch to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\, dr\, d\theta$.
$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \int_0^2\int_0^{2\pi} (7r^2\cos^2\theta - 5r^2\sin^2\theta)r\, d\theta\, dr = \int_0^2\int_0^{2\pi} \left[7\left(\frac{1+\cos 2\theta}{2}\right) - 5\left(\frac{1-\cos 2\theta}{2}\right)\right]r^3\, d\theta\, dr$$
$$= \int_0^2\int_0^{2\pi} (1+6\cos 2\theta)r^3\, d\theta\, dr = \int_0^2 2\pi r^3\, dr = \frac{1}{2}\pi r^4\Big]_0^2 = 8\pi \qquad (2)$$
On S^2, $\mathbf{N} = \mathbf{k}$, and $\mathbf{F}\cdot\mathbf{N} = z = 4$. Thus
$$\iint_{S_2} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_{S_2} 4\, d\sigma = 4(\text{area of circle of radius } 2) = 4\pi(2)^2 = 16\pi \qquad (3)$$
Substituting from (2) and (3) into (1), we have
$$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = 8\pi + 16\pi = 24\pi$$
The right side of Gauss's divergence theorem is
$$\iiint_E \text{div } \mathbf{F}\, dV = \iiint_E \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(z)\right] dV = 3\iiint_E dV$$

We compute the volume of E by the method of slicing. A section perpendicular to the z axis is a circle of radius $\sqrt{z}$ and area πz. Thus
$$\iiint_E \text{div } \mathbf{F}\, dV = 3\int_0^4 \pi z\, dz = 3\cdot\frac{1}{2}\pi z^2\Big]_0^4 = 24\pi$$
Thus we have verified Gauss's divergence theorem for the given **F** and S.

In Exercises 5–8, for the **F** and S of the exercise in Ex. 14.5 find the flux of **F** across S by Gauss's divergence theorem.

5. Ex. 21. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. div $\mathbf{F} = 1 + 1 + 1 = 3$. E is a tetrahedron of volume $\frac{1}{6}\cdot 2\cdot 3\cdot 6 = 6$.

Flux $= \iiint_E 3\ dV = 3\cdot 6 = 18$ and the flux over the faces parallel to the coordinate planes are 0.

6. Ex. 22. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. On the xy plane, $\mathbf{F}\cdot\mathbf{n} = (x\mathbf{i} + y\mathbf{j})\cdot\mathbf{k} = 0$. Let T be the closed hemisphere $x^2 + y^2 + z^2 = 1$ and E the region it bounds, of volume $\frac{2}{3}\pi$.

flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_T \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_E \text{div } \mathbf{F}\ dA = \iiint_E 3\ dA = 2\pi$

7. $\text{div}(x^2\mathbf{i} + xy\mathbf{j} + 2z\mathbf{k}) = 2x + x + 2 = 3x + 2$. Flux $= \int_0^1\int_0^1\int_0^1 (3x+2)dx\ dy\ dz = \int_0^1\int_0^1 \frac{7}{2}\ dy\ dz = \frac{7}{2}$.

8. Exercise 14.5.26. $\mathbf{F}(x,y,z) = 3x\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}$; S is the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$.

▸ We have

$$\begin{aligned}\text{div } \mathbf{F}(x,y,z) &= \mathrm{D}_x(3x) + \mathrm{D}_y(y^2) + \mathrm{D}_z(yz)\\ &= 3 + 2y + y = 3 + 3y\end{aligned}$$

Applying the divergence theorem, we have

$$\text{flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_E \text{div } \mathbf{F}\ dV = \int_0^1\int_0^1\int_0^1 (3+3y)dx\ dz\ dy = \int_0^1 (3+3y)dy = 3y + \tfrac{3}{2}y^2\Big]_0^1 = \tfrac{9}{2}$$

In Exercises 9–16, use Gauss's divergence theorem to evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma$ for **F** and S.

9. $\mathbf{F}(x,y,z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$. div $\mathbf{F} = 2xyz + 2xyz + 2xyz = 6xyz$. E is the rectangular parallelpiped $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$.

Flux $= \iiint_E 6xyz\ dV = 6\int_0^1 x\ dx \int_0^2 y\ dy \int_0^3 z\ dz = 6\cdot\frac{1}{2}x^2\Big]_0^1\cdot\frac{1}{2}y^2\Big]_0^2\cdot\frac{1}{2}z^2\Big]_0^3 = 6\cdot\frac{1}{2}\cdot\frac{4}{2}\cdot\frac{9}{2} = 27$

10. $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_E \text{div}(6x\mathbf{i} + 3y\mathbf{j} + 2z\mathbf{k})dV = \iiint_E 11\ dV = 11\cdot\frac{1}{6}(3)(1)(2) = 11$

11. $\iint_S \mathbf{F}\cdot\mathbf{N}\ d\sigma = \iiint_E \text{div}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})dV = \iiint_E 3\ dV = 3\cdot\frac{4}{3}\pi(2)^2 = 32\pi$

12. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the boundary of the region enclosed on the side by the cylinder $x^2 + y^2 = 9$, below by the xy plane, and above by the plane $z = 4$.

▸ The volume of the region E enclosed by the surface S is

$\mathrm{V} = \pi r^2 h = \pi(3^2)(4) = 36\pi$

Because

div $\mathbf{F} = \mathrm{D}_x(x) + \mathrm{D}_y(y) + \mathrm{D}_z(z) = 1 + 1 + 1 = 3$

then by Gauss's divergence theorem we have

$$\iint_S \mathbf{F}\cdot\mathbf{N}\ d\sigma = \iiint_E \text{div } \mathbf{F}\ dV = \iiint_E 3\ dv = 3(36\pi) = 108\pi$$

13.
$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{N}\ d\sigma &= \iiint_E \text{div}(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})dV = \iiint_E 2(x+y+z)dV\\ &= 2\int_0^3\int_0^4\int_0^{2\pi} (r\cos\theta + r\sin\theta + z)r\ d\theta\ dz\ dr = 2\int_0^3\int_0^4 \Big[r\sin\theta - r\cos\theta + z\theta\ \Big]_0^{2\pi} r\ dz\ \mathrm{dr}\\ &= 4\pi\int_0^3\int_0^4 zr\ dz\ dr = 4\pi\int_0^3 8r\ dr = 4\pi(8)\tfrac{9}{2} = 144\pi\end{aligned}$$

14. $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_E \text{div}(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})dV = \iiint_E (2x + 2y + 2z)dV = 0$ (odd)

15.
$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{N}\ d\sigma &= \iiint_E \text{div}(2x\mathbf{i} + 2yz\mathbf{j} + 3z\mathbf{k})dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (5+2z)dz\ dy\ dx\\ &= \int_0^1\int_0^{1-x}\Big[5z + z^2\Big]_0^{1-x-y} = \int_0^1\int_0^1 [5(1-x-y) + (1-x-y)^2]dx = \int_0^1\Big[-\tfrac{5}{2}(1-x-y)^2 - \tfrac{1}{3}(1-x-y)^3\Big]_0^{1-x} dx\\ &= \int_0^1\Big[\tfrac{5}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3\Big]dx = -\tfrac{5}{6}(1-x)^3 - \tfrac{1}{12}(1-x)^4\Big]_0^1 = \tfrac{5}{6} + \tfrac{1}{12} = \tfrac{11}{12}\end{aligned}$$

16. $\mathbf{F}(x,y,z) = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$

▹ Let

$$M = \frac{x}{x^2+y^2+z^2} \quad N = \frac{y}{x^2+y^2+z^2} \quad R = \frac{z}{x^2+y^2+z^2}$$

Then

$$\frac{\partial M}{\partial x} = \frac{1(x^2+y^2+z^2)-x(2x)}{x^2+y^2+z^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{1(x^2+y^2+z^2)-y(2y)}{x^2+y^2+z^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial R}{\partial x} = \frac{1(x^2+y^2+z^2)-z(2z)}{x^2+y^2+z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

Thus,

$$\text{div }\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Applying Gauss's divergence theorem and using spherical coordinates to evaluate the triple integral, we obtain

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_E \text{div }\mathbf{F}\,dV = \iiint_E \frac{1}{x^2+y^2+z^2}dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{1}{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_1^2 d\rho = -\cos\phi\Big]_0^{\pi}(2\pi)1 = 4\pi$$

In Exercises 17–22, verify Stokes's theorem for **F** and S.

▹ We must show that $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl }\mathbf{F}\cdot\mathbf{N}\,d\sigma$.

17. $\mathbf{F}(x,y,z) = y^2\mathbf{i}+x^2\mathbf{j}+z^2\mathbf{k}$, S is $x^2+y^2+z^2=1$, $z\geq 0$. $\mathbf{R} = \cos t\mathbf{i}+\sin t\mathbf{j}$, $0\leq t\leq 2\pi$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C y^2dx+x^2dy+z^2dz = \int_0^{2\pi}[(\sin^2 t)(-\sin t)+(\cos^2 t)(\cos t)+0]dt = 0 \ \{\text{odd}\}$$

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_S (2x-2y)\mathbf{k}\cdot\mathbf{N}\,d\sigma = \iint_S 2x\,\mathbf{k}\cdot\mathbf{N}\,d\sigma - \iint_S 2y\,\mathbf{k}\cdot\mathbf{N}\,d\sigma = 0 \ \{\text{symmetry}\}$$

18. Same as 17, with direction of circle and **n** reversed.

19. $\mathbf{F}(x,y,z) = y^2\mathbf{i}+x\mathbf{j}+z^2\mathbf{k}$. $\mathbf{R}(t) = \cos t\mathbf{i}+\sin t\mathbf{j}$, $0\leq t\leq 2\pi$.

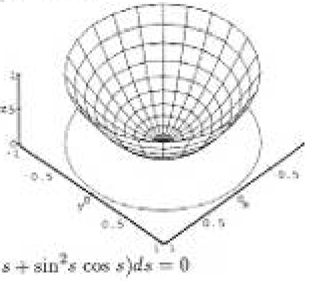
$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C y^2dx+x\,dy+z^2dz = \int_0^{2\pi}[\sin^2 t(-\sin t)+\cos t(\cos t)]dt = \int_0^{2\pi}(-\sin^3 t+\cos^2 t)dt$$
$$= \int_0^{2\pi}(-\sin^3 t+\tfrac{1}{2}+\tfrac{1}{2}\cos 2t)dt = 0+\pi+0 = \pi.$$

Also, from Equation 14.5.9

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_S (1-2y)\mathbf{k}\cdot\mathbf{N}\,d\sigma = \iint_D (-2y)dA = \int_0^1\int_0^{2\pi}(1-2r\sin\theta)r\,d\theta\,dr = \int_0^1 (2\pi)r\,dr = \pi$$

20. $\mathbf{F}(x,y,z) = xy\mathbf{i}+y^2\mathbf{j}+2\mathbf{k}$; S is the portion of the paraboloid $z = x^2+y^2$ below the plane $z=1$.

▹ The figure shows the surface S and its projection D onto the xy plane. D is bounded by the circle $x^2+y^2=1$. The boundary of S is the circle C with center at the point $(0,0,1)$ and radius 1. A vector equation of C with arc length s as parameter is given by



$\mathbf{R}(s) = \cos s\mathbf{i}+\sin s\mathbf{j}+\mathbf{k} \quad 0\leq s\leq 2\pi$

The unit tangent vector to the curve C is

$\mathbf{T}(s) = D_s\mathbf{R}(s) = -\sin s\mathbf{i}+\cos s\mathbf{j}$

Because

$$\mathbf{F}(\mathbf{R}(s)) = \mathbf{F}(\cos s,\sin s,1) = \cos s\sin s\mathbf{i}+\sin^2 s\mathbf{j}+2\mathbf{k}$$

then the left side of Stokes's theorem for the given **F** and S is

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi}(\cos s\sin s\mathbf{i}+\sin^2 s\mathbf{j}+2\mathbf{k})\cdot(-\sin s\mathbf{i}+\cos s\mathbf{j})ds = \int_0^{2\pi}(-\sin^2 s\cos s+\sin^2 s\cos s)ds = 0$$

We evaluate the right side of Stokes's theorem for the given **F** and S.

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 & 2 \end{vmatrix} = -x\mathbf{k}$$

Thus,

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_S -x\mathbf{k}\cdot\mathbf{n} = 0$$

because $-x$ is an odd function and S is symmetric with respect to the yz plane.

Because both sides of Stokes's theorem are zero, we have verified Stokes's theorem for the given **F** and S.

21. $\mathbf{F}(x,y,z) = -3y\mathbf{i} + 3x\mathbf{j} + 2\mathbf{k}$. $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + \mathbf{k}$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C -3y\,dx + 3x\,dy + 2\,dz = \int_0^{2\pi}[-3(3\sin t)(-3\sin t) + 3(3\cos t)(3\cos t) + 2\cdot 0]dt$$

$$= \int_0^{2\pi} 27\,dt = 54\pi. \text{ Also } \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_S 6\mathbf{k}\cdot\mathbf{k}\,d\sigma = 6\iint_S d\sigma = 6(9\pi) = 54\pi$$

22. $\mathbf{F}(x,y,z) = 2z\mathbf{i} + 3x\mathbf{j} + 4z\mathbf{k}$. $\mathbf{R}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C 2z\,dx + 3x\,dy + 4z\,dz = \int_0^{2\pi}[0 + 3(2\cos t)(2\cos t) + 0]dt = 6\int_0^{2\pi}(1+\cos 2t)dt = 12\pi$$

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S (2\mathbf{j} + 3\mathbf{k})\cdot\mathbf{k}\,d\sigma = 3\iint_S d\sigma = 3\cdot\pi(2^2) = 12\pi$$

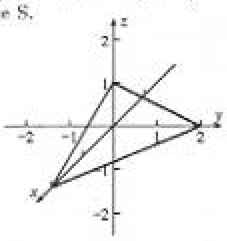
In Exercises 23–28, use Stokes's theorem to evaluate the line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ for **F** and C.

23. $\text{curl}(4y\mathbf{i} - 3z\mathbf{j} + x\mathbf{k}) = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. $\mathbf{N} = \frac{1}{3}\sqrt{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. The equilateral triangle has area $\frac{1}{2}\sqrt{3}$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\,d\sigma = \tfrac{1}{3}\sqrt{3}\iint_S (3-1-4)d\sigma = \tfrac{1}{3}\sqrt{3}(-2)\left(\tfrac{1}{2}\sqrt{3}\right) = -1$$

24. $\mathbf{F}(x,y,z) = (y-x)\mathbf{i} + (x-z)\mathbf{j} + (x-y)\mathbf{k}$; C is the triangle having vertices at $(2,0,0)$, $(0,2,0)$, and $(0,0,1)$.

▸ The figure shows the triangle. We calculate the curl of **F** before choosing the surface S.



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-x & x-z & x-y \end{vmatrix} = -\mathbf{j}$$

To simplify the calculation, we choose S to consist of the three triangles S_x with normal **i**, S_y with normal **j**, and S_z with normal **k** cut off the coordinate planes by the given triangle. Applying Stokes's theorem, we have

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_x} -\mathbf{j}\cdot\mathbf{i}\,d\sigma + \iint_{S_y} -\mathbf{j}\cdot\mathbf{j}\,d\sigma + \iint_{S_z} -\mathbf{j}\cdot\mathbf{k}\,d\sigma$$

$$= 0 - (\text{area of } S_x) + 0 = -1$$

25. $$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl}(-y\mathbf{i} + x\mathbf{j} + z\mathbf{k})\cdot\mathbf{N}\,d\sigma = \iint_S 2\mathbf{k}\cdot\mathbf{k}\,d\sigma = 2\iint_S d\sigma = 2(4\pi) = 8\pi$$

26. $$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl}(yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k})\cdot\mathbf{n}\,d\sigma = \iint_S [(y-z)\mathbf{j} + (y-z)\mathbf{k}]\cdot\mathbf{k}\,d\sigma = \int_0^2\int_0^2 y\,dy\,dx = 4$$

27. P = (1,0,0), Q = (0,1,0), R = (0,0,2). C is triangle PQR.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl}[2y + \sin^{-1}x)\mathbf{i} + e^y\mathbf{j} + (x + \ln(z^2+4))\mathbf{k}]\cdot\mathbf{N}\,d\sigma = \iint_S (-\mathbf{j} - 2\mathbf{k})\cdot\mathbf{N}\,d\sigma$$

$$= \iint_{OPR} -1\,dA + \iint_{OPQ} -2\,dA = (-1)1 + (-2)\tfrac{1}{2} = -2$$

28. $\mathbf{F}(x,y,z) = (2z - e^x)\mathbf{i} + (x^3 + \sin y)\mathbf{j} + (y^2 - \tan z)\mathbf{k}$; C has vector equation $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}, 0 \le t \le 2\pi$.

▸
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z - e^x & x^3 + \sin y & y^2 - \tan z \end{vmatrix} = 2y\mathbf{i} + 2\mathbf{j} + 3x^2\mathbf{k}$$

Because C is in a plane parallel to the xy plane, $\mathbf{N} = \mathbf{k}$. Applying Stokes's theorem, we have

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_S 3x^2\,dA$$

We switch to polar coordinates with $x = r\cos\theta$, $dA = r\,dr\,d\theta$. Thus

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi}\int_0^1 3(r\cos\theta)^2 r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 3r^3\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\tfrac{3}{4}\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}\tfrac{3}{8}(1+\cos 2\theta)\,d\theta = \tfrac{3}{8}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi} = \tfrac{3}{4}\pi$$

Miscellaneous Exercises for Chapter 14

In Exercises 1–4, determine if the vector is a gradient and if it is, find a function having the gradient.

1. We apply Theorem 14.1.1 to the vector $2xe^{x^2}\ln y\mathbf{i} + \frac{e^{x^2}}{y}\mathbf{j} = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$. Then
$\mathrm{M}_y(x,y) = \frac{2x}{y}e^{x^2}$ and $\mathrm{N}_x(x,y) = \frac{2x}{y}e^{x^2}$. $\mathrm{M}_y = \mathrm{N}_x$, so the given vector is a gradient $\nabla f(x,y)$.
$f_x(x,y) = 2xe^{x^2}\ln y$ and $f_y = \frac{e^{x^2}}{y}$. $f(x,y) = e^{x^2}\ln y + g(y)$; $f_y(x,y) = \frac{e^{x^2}}{y} + g'(y) = \frac{e^{x^2}}{y}$.
$g'(y) = 0$; $g(y) = \mathrm{C}$. Therefore $f(x,y) = e^{x^2}\ln y + \mathrm{C}$.

2. $\mathrm{M} = e^x\tan y - \sec y$, $\mathrm{N} = e^x\sec^2 y - x\sec y\tan y$. $\mathrm{M}_y = e^x\sec^2 y - \sec y\tan y = \mathrm{N}_x$
$f_x(x,y) = e^x\tan y - \sec y$, $f(x,y) = e^x\tan y - x\sec y + g(y)$; $f_y(x,y) = e^x\sec^2 y - x\sec y\tan y + g'(y) = \mathrm{N}$
$g'(y) = 0$, $g(y) = \mathrm{C}$. $f(x,y) = e^x\tan y - x\sec y + \mathrm{C}$

3. We apply Theorem 14.1.2 to the vector $\left[\frac{-y}{(x+z)^2} + \frac{1}{x^2}\right]\mathbf{i} + \frac{1}{x+z}\mathbf{j} + \left[\frac{-y}{(x+z)^2} + \frac{2}{z^2}\right]\mathbf{k}$.
Let $\mathrm{M}(x,y,z) = \frac{-y}{(x+z)^2} + \frac{1}{x^2}$, $\mathrm{N}(x,y,z) = \frac{1}{x+z}$, and $\mathrm{R}(x,y,z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$.
$\mathrm{M}_y(x,y,z) = \frac{-1}{(x+z)^2}$ $\quad$ $\mathrm{N}_x(x,y,z) = \frac{-1}{(x+z)^2}$ $\quad$ $\mathrm{R}_x(x,y,z) = \frac{2y}{(x+z)^3}$
$\mathrm{M}_z(x,y,z) = \frac{2y}{(x+z)^3}$ $\quad$ $\mathrm{N}_z(x,y,z) = \frac{-1}{(x+z)^2}$ $\quad$ $\mathrm{R}_y(x,y,z) = \frac{-1}{(x+z)^2}$
Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, the given vector is a gradient $\nabla f(x,y,z)$ where
$f_x(x,y,z) = \frac{-y}{(x+z)^2} + \frac{1}{x^2}$ $\quad$ $f_y(x,y,z) = \frac{1}{x+z}$ $\quad$ $f_z(x,y,z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$
From f_x, $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} + g(y,z)$; $f_y(x,y,z) = \frac{1}{x+z} + g_y(y,z) = \frac{1}{x+z}$. $g_y(y,z) = 0$;
$g(y,z) = \mathrm{h}(z)$. Hence $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} + \mathrm{h}(z)$. $f_z(x,y,z) = \frac{-y}{(x+z)^2} + h'(z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$;
$h'(z) = \frac{2}{z^2}$; $\mathrm{h}(z) = -\frac{2}{z} + \mathrm{C}$. Therefore $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} - \frac{2}{z} + \mathrm{C}$.

4. $y(\cos x - z\sin x)\mathbf{i} + z(\cos x + \sin y)\mathbf{j} - (\cos y - y\cos x)\mathbf{k}$
▷ Let
$\mathrm{M}(x,y,z) = y\cos x - yz\sin x$ $\quad$ $\mathrm{N}(x,y,z) = z\cos x + z\sin y$ $\quad$ $\mathrm{R}(x,y,z) = -\cos y + y\cos x$
We have
$\mathrm{M}_y(x,y,z) = \cos x - z\sin x$ and $\mathrm{N}_x(x,y,z) = -z\sin x$
Because $\mathrm{M}_y(x,y,z) \neq \mathrm{N}_x(x,y,z)$, then the vector is not a gradient.

In Exercises 5 and 6, find a conservative vector field having the potential function f.

5. (a) $\mathbf{F}(x,y) = \nabla f = \nabla(2x^2y + 3xy^3) = (4xy + 3y^3)\mathbf{i} + (2x^2 + 9xy^2)\mathbf{j}$
(b) $\mathbf{F}(x,y,z) = \nabla f = \nabla(xe^y - yze^y) = e^y\mathbf{i} + (xe^y - yze^y - ze^y)\mathbf{j} - ye^y\mathbf{k}$

6. (a) $\mathbf{F}(x,y) = \nabla f = \nabla(e^x\cos y + x\sin y) = (e^x\cos y + \sin y)\mathbf{i} + (-e^x\sin y + x\cos y)\mathbf{j}$
(b) $\mathbf{F}(x,y,z) = \nabla f = \nabla(x^2 + y^2 + z^2)^{-1} = -2(x^2 + y^2 + z^2)^{-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

In Exercises 7–10, prove that the vector field $\mathbf{F}$ is conservative, and find a potential function.

7. $\mathbf{F}(x,y) = \frac{2y^2}{1+4x^2y^4}\mathbf{i} + \frac{4xy}{1+4x^2y^4}\mathbf{j}$ is conservative because $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = \tan^{-1}2xy^2 + \mathrm{C}$.

8. $\mathbf{F}(x,y,z) = (6x - 4y)\mathbf{i} + (z - 4x)\mathbf{j} + (y - 8z)\mathbf{k}$
▷ Let
$\mathrm{M}(x,y,z) = 6x - 4y$ $\quad$ $\mathrm{N}(x,y,z) = z - 4x$ $\quad$ $\mathrm{R}(x,y,z) = y - 8z$
Then
$\mathrm{M}_y(x,y,z) = -4$ $\quad$ $\mathrm{N}_x(x,y,z) = -4$ $\quad$ $\mathrm{R}_x(x,y,z) = 0$
$\mathrm{M}_z(x,y,z) = 0$ $\quad$ $\mathrm{N}_z(x,y,z) = 1$ $\quad$ $\mathrm{R}_y(x,y,z) = 1$
Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, then $\mathbf{F}$ is the gradient of some scalar field, and thus $\mathbf{F}$ is conservative. If ϕ is a potential function, then
$\phi_x(x,y,z) = \mathrm{M}(x,y,z) = 6x - 4y$
Integrating with respect to x, we get
$\phi(x,y,z) = 3x^2 - 4xy + g(y,z)$

Partial-differentiating with respect to y, we have
$\phi_y(x,y,z) = -4x + g_y(y,z)$
Because $\phi_y(x,y,z) = \mathrm{N}(x,y,z)$, then
$-4x + g_y(y,z) = z - 4x$
$g_y(y,z) = z$
$g(y,z) = yz + h(z)$
and
$\phi(x,y,z) = 3x^2 - 4xy + yz + h(z)$
Partial-differentiating with respect to z, we obtain
$\phi_z(x,y,z) = y + h'(z)$
Because $\phi_z(x,y,z) = \mathrm{R}(x,y,z)$, then
$y + h'(z) = y - 8z$
$h'(z) = -8z$
$h(z) = -4z^2 + \mathrm{C}$
and a potential function is
$\phi(x,y,z) = 3x^2 - 4xy + yz - 4z^2 + \mathrm{C}$

9. $\mathbf{F}(x,y,z) = z^2\sec^2 x\mathbf{i} + 2ye^{3z}\mathbf{j} + (3y^2 + 2z\tan x)\mathbf{k}$ is conservative: $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = z^2\tan x + y^2e^{3z} + \mathrm{C}$.

10. $\mathbf{F}(x,y) = (y\sin x - \sin y)\mathbf{i} - (x\cos y + \cos x)\mathbf{j} = \nabla(-y\cos x - x\sin y)$

In Exercises 11–14, find curl $\mathbf{F}$ and div $\mathbf{F}$.

11. $\mathbf{F}(x,y,z) = e^{yz}\mathbf{i} + e^{xz}\mathbf{j} + e^{xy}\mathbf{k}$. div $\mathbf{F} = \dfrac{\partial}{\partial x}(e^{yz}) + \dfrac{\partial}{\partial y}(e^{xz}) + \dfrac{\partial}{\partial z}(e^{xy}) = 0$.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{yz} & e^{xz} & e^{xy} \end{vmatrix} = (xe^{xy} - xe^{xz})\mathbf{i} + (ye^{yz} - ye^{xy})\mathbf{j} + (ze^{xz} - ze^{yz})\mathbf{k}$$

12. $\mathbf{F}(x,y) = \sin y\mathbf{i} + \sin x\mathbf{k}$

▹ $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & 0 & \sin x \end{vmatrix} = -\cos x\mathbf{j} - \cos y\mathbf{k}$$

and

$$\text{div } \mathbf{F} = \mathrm{D}_x(\sin y) + \mathrm{D}_y(0) + \mathrm{D}_z(\sin x) = 0$$

13. $\mathbf{F}(x,y) = \frac{1}{y}\mathbf{i} - \frac{2x}{y}\mathbf{j}$. curl $\mathbf{F} = \left[\dfrac{\partial}{\partial x}\left(-\dfrac{2x}{y}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{1}{y}\right)\right]\mathbf{k} = \left(-\dfrac{2}{y} + \dfrac{1}{y^2}\right)\mathbf{k}$. div $\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{y}\right) + \dfrac{\partial}{\partial y}\left(-\dfrac{2x}{y}\right) = \dfrac{2x}{y^2}$

14. $\mathbf{F}(x,y,z) = \frac{x}{y}\mathbf{i} + \frac{y}{z}\mathbf{j} + \frac{z}{x}\mathbf{k}$. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/y & y/z & z/x \end{vmatrix} = \dfrac{y}{z^2}\mathbf{i} + \dfrac{z}{x^2}\mathbf{j} + \dfrac{x}{y^2}\mathbf{k}$. div $\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{y}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{z}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x}\right) = \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{1}{x}$

In Exercises 15–22, evaluate the line integral over curve C.

15. $\mathbf{F}(x,y) = 3y\mathbf{i} - 4x\mathbf{j}$; $\mathbf{R}(t) = 2t^2\mathbf{i} - t\mathbf{j}$. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 3(-t), -4(2t^2)\rangle \cdot \langle 4t, -1\rangle dt = \int_0^1 -4t^2 dt = -\frac{4}{3}$.

16. $\int_C \mathbf{F}\cdot d\mathbf{R}$: $\mathbf{F}(x,y) = (x+y)\mathbf{i} + (y-x)\mathbf{j}$; C: $\mathbf{R}(t) = t^3\mathbf{i} + t^2\mathbf{j}$ from the point $(8,4)$ to the point $(1,1)$.

▹ Because $\mathbf{R}(2) = 8\mathbf{i} + 4\mathbf{j}$, then $t = 2$ at the point $(8,4)$. Because $\mathbf{R}(1) = \mathbf{i} + \mathbf{j}$, then $t = 1$ at the point $(1,1)$. Furthermore,
$\mathbf{F}(\mathbf{R}(t)) = \mathbf{F}(t^3, t^2) = (t^3 + t^2)\mathbf{i} + (t^2 - t^3)\mathbf{j}$
and
$\mathbf{R}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$
Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_2^1 [(t^3 + t^2)\mathbf{i} + (t^2 - t^3)\mathbf{j}]\cdot(3t^2\mathbf{i} + 2t\mathbf{j})dt = \int_2^1 (3t^5 + 3t^4 + 2t^3 - 2t^4)dt = \tfrac{1}{2}t^6 + \tfrac{1}{5}t^5 + \tfrac{1}{2}t^4\Big]_2^1 = -\tfrac{226}{5}$$

17. $\mathbf{R}(t) = 4\sin t\mathbf{i} - \cos t\mathbf{j}$, $0 \le t \le \frac{1}{2}\pi$. $\displaystyle\int_C (2x + 3y)dx + xy\,dy$

$$= \int_0^{\pi/2} [2(4\sin t) + 3(-\cos t)]4\cos t\,dt + (4\sin t)(-\cos t)(\sin t\,dt)$$

$$= \int_0^{\pi/2} (32\sin t\cos t - 12\cos^2 t - 4\sin^2 t\cos t)dt = 16\sin^2 t - 12\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - \tfrac{4}{3}\sin^3 t\Big]_0^{\pi/2}$$

$$= 16 - 3\pi - \tfrac{4}{3} = \tfrac{44}{3} - 3\pi$$

18. $x = 3\sin t,\ y = 3\cos t.\ \int_C (2x+y)dx + (x-2y)dy = \int_0^{2\pi} [(6\sin t + 3\cos t)3\cos t + (3\sin t - 6\cos t)(-3\sin t)]dt$

$= \int_0^{2\pi} (36 \sin t \cos t + 9\cos^2 t - 9\sin^2 t)dt = 0$

19. $\mathbf{R}(t) = (t-1)\mathbf{i} + (t+1)\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 1.\ \int_C y^2 dx + z^2 dy + x^2 dz$

$= \int_0^1 [(t+1)^2 + (t^2)^2 + (t-1)^2 \cdot 2t]dt = \int_0^1 (t^4 + 2t^3 - 3t^2 + 4t + 1)dt = \frac{1}{5}t^5 + \frac{1}{2}t^4 - t^3 + 2t^2 + t\Big]_0^1 = \frac{27}{10}$

20. $\int_C xe^y dx - xe^z dy + e^z dz$; C is $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, for $0 \le t \le 1$.

▹ Parametric equations of C are

$x = t \quad y = t^2 \quad z = t^3$ for $0 \le t \le 1$

Therefore,

$$\int_C xe^y dx - xe^z dy + e^z dz = \int_0^1 [te^{t^2} dt - te^{t^3}(2t\,dt) + e^{t^3}(3t^2 dt)] = \tfrac{1}{2}e^{t^2} - \tfrac{2}{3}e^{t^3} + e^{t^3}\Big]_0^1$$

$$= \tfrac{1}{2}(e-1) - \tfrac{2}{3}(e-1) + (e-1) = \tfrac{5}{6}(e-1)$$

21. $\mathbf{F}(x,y,z) = 3xy\mathbf{i} + (4y^2 - xz)\mathbf{j} + 6z\mathbf{k}$; $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \langle 3t(t^2), 4t^4 - t^4, 6t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle dt = \int_0^1 (3t^3 + 24t^5)dt = \frac{3}{4}t^4 + 4t^6\Big]_0^1 = \frac{19}{4}$

22. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + z\mathbf{k}.\ \int_C \mathbf{F} \cdot d\mathbf{R} = \int_{(2,0,0)}^{(2,0,2\pi)} 2x\,dx + 3y\,dy + z\,dz = x^2 + \frac{3}{2}y^2 + \frac{1}{2}z^2\Big]_{(2,0,0)}^{(2,0,2\pi)} = 2\pi^2$

In Exercises 23–30, prove that the value of the line integral is independent of the path, and compute the value in any convenient manner. In each exercise C is any sectionally smooth curve from point A to point B.

23. $\mathbf{F}(x,y) = 2xe^y\mathbf{i} + x^2e^y\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^2e^y$. $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(3,2) - \phi(1,0) = 9e^2 - 1$.

24. $\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy$; A is (0, 1) and B is (6, 3).

▹ Let

$$M(x,y) = \frac{1}{y} - y \qquad N(x,y) = -\frac{x}{y^2} - x$$

$$M_y(x,y) = -\frac{1}{y^2} - 1 \qquad N_x(x,y) = -\frac{1}{y^2} - 1$$

Because $M_y(x,y) = N_x(x,y)$, then the line integral is independent of the path. We take a path from A to B consisting of two parts. Let C_1 be the line segment from A(0, 1) to D(0, 3), and let C_2 be the line segment from D(0, 3) to B(6,3). Because $x = 0$ at every point on C_1,

$$\int_{C_1} \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = 0$$

At every point on C_2 we have $y = 3$ and $dy = 0$ and $0 \le x \le 6$. Thus,

$$\int_{C_1} \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2 - x}\right)dy = \int_0^6 (\tfrac{1}{3} - 3)dx = -16$$

By adding the two parts of the line integral, we have

$$\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = 0 + (-16) = -16$$

Alternatively, by inspection

$$\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = \int_{(0,1)}^{(6,3)} d\left(\frac{x}{y} - xy\right) = \frac{x}{y} - xy\Big]_{(0,1)}^{(6,3)} = [\tfrac{6}{3} - 6(3)] - 0 = -16$$

25. $\mathbf{F}(x,y) = (\cos y - y\cos x)\mathbf{i} - (\sin x + x\sin y)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x\cos y - y\sin x$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(\pi,0) - \phi(0,\tfrac{1}{2}\pi) = \pi$

26. $\int_C [(2xy - 2y)\mathbf{i} + (x^2 - 2x + 3y^2)\mathbf{j}] \cdot d\mathbf{R} = x^2y - 2xy + y^3\Big]_{(2,-1)}^{(3,2)} = (18 - 12 + 8) - (-4 + 4 - 1) = 15$

27. $\mathbf{F}(x,y,z) = 3y\mathbf{i} + (3x + 4y)dy\mathbf{j} - 2z\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = 3xy + 2y^2 - z^2$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1,2,0) - \phi(0,1,-1) = 14 - 1 = 13$

28. $\int_C z \sin y\, dx + xz \cos y\, dy + x \sin y\, dz$; A is $(0,0,0)$ and B is $(3,3,\frac{1}{2}\pi)$.

▹ Because integrating $z \sin y$ with respect to x, integrating $xz \cos y$ with respect to y, and integrating $x \sin y$ with respect to z all give $xz \sin y$, we have

$$\int_C z \sin y\, dx + xz \cos y\, dy + x \sin y\, dz = \int_{(0,0,0)}^{(3,3,\pi/2)} d(xz \sin y) = xz \sin y \Big]_{(0,0,0)}^{(2,3,\pi/2)}$$
$$= 2 \cdot \tfrac{1}{2}\pi \sin 3 - 0 = \pi \sin 3$$

29. $\mathbf{F}(x,y,z) = \left(\frac{1}{y} - \frac{2z}{x^2}\right)\mathbf{i} - \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} + \left(\frac{2}{x} + \frac{y}{z^2}\right)\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = \frac{x}{y} + \frac{2z}{x} - \frac{y}{z}$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(4,2,-2) - \phi(2,-1,1) = (2-1+1) - (-2+1-1) = 4$$

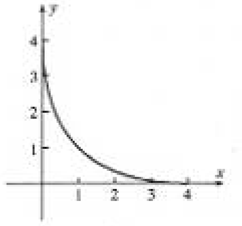
30. $\int_C [(2xy + 3yz)\mathbf{i} + (x^2 - 4yz + 3xz)\mathbf{j} + (3xy - 2y^2)\mathbf{k}] \cdot d\mathbf{R} = x^2y - 2y^2z + 3xyz \Big]_{(0,2,1)}^{(1,-1,4)} = -13$

In Exercises 31–34, use Green's theorem to evaluate the line integral.

31. The ellipse $x^2/3^2 + y^2/4^2 = 1$ has area $\pi(3)(4) = 12\pi$.

$$\oint_C (3x + 2y)dx + (3x + y^2)dy = \iint_R \left[\frac{\partial}{\partial x}(3x + y^2) - \frac{\partial}{\partial y}(3x + 2y)\right] dA = \iint_R dA = 12\pi$$

32. $\oint_C \ln(y+1)dx - \frac{xy}{y+1}dy$, where C is the closed curve determined by the curve $\sqrt{x} + \sqrt{y} = 2$ and the intervals $[0,4]$ on the x and y axes.



▹ The figure shows the region R, bounded below by the x axis and bounded above by the parabola (A.10.35) $y = (2 - \sqrt{x})^2$, $0 \le x \le 4$. Because

$$\mathrm{M}(x,y) = \ln(y+1) \quad \text{and} \quad \mathrm{N}(x,y) = -\frac{xy}{y+1}$$

then

$$\mathrm{N}_x(x,y) - \mathrm{M}_y(x,y) = \frac{-y}{y+1} - \frac{1}{y+1} = -1$$

Applying Green's theorem (14.4.1), we have

$$\oint_C \ln(y+1)dx - \frac{xy}{y+1}dy = \iint_R [\mathrm{N}_x(x,y) - \mathrm{M}_y(x,y)]dA = \iint_R -dA$$
$$= -\int_0^4 \int_0^{(2-\sqrt{x})^2} dy\, dx = -\int_0^4 (4 - 4\sqrt{x} + x)dx = -\left[4x - \tfrac{8}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^4 = -\tfrac{8}{3}$$

33. $\oint_C e^x \sin y\, dx + e^x \cos y\, dy = \iint_R \left[\frac{\partial}{\partial x}(e^x \cos y) - \frac{\partial}{\partial y}(e^x \sin y)\right] dA = \iint_R (e^x \cos y - e^x \cos y) dA = 0$

34. $\oint_C (x^2 - y^3)dx + (y^2 + x^3)dy = \iint_R [\mathrm{D}_x(y^2 + x^3) - \mathrm{D}_y(x^2 - y^3)]dA = \iint_R (3x^2 + 3y^2)dA$

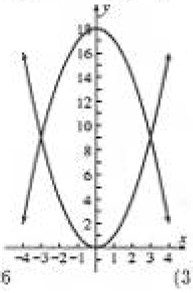
$$= 3\int_0^{2\pi} \int_0^1 r^2 r\, dr\, d\theta = 6\pi \cdot \tfrac{1}{4}r^4 \Big]_0^1 = \tfrac{3}{2}\pi$$

In Exercises 35 and 36, use Theorem 14.4.2 to find the area of the region.

35. $x^2 = x + 2$ gives $x = -1, 2$. C_1: $y = x^2$, $-\mathrm{C}_2$: $y = x + 2$, $-1 \le x \le 2$. By Theorem 14.4.2,

$$\mathrm{A} = \frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_{-1}^{2} x(2x\, dx) - x^2 dx - \frac{1}{2}\int_{-1}^{2} x\, dx - (x+2)dx = \frac{1}{2}\int (x^2 + 2)dx = \frac{1}{2}\left[\frac{1}{2}x^3 + 2x\right]_{-1}^{2} = \frac{9}{2}$$

36. The region enclosed by the two parabolas $y = x^2$ and $x^2 = 18 - y$.



▹ The figure shows the region. The boundary C is the union of C_1 and C_2, where C_1 is the parabola $y = x^2$ from $(-3,9)$ to $(3,9)$, and C_2 is parabola $y = 18 - x^2$ from $(3,9)$ to $(-3,9)$. By Theorem 14.4.2 the area of the region enclosed by C is

$$\mathrm{A} = \tfrac{1}{2}\oint_C x\, dy - y\, dx = \tfrac{1}{2}\left[\oint_{C_1} x\, dy - y\, dx + \oint_{C_2} x\, dy - y\, dx\right] \qquad (1)$$

Because $y = x^2$ on C_1 and $-3 \le x \le 3$, then $dy = 2x$, and

$$\oint_{C_1} x\, dy - y\, dx = \int_{-3}^{3} [x(2x\, dx) - x^2 dx] = \tfrac{1}{3}x^3 \Big]_{-3}^{3} = 18 \qquad (2)$$

Because $y = 18 - x^2$ for C_2 and C_2 is from $(3,9)$ to $(-3,9)$, then

$$\oint_{C_2} x\, dy - y\, dx = \int_{3}^{-3} [x(-2x)dx - (18 - x^2)dx] = \int_{-3}^{3} (x^2 + 18)dx = \tfrac{1}{3}x^3 + 18x \Big]_{-3}^{3} = 126 \qquad (3)$$

Substituting from (2) and (3) into (1), we get

$$\mathrm{A} = \tfrac{1}{2}(18 + 126) = 72$$

The area of the region is 72 square units.

In Exercises 37–40, find the total work done in moving an object along C if the motion is caused by the force field. Length is measured in meters and force is measured in newtons.

37. $\mathbf{F}(x,y) = 2x^2y\mathbf{i} + (x^2+3y)\mathbf{j}$; $y = 3x^2+2x+4$, $0 \le x \le 1$. If W joules work is done

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 2x^2(3x^3+2x+4), x^2+3(3x^2+2x+4)\rangle \cdot \langle 1, 6x+2\rangle dx$$

$$= \int_0^1 (6x^4+64x^3+64x^2+84x+24)dx = \frac{6}{5}t^5 + 16t^4 + \frac{64}{3}t^3 + 42t^2 + 24t\Big]_0^1 = \frac{1568}{15}$$

38. $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \frac{1}{2}\pi$. $W = \int_C xy^2dx - x^2y\,dy = \int_0^{\pi/2} 16[\cos t\sin^2 t(-\sin t) - \cos^2 t\sin t(\cos t)]dt$

$$= -16\int_0^{\pi/2}(\sin^3 t\cos t + \cos^3 t\sin t)dt = -4\Big[\sin^4 t - \cos^4 t\Big]_0^{\pi/2} = -8 \text{ joules}$$

39. $\mathbf{F}(x,y,z) = (xy-z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\mathbf{R}(t) = 4t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$. If W joules work is done

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 4t^2-2t, t, 2t\rangle\cdot\langle 4,1,2\rangle dt = \int_0^1 (16t^2-3t)dt = \frac{16}{3}t^3 - \frac{3}{2}t^2\Big]_0^1 = \frac{23}{6}$$

40. $\mathbf{F}(x,y,z) = xyz\mathbf{i} + e^y\mathbf{j} + (x+z)\mathbf{k}$; C: $\mathbf{R}(t) = 3t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$; $0 \le t \le 3$

▷ If W joules is the work, then

$$W = \int_0^3 \mathbf{F}(3t,t^2,2t)\cdot d\mathbf{R}(t) = \int_0^3 (6t^4\mathbf{i} + e^{t^2}\mathbf{j} + 5t\mathbf{k})\cdot(3\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k})dt = \int_0^3 (18t^4 + 2te^{t^2} + 10t)dt$$

$$= \tfrac{18}{5}t^5 + e^{t^2} + 5t^2\Big]_0^3 = [\tfrac{18}{5}(3^5) + e^9 + 5(3^2)] - e^0 = \tfrac{4594}{5} + e^9$$

The work is approximately 9022 joules.

In Exercises 41 and 42, verify Gauss's divergence theorem and Stokes's theorem in the plane for **F** and R.

41. $\mathbf{F}(x,y) = 4y\mathbf{i} + 3x\mathbf{j}$. C: $\mathbf{R}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \oint_C -3x\,dx + 4y\,dy = \int_0^{2\pi}[-3\cos^3 t\frac{d}{dt}(\cos^3 t) + \sin^3 t\frac{d}{dt}(\sin^3 t)]dt$

$$= -\tfrac{3}{2}(\cos^3 t)^2 + 2(\sin^3 t)^2\Big]_0^{2\pi} = 0 \text{ and } \iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R 0\,dA = 0$$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C 4y\,dx + 3x\,dy = \int_0^{2\pi}[4\sin^3 t(-3\cos^2 t\sin t) + 3\cos^3 t(3\sin^2 t\cos t)]dt$

$$= \int_0^{2\pi} 3\sin^2 t\cos^2 t(3\cos^2 t - 4\sin^2 t)dt = \frac{3}{8}\int_0^{2\pi}\sin^2 2t(7\cos 2t - 1)dt$$

$= \frac{21}{8}\int_0^{2\pi}\sin^2 2t\cos 2t\,dt - \frac{3}{16}\int_0^{2\pi}(1-\cos 4t)dt = 0 - \frac{3}{16}\cdot 2\pi = -\frac{3}{8}\pi$. Also, by Exercise 14.4.25 with $a = 1$, R has area $\frac{3}{8}\pi$. $\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \iint_R(-\mathbf{k})\cdot\mathbf{k}\,dA = -\iint_R dA = -\frac{3}{8}\pi$.

42. $\mathbf{F}(x,y) = 3x^2\mathbf{i} + 4y^2\mathbf{j}$. $x = 4\cos t$, $y = 3\sin t$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C -4y^2dx + 3x^2dy = \int_0^{2\pi}[-36\sin^2 t(-4\sin t) + 48\cos^2 t(3\cos t)]dt = 0$ (odd power)

$$\iint_R \operatorname{div}(3x^2\mathbf{i} + 4y^2\mathbf{j})dA = \iint_R(6x+8y)dA = 0$$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C 3x^2dx + 4y^2dy = \oint_C d(x^3 + \frac{4}{3}y^3) = 0$. $\iint_R \operatorname{curl}(3x^2\mathbf{i} + 4y^2\mathbf{j})dA = \iint_R 0\,dA = 0$

In Exercises 43 and 44, use Green's theorem to find the total work done in moving an object in the counterclockwise direction once around C if the motion is caused by the force field $\mathbf{F}(x,y)$. Length is measured in meters and force is measured in newtons.

43. $W = \oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_R\left[\frac{\partial}{\partial x}(x^2+e^y) - \frac{\partial}{\partial y}(xy^2+\cos x)\right]dA = \iint_R(2x-2xy)dA = 0$ (odd function of x)

44. C is the ellipse $9x^2+y^2 = 9$; $\mathbf{F}(x,y) = (2x-3y)\mathbf{i} + (x+2y)\mathbf{j}$.

▷ The area of the the region R enclosed by the ellipse is $\pi ab = \pi(1)(3) = 3\pi$. Let

$M(x,y) = 2x - 3y \quad N(x,y) = x + 2y$

If W joules is the total work done, then by Green's theorem

$$W = \oint_C M(x,y)dx + N(x,y)dy = \iint_R[N_x(x,y) - M_y(x,y)]dA = \iint_R[1-(-3)]dA = 4\iint_R dA$$

$$= 4(3\pi) = 12\pi$$

The total work is 12 joules.

45. $\mathbf{F} = (4x - 3y)\mathbf{i} + (5y - 4x^2)\mathbf{j}$. The triangle $(0,1)$, $(0,4)$, $(4,4)$ has area $\frac{1}{2} \cdot 3 \cdot 4 = 6$. By the divergence theorem the number of cm^2/sec in the flow is $\oint_C \mathbf{F} \cdot \mathbf{N}\, d\sigma = \iint_R \operatorname{div} \mathbf{F}\, dA = \iint_R (4+5)dA = 9 \cdot 6 = 54$

46. C: $\frac{x^2}{16} + \frac{y^2}{4} = 1$. $\oint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \operatorname{div}[(y^2 + 12x)\mathbf{i} + (4y - x^2)\mathbf{j}]dA = \iint_R (12+4)dA = 16 \cdot \pi(4)(2) = 128\pi$

47. $\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j} = \nabla\phi$ with $\phi(x,y) = -\tan^{-1}\frac{y}{x}$ ($\phi(x,y) = \tan^{-1}\frac{x}{y}$ is not defined on the x axis).

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(-\sqrt{2}, \sqrt{2}) - \phi(\sqrt{2}, \sqrt{2}) = \frac{1}{4}\pi - (-\frac{1}{4}\pi) = \frac{1}{2}\pi$.

48. Apply Green's theorem to compute the area of the quadrilateral having vertices $(0,0), (3,2), (1,5)$, and $(-2,1)$.

▹ Let C be the boundary of the given quadrilateral region R. We choose functions M and N so that $N_x(x,y) - M_y(x,y) = 1$. Then the right side of Green's theorem will give the measure of the area of R, and we may calculate the area by evaluating the integral on the left side of Green's theorem. There are many possible choices for the functions M and N. We let $M(x,y) = 0$ and $N(x,y) = x$. The area of the region R is given by

$$A = \oint_C M(x,y)dx + N(x,y)dy = \oint_C x\, dy$$

The closed curve C consists of four pieces.

We evaluate $\int_C x\, dy$ where C is the segment from (x_1, y_1) to (x_2, y_2). If m is the slope of the segment, then $dy/dx = m$ and $dy = m\, dx$. Thus

$$\int_C x\, dy = \int_{x_1}^{x_2} \frac{y_2 - y_1}{x_2 - x_1} x\, dx = \frac{1}{2} \cdot \frac{y_2 - y_1}{x_2 - x_1} x^2 \Big]_{x_1}^{x_2} = \frac{1}{2} \cdot \frac{y_2 - y_1}{x_2 - x_1}({x_2}^2 - {x_1}^2) = \frac{1}{2}(y_2 - y_1)(x_2 + x_1) \qquad (1)$$

Applying (1) to the segment from $(0,0)$ to $(3,2)$, we get

$$\int_{C_1} x\, dy = \tfrac{1}{2}(2-0)(3+0) = 3$$

Applying (1) to the segment from $(3,2)$ to $(1,5)$, we get

$$\int_{C_2} x\, dy = \tfrac{1}{2}(5-2)(1+3) = 6$$

Applying (1) to the segment from $(1,5)$ to $(-2,1)$, we get

$$\int_{C_3} x\, dy = \tfrac{1}{2}(1-5)(-2+1) = 2$$

Applying (1) to the segment from $(-2,1)$ to $(0,0)$, we get

$$\int_{C_4} x\, dy = \tfrac{1}{2}(0-1)(0-2) = 1$$

Therefore,

$$A = \oint_C x\, dy = \oint_{C_1} x\, dy + \oint_{C_2} x\, dy + \oint_{C_3} x\, dy + \oint_{C_4} x\, dy = 3 + 6 + 2 + 1 = 12$$

The area of the quadrilateral is 12 square units.

49. $f(x,y) = z = 3x + 2y$. $f_x(x,y) = 3$, $f_y(x,y) = 2$. Triangle D has side $3x + 2y = 6$.

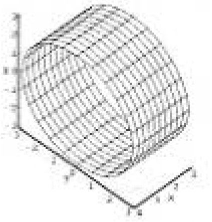
$\iint_S xy\, d\sigma = \iint_D xy\sqrt{2^2 + 3^2 + 1}\, dA = \sqrt{14}\int_0^2 \int_0^{(6-3x)/2} xy\, dy\, dx = \sqrt{14}\int_0^2 \frac{1}{2}xy^2\Big]_0^{(6-3x)/2} dx$

$= \frac{1}{8}\sqrt{14}\int_0^2 x(6-3x)^2 dx = \frac{1}{8}\sqrt{14}\Big[-\frac{1}{9}x(6-3x)^2 - \frac{1}{108}(6-3x)^4\Big]_0^2 = \frac{1}{8}\sqrt{14} \cdot \frac{6^4}{108} = \frac{3}{2}\sqrt{14}$

50. See Exercise 14.5.6

51. $f(x,y) = z = 9 - x^2$. $f_x(x,y) = -2x$, $f_y(x,y) = 0$. $\iint_S x\, d\sigma = \iint_D x\sqrt{4x^2+1}\, dA = \frac{1}{8}\int_0^2 \int_0^3 8x(4x^2+1)^{1/2} dx\, dy$

$= \frac{1}{8}\int_0^2 \frac{2}{3}(4x^2+1)^{3/2}\Big]_0^3 dy = \frac{1}{12}\int_0^2 [(37)^{3/2} - 1]dy = \frac{1}{6}[(37)^{3/2} - 1]$



52. Evaluate the surface integral $\iint_S xyz\, d\sigma$, where S is the portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 1$ and $x = 4$.

▹ The surface is shown in the figure. Because the surface is symmetric with respect to the xz plane $y = 0$, and the integrand xyz is an odd function of y, then the value of the integral is 0.

53. $\rho(x,y,z)=kz^2$. $f(x,y)=z=\sqrt{4-x^2-y^2}$, $f_x=\dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y=\dfrac{-y}{\sqrt{4-x^2-y^2}}$. The mass is M kg.

$$\mathrm{M}=\iint_S \rho(x,y,z)d\sigma=\iint_D \mathrm{k}(4-x^2-y^2)\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\ \mathrm{dA}=2\mathrm{k}\iint_D\sqrt{4-x^2-y^2}\ \mathrm{dA}$$
$$=2\mathrm{k}\int_0^{\pi/2}\int_0^2 r\cdot r\ dr\ d\theta=2\mathrm{k}\int_0^{\pi/2}\frac{8}{3}\ d\theta=2\mathrm{k}\cdot\frac{\pi}{2}\cdot\frac{8}{3}=\frac{8}{3}k\pi$$

54. $z=\sqrt{4-x^2-y^2}$. $\mathrm{M}=\iint_S(4-z)d\sigma=4\iint_S d\sigma-\iint_D\sqrt{4-x^2-y^2}\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\mathrm{dA}$
$$=4\iint_S d\sigma-2\iint_D \mathrm{dA}=4\cdot 2\pi(2^2)-2\cdot\pi(2)^2=24\pi$$

55. $f(x,y)=z=\sqrt{x^2+y^2}$. $f_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$. $d\sigma=\sqrt{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2+y^2}+1}\,\mathrm{dA}=\sqrt{2}\,\mathrm{dA}$.
D is an annulus of radii 1 and 4. $\rho(x,y,z)=(10-z)\mathrm{kg/m^2}$. M kg is the funnel's mass.
$$\mathrm{M}=\iint_S(10-z)d\sigma=\iint_D[10-\sqrt{x^2+y^2}]\sqrt{2}\ \mathrm{dA}=\int_0^{2\pi}\int_0^4(10-r)r\ dr\ d\theta=\sqrt{2}\theta\Big]_0^{2\pi}\Big[5r^2-\frac{1}{3}r^3\Big]_0^4=108\sqrt{2}\pi$$

56. S is that part of the sphere $x^2+y^2+z^2=9$ that is above the region D in the xy plane enclosed by the circle $x^2+y^2=1$. If the velocity field of a fluid is given by $\mathbf{F}(x,y,z)=-y\mathbf{i}+x\mathbf{j}+3\mathbf{k}$, find the flux of **F** across S.

▷ We have
$\mathrm{M}(x,y,z)=-y\quad \mathrm{N}(x,y,z)=x\quad \mathrm{R}(x,y,z)=3$
solving the equation of the sphere for z, we obtain
$$z=f(x,y)=\sqrt{9-x^2-y^2}$$
Thus
$$f_x(x,y)=\frac{-x}{\sqrt{9-x^2-y^2}}\quad f_y(x,y)=\frac{-y}{\sqrt{9-x^2-y^2}}$$
If **n** is a unit upper normal vector of S, the flux of **F** across S is given by
$$\iint_S\mathbf{F}\cdot\mathbf{n}\ d\sigma=\iint_D(-\mathrm{M}f_x-\mathrm{N}f_y+\mathrm{R})\mathrm{dA}=\iint_D\Big(\frac{-xy}{\sqrt{9-x^2-y^2}}+\frac{xy}{\sqrt{9-x^2-y^2}}+3\Big)\mathrm{dA}=2\iint_D\mathrm{dA}$$
Because the area of the region D is $\pi r^2=\pi$, then the flux of **F** across S is 3π cubic units per unit of time.

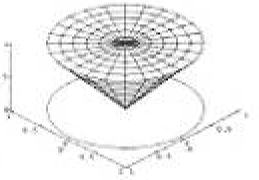
57. $\mathbf{F}(x,y,z)=2x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k}$. $f(x,y)=z=4-x^2-y^2$. $f_x(x,y)=-2x$. $f_y(x,y)=-2y$. D is $x^2+y^2\le 4$
$$\text{flux}=\iint_S\mathbf{F}\cdot\mathbf{N}\ d\sigma=\iint_D[(-2x)(-2x)+(-2y)(-2y)+3(4-x^2-y^2)]d\mathrm{A}=\iint_D(x^2+y^2+12)d\mathrm{A}$$
$$=\int_0^{2\pi}\int_0^2(r^2+12)r\ dr\ d\theta=\int_0^{2\pi}d\theta\int_0^2(r^3+12r)dr=\theta\Big]_0^{2\pi}\Big[\frac{1}{4}r^4+6r^2\Big]_0^2=2\pi\cdot 28=56\pi$$

58. $\mathbf{F}(x,y,z)=\frac{1}{2}z\mathbf{i}$, S: $x^2+y^2+z^2=4$.
$\iint_S\frac{1}{2}z\mathbf{i}\cdot\mathbf{n}\ d\sigma=0$ because $\mathbf{i}\cdot\mathbf{n}$ is even, z is odd with respect to xy plane. $\iiint_E\mathrm{div}(\frac{1}{2}z\mathbf{i})d\mathrm{V}=\iiint_E 0\ d\mathrm{V}=0.$

In Exercises 59 and 60, Use Gauss's divergence theorem to evaluate the surface integral $\iint_S\mathbf{F}\cdot\mathbf{N}\ d\sigma$ for **F** and S.

59. $\mathbf{F}(x,y,z)=2x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$. $\iint_S\mathbf{F}\cdot\mathbf{N}\ d\sigma=\iiint_E\mathrm{div}\ \mathbf{F}\ \mathrm{dV}=\iiint_E 5\ \mathrm{dV}=5(16\pi)(2)=160\pi.$

60. $\mathbf{F}(x,y,z)=x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$; S is the boundary of the region enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=1$.



▷ Refer to the figure. By Gauss's divergence theorem
$$\iint_S\mathbf{F}\cdot\mathbf{N}\ d\sigma=\iiint_E\mathrm{div}\ \mathbf{F}\ dV=\iiint_E(2x+2y+2z)\ dV$$
Because the region is symmetric with respect to the planes $x=0$ and $y=0$, the integrals of the first two terms are 0. To evaluate the third, we switch to cylindrical coordinates with $dV=r\ dr\ d\theta\ dz$. Thus
$$\iint_S\mathbf{F}\cdot\mathbf{N}\ d\sigma=\int_0^1\int_0^{2\pi}\int_0^z 2zr\ dr\ d\theta\ dz=\int_0^1\int_0^{2\pi}r^2z\Big]_0^z d\theta\ dz$$
$$=\int_0^1\int_0^{2\pi}z^3\ d\theta\ dz=\int_0^1 2\pi z^3\ dz=\frac{1}{2}\pi z^4\Big]_0^1=\frac{1}{2}\pi$$

In Exercises 61 and 62, verify Stokes's theorem for F and S.

61. C: $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$. $\mathbf{F} = z\mathbf{i} + 4x\mathbf{j} + 2z\mathbf{k} = 12\cos t\mathbf{j}$. By equation 14.4.11

$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi} [0 + (12\cos t)(3\cos t)]dt = 36\int_0^{2\pi} \cos^2 t\,dt = 36\cdot\frac{1}{2}\cdot 2\pi = 36\pi$. Also

curl $\mathbf{F} = 1\mathbf{j} + 4\mathbf{k}$, $f(x,y) = z = 9 - x^2 - y^2$. From equation 19.5.9,

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_D (0 - 1\cdot 2y + 4)dA = \iint_D 4\,dA = 4(9\pi) = 36\pi$$

62. C: $\mathbf{R}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$, $0 \le t \le 2\pi$. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C xy\,dx = 0$ because y is odd.

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S (-y\mathbf{i} - z\mathbf{j} - x\mathbf{k})\cdot\tfrac{1}{4}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})d\sigma = -\tfrac{1}{4}\iint_S (xy + yz + xz)d\sigma = 0 \text{ (odd in } x \text{ and } y)$$

In Exercises 63 and 64, use Stokes's theorem to evaluate the line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ for F and C.

63. S is $x^2 + y^2 \le 1$, $z = 1$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl}[(z + \ln(x^2+1))\mathbf{i} + (\cos y - x^2)\mathbf{j} + (3y^2 - e^z)\mathbf{k}]\cdot\mathbf{N}\,d\sigma$

$= \iint_S (6y\mathbf{i} - \mathbf{j} - 2x\mathbf{k})\cdot\mathbf{k}\,dA = \iint_S -2x\,dA = 0$ because x is odd.

64. $\mathbf{F}(x,y,z) = -2y\mathbf{i} + 3x\mathbf{j} + z\mathbf{k}$; C is the circle $x^2 + y^2 = 1$ in the xy plane.

▹
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y & 3x & z \end{vmatrix} = 5\mathbf{k}$$

Applying Stokes's theorem, we have

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 5\mathbf{k}\cdot\mathbf{n}\,d\sigma$$

Because S is any sectionally smooth surface whose boundary is C, we may take S to be that portion of the xy plane enclosed by the given circle C. Because **n** is a unit upper normal vector of S, then $\mathbf{n} = \mathbf{k}$ and $\mathbf{k}\cdot\mathbf{n} = 1$. The surface area of S is $\pi(1)^2 = \pi$. We conclude that

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 5\iint_S d\sigma = 5\pi$$

In Exercises 65 and 66, prove the identity if f is a real-valued function and **v** is a vector-valued function.

▹ Let $\mathbf{v} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$.

65.
$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= D_x(fp) + D_y(fq) + D_z(fr) = f_xp + fp_x + f_yq + fq_y + f_zr + fr_z \\ &= (f_xp + f_yq + f_zr) + f(p_x + q_y + r_z) = (\nabla f)\cdot\mathbf{v} + f(\nabla\cdot\mathbf{v})\end{aligned}$$

66.
$$\begin{aligned}\nabla\times(f\mathbf{v}) &= \nabla\times(fp\mathbf{i} + fq\mathbf{j} + fr\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fp & fq & fr \end{vmatrix} = \left[\frac{\partial}{\partial y}(fr) - \frac{\partial}{\partial z}(fq)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(fr) - \frac{\partial}{\partial z}(fp)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(fq) - \frac{\partial}{\partial y}(fp)\right]\mathbf{k} \\ &= [f_yr - f_zq]\mathbf{i} - [f_xr - f_zp]\mathbf{j} + [f_xq - f_yp]\mathbf{k} + [fr_y - fq_z]\mathbf{i} - [fr_x - fp_z]\mathbf{j} + [fq_x - fp_y]\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ p & q & r \end{vmatrix} + f\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ p & q & r \end{vmatrix} = \nabla f\times\mathbf{v} + f(\nabla\times\mathbf{v})\end{aligned}$$